Extended MHD Benchmarking and Validation

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Important Target Problems

- Non-linear ELM evolution
- Neo-classical tearing modes/island dynamics
- Giant sawteeth
- RF/MHD coupling
- Plasma relaxation: characteristic fields and flows

All are extremely complicated and require "extended MHD"

How do we know we're getting the "right" answer?

Extensions to Resistive MHD

- Anisotropic heat flux
- 2-fluid Ohm's law
- Anisotropic parallel viscosity
- Ion gyro-viscosity*
- Neo-classical stress tensor for ions* and electrons*
- Energetic ion species

**There is no general agreement on the form of several of these terms*

How do we know we're getting the "right" answer?

Algorithms are Complicated

Momentum

$$m_i n^{j+1/2}\left(\frac{\Delta \mathbf{V}}{\Delta t} + \boxed{\underbrace{\frac{1}{2}\mathbf{V}^{j}\cdot\nabla\Delta\mathbf{V} + \frac{1}{2}\Delta\mathbf{V}\cdot\nabla\mathbf{V}^{j}}_{\textit{Implicit advection}}}\right) - \boxed{\underbrace{\Delta t L^{j+1/2}(\Delta\mathbf{V})}_{\textit{SI MHD}}} + \boxed{\underbrace{\nabla\cdot\Pi_i(\Delta\mathbf{V})}_{\textit{Includes ALL stresses}}} = \mathbf{J}^{j+1/2}\times\mathbf{B}^{j+1/2} - m_i n^{j+1/2}\mathbf{V}^{j}\cdot\nabla\mathbf{V}^{j} - \nabla p^{j+1/2} - \nabla\cdot\Pi_i\left(\mathbf{V}^{j}\right)$$

Continuity

$$\frac{\Delta n}{\Delta t} + \boxed{\frac{1}{2}\mathbf{V}^{j+1}\cdot\nabla\Delta n} = -\nabla\cdot\left(\mathbf{V}^{j+1}\cdot n^{j+1/2}\right)$$

Energy

$$\frac{3n}{2}\left(\frac{\Delta T_\alpha}{\Delta t} + \boxed{\frac{1}{2}\mathbf{V}_\alpha^{j+1}\cdot\nabla\Delta T_\alpha}\right) + \boxed{\frac{1}{2}\underbrace{\nabla\cdot\mathbf{q}_\alpha(\Delta T_\alpha)}_{\textit{Anisotropic thermal conduction}}} = -\frac{3n}{2}\mathbf{V}_\alpha^{j+1}\cdot\nabla T_\alpha^{j+1/2} - nT_\alpha^{j+1/2}\nabla\cdot\mathbf{V}_\alpha^{j+1} - \nabla\cdot\mathbf{q}_\alpha\left(T_\alpha^{j+1/2}\right) + Q_\alpha^{j+1/2}$$

Maxwell/Ohm

$$\frac{\Delta \mathbf{B}}{\Delta t} + \boxed{\frac{1}{2}\mathbf{V}^{j+1}\cdot\nabla\Delta\mathbf{B}} + \boxed{\underbrace{\frac{1}{2}\nabla\times\frac{1}{ne}\left(\mathbf{J}^{j+1/2}\times\Delta\mathbf{B} + \Delta\mathbf{J}\times\mathbf{B}^{j+1/2}\right)}_{\textit{Implicit HALL term}}} + \boxed{\underbrace{\frac{1}{2}\nabla\times\eta\Delta\mathbf{J}}_{\textit{Implicit resistive term}}} = -\nabla\times\left[\frac{1}{ne}\left(\mathbf{J}^{j+1/2}\times\mathbf{B}^{j+1/2} - \nabla p_e\right) - \mathbf{V}^{j+1}\times\mathbf{B}^{j+1/2} + \eta\mathbf{J}^{j+1/2}\right]$$

How do we know we're getting the "right" answer?

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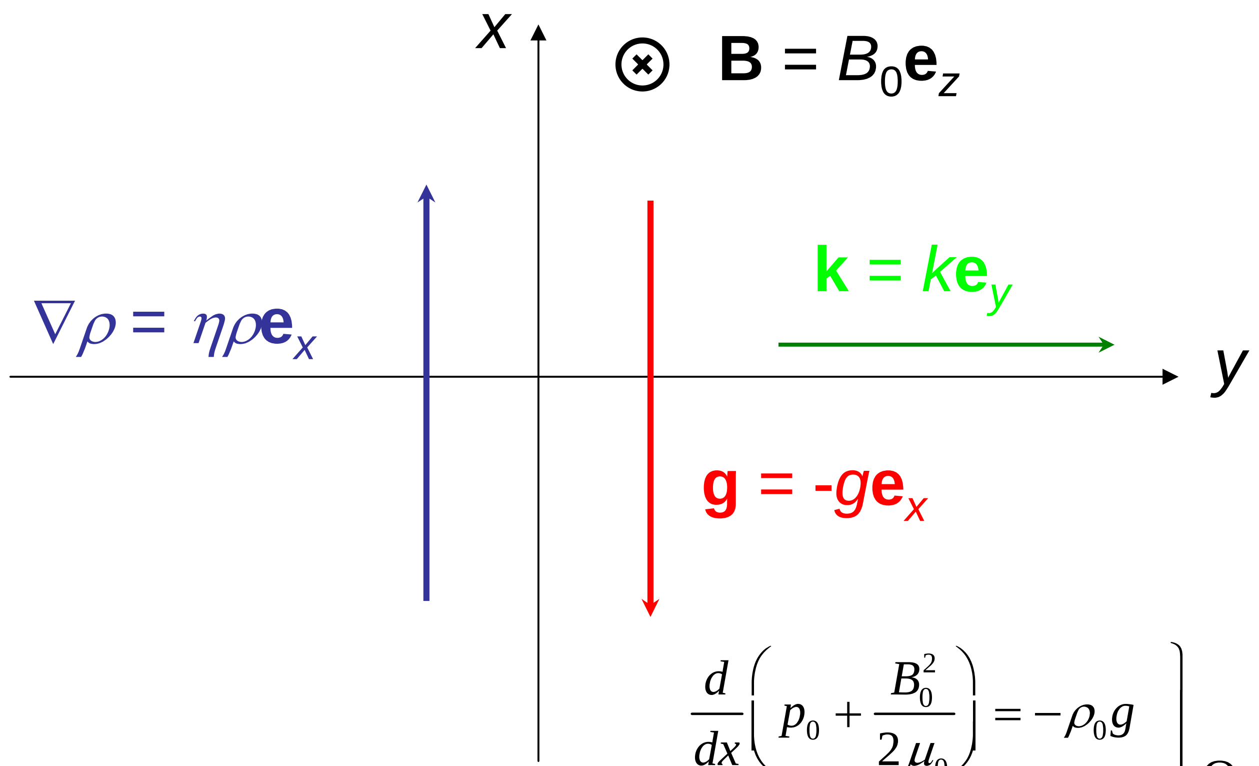
Need Simple Problems with Known Solutions

- Simple geometry, but capture essential physics
- Analytic solution preferred, but independent numerical results useful
- Start simple => add complications
- Linear is good, non-linear better (but attainable?)
- *Need help from theory*

Extended MHD Validation Problems

• *g*-mode interchange in a slab (Rayleigh-Taylor-Parker-Roberts-Taylor)	• MHD • 2-fluid stabilization • Gyro-viscous stabilization
• Collisional drift waves in a slab (Coppi, et al.)	• 2-fluid terms (Hall) • Collisional effects • Stability thresholds
• GEM reconnection problem (slab)	• 2-fluid reconnection • Comparison with MHD • Well documented numerical results • Non-linear
• Critical island width for temperature flattening (Fitzpatrick)	• Anisotropic thermal conduction
• Destabilization of neo-classical tearing mode (Gianakon, Kruger, Hegna)	• Models for neo-classical closures • Linear
• Kink stabilization by energetic particles (Cheng, Fu, Kim)	• Energetic particle ion closures schemes • Linear • Numerical results

Example: *g*-mode Stability (*Roberts and Taylor*, 1963)



$$\left.\begin{aligned} &\frac{d}{dx}\left(p_0+\frac{B_0^2}{2\mu_0}\right) = -\rho_0 g \\ &\frac{d\rho_0}{dx} = \eta\rho_0 \quad , \quad \eta \equiv 1/L_n \end{aligned}\right\} \text{Only } p_T = p_0 + \frac{B_0^2}{2\mu_0} \text{ matters}$$

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2-fluid/Gyro-viscous Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \rho \mathbf{g} - \nabla \cdot \Pi$$

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{M}{\rho e} \left[\rho \frac{d\mathbf{V}}{dt} + \nabla p_i - \rho \mathbf{g} + \nabla \cdot \Pi \right]$$

$$\Pi_{xx} = -\Pi_{yy} = -\rho \nu \left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \right)$$

$$\Pi_{xy} = \Pi_{yx} = \rho \nu \left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} \right)$$

$$\nu = \rho_i{}^2 \Omega / 2 \qquad \rho_i^2 = V_{th}^2 / \Omega^2$$

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Simplifying Assumptions

Only variations in $p_T = p + B^2 / 2\mu_0$ affect dynamics

$\Rightarrow$ Ignore perturbations to B

$\Rightarrow$ $\nabla \times \mathbf{E} = 0$ (low β, electrostatic)

Assume ions are barotropic, $p_i = p_i(\rho)$

$\Rightarrow$ Simplifies Ohm's law

Variation in x much weaker than variation in y

$\Rightarrow \eta^2 << k^2$

$\Rightarrow$ Can ignore explicit x-dependence of equilibrium

Assume $\exp(i\omega t + iky)$ dependence

$\Rightarrow$ Linearized equations are algebraic

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Final *g*-mode Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p_T + \rho \mathbf{g} - \nabla \cdot \Pi$$

$$\nabla \cdot \mathbf{V} + \frac{1}{\Omega} \nabla \times \left[\frac{d\mathbf{V}}{dt} - \frac{1}{\rho^2} \nabla \rho \times \nabla \cdot \Pi \right] = 0$$

Plus definition of Π

4 equations in 4 unknowns: $\rho, \mathbf{V}, p_T$

Last equation serves as "equation of state", or closure

Stability Results

$$\omega^2 - \omega_* \omega + \gamma_{MHD}^2 = 0$$

$$\omega_* = \omega_{*2F} + \omega_{*GV}$$

$$\omega_{*2F} = \frac{gk}{\Omega} \qquad \omega_{*GV} = \frac{1}{2}\frac{\rho_i^2 k^2}{kL}\Omega \qquad \gamma_{MHD}^2 = \frac{g}{L} + \underbrace{\frac{g^2}{V_A^2 + C_s^2}}_{\text{Compressible Correction}}$$

$$\omega = \frac{1}{2}\left(\omega_* \pm \sqrt{\omega_*^2 - 4\gamma_{MHD}^2}\right)$$

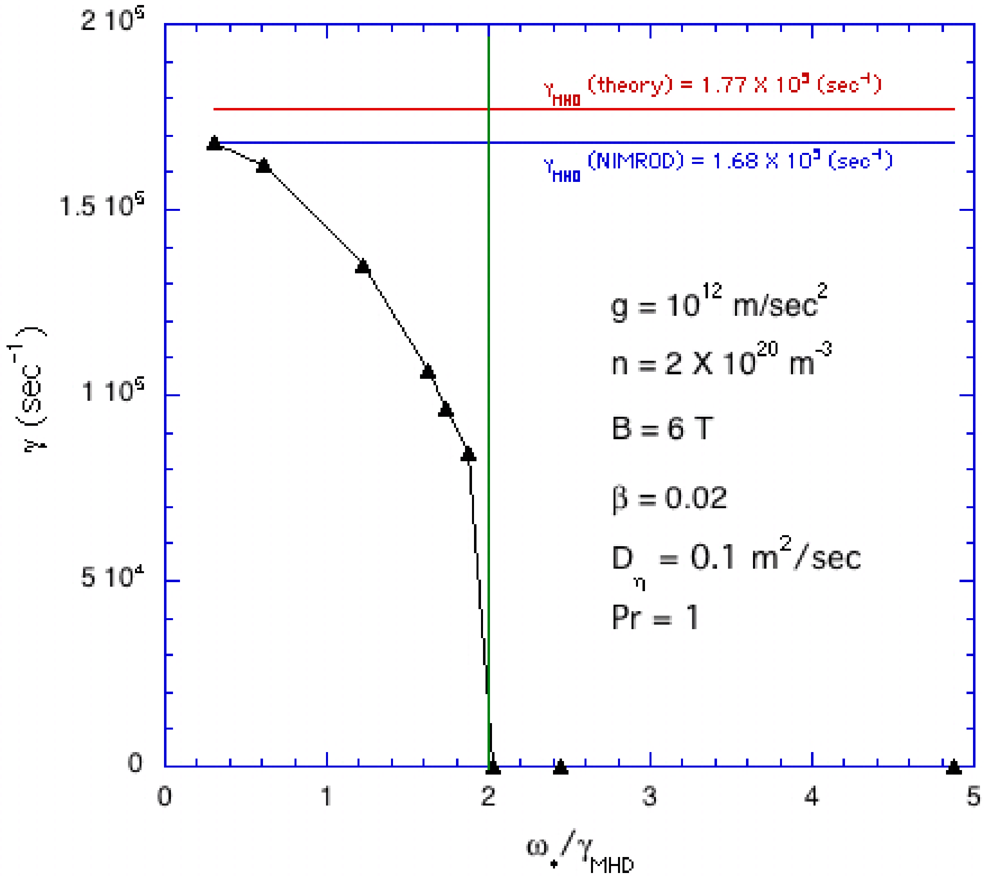
Stable if: $\omega_* > 2\gamma_{MHD}$

2-fluid *g*-mode in NIMROD

- Validation of NIMROD on *g*-mode problem
- 2-fluid only
- Fully compressible
- Walls placed far away
- Get good agreement with theory on both 2-fluid stability threshold and MHD growth rate
- Found heuristic time step CFL limit:

  $\omega_{*MAX}\Delta t < 1/4$

- Still working on GV validation



Problem can be "Extended"

• Add transverse component of magnetic field (B_y)	• $k_{\parallel}$ effects • Stabilization • Whistlers and KAWs
• Add sheared transverse field ($B_y(x)$)	• Mode localization
• Move walls closer	• Boundary conditions (not trivial for 2-fluid model)
• Scaling with resistivity	• 2-fluid effects on resistive *g*-modes

Need analytic solutions of these problems

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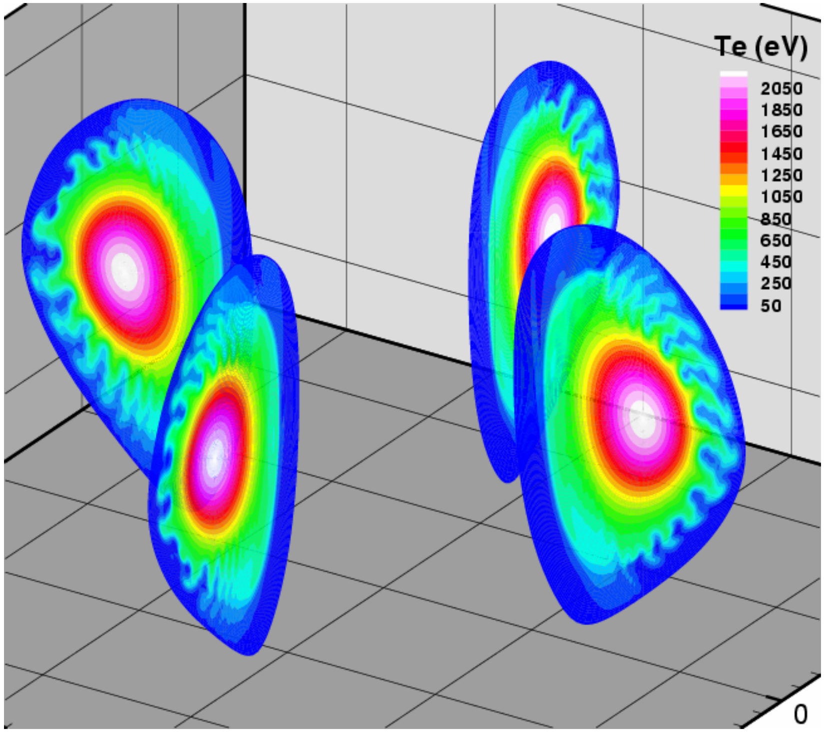
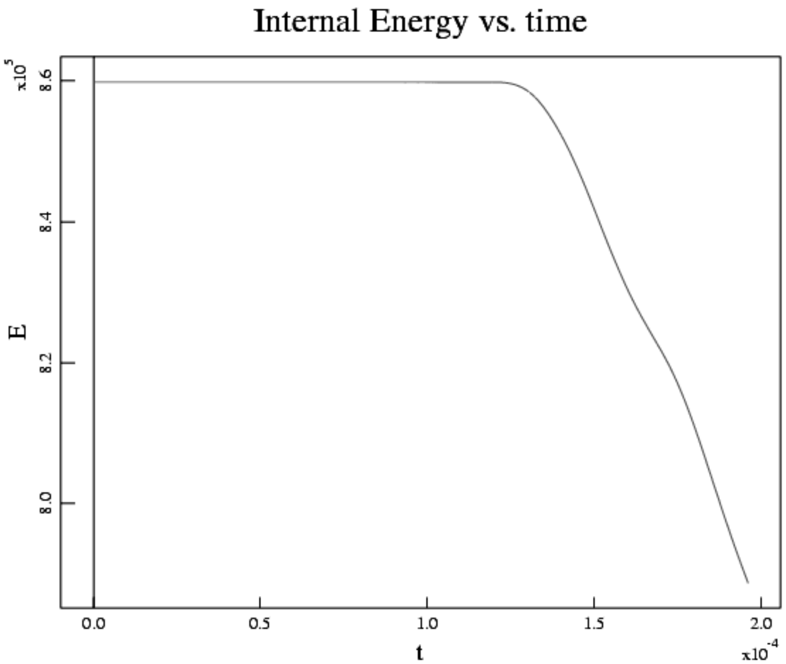
Nonlinear ELM Evolution
(Where we'd like accurate 2-fluid models)





- Resistive MHD
- Anisotropic thermal conduction
- ELM interaction with wall

- 70 kJ lost in 60 μsec

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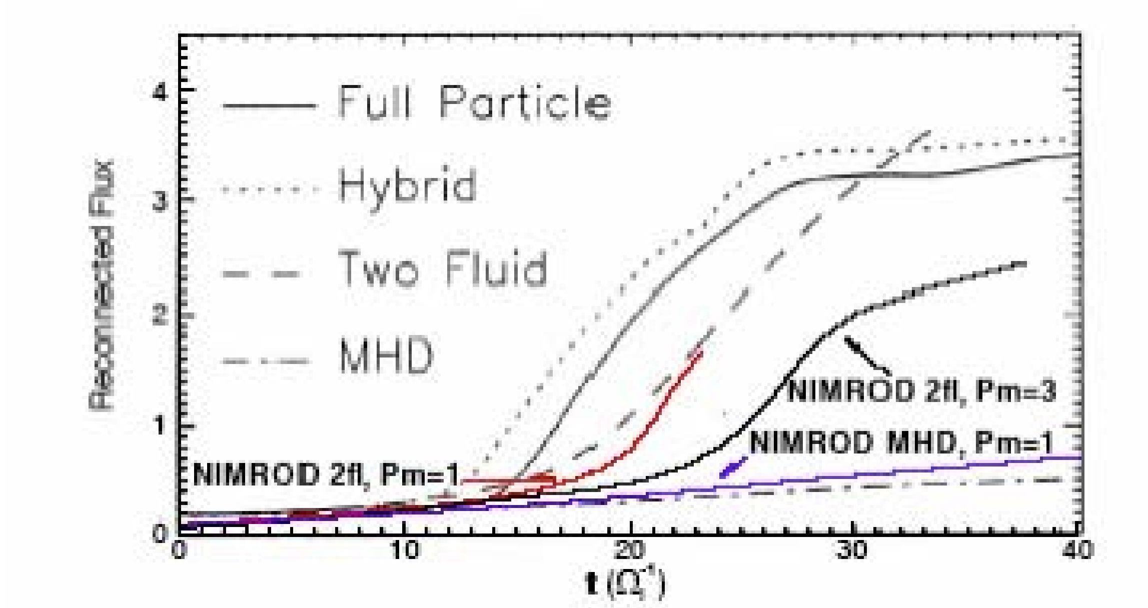
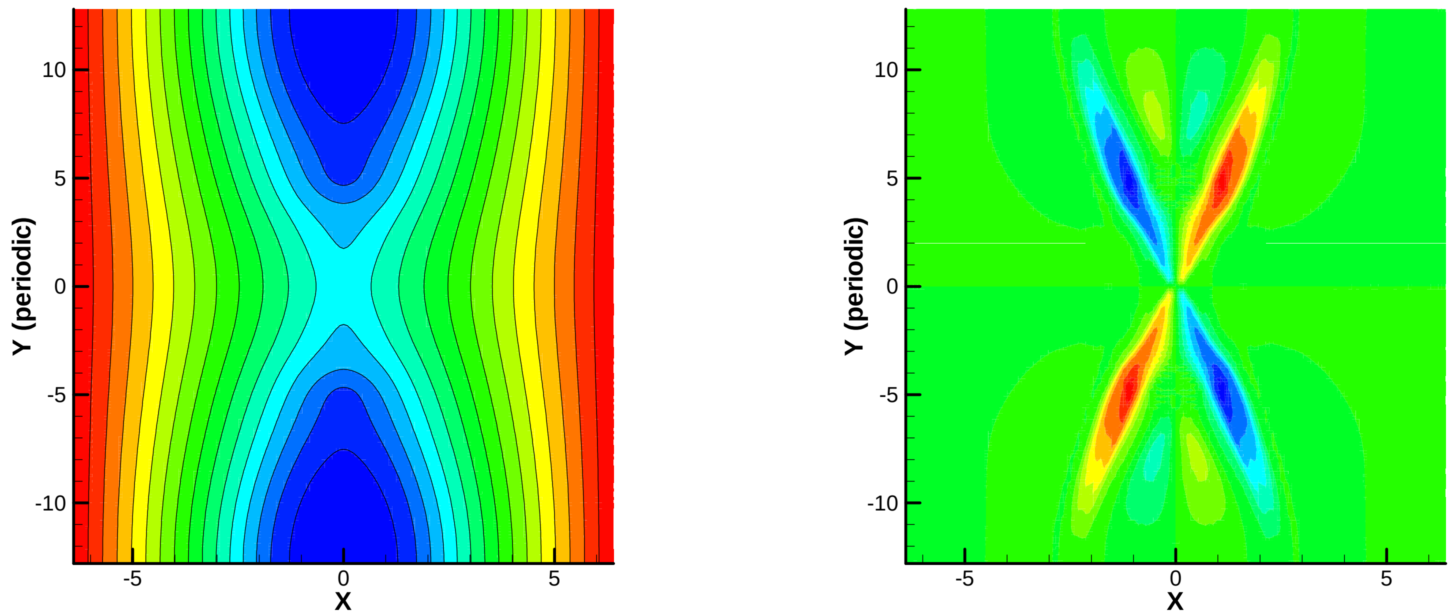
Two-fluid Reconnection
GEM Problem



- 2-D slab
- $\eta = 0.005$
- Good agreement with many other calculations
- Computed with same code used for tokamaks, spheromaks, RFPs

The NIMROD Hall-MHD computation shows important characteristics of two-fluid reconnection.



The characteristic results from t=23 Ω^{-1} are the open geometry of the reconnecting magnetic flux (left) and the quadrupole out-of-plane magnetic field (right).

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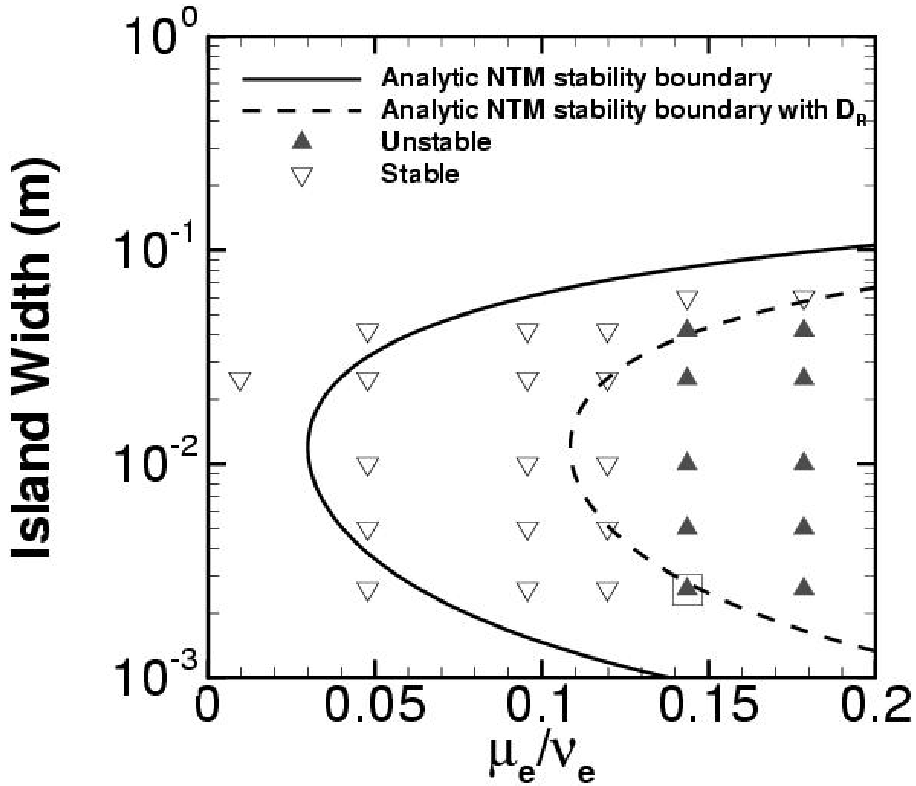
Test of "Heuristic Closure" for Neoclassical Physics



(Gianakon et.al., Phys. Plasmas **9**, 536 (2002)

Neo-classical theory gives flux surface average

Local form for stress tensor forces:

$$\nabla \cdot \Pi_\alpha = \rho_\alpha \mu_\alpha \left\langle B^2 \right\rangle \frac{\mathbf{V}_\alpha \cdot \mathbf{e}_\theta}{\left(B_\alpha \cdot \mathbf{e}_\theta\right)^2} \mathbf{e}_\theta$$

- Valid for both ion and electrons
- Energy conserving and entropy producing
- Gives:
  - bootstrap current
  - neoclassical resistivity
  - polarization current enhancement

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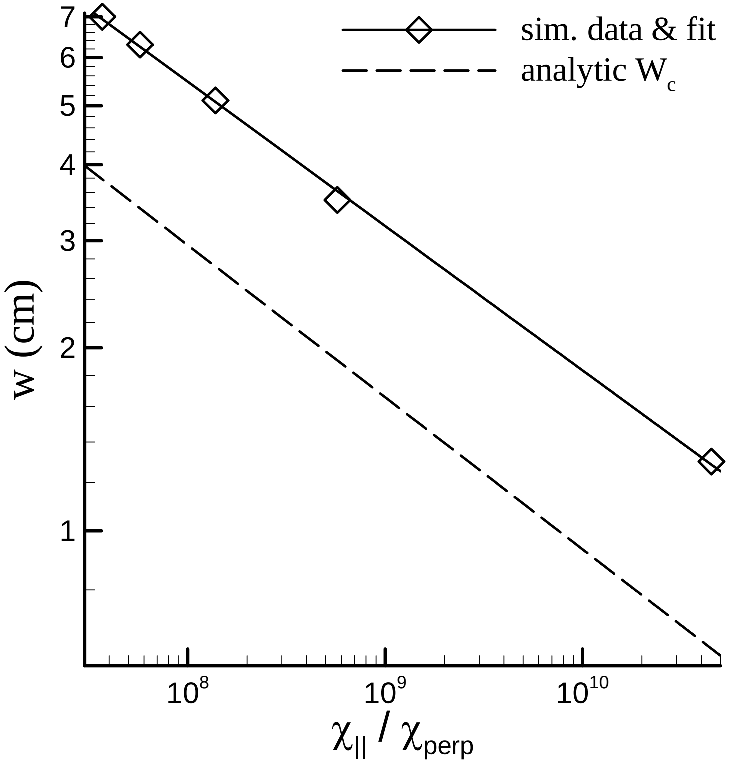
Testing Anisotropic Heat Conduction

- Critical island width for temperature flattening
- Dealing with extreme anisotropy
- Agreement on scaling (Fitzpatrick)



Beyond Extended MHD: Parallel Kinetic closures

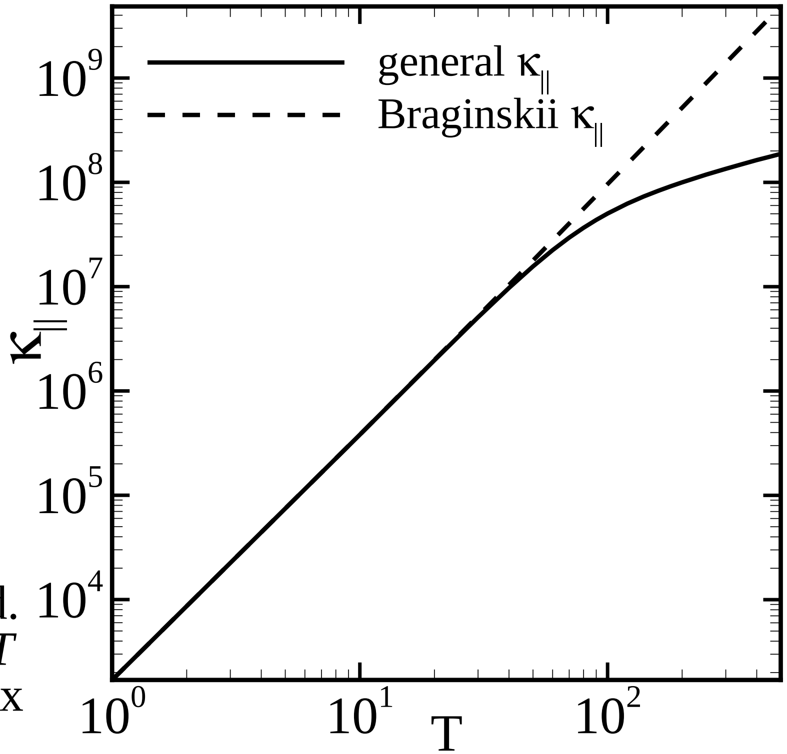
- Parallel closures for $q_{\|}$ and $\Pi_{\|}$ derived using Chapman-Enskog-like approach.

- Non-local; requires integration along perturbed field lines.

- General closures map continuously from collisional to nearly collisionless regime.

- General $q_{\|}$ closure predicts collisional response for heat flow inside magnetic island. As plasma becomes moderately collisional (T > 50 eV), general closure predicts correct flux limited response.

- Incorporated into global extended MHD algorithms.



Thermal diffusivity as function of T showing $T^{5/2}$ response of Braginskii and general closure.

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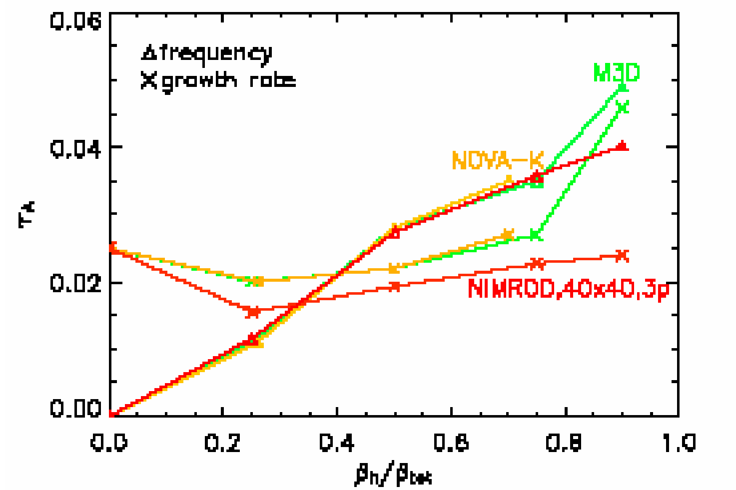
Beyond Extended MHD: Kinetic Minority Species

- Minority ions species affects bulk evolution:

$$n_h << n_0 \quad , \qquad \beta_h \sim \beta_0$$

$$Mn\frac{d\mathbf{V}}{dt} = \mathbf{J}\times\mathbf{B} - \underbrace{\nabla\cdot\Pi}_{\textit{Bulk Plasma}} - \underbrace{\nabla\cdot\Pi_h}_{\textit{Hot Minority Ion Species}}$$

$$\delta\Pi_h = \int M(\mathbf{v}-\mathbf{V}_h)(\mathbf{v}-\mathbf{V}_h)\delta f(\mathbf{x},\mathbf{v})d^3\mathbf{v}$$



- δf determined by kinetic particle simulation in evolving fields
- Demonstrated transition from internal kink to fishbone
- Benchmark of three codes

Form of the Gyro-viscous Stress (Hooke's Law for a Magnetized Plasma)

- Braginskii: $\Pi_{\wedge} = \Pi^{gv} = \dfrac{p}{4\Omega}\left[(\mathbf{b}\times\mathbf{W})\cdot(\mathbf{I}+3\mathbf{bb}) + transpose\right]$

$$\mathbf{W} = \nabla\mathbf{V} + \nabla\mathbf{V}^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{V}$$

- Suggested modifications for consistency (*Mikhailovskii and Tsypin*, *Hazeltine and Meiss*, *Simakov and Catto*, *Ramos*) involve adding term proportional to the ion heat rate of strain:

$$\Pi_{\wedge q} = \frac{2}{5\Omega}\left[\mathbf{b}\times\mathbf{W}_q\cdot(\mathbf{I}+3\mathbf{bb}) + transpose\right]$$

$$\mathbf{W}_q = \nabla\mathbf{q}_i + \nabla\mathbf{q}_i^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{q}_i$$

- Implicit numerical treatment difficult (new coupling between momentum and energy equations)
- What is the effect of this term on dispersion and stability?
  - *Does it introduce new normal modes?*
  - *Does it alter stability properties?*

Effect of Ion Heat Stress on Important Dynamics

$$\Pi_{\wedge q} = \frac{2}{5\Omega}\left[\mathbf{b}\times\mathbf{W}_q\cdot\left(\mathbf{I}+3\mathbf{bb}\right)+transpose\right]$$

$$\mathbf{W}_q = \nabla\mathbf{q}_i + \nabla\mathbf{q}_i^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{q}_i \qquad \mathbf{q} = -\kappa_{\|}\nabla_{\|}T - \kappa_{\perp}\nabla_{\perp}T - \kappa_{\wedge}\mathbf{b}\times\nabla_{\perp}T$$

$$\rho_0\frac{\partial\mathbf{V}}{\partial t} = -\nabla p - \nabla\cdot\Pi_{\wedge q}$$

$$\frac{\partial p}{\partial t} = -\gamma p_0\nabla\cdot\mathbf{V}$$

$$\omega^2 = C_s^2 k^2\left\lfloor 1+f(\theta)\left(\rho_i k\right)^2\right\rfloor \qquad f(0)=0 \qquad f(\pi/2)=1$$

- Dispersive effect on compressional waves, but……
- Negligible effect on *g*-mode stability
- Prioritization: *ignore these terms (for now!)*
- Open to counter arguments

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Theory and Computation

- Theory and computation are synergistic
  - Just different tools for solving problems
- Theory needs guidance from the needs of large scale computations
- Computations need guidance for relevant equations and expectations from theory
- Closer collaboration between theory and computation required for the success of either program
- *Attempts to promote one at the expense of the other are unwise*

Summary

- Extended MHD problems are extremely difficult and complex
- Must be studied with equally complex algorithms
- Basic processes often masked (and also influenced) by geometry
- *How do we learn to trust the computational models?*
  - Define program of simple problems with known solutions to test aspects of the models
    - Hierarchy of problems from simple to more complex
    - Require all codes to run these problems
    - Urge theorists to propose relevant test problems
  - Cross your fingers when studying non-linear tokamaks!