A Physics-based Implicit Method for M3D

G.Y. Fu, J. Breslau, J. Chen and M3D group

FDM3D meeting, PPPL, March. 19-21, 2007

An implicit method for M3D

- In the current version of M3D, only compressional Alfven waves are advanced implicitly. Thus, the time step size is limited by CFL condition due to shear Alfven waves.
- We have recently developed an implicit method which is valid for the full resistive MHD equations.

Resistive MHD equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0; \\
\rho \frac{d\mathbf{v}}{dt} &= \mathbf{J} \times \mathbf{B} - \nabla p + \mu \nabla^2 \mathbf{v} \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}; \\
\frac{dp}{dt} &= -\gamma p \nabla \cdot \mathbf{v} + \rho \nabla \cdot \kappa \cdot \nabla \frac{p}{\rho} \\
\mathbf{J} &= \nabla \times \mathbf{B}; \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= \eta \mathbf{J} \\
\nabla \cdot \mathbf{B} &= 0.
\end{aligned}
$$

In (R, Z, φ) coordinates

2.1 Decompose v into (u, χ, v_φ), we have

$$\mathbf{v} = R^2 \epsilon \nabla u \times \nabla \varphi + \nabla_\perp \chi + v_\varphi \hat{\varphi}$$

2.2 Decompose B into (ψ, I), we have

$$\mathbf{B} = \nabla \psi \times \nabla \varphi + \frac{1}{R} \nabla_\perp F + R_0 I \nabla \varphi$$

2.3 F equation

$$\nabla_\perp^2 F = -\frac{a}{R} \tilde{I}', \ \ \text{with} F \equiv \frac{\partial f}{\partial \varphi}, I = \frac{R}{R_0} B_\varphi = 1 + \epsilon \tilde{I}.$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\Delta^\dagger U \quad = \quad & \epsilon R\nabla_\perp U \times \nabla_\perp(\Delta^\dagger U)\cdot\hat{\varphi} - \nabla_\perp\chi\cdot\nabla_\perp(\Delta^\dagger U) - \Delta^\dagger U(2\epsilon\frac{\partial U}{\partial Z} + \Delta^\dagger\chi) \\
& -\frac{v_\varphi}{R}\frac{\partial}{\partial\varphi}\Delta^\dagger U - \nabla_\perp(\frac{v_\varphi}{R})\cdot\nabla_\perp(\frac{\partial U}{\partial\varphi}) \\
& +2R_0\frac{v_\varphi}{R}\frac{\partial}{\partial Z}\frac{v_\varphi}{R} + \frac{R_0}{R}\nabla_\perp(\frac{v_\varphi}{R})\times\nabla_\perp(\frac{\partial\chi}{\partial\varphi})\cdot\hat{\varphi} \\
& +R_0[\mathbf{B}\cdot\nabla(\frac{C}{d}) + \mathbf{J}\cdot\nabla(\frac{I}{d})] \\
& +\frac{2}{d}\frac{\partial p}{\partial Z} + R\nabla_\perp\frac{1}{d}\times\nabla_\perp p\cdot\hat{\varphi} \\
& -R_0\nabla\varphi\cdot\nabla\times(\mu\frac{R^2}{d}\nabla^2\mathbf{v}) \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C_a}{\partial t} \quad = \quad & \epsilon R[\nabla_\perp(\Delta^* U)\times\nabla_\perp\psi + \nabla_\perp U\times\nabla_\perp C_a + 2\nabla_\perp(\frac{\partial U}{\partial R})\times\nabla_\perp(\frac{\partial\psi}{\partial R}) + 2\nabla_\perp(\frac{\partial U}{\partial Z})\times\nabla_\perp(\frac{\partial\psi}{\partial Z})]\cdot\hat{\varphi} \\
& \epsilon R[\nabla_\perp(\Delta^* U)\cdot\nabla_\perp F + \nabla_\perp U\cdot\nabla_\perp(\Delta^* F) + 2\nabla_\perp(\frac{\partial U}{\partial R})\cdot\nabla_\perp(\frac{\partial F}{\partial R}) + 2\nabla_\perp(\frac{\partial U}{\partial Z})\cdot\nabla_\perp(\frac{\partial F}{\partial Z})] \\
& -\nabla_\perp(\Delta^*\chi)\cdot\nabla_\perp\psi - \nabla_\perp\chi\cdot\nabla_\perp C_a - 2\nabla_\perp(\frac{\partial\chi}{\partial R})\cdot\nabla_\perp(\frac{\partial\psi}{\partial R}) - 2\nabla_\perp(\frac{\partial\chi}{\partial Z})\cdot\nabla_\perp(\frac{\partial\psi}{\partial Z}) \\
& [\nabla_\perp(\Delta^*\chi)\times\nabla_\perp F + \nabla_\perp\chi\times\nabla_\perp(\Delta^* F) + 2\nabla_\perp(\frac{\partial\chi}{\partial R})\times\nabla_\perp(\frac{\partial F}{\partial R}) + 2\nabla_\perp(\frac{\partial\chi}{\partial Z})\times\nabla_\perp(\frac{\partial F}{\partial Z})]\cdot\hat{\varphi} \\
& +\frac{\partial}{\partial\varphi}\Delta^*\Phi \\
& +\eta(\Delta^* C_a + \frac{1}{R}\frac{\partial}{\partial Z}\Delta^* F) \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \tilde{I}}{\partial t} &= \epsilon R \nabla_\perp U \times \nabla_\perp \tilde{I} \cdot \hat{\varphi} - \nabla_\perp \chi \cdot \nabla_\perp \tilde{I} - \frac{v_\varphi}{R} \frac{\partial \tilde{I}}{\partial \varphi} \\
&+ R \nabla_\perp (\frac{v_\varphi}{R}) \times \nabla_\perp \psi \cdot \hat{\varphi} + R \nabla_\perp F \cdot \nabla_\perp (\frac{v_\varphi}{R}) \\
&- (\frac{1}{\epsilon} + \tilde{I}) \Delta^* \chi + \eta [\Delta^* \tilde{I} - \frac{1}{R} \nabla_\perp^2 F' + \frac{2}{R^2} (\frac{\partial F'}{\partial R} + \frac{\partial \psi'}{\partial Z})] \\
&+ \nabla_\perp \eta \cdot [\nabla_\perp \tilde{I} - \frac{1}{R} \nabla_\perp F' - \nabla_\perp \psi' \times \nabla \varphi]
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial \chi}{\partial R}) &= -\epsilon R \frac{\partial}{\partial z} \left(\frac{\partial U}{\partial t} \right) - \mathbf{v}_\perp \cdot \nabla_\perp \left(\frac{\partial \chi}{\partial R} + \epsilon R \frac{\partial U}{\partial z} \right) \\
&- \epsilon v_\varphi \frac{\partial U'}{\partial z} - \frac{v_\varphi}{R} \frac{\partial \chi'}{\partial R} + \frac{v_\varphi^2}{R} - \frac{R^2}{d} \frac{\partial p}{\partial R} \\
&+ \frac{1}{d} \left(\frac{1}{\epsilon} + \tilde{I} \right) \left[\frac{1}{R} \left(\frac{\partial F'}{\partial R} + \frac{\partial \psi'}{\partial z} \right) - \frac{\partial \tilde{I}}{\partial R} \right] + \frac{C}{d} \left(\frac{\partial F}{\partial z} - \frac{\partial \psi}{\partial R} \right) + \hat{R} \cdot \mu \frac{R^2}{d} \nabla^2 \mathbf{v}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial \chi}{\partial Z}) &= \epsilon R \frac{\partial}{\partial R} \left(\frac{\partial U}{\partial t} \right) - \mathbf{v}_\perp \cdot \nabla_\perp \left(\frac{\partial \chi}{\partial z} - \epsilon R \frac{\partial U}{\partial R} \right) \\
&+ \epsilon v_\varphi \frac{\partial U'}{\partial R} - \frac{v_\varphi}{R} \frac{\partial \chi'}{\partial z} - \frac{R^2}{d} \frac{\partial p}{\partial z} \\
&+ \frac{1}{d} \left(\frac{1}{\epsilon} + \tilde{I} \right) \left[\frac{1}{R} \left(\frac{\partial F'}{\partial z} - \frac{\partial \psi'}{\partial R} \right) - \frac{\partial \tilde{I}}{\partial z} \right] - \frac{C}{d} \left(\frac{\partial F}{\partial R} + \frac{\partial \psi}{\partial z} \right) + \hat{z} \cdot \mu \frac{R^2}{d} \nabla^2 \mathbf{v}
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= \epsilon R \nabla_\perp U \times \nabla_\perp p \cdot \hat{\varphi} - \nabla_\perp \chi \cdot \nabla_\perp p - \frac{v_\varphi}{R} \frac{\partial p}{\partial \varphi} \\
&- \gamma p [\Delta^\dagger \chi + 2\epsilon \frac{\partial U}{\partial Z} + \frac{1}{R} \frac{\partial v_\varphi}{\partial \varphi}] \\
&+ d \nabla \cdot \kappa \cdot \nabla \left(\frac{p}{d} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v_\varphi}{\partial t} &= \epsilon R \nabla_\perp U \times \nabla_\perp v_\varphi \cdot \hat{\varphi} - \nabla_\perp \chi \cdot \nabla_\perp v_\varphi - \frac{v_\varphi}{R}[\epsilon R \frac{\partial U}{\partial Z} + \frac{\partial \chi}{\partial R} + \frac{\partial v_\varphi}{\partial \varphi}] \\
&+ \frac{1}{d}[\nabla_\perp \tilde{I} \cdot \nabla_\perp F - \frac{1}{R}(\nabla_\perp F' \cdot \nabla_\perp F + \nabla_\perp \psi' \cdot \nabla_\perp \psi) \\
&+ \nabla_\perp \tilde{I} \times \nabla_\perp \psi \cdot \hat{\varphi} + \frac{1}{R}\nabla_\perp \psi' \times \nabla_\perp F \cdot \hat{\varphi} + \frac{1}{R}\nabla_\perp F' \times \nabla_\perp \psi \cdot \hat{\varphi}] \\
&- \epsilon \frac{R}{d}\frac{\partial p}{\partial \varphi} + \hat{\varphi} \cdot (\mu \frac{R^2}{d}\nabla^2 \mathbf{v}) \qquad (15)
\end{aligned}
$$

$$\frac{\partial}{\partial t} w = R_0 B \cdot \nabla \left(\frac{c}{d}\right) + \ldots\ldots$$

$$\frac{\partial}{\partial t} c = \varepsilon R^2 B \cdot \nabla (w) + \ldots\ldots$$

$$\frac{\partial}{\partial t} \tilde{I} = -\left(\frac{1}{\varepsilon} + \tilde{I}\right) \Delta^* \chi + R^2 B_p \cdot \nabla \left(\frac{v_\phi}{R}\right) + \ldots\ldots$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \chi}{\partial R}\right) = -\frac{1}{d} \left(\frac{1}{\varepsilon} + \tilde{I}\right) \frac{\partial \tilde{I}}{\partial R} + \ldots\ldots$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \chi}{\partial z}\right) = -\frac{1}{d} \left(\frac{1}{\varepsilon} + \tilde{I}\right) \frac{\partial \tilde{I}}{\partial z} + \ldots\ldots$$

$$\frac{\partial}{\partial t} p = -\gamma p \Delta^+ \chi - \gamma p \frac{1}{R} \frac{\partial v_\phi}{\partial \phi} + \ldots\ldots$$

$$\frac{\partial}{\partial t} v_\phi = R B_p \cdot \nabla \tilde{I} - \varepsilon \frac{R}{d} \frac{\partial p}{\partial \phi} + \ldots\ldots$$

Implicit operator for shear Alfven waves

$$\frac{dw}{dt} = R_0 B \cdot \nabla(c/d) + other\ terms$$

$$\frac{dc}{dt} = \epsilon R^2 B \cdot \nabla w + other\ terms$$

Thus implicit equation for w can be written as

$$w^{n+1} - (\Delta t)^2 B \cdot \nabla(R^2/d) B \cdot \nabla w^{n+1} = RHS$$

Implicit operator for V_ϕ

$$V_\phi^{n+1} - (\Delta t)^2 (1/d) R B_p \cdot \nabla R B_p \cdot \nabla V_\phi^{n+1} = RHS$$

Main Results

- Initial results are very encouraging !
- Full MHD equations can be advanced stably for time step well over the shear Alfven CFL limit at zero resistivity and viscosity (for both linear and nonlinear runs).

Discussions

- Implicit method is necessary for high-resolution computation;
- The implicit method in this work is only partial. But it can serve an effective pre-conditioner for full implicit method;
- Future work will consider full linear and nonlinear implicit method.