Finite Time Lyapunov Exponents for Magnetically Confined Plasmas

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Topics

- Finite time Lyapunov Exponent – modern version
  - Vector field “tracer” for fluid velocity field
  - Mathematical relation to 3D vector field structure, including time dependence, turbulence
- Plasma has multiple vector fields: **B**, **V**
  - Elsässer variables (incompressible MHD) **V**±**B**
- Example: Sawtooth crash with full 3D MHD from M3D code

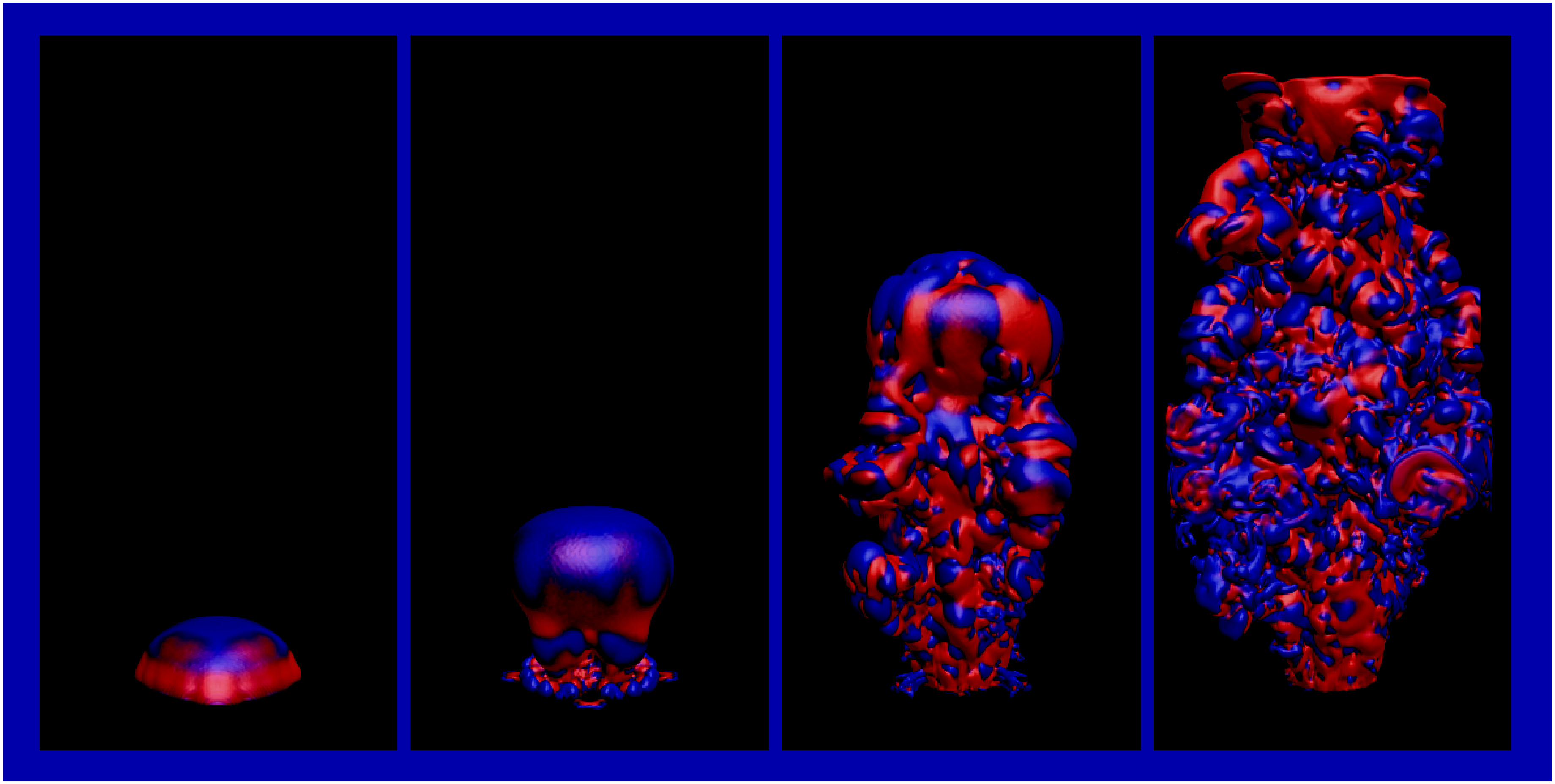
Finite Time Lyapunov Exponent

- Finite Time Lyapunov Exponent (FTLE) measures the local divergence or convergence of adjacent field lines of a vector field over a chosen finite distance or time interval (strain)
  - Characterize 3D structure in terms of a scalar function of space
  - Pick out boundaries in flow, including vortices in turbulent fluids
- Time-dependence: Lagrangian Coherent Structures in a fluid velocity field represent the larger scale structure of the field
  - Large FTLE values correspond to the LCS: mathematical relation
- Recent theoretical, computational, and experimental developments, starting with G. Haller (2001, 2002) have made FTLEs a practical tool for the study of fluid motion
  - Mathematical foundation in 3D by Lekien, Shadden, et al (2007)
  - Experimental tests in fluids (tracer particles, eg Mathur, PRL 2007)
  - Rapidly evolving area: methods, interpretation, fluid studies and other applications, computer science

FTLE Selected References

- G. Haller, ``Distinguished material surfaces and coherent structures in three-dimensional flow,'' Physica D, **149** 248 (2001).
- G. Haller, ``Lagrangian coherent structures from approximate velocity data,'' Phys. Fluids **14** 1851 (2002).
- S.C. Shadden, F. Lekien, J.E. Marsden, ``Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows,'' Physica D **212** 271 (2005).
- F. Lekien, S.C. Shadden, J.E. Marsden,``Lagrangian coherent structures in n-dimensional systems," J. Math. Phys. **48** 065404 (2007).
- M. Mathur, et al., ``Uncovering the Lagrangian skeleton of turbulence,'' Phys. Rev. Lett. **98** 144502 (2007).
- T.M. Özgökmen, et al., ``On multi-scale dispersion under the influence of surface mixed layer instabilities and deep flows,'' Ocean Modelling (2012).

Fluid jet with turbulence: FTLE picture



FTLE rendering for a jet of fluid moving upward into a stationary fluid. Forward time (diverging flow) regions are red, and backward time (converging) in blue (C. Garth et al., *SciDAC Review* **15** December (2009))

FTLE for plasma

- Apply to MHD plasma **B** and **V** fields, at one time.
- H. Krishnan and H. Childs, LBNL, have implemented the FTLE calculation in the VisIt visualization package.
  - VisIt method was first developed, tested on an ocean circulation problem (Özgökmen/Krishnan et al. 2012)
  - Plasma calculation is specific for M3D output, but will become a general routine in VisIt.
- High accuracy, independent of original data grid
  - Field lines traced starting from a grid of 4 points located around each point in space. Divergence of these field lines over given tracing distance is converted to an exponential rate.
  - Convergence is calculated by tracing field lines in opposite direction.
- First application to plasma – sawtooth crash

Sawtooth crash is good test case

- 1/1 kink mode in torus has strong structure, larger scale, 3D effects; large structure easier to study than $m \geq 2$ magnetic islands
- New 3D sawtooth simulations are badly needed
  - Nonlinear analytical models are based on essentially 2D simulations
    - Cylindrical or reduced aspect ratio RMHD
    - Older results at unrealistically high resistivity
  - Experimental observations do not fit existing simple models
- What role does stochastic **B**, turbulent **V** play in the crash at low resistivity?
- Relation between the two fields, role of compressibility: equivalent Elsässer variables for incompressible MHD $\mathbf{z}^{\pm} = \mathbf{V} \pm \mathbf{B}$

Numerical Simulation

- Full compressible MHD with density evolution and strong parallel thermal conductivity, relatively low resistivity, using M3D initial value code
  - First step: MHD only
- Axisymmetric toroidal plasma configuration from experiment, with the X-point separatrix taken to be a rigid conducting boundary
  - Alcator C-Mod-like shape, but with low q_0 ~0.5^+ => strong 1/1 mode
- Resistivity $\eta=10^{-6}$-10^{-8} crashes are qualitatively similar, more violent at lower η. Case $\eta=10^{-8}$, $\mu=10^{-5}$ shown.
  - Started with initial 1/1 density perturbation $\delta n/n_0=0.04$, but the later crash here is similar to cases with small initial $\delta n/n_0=0.001$.

Strong sawtooth crash is 3D, not 2D

- Crash is fully 3D with strong poloidal (θ) variation, unlike RMHD or cylindrical plasma with 2D helical symmetry models where only harmonics are m(θ-φ)
  - High-field side and low-field side reconnection different
  - Perpendicular **V**, **B** not completely described by poloidal stream functions
- $\partial n/\partial t$ and $\partial T/\partial t$ evolve differently; not just $\partial p/\partial t$
  - Temperature rapidly moves out into a thin layer near the X-point, then re-peaks into the plasma center from there; transient 'cold bubble', not related to a 1/1 magnetic bubble
  - Density moves through X-point more slowly, eventually ends up on poloidally opposite side from X-point, strongly localized in (θ,φ); the part outside q=1 is lost in a second phase of high-m ballooning driven by the density concentration
- Very fast crash to complete loss of central temperature, after peak T begins to move off-axis
  - “Stochastic” magnetic layer surrounds the hot core at large displacement

- Central *q* rises to 1 at *T* crash
- Crash continues after initial loss of central *T*; slower central density loss; density evolution continues after *T* and magnetic surfaces reform centrally
- Not Kadomtsev reconnection

- Notes :
  - $\varphi=0$, $\varphi=\pi$ views represent a single cut through the torus, seen from same side. $\varphi=0$ has high field side (HFS) reconnection, $\varphi=\pi$ low field side (LFS)
  - Density color scale is absolute, but temperature scale is based on maximum T at each time. Final central T is approx 2/3 original peak value.

Results: Magnetic field

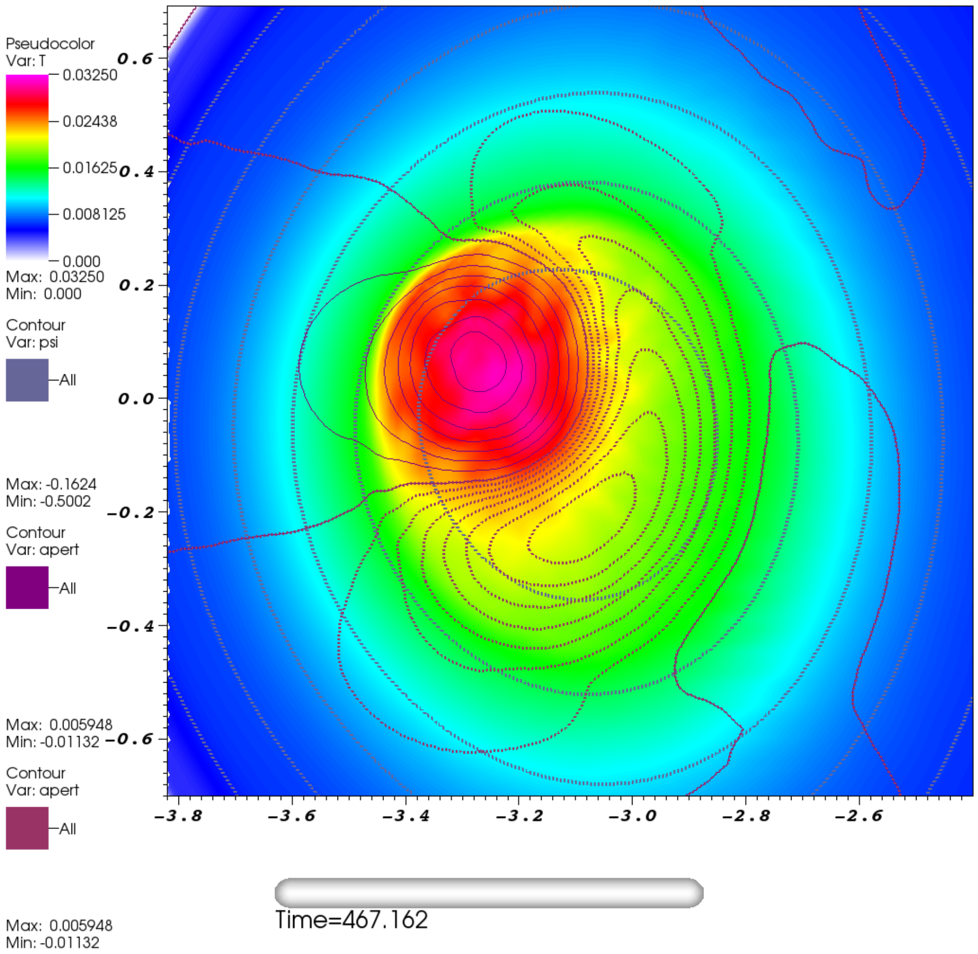
- FTLEs for sawtooth crash show strong **B**, **V** structures
- FTLE(**B**) structure is related to non-axisymmetric poloidal magnetic flux $\tilde{\psi}$.
  - Field lines traced the equivalent of once around the 1/1 kink core.
- Large FTLE band does not exactly follow $\tilde{\psi}$, also crosses magnetic puncture plot boundaries.
- FTLE has two points with very large FTLE on the outermost good magnetic surface bounding the kink core. Field line tracing shows these are a 2/1 X-point line.
  - $\tilde{\psi}$ does not follow magnetic structure of magnetic puncture plot near outer q=1 boundary (higher order island chains).
- FTLE also picks up 3/1 island near boundary of plasma (again, does not follow island structure of puncture plot)

Results: Velocity

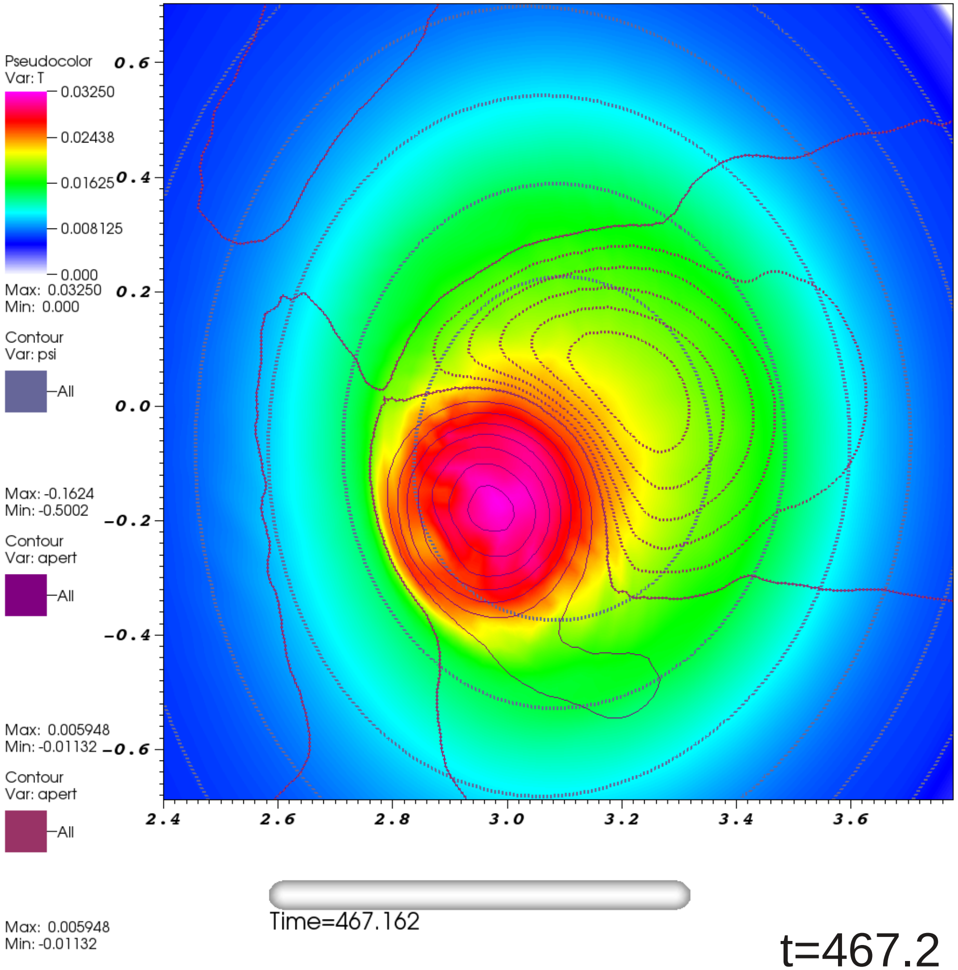
- FTLE(**V**) shows main 1/1 kink convective cells in the poloidal velocity stream function U
  - Traced for same time as **B**, ie, shorter length L=tV≈2πRV/B.
  - Overall structure follows U poloidal stream lines
  - Progressively finer vortical structure as field lines are followed for longer distances (typical FTLE behavior for conventional fluid velocity).
- FTLE(-**B**) and (-**V**) show similar structures to FTLEs of the +direction fields. (FTLE(-X) structures can overlap +X ones.)
- FTLE(**V+B**) and (**V-B**) also have structure over $q \leq 1$, different orientation than individual fields
  - Due to macroscopic effects, rotation of **B** structure by **V**
  - Fit to lowest order approximation $U \pm \tilde{\psi}/R_0$ is OK, but less good.

Temperature: displaced core

φ=π (LFS)



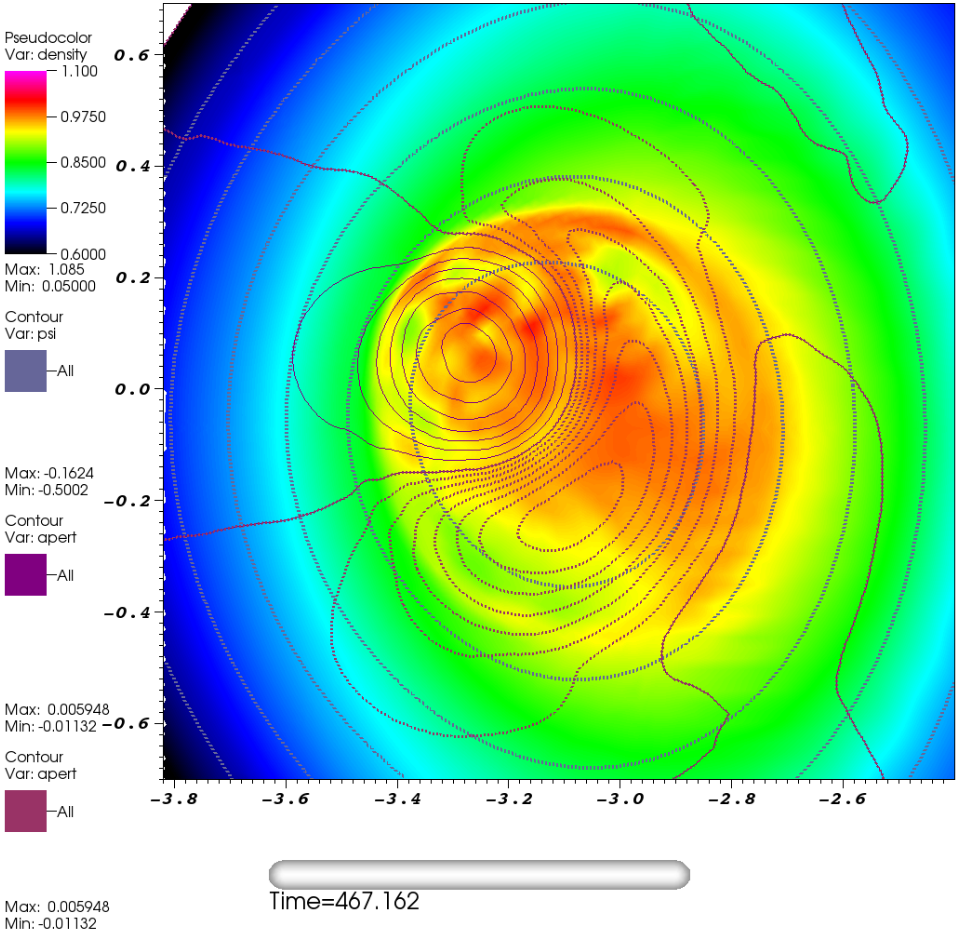
φ=0 (HFS)



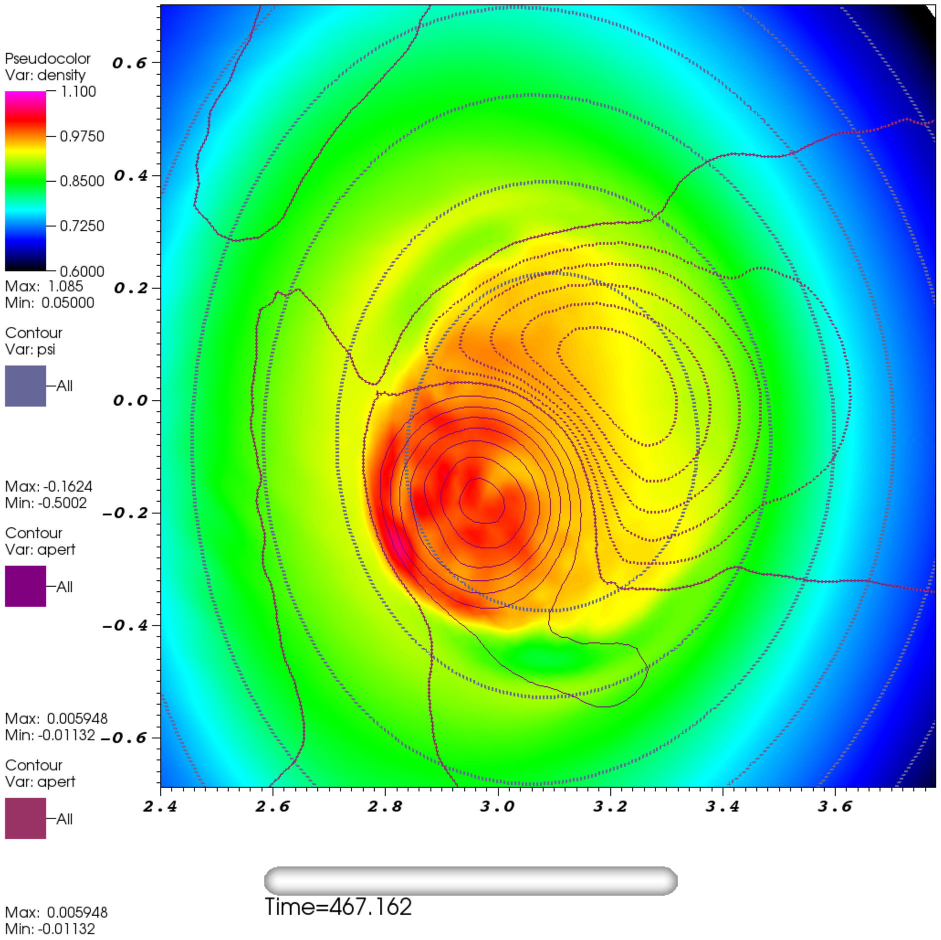
t=467.2

Density: displaced core

φ=π (LFS)

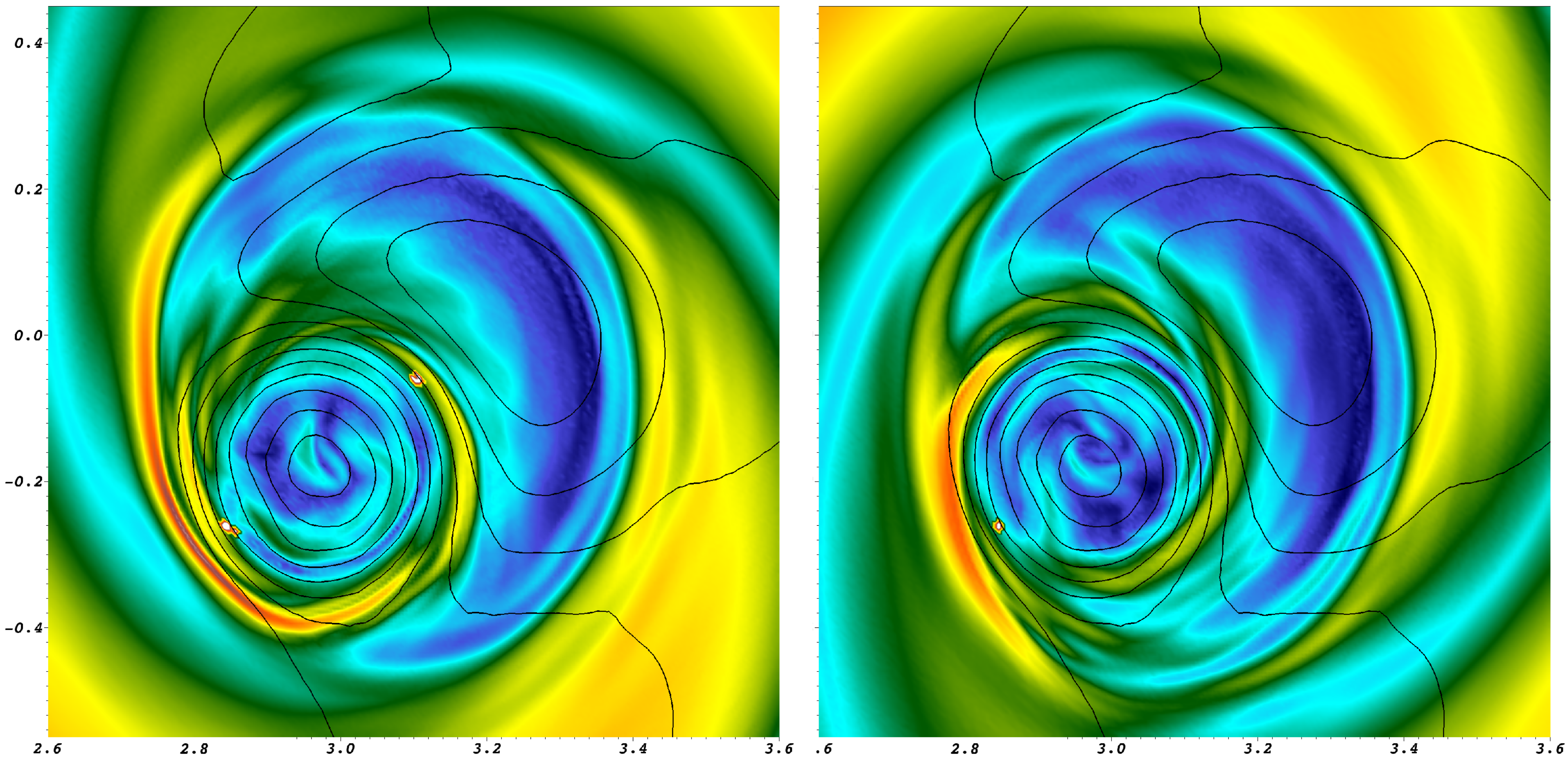


φ=0 (HFS)



t=467.2

Finite Time Lyapunov Exponent: Magnetic Field


FTLE(+**B**) with perturbed poloidal magnetic flux $\tilde{\psi}$ (lines)

FTLE(-**B**)

- Magnetic core in lower half of central q<1 region, island in upper half
- Singular FTLE point at magnetic 'X-point'
- Poloidal magnetic flux ψ does not completely describe **B**-field

$\varphi=0$ (HFS), $t=467.2$