# Higher Order Spectral Elements

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# What are spectral elements

- My working definition:
  - High-order elements of variable order (i.e., given an arbitrary order, we have a prescription for elements)
  - using Gaussian quadrature
  - with (mapped/transformed) tensor product basis
- Usually: diagonal mass matrices (not necessarily; my codes implements arbitrary integration order by interpolating)
- My choice: nodal basis (i.e., parametrization by point values). Alan Glasser's choice: modal basis.

# Why spectral elements, I

- Exponential convergence for many problems – very efficient discretization.
- $p$-refinement (increasing the degree) is very easy to implement.
- Works on any quadrilateral mesh, even with curved elements.
- Fast application and assembly of stiffness and mass matrix. Fast solvers for special situations. Direct solvers for symmetric positive (or negative) definite problems. Much work on preconditioning for standard problems such as Poisson equation, Helmholtz equation, and Maxwell equation. (But that work needs to be adapted or applied to the M3D problems.)
- Straightforward discretization and implementation.
- Fast solvers and methods for regular domains/problems can be used for the development and debugging of methods for mapped and/or curved domains.

# Why spectral elements, II

- General, variable order discretization that can be implemented using a few different types of matrices and modules (don't need to implement different elements and modules for different orders and problems)
- Easy derivation and computation of discretizations and matrices (PDE $\mapsto$ weak $\mapsto$ discretization using standard matrices $\mapsto$ optimized formulation).
- Lends itself to a modular implementation.
- A nodal representation is especially adapted to operations in the physical domain (Poisson bracket as a pointwise operation can be discretized easily). Can be more easily debugged and understood. Can be transformed easily into modal representation. Modal representation might lend itself to easier frequency-space filtering, resolution or truncation error analysis.

# Spectral elements in words

- Discrete problem is a subassembled version of element-wise discretizations, consisting of combinations of derivatives (differentation matrices), interpolation (interpolation matrices), and integrations (mass matrices).

- The fundamental matrices (derivative, interpolation, mass, even mapping to $C^1$ or modal representation) on the reference element are tensor product matrices or sums thereof – easy to implement, fast to apply or assemble. Application of tensor product matrices turns into dense matrix-matrix (respective tensor-matrix) multiplications which run at close to peak.

- Mappings and curved elements bring in geometry factors inside of diagonal matrices (or, in the application, pointwise multiplications). The geometry factors themselves can be computed using the standard matrices.

# Available framework/implemented modules

- Subassemble the global matrix
- Apply the global matrix in a matrix-vector product
- Static condensation (i.e., subassemble the global Schur complement system)
- Global sparse solve [block-sparse methods could be useful] for the Schur complement system and the needed local solves for the static condensation.
- Computing element matrices for Laplacian, Helmholtz, including weighting etc.
- Computing geometry factors for straightline/curved elements.
- Standard spectral element matrices and algorithms.

# Preview

To come tomorrow:

- Overview of the framework and the program package
- Further details for the implemented elements (including symbolic representations of the matrices)
- Integration of nodal spectral elements and direct solvers for them into M3D.