Tearing Mode Interaction with Drift Wave Turbulence

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Motivation

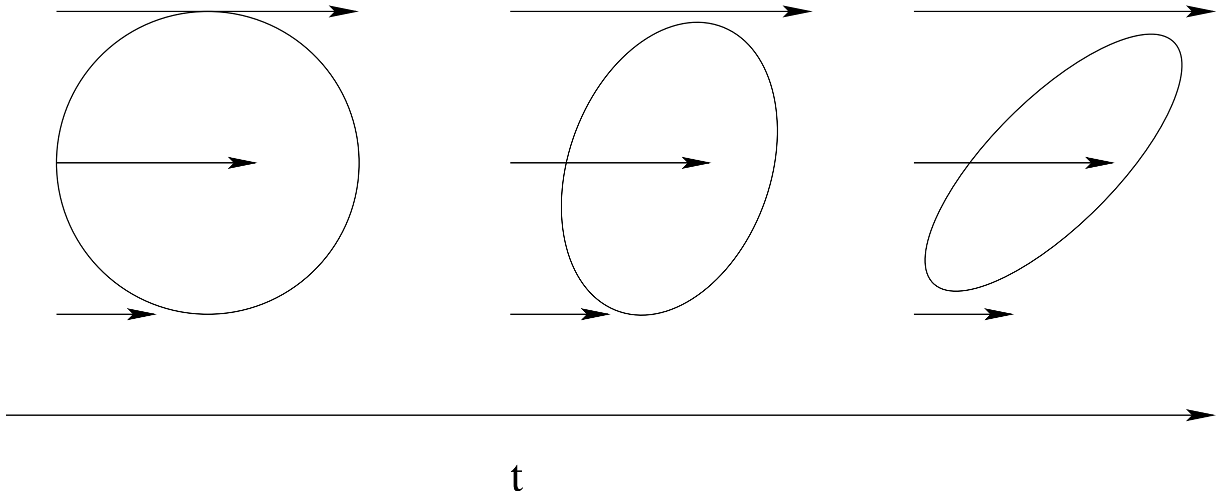
- Existing studies of drift wave turbulence have demonstrated strong non-local interactions with large scale structures
- The evolution of the large scale structure is dictated by *both* stresses exerted on the large scales by drift wave turbulence, as well as shearing of turbulence via large scale flows
- Description necessitates treating evolution of *both* large scales and small scales on equal footing
- Here we seek a *minimal* self-consistent description of the coupled evolution of a tearing mode with drift wave turbulence

Outline

- Brief Overview
- Formulation:
  - Wave kinetics and adiabatic theory
  - Mean field equations for large scales
- Linear Theory:
  - Tearing mode in presence of background of drift waves
- Summary

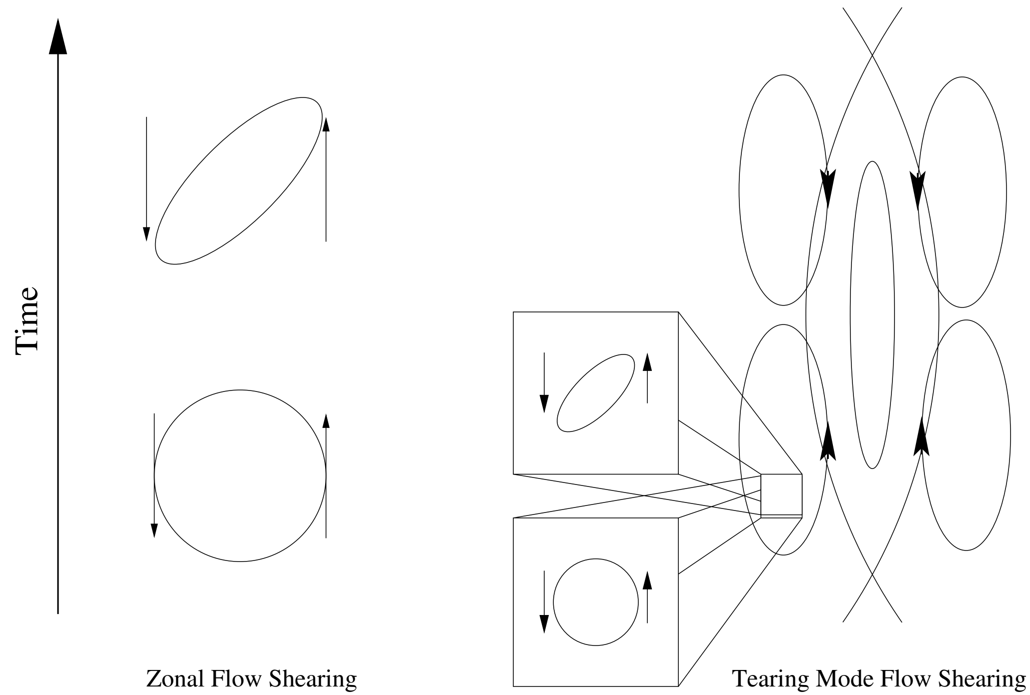
Drift Wave Zonal Flow Interaction

- Zonal flows are a secondary instability driven via a nonlocal cascade of energy from small scale drift wave turbulence
- Zonal flow generation can be understood via modulational instability



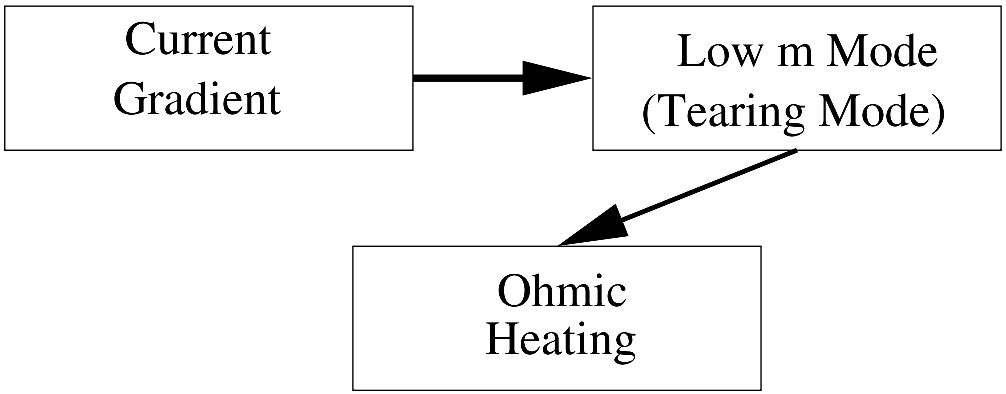
- Drift waves enhance initial seed shear flow leading to instability

Tearing Mode Interaction w/ Drift Waves (I)

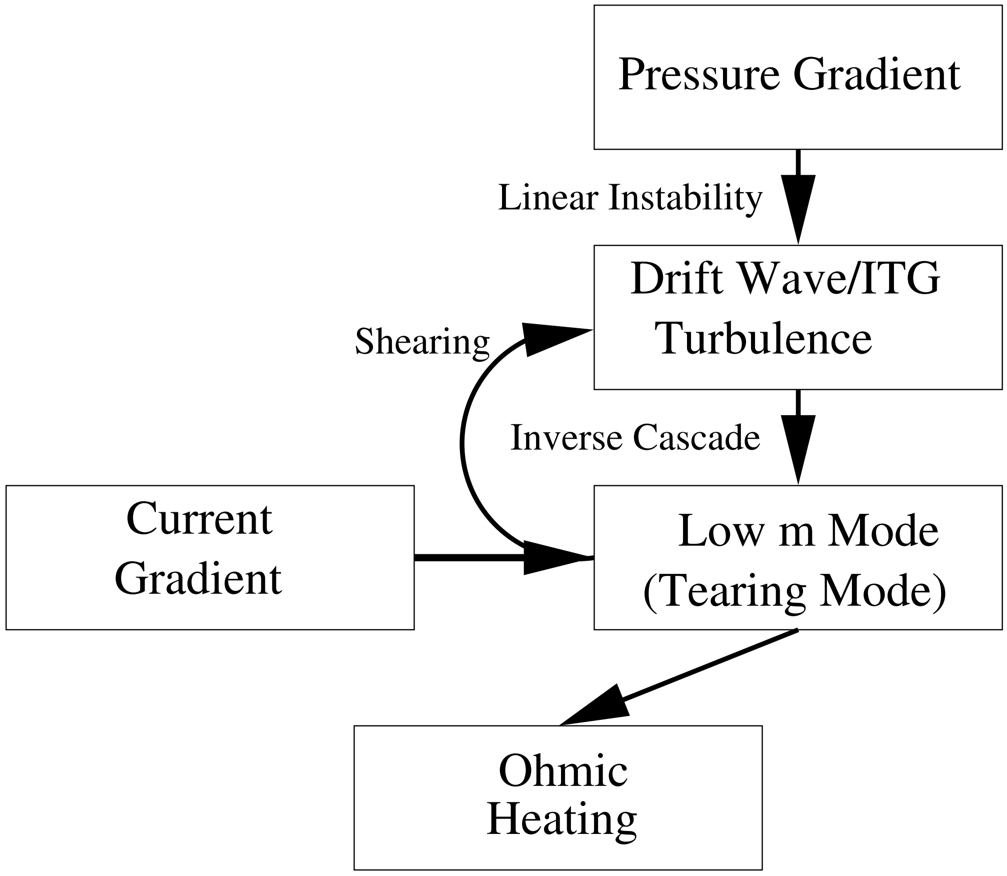


- Velocity shear from low-m mode similar to drift wave-zonal flow interaction
- Expect drift waves to enhance shear flow induced by tearing, i.e. modulational instability.

Tearing Mode Interaction w/ Drift Waves (II)



Tearing Mode Interaction w/ Drift Waves (II)



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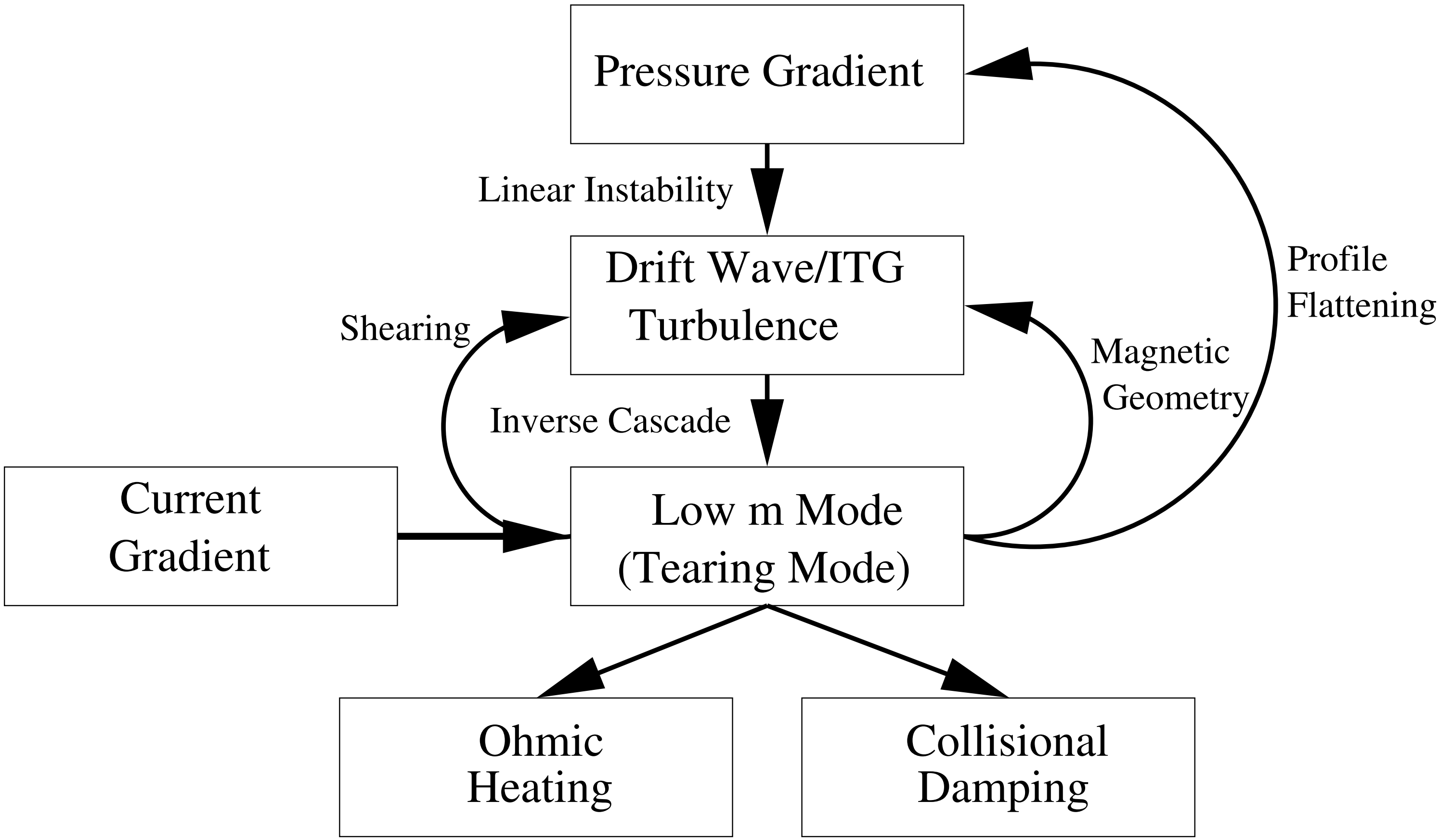
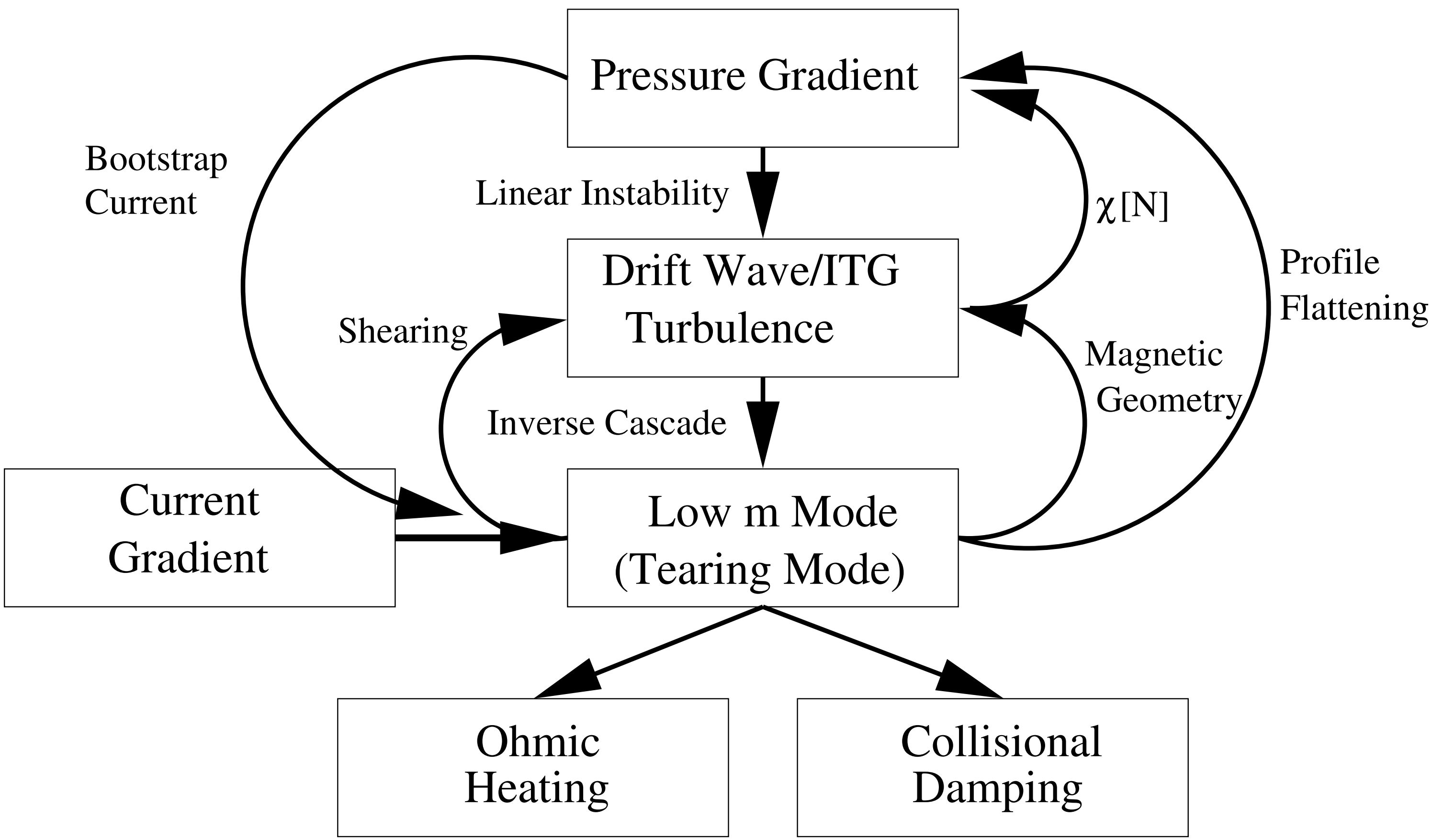


Tearing Mode Interaction w/ Drift Waves (II)



Model

- Seek *minimal* description of coupled evolution of tearing mode-drift wave dynamics within cylinder

Small Scales$\Rightarrow$Wave Kinetics

- Self-consistent evolution of intensity of drift wave turbulence
- Nonlocal interaction with large scales via shearing and advection

Large Scales$\Rightarrow$RMHD

- Provides *simple* model for large scale evolution and coupling to small scales via stresses

Wave Kinetics (I)

- We describe drift waves via the Hasegawa-Mima equation

$$0 = \left(\frac{\partial}{\partial t} + \frac{c}{B_0}\left(\hat{\mathbf{z}} \times \nabla \phi^<\right) \cdot \nabla\right) \frac{e\phi^>}{T_e} + v_e^* \frac{\partial}{\partial y} \frac{e\phi^>}{T_e} - \rho_s^2 \left(\frac{\partial}{\partial t} + \frac{c}{B_0}\left(\hat{\mathbf{z}} \times \nabla \phi^<\right) \cdot \nabla\right) \nabla_\perp^2 \frac{e\phi^>}{T_e}$$

Tools to simplify analytic calculation:

- Spatial separation between drift wave turbulence and tearing mode fields. Built in expansion parameter

- Slow temporal evolution of mean fields versus drift waves. Facilitates use of adiabatic theory.

Thus, seek description of drift wave turbulence via adiabatic varying quantity.

Wave Kinetics (II)

- Here we are interested in the evolution of the spatially modulated intensity of drift wave turbulence $|\phi_k^>|^2(\mathbf{x},t)$
- Thus, useful to write Hasegawa-Mima equation in terms of Wigner functions defined by:

$$I_k(\mathbf{x},t) \equiv \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} \left\langle \phi_{k+q}^> \phi_{-k}^> \right\rangle$$

- After expanding in $|\mathbf{q}|/|\mathbf{k}| \ll 1$, a conservation law for the drift wave enstrophy density can be derived (**?**)

$$\frac{\partial}{\partial t}N_k + \overbrace{\frac{\partial}{\partial \mathbf{k}}(\omega_k + \mathbf{k}\cdot\mathbf{V}_0)\cdot\frac{\partial}{\partial \mathbf{x}}N_k}^{\text{Spatial Dynamics}} - \overbrace{\frac{\partial}{\partial \mathbf{x}}(\omega_k + \mathbf{k}\cdot\mathbf{V}_0)\cdot\frac{\partial}{\partial \mathbf{k}}N_k}^{\text{Shearing}} = S$$

$$S = \underbrace{\gamma_k N_k}_{\text{Linear Growth}} - \underbrace{\Delta\omega N_k^2}_{\text{Nonlinear Saturation}} \quad , \quad \underbrace{N_k = \left(1+\rho_s^2 k_\perp^2\right)^2 I_k}_{\text{Drift Wave Population}}$$

- Thus, drift wave population advected and refracted by mean flows.

Wave Kinetics (III)

Interesting to consider effects of radial eigenmode structure on wave kinetic formalism

- Presence of magnetic shear at resonant surfaces introduces radial mode structure
- Thus, necessary to consider $\phi(\mathbf{x}, t)$ of the form

$$\phi(\mathbf{x}, t) = \sum_{k_y, k_z} a_{k_y, k_z}(t)\, e^{-i\omega_k t - ik_y y - ik_z z} \phi^{>}_{k_y, k_z}(x)$$

- Where $\phi^{>}_{k_y, k_z}(x)$ corresponds to a linear eigenmode located on the r_{k_y, k_z} resonant surface
- Here $a_{k_y, k_z}(t)$ represents the slowly varying amplitude of the linear eigenmodes

Wave Kinetics (IV)

- Again useful to formulate problem in terms of Wigner functions defined for the inhomogeneous case by:

$$I_{k_y,k_z}(y,t) = \sum_{q_y} e^{iq_y y} \left\langle a_{k+q}(t)\, e^{-i\omega_{k+q}t} a_{-k}(t)\, e^{-i\omega_{-k}t} \right\rangle$$

- Where $I_{k_y,k_z}(y,t)$ represents the local intensity of turbulence at a resonant surface r_{k_y,k_z}
- Following a similar procedure as in the homogenous case (**?**; **?**), a W.K.E. can be derived, describing the evolution of the wave action in the presence of large scale flows

Mean Flows (I)

- Here we consider mean flow equations, interacting with electrostatic micro turbulence

- Note that for electrostatic turbulence, coupling to micro turbulence is primarily through the polarization nonlinearity

$$0 = \frac{\partial}{\partial t}\psi^{<} + \frac{c}{B_0}\left(\hat{\mathbf{z}} \times \nabla\phi^{<}\right) \cdot \nabla\psi^{<} - v_A \frac{\partial}{\partial z}\phi^{<} - \eta_c \nabla_\perp^2 \psi^{<}$$

$$\begin{aligned} 0 \ &= \ \frac{\partial}{\partial t}\nabla_\perp^2\phi^{<} + \frac{c}{B_o}\left(\hat{\mathbf{z}} \times \nabla\phi^{<}\right) \cdot \nabla\nabla_\perp^2\phi^{<} - v_A \frac{\partial}{\partial z}\nabla_\perp^2\psi^{<} \\ &- \ \frac{c}{B_0}\left(\hat{\mathbf{z}} \times \nabla\psi^{<}\right) \cdot \nabla\nabla_\perp^2\psi^{<} - \underbrace{\left\langle\left(\hat{\mathbf{z}} \times \nabla\phi^{>}\right) \cdot \nabla\nabla_\perp^2\phi^{>}\right\rangle}_{\text{Coupling to Micro Turbulence}} \end{aligned}$$

- Where the average $\langle ... \rangle$ is over fast spatial and temporal scales.

- Absence of turbulent resistivity in mean field equations results from drift waves decoupling from Alfven modes in low β plasma.

Mean Flows (II)

Integrating by parts and Fourier transforming the stress term:

$$\left\langle \left(\hat{\mathbf{z}} \times \nabla \phi^{>}\right) \cdot \nabla \nabla_{\perp}^{2} \phi^{>} \right\rangle = -\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \int d\mathbf{k} \frac{k_x k_y}{\left(1 + \rho_s^2 k_{\perp}^2\right)^2} N_k + \frac{\partial^2}{\partial x \partial y} \int d\mathbf{k} \frac{\left(k_x^2 - k_y^2\right)}{\left(1 + \rho_s^2 k_{\perp}^2\right)^2} N_k$$

- Right hand side goes to zero in absence of anisotropy of micro turbulence
- To understand response of micro turbulence we calculate response of micro turbulence to “seed” asymmetry, symbolically:

$$\left\langle \left(\hat{\mathbf{z}} \times \nabla \phi^{>}\right) \cdot \nabla \nabla_{\perp}^{2} \phi^{>} \right\rangle \sim \frac{\partial^2}{\partial x^2} \int d\mathbf{k} M\left(\mathbf{k}\right) \frac{\delta N_k}{\delta \phi^{<}} \phi^{<}$$

Closure of Drift Wave-Tearing Mode Equations

$$\delta N_k = \frac{c}{B_0}\frac{-1}{\left(\omega_q - \mathbf{q}\cdot\mathbf{v}_{gr}\right) + i\left(\gamma_q + \gamma_k\right)}\left\{q_y\frac{\partial N_k^0}{\partial x} - i\left(\mathbf{k}\times\mathbf{q}\right)_z \mathbf{q}\cdot\frac{\partial N_k^0}{\partial \mathbf{k}}\right\}\phi^<$$

- Shearing term dominates for $q_x^2 > q_y/L_I$. ω_q and γ_q correspond to large scale inverse time scales. Hence, can be dropped in comparison to the fast (small) scales

$$\begin{aligned}\left\langle\left(\hat{\mathbf{z}}\times\nabla\phi^>\right)\cdot\nabla\nabla_\perp^2\phi^>\right\rangle \;&\approx\; -c_s^2\int d\mathbf{k}\frac{\rho_s^2k_y^2}{\left(1+\rho_s^2k_\perp^2\right)^2}\frac{\gamma_k}{\left(\gamma_k^2+\left(\mathbf{q}\cdot\mathbf{v}_{gr}\right)^2\right)}k_x\frac{\partial N_k^0}{\partial k_x}\frac{\partial^4\phi^<}{\partial x^4}\\ &=\; -\nu_T\frac{d^4\phi^<}{dx^4}\end{aligned}$$

- Thus, for $k_x\frac{dN_k^0}{dk_x} < 0$:

$$\nu_T < 0$$

Magnitude of Anomalous Viscosity

- We estimate the magnitude of ν_T via a mixing length argument
- For drift waves we estimate $\ell_m \sim \rho_s$, roughly the peak of the linear growth rate, yielding:

$$|\nu_T| \approx \frac{\rho_s}{L_n}\omega_{ci}\rho_s^2 = D_{GB}$$

- Comparing strength of inertial term within tearing mode equations with viscous piece

$$|\nu_T|\,\nabla_\perp^2\nabla_\perp^2\phi \sim D_{GB}\nabla_\perp^2\nabla_\perp^2\phi > \gamma_T\nabla_\perp^2\phi$$

$$\Rightarrow \underbrace{(\omega_{ci}\tau_\eta)}_{O(10^{10})}\underbrace{(\rho_s/a)^2}_{O(10^{-6})} > \underbrace{(L_s/\rho_s)\,(1/S)^{2/5}}_{O(10)}\underbrace{(L_s/am)^{2/5}\,(\Delta' a)^{6/5}}_{O(1)}$$

- Thus, for practical purposes turbulent negative viscosity nearly always dominates inertia

Tearing Mode Equations

- Considering the limit where $\partial/\partial x \gg \partial/\partial y$, the linearized tearing mode equations are given by:

$$\overbrace{\gamma_q \frac{\partial^2 \phi^<}{\partial x^2} = i q_y v_A \frac{x}{L_s} J + \nu_T \frac{\partial^4 \phi^<}{\partial x^4}}^{\text{Vorticity Equation}}$$

$$\underbrace{\eta_c J = \gamma_q \psi^< - i q_y v_A \frac{x}{L_s} \phi^<}_{\text{Ohm's Law}}$$

Useful simplification:

- for $\gamma_T \tau_\eta^{(T)} < 1$, magnetic field is able to diffuse into visco-resistive layer. Thus can approximate $\psi^< \rightarrow \psi_0 = \text{const}$

Tearing Mode (I)

- Leads to following set of interior equations (in dimensionless units)

$$0 = -\frac{\partial^4 \Phi}{\partial \sigma^4} - \frac{1}{\alpha}\frac{\partial^2 \Phi}{\partial \sigma^2} + \sigma\left(1 + \sigma\Phi\right) \qquad \Delta' = -\frac{i\omega_q}{\eta_c} x_\nu \int d\sigma\left(1 + \sigma\Phi\right)$$

$$\alpha = i\left|\nu_T\right| / \left(x_T^2 \omega_q\right) \qquad \Delta' = \left(\psi'\left(0^+\right) - \psi'\left(0^-\right)\right)/\psi_0$$

- Further simplification:
  - for $|\alpha| \gg 1$, eigenmode equation reduces to

$$0 = -\frac{\partial^4 \Phi}{\partial \sigma^4} + \sigma\left(1 + \sigma\Phi\right)$$

Useful to examine asymptotic behavior of eigenmode equation

Tearing Mode (II)

- Structure of equation can easily be seen to lead to formation of strongly oscillating solutions.
- Motivates looking for solutions of the form:

$$\Phi(\sigma) = \overbrace{f(\sigma)}^{\text{slowly varying}} \cdot \overbrace{e^{i\psi(\sigma)}}^{\text{rapidly varying}} - \overbrace{1/\sigma}^{\text{algebraic decay}}$$

- Plugging in and taking derivatives, gives for the real and imaginary pieces:

$$0 = -f(\psi')^4 + \sigma^2 f$$

$$0 = 4f'(\psi')^3 + 6f(\psi')^2\psi''$$

Tearing Mode (III)

Solving:

$$\Phi(\sigma) = sgn(\sigma)\frac{D}{|\sigma|^{3/4}}exp\left(i\frac{2}{3}|\sigma|^{3/2} + i\phi_D\right) + sgn(\sigma)\frac{E}{|\sigma|^{3/4}}exp\left(-i\frac{2}{3}|\sigma|^{3/2} + i\phi_E\right) - \frac{1}{\sigma}$$

- Thus, oscillating portion of solution dies off slower than $-1/\sigma$, hence will not be able to effect match with exterior
- Can formulate problem more conveniently in Fourier space

$$\frac{d^2\Phi(q_x)}{dq_x^2} - sgn(\nu_{xx})\, q_x^4\Phi(q_x) = 2\pi i\frac{d}{dq_x}\delta(q_x)$$

$$\Delta' = -\frac{i\omega_q}{\eta_c}x_\nu\left(2\pi\delta(q_x) + i\frac{d\Phi(q_x)}{dq_x}\,|_{q_x=0}\right)$$

Tearing Mode (IV)

- Convenient to introduce solution of the form (**?**)

$$\Phi\left(q_x\right) = i\pi sgn\left(q_x\right)\frac{\Phi_{hom}\left(\left|q_x\right|\right)}{\Phi_{hom}\left(0\right)}$$

- Leads to simplified eigenvalue and eigenmode equations:

$$\Delta' = -i\pi\frac{\omega_q}{\eta_c}x_\nu\frac{1}{\Phi_{hom}\left(0\right)}\frac{d\Phi_{hom}}{dq_x}\Big|_{q_x=0}$$

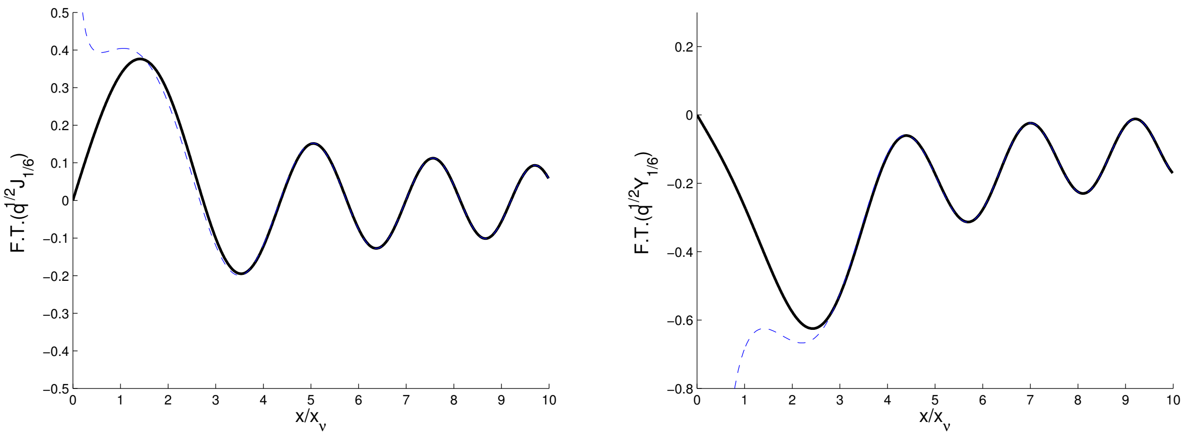
$$\frac{d^2\Phi_{hom}\left(q_x\right)}{dq_x^2} + q_x^4\Phi_{hom}\left(q_x\right) = 0$$

- Solution to homogeneous equation:

$$\Phi_{hom}\left(q_x\right) = A\sqrt{q_x}J_{\frac{1}{6}}\left(\frac{q_x^3}{3}\right) + B\sqrt{q_x}Y_{\frac{1}{6}}\left(\frac{q_x^3}{3}\right)$$

Tearing Mode (V)

- In real space the solutions have the form



Note that oscillations are $\pi/2$ out of phase. Hence, setting amplitudes A and B corresponds to setting the phase of oscillations of solution. Can't be set by exterior solution

Tearing Mode (VI)

- The dispersion relation can now be written

$$\Delta' = -i\frac{\pi}{2}\frac{1}{6^{1/6}}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{7}{6}\right)}\frac{\omega_q}{\eta_c}x_\nu\left(\frac{3^{1/3}}{2^{1/6}}+\frac{1}{6^{1/6}}\frac{A}{B}\right)$$

- In order to control the oscillations induced by the negative viscosity, it is necessary to impose a condition on the wave energy flux
- The only physically plausible condition is outgoing wave boundary conditions
- Matching the solution to the asymptotic solutions leads to A/B being pure imaginary
- Thus, a real frequency is self-consistently induced, allowing for wave absorption in the exterior region

Tearing Mode (VII)

- In order to calculate the ratio A/B, necessary to match the exact solution the outgoing

$$\tilde{\Phi}\left(\sigma\right) = sgn\left(\sigma\right)\frac{D}{\left|\sigma\right|^{3/4}}e^{ik_x|x|+i\phi_D} + sgn\left(\sigma\right)\frac{E}{\left|\sigma\right|^{3/4}}e^{-ik_x|x|+i\phi_E}\ , \tag{1}$$

- where k_x is defined as $k_x = (2/3)\sqrt{|x|}/x_\nu^{3/2}$.
- k_x can be related to the frequency through the dispersion relation $Re\left(\omega_q\right) \sim \left(\eta_c \Delta'\right)/x_\nu$, which yields $k_x \sim \sqrt{|x|}\omega_q^{3/2}/\left(\eta_c \Delta'\right)^{3/2}$.
- Thus, the sign of v_{gr} can be determined from $v_{gr}^{-1} = \partial k_x/\partial \omega_q$.

$$\tilde{\Phi}\left(x\right) = sgn\left(x\right)\frac{D}{\left|\sigma\right|^{3/4}}e^{ik_x|x|+i\phi_D}\ . \tag{2}$$

Tearing Mode (VIII)

The dispersion relation is then given by:

$$\gamma_q \sim \frac{\eta_c}{x_\nu}\Delta' \sim \frac{\eta_c^{5/6}}{\left|\nu_T\right|^{1/6}}\left(\frac{q_y v_A}{L_s}\right)^{1/3}\Delta'$$

$$Re\left(\omega_q\right) \sim \frac{\eta_c^{5/6}}{\left|\nu_T\right|^{1/6}}\left(\frac{q_y v_A}{L_s}\right)^{1/3}\Delta'$$

- Requires $\Delta' > 0$ for instability
- Modes are not strongly localized to visco-resistive layer

Summary

- Self-consistent formulation of interaction of a tearing mode with drift wave turbulence
- Identification of the negative turbulent viscosity as the dominant effect on low-m tearing mode
- Calculation of linear growth rate of tearing mode in the presence of negative viscosity