Princeton Plasma Physics Laboratory

Coupling a resistive wall to-M3D*

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Introduction

- The vacuum equations are intrinsically linear so that the solutions obtained with the 2 dimensional VACUUM code are still applicable for nonlinear problems *provided that the boundary conditions are still approximately linearized, and the "background" equilibrium is still approximately two-dimensional*.

- One way to perhaps accomplish this is with a "buffer zone" between the fully developed nonlinear plasma and the vacuum. In this zone would be a transition from the nonlinear regime to an approximately linearized, two-dimensional boundary outside of which the vacuum solution is valid and can be applied as outlined below to establish the outer boundary conditions. M3D or NIMROD would treat both the plasma core and the buffer zone with the VACUUM code treating the region to infinity or to a conducting shell.

- This presentation will assume that the buffer zone is bounded by a toroidally symmetric resistive shell of reasonably arbitrary poloidal cross section. The interior region is solved by the nonlinear codes which couples the toroidal harmonics. In the exterior region, the VACUUM code can be applied and the solution is diagonal in the toroidal harmonics.

Outline

- The Method of solution for the scalar potential, χ, of the magnetic field in the vacuum will be briefly described.[1]

- Then the matching across the (thin) resistive shell will be calculated. [NIMROD]

- The scalar potential only give the non-secular $(n \neq 0)$ solutions.
  - Will need the "Lüst–Martensen" terms[2] for $(n = 0)$ to complete the solution. These are essentially the Grad-Shafranov solutions for the perturbations.

Note: The M3D code already has the a version of the vacuum solutions implemented by Alex Pletzer and Hank Strauss based on the method outlined here, but the intention is to relace it with the method outlined here.

[1] M.S. Chance, Phys. Plasmas, 4(1997) 2161R.

[2] Lüst and E. Martensen, Zeitschrift für Naturforschung 15a, 706–13 (1960).

Cross-Section of the System

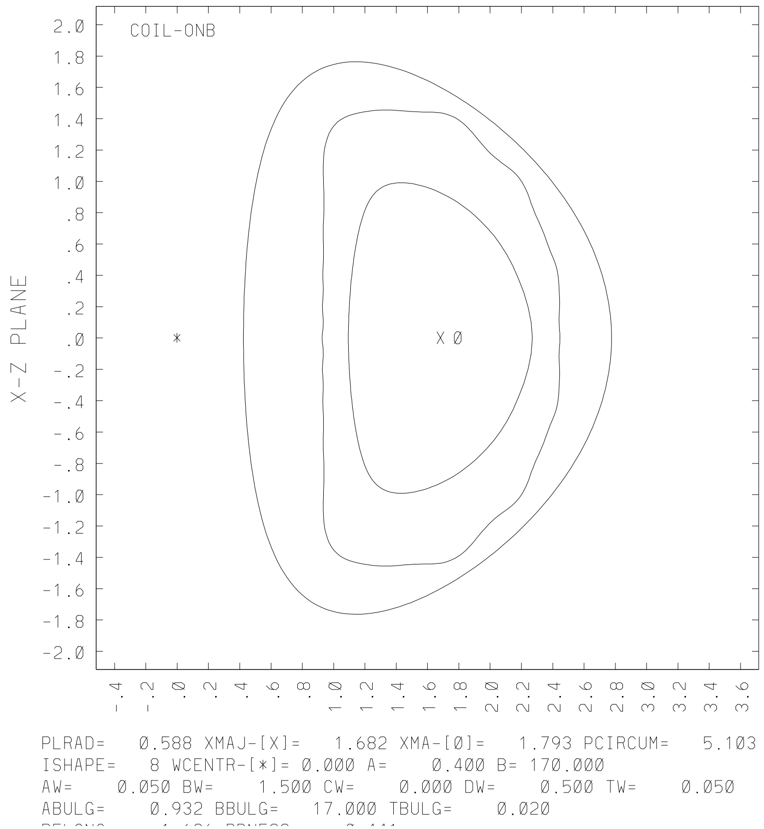

Figure 1: The Plasma, Resistive Shell and Conducting Wall Cross-sections.

Magnetic field representation in M3D

In the plasma region

$$\mathbf{B} = \nabla\psi \times \nabla\phi + \frac{1}{X}\nabla_{\perp}F + R_0 I \nabla\phi \quad (1)$$

where

$$\nabla_{\perp}F \equiv \nabla F - \nabla\phi\frac{\partial F}{\partial\phi} \quad (2)$$

The 2nd and 3rd terms are related through the divergence-free condition.

The vacuum has a somewhat similar representation for the magnetic field, $\mathbf{Q}$:

$$\mathbf{Q} = \nabla\chi + \nabla\psi \times \nabla\phi + I_v \nabla\phi \quad (3)$$

where χ is the magnetic scalar potential, satisfing $\nabla^2\chi = 0$ and will be restricted to $n \neq 0$.

The other terms are the $n = 0$ axisymmetric terms. These are the Lüst-Martensen terms and, written in this form, they satisfy the boundary requirements for their solutions. They have the form of the axisymmetric representation of the magnetic field in the vacuum and so can be solved via the Grad-Shafranov solution in a vacuum.

The solution for both χ and ψ can be solved using a Green's function technique.

The Magnetic Scalar Potential

The magnetic fields on the surfaces are represented by the scalar potential, χ, with $\mathbf{B} = \nabla\chi$. Using Green's second theorem,

$$4\pi\bar{\chi}(\mathbf{r}) + \int_S \bar{\chi}(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S}' = \int_{S_p+S_w} G(\mathbf{r},\mathbf{r}')\nabla'\bar{\chi}(\mathbf{r}')\cdot d\mathbf{S}', \tag{4}$$

where $\bar{\chi}$ satisfies $\nabla^2\bar{\chi} = 0$ in the vacuum region. $G(\mathbf{r},\mathbf{r}')$ is chosen to be the free space Green's function for the Laplacian, i.e., $\nabla^2 G(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}')$. Barred quantities here denote that the quantities contain their ϕ dependence. Writing $\bar{\chi}(\mathbf{r}) = \chi(\boldsymbol{\rho})\exp(-in\phi)$, we get

$$2\chi(\boldsymbol{\rho}) + \frac{1}{2\pi}\int_S e^{in(\phi-\phi')}\chi(\boldsymbol{\rho}')\nabla' G(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S}' = \frac{1}{2\pi}\int_{S_p+S_w} e^{in(\phi-\phi')} G(\mathbf{r},\mathbf{r}')\nabla'\chi(\boldsymbol{\rho}')\cdot d\mathbf{S}'. \tag{5}$$

The right hand side is treated as a known inhomogeneous quantity so that one can solve for χ on S.

The magnetic scalar potential, *– cont'd*

Using $d\mathbf{S}' = \mathcal{J}\nabla'\mathcal{Z}\; d\theta' d\phi'$, where $\nabla'\mathcal{Z}$ is normal to the vacuum surface in a coordinate system with $\mathcal{J} = (\nabla\mathcal{Z} \times \nabla\theta\cdot\nabla\phi)^{-1}$, one can write

$$2\chi(\boldsymbol{\rho}) + \int_C \chi(\theta')\mathcal{K}(\boldsymbol{\rho},\theta')\, d\theta' = \int_{C_p+C_w} \mathcal{G}(\boldsymbol{\rho},\theta')\mathcal{B}(\theta')\, d\theta', \tag{6}$$

The function $\mathcal{G}$ is now a two-dimensional Green's function:

$$\mathcal{G}(\boldsymbol{\rho},\theta') \equiv \frac{1}{2\pi}\oint G(\mathbf{r},\mathbf{r}')e^{in(\phi-\phi')}\, d\phi', \tag{7}$$

$$\mathcal{K}(\boldsymbol{\rho},\theta') \equiv \frac{1}{2\pi}\oint \mathcal{J}\nabla' G(\mathbf{r},\mathbf{r}')\cdot\nabla'\mathcal{Z}e^{in(\phi-\phi')}\, d\phi', \tag{8}$$

$\mathcal{B}(\theta') = \mathcal{J}\nabla'\chi\cdot\nabla'\mathcal{Z}$ can be expanded in a set of suitably chosen (orthonormal) basis functions appropriate for the source surface,

$$\mathcal{B}(\theta) = \sum_l \mathcal{B}_l\varphi_l(\theta).$$

The plasma response, $\mathcal{C}$, to the coefficients of the basis functions, $\mathcal{B}_l$ is thus

$$\chi(\theta_i) = \sum_l \mathcal{C}_l(\theta_i)\mathcal{B}_l, \tag{9}$$

Matching Across a Thin Resistive Shell - (nimrod)

Assuming a thin shell with resistivity η, and thickness Δ, Ampere's and Faraday's law yield the relations between the solutions at the inner and outer side of the shell:

Ampere's Law for the skin current, $\mathbf{K}$, across the cross section of the shell:

$$\mathbf{n} \times \langle \mathbf{Q} \rangle = \mathbf{K} \tag{10}$$

$$\text{where} \qquad \mathbf{K} = \Delta \mathbf{J}, \tag{11}$$

and $\langle \mathbf{Q} \rangle$ is the discontinuity in $\mathbf{Q}$ across the shell.

Faraday's Law:

$$\frac{\partial \mathbf{Q}}{\partial t} = -\nabla \times \mathbf{E} \tag{12}$$

$$= -\frac{\eta}{\Delta} \nabla \times \mathbf{K} \tag{13}$$

Or,

$$\frac{\partial \mathbf{Q}}{\partial t} = -\frac{\eta}{\Delta} \nabla \times \left(\mathbf{n} \times \langle \mathbf{Q} \rangle \right). \tag{14}$$

– *cont'd* $\longrightarrow$

Matching Across a Thin Resistive Shell, *– cont'd*

The normal component is,

$$\frac{\partial Q_n}{\partial t} = -\frac{\eta}{\Delta}\nabla\cdot\{(\mathbf{n}\times\langle\mathbf{Q}\rangle)\times\mathbf{n}\}. \tag{15}$$

This relation steps Q_n forward in time at the shell. NIMROD calculates Q_n^- and the (RHS)$^-$. VACUUM calculates the (RHS)$^+$ in the vacuum region as a response of Q_n which is assumed continuous across the shell.

Eq. (15) is the surface Laplacian of χ, where $\mathbf{Q}_v = \nabla\chi$. It can be written as:

$$\nabla\cdot\{(\mathbf{n}\times\langle\mathbf{Q}\rangle)\times\mathbf{n}\} = \tag{16}$$

$$\frac{1}{X\left(X_\theta^2+Z_\theta^2\right)^{1/2}}\left(\frac{\partial}{\partial\theta}\frac{X}{\left(X_\theta^2+Z_\theta^2\right)^{1/2}}\frac{\partial\chi}{\partial\theta}\right)+\frac{1}{X^2}\frac{\partial^2\chi}{\partial\phi^2}. \tag{17}$$

Using

$$\chi(\theta,\phi) = \sum_k \mathcal{C}_k(\theta)\mathcal{B}_k e^{-in\phi} \tag{18}$$

we get a response function, $\mathcal{C}_k^K(\theta)$, for the vacuum contribution:

$$\frac{\partial {Q_n}^+}{\partial t} = -\frac{\eta}{\Delta}\sum_k \mathcal{C}_k^K(\theta)\mathcal{B}_k e^{-in\phi}. \tag{19}$$

The Response Function

$$\mathcal{C}_k^K(\theta) \equiv \frac{1}{X\left(X_\theta^2 + Z_\theta^2\right)^{1/2}} \frac{\partial}{\partial\theta}\left(\frac{X}{\left(X_\theta^2 + Z_\theta^2\right)^{1/2}} \frac{\partial \mathcal{C}_k(\theta)}{\partial\theta}\right) - \frac{n^2}{X^2}\mathcal{C}_k(\theta) \qquad (20)$$

- $\mathcal{C}_k^K(\theta)$ depends only on the geometry of the system and needs to be calculated only once, provided the shell which is parameterized by $X(\theta), Z(\theta)$ is stationary.
- Only surface derivatives on $\mathcal{C}(\theta,\phi)$ are needed. This is convenient since the method of solution provides $\mathcal{C}(\theta,\phi)$ directly on the bounding surfaces. These derivatives are easily obtained.
- Since we know $\mathcal{C}(\theta,\phi)$, other attributes of the system can be calculated, such as the induced currents in the surfaces and the Mirnov loop signals in the vacuum region.

The axisymmetric terms

$$\mathbf{Q}_A = \nabla\psi \times \nabla\phi + I_v \nabla\phi \tag{21}$$

The boundary conditions are such that

$$\mathbf{n}\cdot\mathbf{Q}_A = 0 \tag{22}$$

individually for both terms.

In the currentless vacuum,

$$\nabla\phi \cdot \nabla \times \mathbf{Q}_A = 0 = \nabla\cdot[(\nabla\psi \times \nabla\phi) \times \nabla\phi] \tag{23}$$

$$\text{or} \qquad \nabla\cdot\frac{1}{X^2}\nabla\psi = 0. \tag{24}$$

A relation equivalent to Green's second theorem for ψ can be derived:

$$4\pi\bar{\psi}(\mathbf{r}) + X^2 \int_S \frac{\bar{\psi}(\mathbf{r}')}{X^2}\nabla' G_A(\mathbf{r},\mathbf{r}')\cdot d\mathbf{S}' = X^2 \int_{S_p+S_w} \frac{G_A(\mathbf{r},\mathbf{r}')}{X^2}\nabla'\bar{\psi}(\mathbf{r}')\cdot d\mathbf{S}', \tag{25}$$

– *cont'd* →

The axisymmetric terms, *– cont'd*

$G_A(\mathbf{r}, \mathbf{r}')$ is chosen to be such that

$$\nabla \cdot \frac{1}{X^2} G_A = -\frac{4\pi}{X^2} \delta(\mathbf{r} - \mathbf{r}'). \tag{26}$$

Barred quantities here again denote that the quantities contain their ϕ dependence.

- The axisymmetric equation can be solved in similar way as for the scalar potential χ.

- The two dimensional Green's function used for $n \neq 0$ involves the associated Legendre function of the first kind, $P^n_{-1/2}$, and for the $n = 0$ terms it involves $P^1_{-1/2}$. Both of these functions and their derivatives contain singularities and a significant effort is expended in dealing with with these in an efficient manner.

- These terms and the interface to the codes will be calculated soon.

References

[1] M. S. Chance, Phys. Plasmas, 4 (1997) 2161.

[2] Lüst and E. Martensen, Zeitschrift für Naturforschung 15a, 706–13 (1960).