Progress report on the two-fluid theory of the tearing mode

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These notes report on the status of theoretical work on the two-fluid resistive tearing instability, aimed at deriving analytic results for the verification tasks proposed by CEMM. This work is being carried out in collaboration with E. Ahedo from the Polytechnic University of Madrid, on sabbatical leave at M.I.T.

1. Basic two-fluid system.

We propose to base the resistive tearing mode analysis on a two-fluid model with massless electrons, including all the diamagnetic effects for finite density and temperature gradients, closed by setting to zero the pressure anisotropies and the parallel heat fluxes (i.e. all the "parallel closure" terms). This is the simplest model that includes all the two-fluid effects (Hall and diamagnetic) we are interested in and is already implemented in the latest versions of M3D and NIMROD:

$$\frac{\partial n}{\partial t} \;+\; \nabla\cdot(n\mathbf{u}) \;=\; 0\;,$$

$$\mathbf{j} \;=\; \nabla\times\mathbf{B}\;,$$

$$\frac{\partial \mathbf{B}}{\partial t} \;+\; \nabla\times\mathbf{E} \;=\; 0\;,$$

$$\mathbf{E} \;=\; -\mathbf{u}\times\mathbf{B} \;+\; \frac{1}{en}\Big(\mathbf{j}\times\mathbf{B}-\nabla p_e\Big) \;+\; \eta\mathbf{j}\;,$$

$$m_\iota n\Big[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\Big] \;+\; \nabla(p_\iota+p_e) \;+\; \nabla\cdot\mathbf{P}_\iota^{GV} \;-\; \mathbf{j}\times\mathbf{B} \;=\; 0\;,$$

$$\frac{3}{2}\Big(\frac{\partial p_\iota}{\partial t}+\mathbf{u}\cdot\nabla p_\iota\Big) \;+\; \frac{5}{2}p_\iota\nabla\cdot\mathbf{u} \;+\; \nabla\cdot\mathbf{q}_{\iota\perp} \;=\; 0\;,$$

$$\frac{3}{2}\Big[\frac{\partial p_e}{\partial t}+\Big(\mathbf{u}-\frac{1}{en}\mathbf{j}\Big)\cdot\nabla p_e\Big] \;+\; \frac{5}{2}p_e\nabla\cdot\Big(\mathbf{u}-\frac{1}{en}\mathbf{j}\Big) \;+\; \nabla\cdot\mathbf{q}_{e\perp} \;=\; 0\;.$$

Here, η is taken as constant and the ion gyroviscosity and the diamagnetic perpendicular heat fluxes are:

$$\mathsf{P}^{GV}_{\iota,jk} \;=\; \frac{m_\iota p_\iota}{4eB}\; \epsilon_{[jlm} b_l \left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n}\right)\left(\delta_{nk]} + 3 b_n b_{k]}\right) \;,$$

$$\mathbf{q}_{\iota\perp} \;=\; \frac{5p_\iota}{2eB}\mathbf{b}\times\nabla\left(\frac{p_\iota}{n}\right) \;,$$

$$\mathbf{q}_{e\perp} \;=\; -\frac{5p_e}{2eB}\mathbf{b}\times\nabla\left(\frac{p_e}{n}\right) \;.$$

Additional dissipative terms that may be needed in numerical simulations are expected to be sufficiently small to allow a meaningful extrapolation to zero.

2. One-dimensional equilibrium and two-dimensional linear perturbation.

We will assume a stationary equilibrium in a one-dimensional slab geometry, with variation along the x-direction. We will allow in principle for independent equilibrium density, ion temperature and electron temperature gradients, and for general equilibrium flows. The corresponding equilibrium functions are therefore:

$$n_0 = n_0(x)$$

$$p_{\iota 0} = p_{\iota 0}(x)$$

$$p_{e0} = p_{e0}(x)$$

$$\mathbf{u}_0 = \mathbf{u}_0(x) = u_{0y}(x)\mathbf{e}_y + u_{0z}(x)\mathbf{e}_z,$$

$$\mathbf{B}_0 = \mathbf{B}_0(x) = B_{0y}(x)\mathbf{e}_y + B_{0z}(x)\mathbf{e}_z,$$

$$\mathbf{j}_0 = \mathbf{j}_0(x) = -B'_{0z}(x)\mathbf{e}_y + B'_{0y}(x)\mathbf{e}_z.$$

These satisfy identically the continuity, Ampere and pressure equations, as well as the y and z components of the momentum equation. The generalized Ohm's law is satisfied with the equilibrium electric field:

$$E_{0x}(x) = -u_{0y}(x)B_{0z}(x) + u_{0z}(x)B_{0y}(x) - \frac{1}{en_0(x)}\Big[\frac{1}{2}B_0^2(x) + p_{e0}(x)\Big]' \ ,$$

$$E_{0y}(x) = \eta j_{0y}(x) \ ,$$

$$E_{0z}(x) = \eta j_{0z}(x) \ ,$$

and, anticipating that η will be assumed to be very small, the resistive diffusion of the magnetic field is neglected: $\partial \mathbf{B}_0(x)/\partial t = -\eta \mathbf{B}_0''(x) \simeq 0$. The x component of the momentum equation yields the following condition on the magnitude of the equilibrium magnetic field, $B_0(x) = [B_{0y}^2(x) + B_{0z}^2(x)]^{1/2}$:

$$\frac{1}{2}B_0^2(x) + p_{e0}(x) + p_{\iota 0}(x) - \frac{m_\iota p_{\iota 0}(x)}{2eB_0^2(x)}\Big[u_{0y}'(x)B_{0z}(x) - u_{0z}'(x)B_{0y}(x)\Big] = constant \ .$$

Specific forms of the $B_{0y}(x)$, $n_0(x)$, $p_{e0}(x)$ and $p_{\iota 0}(x)$ profiles to be considered are:

$$B_{0y}(x) = B_{0y}^{\infty} \tanh\Big(\frac{x}{L_B}\Big) \ ,$$

$$n_0(x) = n_0^0 + (n_0^{\infty} - n_0^0)\tanh\Big(\frac{x}{L_n}\Big) \ ,$$

$$\frac{p_{e0}(x)}{n_0(x)} = T_{e0}^0 + (T_{e0}^{\infty} - T_{e0}^0)\tanh\Big(\frac{x}{L_{Te}}\Big) \ ,$$

$$\frac{p_{\iota 0}(x)}{n_0(x)} = T_{\iota 0}^0 + (T_{\iota 0}^{\infty} - T_{\iota 0}^0)\tanh\Big(\frac{x}{L_{T\iota}}\Big) \ .$$

The linear stability analysis will be carried out assuming linearized quantities of two-dimensional form, independent of z and periodic in y:

$$Q(\mathbf{x}, t) = Q_0(x) + Q_1(x) \exp(iky + \gamma t) \ .$$

3. Results in the absence of equilibrium flow and density or temperature gradients.

As a first step we have considered the case of an equilibrium without flow, $\mathbf{u}_0 = 0$, and constant n_0, p_{e0} and $p_{\iota 0}$. The equilibrium force balance requires then that the magnitude of the magnetic field B_0 be constant too. Thus, as far as the two-fluid physics is concerned, diamagnetic effects are absent and only the Hall effect is retained. A recent study of this problem is that of Mirnov, Hegna and Prager, Phys. Plasmas **11**, 4468 (2004), who considered in detail different asymptotic parameter regimes but assumed always a strong guide field limit (or large aspect ratio in the tokamak analogy), $B_{0y}^{\infty}/B_0 \ll 1$. Here we consider a slightly more limited range for the plasma beta and the instability index Δ' (that still covers the regime of interest for tokamak plasmas), but allow instead arbitrary B_{0y}^{∞}/B_0 ratios, more in line with modern finite aspect ratio tokamaks. Our analysis assumes the resistivity to be the only basic small parameter and, within its regime of applicability, yields an exact dispersion relation without the recourse to any asymptotic treatment of other independent variables.

The dimensionless parameters involved in this problem and the way they are treated in our analysis are as follows:

1). The normalized resistivity or inverse magnetic Reynolds number defined in terms of the k^{-1} length and the B_{0y}^{∞} magnetic field, which is our single basic expansion parameter:

$$\epsilon_\eta = \frac{\eta k (m_\iota n_0)^{1/2}}{B_{0y}^{\infty}} \ll 1 \ .$$

2). The instability growth rate, normalized to the Alfven time defined in terms of k^{-1} and B_{0y}^{∞}:

$$\epsilon_\gamma = \frac{\gamma (m_\iota n_0)^{1/2}}{k B_{0y}^{\infty}} \ ,$$

which is the eigenvalue of the problem and, as an output, turns out to be also much less than unity, scaling like some positive fractional power of ϵ_η.

3). The Hall parameter:

$$\epsilon_H = kd_\iota = \frac{km_\iota^{1/2}}{en_0^{1/2}} \ ,$$

which will be treated as arbitrary, ranging from the single-fluid limit $\epsilon_H \to 0$ to $\epsilon_H = O(1)$.

4). The magnetic component ratio:

$$\epsilon_B = \frac{B_{0y}^\infty}{B_0} \ ,$$

which will be treated as arbitrary. When expressed in terms of our normalized parameters, our dispersion relation will be independent of ϵ_B.

5). The product kL_B, whose allowed range will be such that the normalized instability index:

$$\bar{\Delta}'(kL_B) = \frac{\Gamma(1/4)}{2\pi\Gamma(3/4)\ k^{3/2}L_B^{1/2}} \ \frac{B_{1x}'(0+) - B_{1x}'(0-)}{B_{1x}(0)} = \frac{\Gamma(1/4)}{\pi\Gamma(3/4)}\Big[(kL_B)^{-5/2} - (kL_B)^{-1/2}\Big] \ ,$$

is positive and comparable to or less than unity. Only the extreme limit $\bar{\Delta}'(kL_B) \gg 1$ is excluded.

6). The ratio between kinetic and magnetic pressures:

$$\beta = \frac{2(p_{e0} + p_{\iota 0})}{B_0^2} \ ,$$

which will be required to satisfy $\epsilon_\eta^{2/5} \ll \beta \leq 1$. Thus our analysis excludes the very low or zero beta limits but, for realistic values of β and the resistivity in fusion-relevant tokamak plasmas, our condition on β is well satisfied. Within this range of applicability, and for $\bar{\Delta}'(kL_B) \lesssim 1$, our dispersion relation will be independent of β.

Under the above discussed applicability conditions, our Hall-tearing mode dispersion relation has the compact form:

$$f\Big(\frac{\epsilon_H^2 \epsilon_\gamma}{\epsilon_\eta}\Big) \ \frac{\epsilon_\gamma^{5/4}}{\epsilon_\eta^{3/4}} \ = \ \bar{\Delta}'(kL_B) \ ,$$

where f is the function of a single variable

$$f(w) \ = \ \Big[1 + \frac{w}{2} + \Big(w + \frac{w^2}{4}\Big)^{1/2}\Big]^{-1/4}\Big[\frac{1}{2} + \frac{1}{4}\Big(\frac{1}{w} + \frac{1}{4}\Big)^{-1/2}\Big] +$$

$$+ \Big[1 + \frac{w}{2} - \Big(w + \frac{w^2}{4}\Big)^{1/2}\Big]^{-1/4}\Big[\frac{1}{2} - \frac{1}{4}\Big(\frac{1}{w} + \frac{1}{4}\Big)^{-1/2}\Big] \ ,$$

having the asymptotic behaviors $f(w) \to 1 \; for \; w \ll 1$ and $f(w) \to w^{-1/4} \; for \; w \gg 1.$ This expression defines implicitly a normalized growth rate independent of β and ϵ_B of the form

$$\epsilon_\gamma = \bar{\Delta}'^2 \; F(\epsilon_\eta \bar{\Delta}'^{-2}, \epsilon_H) \; ,$$

where F is a function of two variables with the asymptotic behaviors $F(v,w) \to v^{3/5} \; for \; w \ll v^{1/5}$ and $F(v,w) \to v^{1/2} w^{1/2} \; for \; w \gg v^{1/5}$.