Solver Strategies Used in SEL

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Solver Strategies Used in SEL

- Now: Static condensation, Schur complement.
  - Small local direct solves for grid cell interiors.
  - Preconditioned GMRES for Schur complement.
- Eventual: Domain substructuring, FETI-DP. Improved precondition, scalability

Static Condensation, Schur Complement

Partition into Subdomains (Grid Cells) Ω_i

I: Interiors
Γ: Interface: (faces) + edges + vertices.

Block Matrix Form

$$\mathsf{L}\mathbf{u} = \mathbf{r}, \quad \mathsf{L} = \begin{pmatrix} \mathsf{L}_{II} & \mathsf{L}_{I\Gamma} \\ \mathsf{L}_{\Gamma I} & \mathsf{L}_{\Gamma\Gamma} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \mathbf{r}_I \\ \mathbf{r}_\Gamma \end{pmatrix}$$

Solution for $\mathbf{u}_I$

$$\mathbf{u}_I = \mathsf{L}_{II}^{-1}\left(\mathbf{r}_I - \mathsf{L}_{I\Gamma}\mathbf{u}_\Gamma\right)$$

Schur Complement

$$\mathsf{S} \equiv \mathsf{L}_{\Gamma\Gamma} - \mathsf{L}_{\Gamma I}\mathsf{L}_{II}^{-1}\mathsf{L}_{I\Gamma}, \quad \mathsf{S}\mathbf{u}_\Gamma = \mathbf{r}_\Gamma - \mathsf{L}_{\Gamma I}\mathsf{L}_{II}^{-1}\mathbf{r}_I$$

- L_{II}^{-1}: small local direct solves, LU factorization and back substitution.
- S^{-1}: global solve, preconditioned GMRES.

The Benefits of Static Condensation

nx = number of grid cells in x direction
ny = number of grid cells in y direction
np = degree of polynomials in x and y
$nqty$ = number of physical quantities

N = order of global matrix to be solved

Without static condensation: $N = nx\ ny\ nqty\ np^2$
With static condensation: $N = nx\ ny\ nqty\ (2\ np - 1)$

Surface to volume ratio.
Substantial reduction of condition number.

FETI-DP

Finite Element Tearing and Interconnecting, Dual-Primal
Domain decomposition, non-overlapping, Schur complement

Axel Klawonn and Olof B. Widlund,
"Dual-Primal FETI Methods for Linear Elasticity,"
Comm. Pure Appl. Math. **59**, 1523-1572 (2006).

Partition

- I: Interior points, inside each subdomain (grid cell) Ω_i.
- Δ: Dual interface points, continuity imposed by Lagrange multipliers.
- Π: Primal interface points, continuity imposed directly.

Initial Block Matrix Form

$$\mathsf{L}\mathbf{u} = \mathbf{r}, \quad \mathsf{L} = \begin{pmatrix} \mathsf{L}_{II} & \mathsf{L}_{I\Delta} & \mathsf{L}_{I\Pi} \\ \mathsf{L}_{\Delta I} & \mathsf{L}_{\Delta\Delta} & \mathsf{L}_{\Delta\Pi} \\ \mathsf{L}_{\Pi I} & \mathsf{L}_{\Pi\Delta} & \mathsf{L}_{\Pi\Pi} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_\Pi \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \mathbf{r}_I \\ \mathbf{r}_\Delta \\ \mathbf{r}_\Pi \end{pmatrix}$$

Solution Strategy

- Small dense block matrices of $\mathbf{L}_{BB}$ solved locally by LAPACK.
- Sparse global, primal matrix $\mathbf{S}_{\Pi\Pi}$ solved in parallel by SuperLU_dist.
- Global Schur complement matrix $\mathbf{F}$ solved by parallel preconditioned Krylov method, *e.g.* GMRES. Requires preconditioner for adequate rate of convergence.
- Choose primal interface constraints to provide coarse global problem, ensure scalability. 2D: vertices. 3D: more complicated.
- The scalability of $\mathbf{F}$ is accomplished by the coarse, primal solver. The quality of the preconditioner determines the rate of convergence but not the scalability.
- Scalability has been proven analytically for a limited range of simple problems: Poisson, linear elasticity, Navier-Stokes. More general: empirical.

Preconditioning and Scalability

Definitions For Each Subdomain Ω_i

$$\mathbf{B}_{D,\Delta}^{(i)} \equiv \text{ scaled jump matrix}$$

$$\mathbf{R}_{\Gamma\Delta}^{(i)} \equiv \text{ restriction matrix from full interface to dual variables}$$

$$\mathbf{S}_{\varepsilon}^{(i)} \equiv \text{ Schur complement obtained by eliminating interior variables}$$

Preconditioner

$$\mathbf{M}^{-1} = \sum_{i=1}^{n} \mathbf{B}_{D,\Delta}^{(i)} \mathbf{R}_{\Gamma\Delta}^{(i)} \mathbf{S}_{\varepsilon}^{(i)} \mathbf{R}_{\Gamma\Delta}^{(i)T} \mathbf{B}_{D,\Delta}^{(i)T}, \quad \mathbf{M}^{-1}\mathbf{F}\lambda = \mathbf{M}^{-1}\mathbf{d}$$

Condition Number

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i, \quad \kappa(\mathbf{A}) \equiv \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right|$$

Scalability

A method is scalable if the condition number of the matrix, and hence the number of Krylov iterations to convergence, is independent of the number of subdomains.