Ordered Fluid Equations

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1. Introduction

The MHD model describes perpendicular dynamics of a continuous electrically conducting fluid medium permeated by a magnetic field. The equations of the model can be derived by without recourse to kinetic theory[1]. A basic assumption of the model is that all physical quantities are "averaged over elements of volume that are 'physically infinitesimal', ignoring the microscopic variations...that result from the molecular structure of matter"[1]. In a plasma, this model cannot describe effects due to the cyclotron gyration of individual charged particles in a magnetic field (so called FLR, or finite Larmor radius, effects[4]), or the average drift of these orbits relative to the magnetic field. Further, in a hot confined plasma the particle mean free path parallel to the magnetic field often exceeds all other macroscopic scale lengths, thus negating the basic MHD assumption and dictating a kinetic treatment. For these parameter regimes one must apply a fluid theory rigorously derived from the moments of the underlying kinetic equation[2,3,5]. This procedure produces a hierarchy of equations for successively higher velocity moments of the distribution function. In principle, this infinite set of equations is equivalent to the underlying kinetic description. In practice, the set must be truncated at some order by an independent expression (closure relation) for the highest order velocity moment. The resulting finite set of equations contains several non-dimensional parameters whose relative ordering (i.e., large or small) can further simplify the model and isolate specific physical effects.

In this note we examine the role played by these non-dimensional parameters in elucidating the form of the fluid equations. We will concentrate on the fluid momentum balance equation (derived from the first moment of the kinetic equation), as it illustrates the various forms found in the literature. We will not be concerned so much with specific closure expressions as with the structure of the equations themselves. For our purposes it is sufficient to know that appropriate closure relations exist. (In fact, one may have several from which to choose.) The goal is to illustrate self-consistent expressions of fluid momentum balance that incorporate the effects of finite ion Larmor radius[4-10].

The non-dimensional parameters that characterize the fluid equations are $e = w/\Omega_i$, the ratio of the characteristic frequency to the ion gyro-frequency, $x = V_0/V_{thi}$, the ratio of the characteristic flow velocity to the ion thermal speed, $d = r_i/L$, the ratio of the ion Larmor (gyro-) radius to the macroscopic scale length, n/Ω_i, the ratio of the collision frequency to the ion gyro-frequency, and $b = (V_{thi}/V_A)^2$, the square of the ratio of the thermal speed to the Alfvén speed. This latter parameter is not independent of the others since $b = d/x$. In a magnetized plasma, one parameter, d, can be considered small. The other parameters are ordered large or small relative to d. The resulting equations will have terms that are proportional to various powers of d. By systematically ignoring terms that are smaller than some order of d we will arrive at a hierarchy of fluid models that are valid for describing different types of plasma dynamics.

We distinguish between two-fluid effects and FLR effects. Two-fluid effects arise simply because the electrons and ions flow as distinct fluids, as manifested by the relationship $\mathbf{J} = ne(\mathbf{V}_i - \mathbf{V}_e)$. An example is the Hall term in Ohm's law, which appears when this expression is used to eliminate the electron velocity in favor of the ion velocity in the electron equation of motion. FLR effects arise because of terms that appear in the moment equations when the underlying kinetic equation is solved assuming an expansion in r_i / L. An example is the gyro-viscous (or cross) component of the ion stress tensor. Generally, both effects are required to obtain a self-consistent description of plasma dynamics at some desired order of accuracy.

Hall MHD appears when we allow for large flows relative to the thermal speed ($V_0 \sim V_{th} / d$) and high frequencies ($w \sim \Omega_i$). This model applies to very low-b conditions ($b \sim d^2$). In this model the consistent Ohm's law contains the Hall term ($\mathbf{J} \times \mathbf{B}$) but not the electron diamagnetic term (∇p_e). Restricting to smaller flows ($V_0 \sim V_{th}$) but lower frequencies ($w \sim d\Omega_i$) yields the usual MHD model, applicable when $b \sim d$. In this model Ohm's law contains neither the Hall term nor electron diamagnetic effects; whistler waves are too fast to be captured by the ordering.

The drift ordering[2,11-13] is restricted to slow flows ($V_0 \sim dV_{th}$) and even lower frequencies ($w \sim d^2\Omega_i$), but allows higher b ($\sim O(1)$). The resulting equations exhibit force balance to $O(d^2)$. At $O(1)$ in this ordering we get the transport model, which ignores inertia and yields a diffusive description of the perpendicular flows. Retaining corrections that are $O(d^2)$ in the ion equation, and $O(d)$ in the electron equation (Ohm's law), yields the drift model. As in MHD, Ohm's law does not contain the Hall term, but the electron diamagnetic term remains.

The Hall MHD, MHD, and drift models are distinguished by the degree of force imbalance that is allowed. In Hall MHD, unbalanced forces appear at $O(1)$, in MHD at $O(d)$, and in the drift model at $O(d^2)$. In all models the ions and electrons flow as distinct fluids. The Hall term ($\mathbf{J} \times \mathbf{B}$) formally enters the generalized Ohm's law at the same order as the Alfvén term ($\mathbf{V}_i \times \mathbf{B}$). The diamagnetic term (∇p_e) may appear at a different order. In the MHD and drift models the Hall term is eliminated from Ohm's law by force balance considerations that must be deduced separately from the momentum equation, and not by any ordering of the electron dynamics. This is a direct consequence of the low frequency assumption made in both models.

The drift model can be put into a useful form by expressing the individual fluid velocities in terms of the common $\mathbf{E} \times \mathbf{B}$ drift of the ions and electrons. In this form a remarkable cancellation (called the gyro-viscous cancellation[2,5,6,8,12,14-16]) occurs between the advective momentum flux and the gyro-viscous force. This results in a relatively simple model in which the new terms appear as small corrections to ideal MHD. This standard drift model can be generalized[17] to make Ohm's law exact to all orders of d.

Computational implementation of two-fluid models has been hindered by the high frequency whistler modes that are introduced by two-fluid effects. These dispersive modes can be stabilized numerically by a non-symmetric semi-implicit operator[18], at the expense of an extra matrix inversion every time step. Whistler waves are eliminated analytically from both the MHD and drift models by the assumption of low frequency. However, in the drift model the dispersive kinetic Alfvén wave [19] (KAW) remains. The KAW modes are described by a wave equation that is second order in time and fourth order in space, and is quite similar in structure to the whistler wave equation. This suggests (but does not prove) that the KAW branch may be stabilized by algorithms that have already been developed for the whistler waves[18].

As a result of these considerations, a two-fluid model for extended MHD modeling of tokamak plasmas is proposed. This model consists of the generalized drift model[17] supplemented by a semi-implicit operator to stabilize the remaining KAW branch. Since the equations of this model bear a close resemblance to the MHD equations, implementation of the explicit terms should be relatively straight forward. This model has the following desirable properties: 1) Both the Ohm's law and the ion drift velocity are exact, i.e., they include corrections to all orders in δ (although the equation of motion is accurate to $O(\delta^2)$); 2) The whistler branch is eliminated; 3) Fast flow are allowed; 4) The correction terms appear as simple additive corrections to the usual MHD model; and, 5) It is possible that the remaining high frequency dispersive waves may be stabilized at arbitrary time step by existing algorithms[18]. Resistivity and the thermal force can be implemented in a straight forward manner.

This note is organized as follows. In Section 2 we present the non-dimensional two-fluid equations and introduce the parameters of the model. In Section 3 we discuss various issues related to the stress tensor. In Section 4 we present the Hall MHD model, and in Section 5 the "ideal" MHD model. The drift ordering is introduced in Section 6. Section 7 briefly discusses the transport model, Section 8 discusses the standard drift model, including the gyro-viscous cancellation, and Section 9 introduces the generalized drift model. Section 10 investigates the wave equations and dispersion relations for both whistler waves and kinetic Alfvén waves, and suggests that each may be numerically stabilized same (or similar) semi-implicit operator(s). Section 11 shows that the drift ordering can be further generalized to accommodate large flows. A final discussion of the proposed two-fluid model is found in Section 12.

2. Non-dimensional Fluid Equations

The fluid moment equations are a shortcut to obtaining an approximate solution of the kinetic equation. Understanding the role of the various terms the fluid equations is facilitated by writing them in non-dimensional form. Assuming quasi-neutrality and neglecting the electron mass, the continuity and momentum balance equations for ions and electrons are

$$\varepsilon \frac{\partial n}{\partial t} = -\xi\delta\nabla\cdot n\mathbf{V}_i = -\xi\delta\nabla\cdot n\mathbf{V}_e \quad , \tag{2.1a,b}$$

$$\varepsilon\xi \frac{\partial \mathbf{V}_i}{\partial t} + \xi^2\delta\mathbf{V}_i\cdot\nabla\mathbf{V}_i = -\frac{1}{n}\delta\left(\nabla p_i + \frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i\right) + \xi\left(\mathbf{E} + \mathbf{V}_i\times\mathbf{B}\right) \;, \tag{2.1c}$$

and

$$x\mathbf{E} = -x\mathbf{V}_e \times \mathbf{B} - \frac{1}{n} d\left(\nabla p_e + \frac{\Pi_{e0}}{p_0} \nabla \cdot \Pi_e \right) \quad . \tag{2.1d}$$

The friction force has been neglected for simplicity, and it is assumed that there exist independent expressions or equations for the scalar pressures p_a and the stress tensors Π_a. The stress tensor will be discussed further in Section 3. The non-dimensional "pre-Maxwell" equations are

$$e\frac{\P\mathbf{B}}{\P t} = -xd\nabla \times \mathbf{E} \quad , \tag{2.2a}$$

and

$$\mathbf{J} = x\nabla \times \mathbf{B} \quad , \tag{2.2b}$$

along with the constituitive relation

$$\mathbf{J} = n\left(\mathbf{V}_i - \mathbf{V}_e\right) \quad . \tag{2.3}$$

In Equations (2.1-3), the velocity is measured in units of V_0 (m/sec), the magnetic is measured in units of B_0 (Tesla), the density in units of n_0 (m^{-3}), the electric field in units of $E_0 = V_0 B_0$ (Volts/m), the current density in units of $J_0 = n_0 e V_0$ (Amp/m^2), pressure is measured in units of $m_i n_0 V_{thi}^2$ (Pa, where $V_{thi} = \sqrt{2T_i / m_i}$ is the ion thermal speed), distance in units of $L \sim \nabla_\perp^{-1}$ (m), and time in units of $t_0 = w^{-1}$ (sec), where w is some characteristic frequency. This normalization allows the relative units of the electric field and current density to vary with V_0, thus keeping the non-dimensional dependent variables in scale. The stress tensor is measured in units of Π_{a0} (Pa); choices for this factor will be discussed in Section 3. The dimensionless constants appearing as coefficients in Equations (2.1-3) are

$$e = \frac{w}{\Omega} \quad , \tag{2.4a}$$

$$x = \frac{V_0}{V_{thi}} \quad , \tag{2.4b}$$

and

$$d = \frac{r_i}{L} \quad , \tag{2.4c}$$

where $\Omega = eB_0 / m$ is the ion cyclotron frequency and $r_i = V_{thi} / \Omega$ is the ion Larmor radius. Note that the parameter $b = \left(V_{thi} / V_A\right)^2$, which is related to the normalized plasma pressure, is not arbitrary, but is given by $b = d / x$.

The ion and electron fluid velocities can be decomposed into perpendicular and parallel parts:

$$\mathbf{V}_a = \mathbf{V}_{\perp a} + \mathbf{V}_{\| a} \quad . \tag{2.5}$$

From Equations (2.1c,d), the perpendicular parts can be expressed as

$$x\mathbf{V}_{\perp i} = x\frac{\mathbf{E}\times\mathbf{B}}{B^2} + d\frac{1}{nB^2}\mathbf{B}\times\left(\nabla p_i + \frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i\right)$$
$$+x\frac{1}{B^2}\mathbf{B}\times\left(e\frac{\P\mathbf{V}_i}{\P t} + xd\mathbf{V}_i\cdot\nabla\mathbf{V}_i\right) \quad , \tag{2.6a}$$

and

$$x\mathbf{V}_{\perp e} = x\frac{\mathbf{E}\times\mathbf{B}}{B^2} - d\frac{1}{nB^2}\mathbf{B}\times\left(\nabla p_e + \frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_e\right) \quad . \tag{2.6b}$$

The perpendicular ion and electron velocities have a common part, $\mathbf{V}_E = \mathbf{E}\times\mathbf{B}/B^2$. (This is sometimes called the MHD velocity.) The determination of the parallel velocities may require a kinetic theory.

Finally, Equations (2.1c,d) and (2.3) may be combined to yield

$$n\left(ex\frac{\P\mathbf{V}_i}{\P t} + x^2 d\mathbf{V}_i\cdot\nabla\mathbf{V}_i\right) = x\mathbf{J}\times\mathbf{B} - \frac{1}{n}d\left(\nabla p + \frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i\right) \quad . \tag{2.7}$$

This is the equation of motion for the momentum carrying component of the fluid plasma. In Equation (2.7), we have set $p = p_e + p_i$. We can also substitute Equation (2.3) into Equation (2.1d) to obtain the generalized Ohm's law

$$x\mathbf{E} = -x\mathbf{V}_i\times\mathbf{B} + x\frac{1}{n}\mathbf{J}\times\mathbf{B} - d\frac{1}{n}\left(\nabla p_e + \frac{\Pi_{e0}}{p_0}\nabla\cdot\Pi_e\right) \quad . \tag{2.8}$$

Note that the terms $\mathbf{V}_i\times\mathbf{B}$ and $\mathbf{J}\times\mathbf{B}$ appear formally at the same order as $\mathbf{E}$. The relative importance of these terms will be determined by force balance considerations (see Equation (2.7)).

The importance the various terms in Equations (2.1-8) is determined by the relative sizes of the dimensionless constants given in Equations (2.4a-d). For the strongly magnetized plasma considered here, we can always assume $d \ll 1$. Different fluid models emerge depending on how we order the remaining dimensionless variables with respect to d.

First we must determine how to order the stress tensor.

3. The Stress Tensor

Further progress requires some knowledge of the ion stress tensor Π_i. A well known formulation is that given by Braginskii[20], which assumes small parallel mean free path, small Larmor radius, and collision frequency large compared with the characteristic frequency (see Callen[3] for many useful details). The stress tensor can be written in terms of its parallel ($\mathbf{bb}$), cross ($\mathbf{b}\times\mathbf{I}$), and perpendicular ($\mathbf{I}-\mathbf{bb}$) components as

$$\Pi_i = \Pi_{\parallel} + \Pi_{\wedge} + \Pi_{\perp} \quad , \tag{3.1}$$

where

$$\Pi_{\|} = -\frac{3}{2} h_0 (\mathbf{b} \cdot \mathbf{W} \cdot \mathbf{b}) \left(\mathbf{I} - \frac{1}{3} \mathbf{bb} \right) \quad , \tag{3.2a}$$

$$\Pi_{\wedge} = \frac{h_3}{2} \left[\mathbf{b} \times \mathbf{W} \cdot (\mathbf{I} + 3\mathbf{bb}) + transpose \right] \quad , \tag{3.2b}$$

$$\begin{aligned} \Pi_{\perp} = -h_1 \Big\{ & (\mathbf{I} - \mathbf{bb}) \cdot \mathbf{W} \cdot (\mathbf{I} - \mathbf{bb}) - \frac{1}{2} (\mathbf{I} - \mathbf{bb})(\mathbf{I} - \mathbf{bb}) : \mathbf{W} \\ & + 4 \left[(\mathbf{I} - \mathbf{bb}) \cdot \mathbf{W} \cdot \mathbf{bb} + transpose \right] \Big\} \quad , \end{aligned} \tag{3.2c}$$

$$\mathbf{W} = \nabla \mathbf{V}_i + \nabla \mathbf{V}_i^T - \frac{2}{3} \mathrm{I} \nabla \cdot \mathbf{V}_i \quad , \tag{3.2d}$$

and

$$h_0 = 0.96 \frac{n T_i}{n} \quad , \tag{3.3a}$$

$$h_1 = \frac{3}{10} \frac{n T_i n}{\Omega^2} \quad , \tag{3.3b}$$

$$h_3 = \frac{n T_i}{2\Omega} \quad , \tag{3.3c}$$

where n is the collision frequency. Equations (3.2a-d) express the stress tensor in terms of gradients of the velocity, and therefore close the system of equations.

The components of the stress tensor represent fluxes of the three independent vector components of the momentum in the three independent spatial directions; e.g., Π_{xy} represents the flow x-momentum in the y-direction. Since it is a fundamental law of nature that the stress tensor be symmetric, this component also represents the flow of y-momentum in the x-direction, $\Pi_{xy} = \Pi_{yx}$. Conversely (and equivalently), a flow of x-momentum in the y-direction implies that there must be a corresponding flow of y-momentum in the x-direction.

The parallel stress, Equation (3.2a), gives the flux of momentum due to collisions between particles moving along the magnetic field, so that the force $\nabla \cdot \Pi_{\|}$ is dissipative. This is the viscous force that survives in the absence of a magnetic field; it is just the "hydrodynamic" viscous force. The coefficient h_0, Equation (3.3a), is inversely proportional to the collision frequency. Therefore, counter to intuition, high collisionality implies low collisional viscosity, and vice versa. As the mean free path l between collisions increases (n decreases) the streaming particles can carry their momentum farther before collisional transfer, thus increasing the effective transport of momentum.

The perpendicular stress, Equation (3.2c), gives the flux of momentum due to collisions between particles that are constrained to gyrate about the magnetic field. The force $\nabla \cdot \Pi_{\perp}$ is thus also dissipative. The coefficient $h_1 \sim h_0 (n / \Omega)^2$ is proportional to the collision frequency, so that this component of the viscous stress vanishes in the limit of low collisionality. Physically, the particles are now tied to particular magnetic field lines, so that the effective step size for momentum transfer is the Larmor radius ($\sim \Omega^{-1}$) while the mean time between collisions remains n^{-1}.

The cross stress, Equation (3.2b), gives the flux of momentum due to spatial gradients in the distribution of particle guiding centers. As this is strictly a kinematic effect, it is independent of collisions, and the force $\nabla \cdot \Pi_{\wedge}$ is not dissipative even though coefficient h_3 given in Equation (3.3c) is formally $\sim h_0(n/\Omega)$. This coefficient is often called the *gyro-viscosity*. The corresponding component of the stress tensor is called the *gyro-viscous stress*, denoted by Π_{gv}, and $\nabla \cdot \Pi_{gv}$ is the *gyro-viscous force.*

The stress tensor can be written in non-dimensional form using the normalization given in Section 2. The result is

$$\frac{\Pi_{i0}}{p_{i0}} \nabla \cdot \Pi_i = xd \left[\frac{1}{n/\Omega} \nabla \cdot \Pi_{\|} + \nabla \cdot \Pi_{gv} + \frac{n}{\Omega} \nabla \cdot \Pi_{\perp} \right] \quad . \tag{3.4}$$

The classical Braginskii expression for the parallel viscous force, which decreases with increasing collisionality, is seldom valid in a hot confined plasma such as a tokamak. Instead, this force determined from neo-classical theory[21], which provides expressions for the parallel force for all collisionality regimes. This force (in reality, its flux surface average) is expressed as[19]

$$\left\langle \mathbf{B} \cdot \nabla \cdot \Pi_i^{nc} \right\rangle = mn \left\langle B^2 \right\rangle m_i \frac{V_{qi}}{B_q} \mathbf{e}_q \quad , \tag{3.5}$$

where m_i is the neo-classical damping coefficient and $\mathbf{e}_q$ is a unit vector in the poloidal direction. Introducing the dimensionless variables of Section 2 into this expression, we find the normalization

$$\frac{\Pi_0^{nc}}{p_0} = \frac{x}{d} \frac{m}{\Omega} \quad . \tag{3.6}$$

The so-called "banana regime" that is of interest in a low collisionality tokamak occurs when $n/\Omega << e_A^{3/2}(w_b/\Omega) \sim e_A^{3/2} d/q$, where e_A is the inverse aspect ratio of the device, w_b is the bounce frequency, and q is the local safety factor. It is therefore consistent to order $n/\Omega \sim d^2$. In this regime neo-classical theory shows that $m \sim e_A^{1/2} n$, so that Equation (3.6) becomes

$$\frac{\Pi_0^{nc}}{p_0} = \frac{x}{d} e_A^{1/2} \frac{n}{\Omega} \sim e_A^{1/2} xd \quad . \tag{3.7}$$

Despite the previous discussion, it is common in non-linear computational models to introduce an isotropic artificial viscosity of the form

$$\frac{\Pi_0^{visc}}{p_0} \nabla \cdot \Pi^{visc} = -nm_A \nabla^2 \mathbf{V} \quad . \tag{3.8}$$

This expression resembles the zero field limit of the Braginskii model, and is not valid for a highly magnetized plasma. However, it is relatively simple to implement and is widely used nonetheless. We will consider the artificial viscosity coefficient m_A to be of adjustable order in d, meaning that we can choose its order so that it appears wherever we wish in the final equations.

With these caveats, the non-dimensional viscous force in the banana regime is

$$\frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i = -nm_A\nabla^2\mathbf{V}_i + xd\left[\nabla\cdot\Pi_i^{gv} + e_A^{1/2}\mathbf{b}\cdot\nabla\cdot\Pi_i^{nc} + d^2\nabla\cdot\Pi_{\perp i}\right] \quad . \qquad (3.9)$$

The Braginskii expression for the electron viscous force is proportional to the electron mass, and is therefore ignored when it appears in concert with the ion stress. However, the parallel viscous stress for the electrons will taken to be neo-classical.

For reference, it is instructive to write down the full momentum balance equation, including the viscous stress, without any ordering assumptions. The result is

$$\begin{aligned} x\mathbf{J}\times\mathbf{B} - d\nabla p = n&\left(ex\frac{\P\mathbf{V}_i}{\P t} + x^2 d\mathbf{V}_i\cdot\nabla\mathbf{V}_i\right) \\ &+ xd^2\left(\frac{1}{n/\Omega}\nabla\cdot\Pi_{\|} + \nabla\cdot\Pi_i^{gv} + \frac{n}{\Omega}\nabla\cdot\Pi_{\perp}\right) \qquad (3.10) \\ &+ x\frac{m}{\Omega}\mathbf{b}\cdot\nabla\cdot\Pi_i^{nc} - nm_A\nabla^2\mathbf{V}_i \quad . \end{aligned}$$

The terms on left hand side contain the "equilibrium" forces. The terms on the right hand side are all proportional to the velocity and are the plasma response when the system deviates from force balance. As stated following Equation (3.8), the artificial viscous parameter m_A is adjustable with respect to d and will be chosen for convenience in the later sections.

Having ordered the stress tensor according to the relative collisionality, we are ready to resume the discussion of Section 2.

4. The Hall MHD Ordering

In general, we would like to permit flows of arbitrary speeds and large accelerations, as might occur, for example, in shock tubes, fast Z-pinches, fast opening switches, and the inner heliosphere. We are thus led to order the frequency and flow as

$$e \sim 1 \quad , \qquad x \sim 1/d \quad . \qquad (4.1)$$

With $n/\Omega \sim m/\Omega \sim d$, as is assumed in the Braginskii closure and is valid in the plateau collisionality regime, the continuity, ion, and electron equations become

$$\frac{\P n}{\P t} = -\nabla\cdot n\mathbf{V}_i \quad , \qquad (4.2a)$$

$$\mathbf{J}\times\mathbf{B} = n\frac{d\mathbf{V}_i}{dt} + d\left(\nabla\cdot\Pi_{\|} + \mathbf{b}\cdot\nabla\cdot\Pi_i^{nc}\right) + \frac{1}{n}d^2\left(\nabla p + \nabla\cdot\Pi_{gv}\right) + O(d^3) \quad , \qquad (4.2b)$$

(where $d/dt = \P/\P t + \mathbf{V}_i\cdot\nabla$) and

$$\mathbf{E} = -\mathbf{V}_e\times\mathbf{B} - \frac{1}{n}d^2\nabla p_e \quad , \qquad (4.2c)$$

along with

$$d\mathbf{J} = \nabla \times \mathbf{B} \quad , \tag{4.3}$$

$$\frac{\P\mathbf{B}}{\P t} = -\nabla \times \mathbf{E} \quad , \tag{4.4}$$

and

$$\mathbf{J} = n(\mathbf{V}_i - \mathbf{V}_e) \quad . \tag{4.5}$$

In this ordering, $b = d / x \sim d^2$, so that this model is restricted to very low-b plasmas. From Equation (4.2b), we see that there is force imbalance at $O(1)$. This allows for very fast flows and rapid acceleration. The lowest order correction is from the parallel viscous and neo-classical forces. Pressure and gyro-viscous forces enter at $O(d^2)$, as is to be expected at this order in b. In this model the dynamics are dominated by the Lorentz force. The electron velocity is $\mathbf{V}_e = \mathbf{V}_i - \mathbf{J}/n$, so that the ions and electrons flow as separate fluids, and Ohm's law must now be written as

$$\mathbf{E} = -\mathbf{V}_i \times \mathbf{B} + \frac{1}{n}\mathbf{J} \times \mathbf{B} + O(d^2) \quad . \tag{4.7}$$

Since $\mathbf{J} \times \mathbf{B} \sim O(1)$ (see Equation (4.2b)), the whistler dynamics enter at lowest order and must be retained to describe very fast flows in a very low-b plasma, thus the term Hall MHD

Ignoring terms that are $O(d^2)$ in the equation of motion results in the force-free model that has been commonly used to study the dynamo in reversed-field pinch (RFP) plasmas. In light of Equation (4.7), this model seems to be consistent only if the whistler term is retained in Ohm's law. Pressure corrections to the ion dynamics formally require inclusion of the gyro-viscous stress, and the viscous stress should be considered to include neo-classical effects.

In the banana collisionality regime $(n/\Omega_i \sim m/\Omega \sim d^2)$, Equation (4.2b) becomes

$$\mathbf{J} \times \mathbf{B} = n\frac{d\mathbf{V}_i}{dt} + \nabla \cdot \Pi_{\|} + d\mathbf{b} \cdot \nabla \cdot \Pi_i^{nc} + \frac{1}{n}d^2\left(\nabla p + \nabla \cdot \Pi_{gv}\right) + O(d^4) \quad . \tag{4.8}$$

The promotion of the parallel viscous stress to $O(1)$ is unphysical, representing a breakdown in the assumptions of the Braginskii model. It can be replaced by the artificial isotropic viscous stress at $O(d)$.

5. The MHD Ordering

Supersonic flows rarely occur in a confined plasma, such as a tokamak, so for such cases we can restrict the flows to be on the order of the thermal velocity, and require low frequencies (compared with the cyclotron frequency). We thus define the MHD ordering[2] as

$$e \sim d, \quad x \sim 1 \quad . \tag{5.1}$$

Then $b \sim O(d)$, so that this ordering allows for somewhat higher (although still small) b than the fast ordering of Section 4. The relevant equations are then

$$\frac{¶n}{¶t} + \nabla \cdot n\mathbf{V}_i = 0 \quad , \tag{5.2a}$$

$$nd\frac{d\mathbf{V}_i}{dt} = \mathbf{J} \times \mathbf{B} - d\left(\nabla p + \frac{\Pi_{i0}}{p_0} \nabla \cdot \Pi_i \right) \quad , \tag{5.2b}$$

$$\mathbf{E} = -\mathbf{V}_e \times \mathbf{B} - d\frac{1}{n}\left(\nabla p_e + \frac{\Pi_{e0}}{p_0} \nabla \cdot \Pi_e \right) \quad , \tag{5.2c}$$

$$\frac{¶\mathbf{B}}{¶t} = -\nabla \times \mathbf{E} \quad , \tag{5.2d}$$

and

$$\mathbf{J} = \nabla \times \mathbf{B} \quad , \tag{5.2e}$$

where $d / dt = ¶ / ¶t + \mathbf{V}_i \cdot \nabla$. In the plateau regime the momentum balance equation becomes

$$\mathbf{J} \times \mathbf{B} = d\left(n\frac{d\mathbf{V}_i}{dt} + \nabla p + \nabla \cdot \Pi_{\|} + \mathbf{b} \cdot \nabla \cdot \Pi_i^{nc} \right) + d^2 \nabla \cdot \Pi_i^{gv} + O(d^4) \quad . \tag{5.3a}$$

Deviations from a force-free ($\mathbf{J} \times \mathbf{B} = 0$) state now only appear at $O(d)$. Both the Braginskii and neo-classical viscous stresses appear formally at this order. The electron equation is

$$\mathbf{E} = -\mathbf{V}_e \times \mathbf{B} + O(d) \quad . \tag{5.3b}$$

We will ignore terms that are $O(d^2)$ in the equation of motion, and terms that are $O(d)$ in Ohm's law.

Using Equation (2.6a), the ion velocity is

$$\mathbf{V}_i = \mathbf{V}_{\|i} + \mathbf{V}_E + O(d) \quad . \tag{5.4a}$$

Consistent with Equation (5.1), we ignore the terms in the ion velocity that are $O(d)$. Then the electron velocity is

$$\mathbf{V}_e = \mathbf{V}_i - \frac{1}{n}\mathbf{J} \quad . \tag{5.4b}$$

Thus in MHD the electrons and ions flow as separate fluids, with the differences in their parallel and perpendicular velocities contained in the current density $\mathbf{J}$. The generalized Ohm's law is then

$$\mathbf{E} = -\mathbf{V}_i \times \mathbf{B} + \frac{1}{n}\underbrace{\mathbf{J} \times \mathbf{B}}_{O(d)} - d\frac{1}{n}\nabla p_e \quad . \tag{5.5}$$

However, from Equations (5.2e) and (5.3a), the second term on the right had side is $O(d)$ and can be ignored. Thus the MHD Ohm's law becomes

$$\mathbf{E} = -\mathbf{V}_i \times \mathbf{B} \quad . \tag{5.6}$$

The lack of whistler and diamagnetic terms in the MHD model comes about from the force balance requirement $\mathbf{J}\times\mathbf{B} \sim O(d)$, and not because the ions and electrons flow as a "single" fluid.

In the banana regime, Equation (5.3a) is

$$\mathbf{J}\times\mathbf{B} = \nabla\cdot\Pi_{\|} + d\left(n\frac{d\mathbf{V}_i}{dt} + \nabla p + \mathbf{b}\cdot\nabla\cdot\Pi_i^{nc} \right) \quad . \tag{5.7}$$

Again, the promotion of the Braginskii parallel viscous stress by one order in d represents a breakdown in the model.

6. The Drift Ordering

A consequence of the MHD ordering, Equation (5.1), is that force imbalance (in this case, deviation from a force-free state) can occur at first order in d. In the drift ordering we envision a situation in which the equilibrium pressure and Lorentz forces are of the same order and remain in almost perfect balance for all times. Motions away from this state will be slow and exceedingly low frequency. This leads to the choice

$$e \sim d^2, \quad x \sim d \quad . \tag{6.1}$$

This allows $b \sim O(1)$. Then the form of the continuity equation remains unchanged from the MHD model, Equation (5.2a), while the expressions of total and electron momentum in the banana regime become

$$-\nabla p + \mathbf{J}\times\mathbf{B} = d^2\left(n\frac{d\mathbf{V}_i}{dt} + \nabla\cdot\Pi_i^{gv} + \mathbf{b}\cdot\nabla\cdot\Pi_i^{nc} - nm_A\nabla^2\mathbf{V}_i \right) + O(d^4) \quad , \tag{6.2a}$$

and

$$\mathbf{E} = -\mathbf{V}_e\times\mathbf{B} - \frac{1}{n}\nabla p_e \quad . \tag{6.2b}$$

Here we have ordered the artificial viscosity as $m_A \sim d^2$ so that it appears at the same order as the acceleration in Equation (6.2a). As anticipated, the pressure and Lorentz forces are now comparable and deviations from force balance occur at $O(d^2)$. Note that the neo-classical ion parallel viscous force enters at the same order as the gyro-viscous stress.

Ampere's law is now

$$\mathbf{J} = d\nabla\times\mathbf{B} \quad . \tag{6.3}$$

The ion velocity in the drift ordering is

$$\mathbf{V}_i = \mathbf{V}_E + \mathbf{V}_{*i} + \mathbf{V}_{pi} + \mathbf{V}_{\|i} + O(d^2) \quad , \tag{6.4a}$$

where

$$\mathbf{V}_{*i} = \frac{1}{nB^2}\mathbf{B}\times\nabla p_i \quad , \tag{6.4b}$$

and

$$\mathbf{V}_{pi} = \frac{1}{nB^2}\mathbf{B}\times\nabla\cdot\Pi_i^{gv} \quad , \tag{6.4c}$$

are the ion diamagnetic and "stress" drift velocities, respectively. The electron velocity is

$$\mathbf{V}_e = \mathbf{V}_i - \frac{1}{n}\mathbf{J} \quad . \tag{6.4d}$$

The appearance of the gyro-viscous stress in ion momentum equation is consequence of the FLR corrections. Note that the gyro-viscous force appears at the same order as the advective acceleration. In the drift ordering it is inconsistent to retain $\mathbf{V}\cdot\nabla\mathbf{V}$ while ignoring $\nabla\cdot\Pi_{gv}$.

7. The Transport Model

The transport model is a special case of the drift ordering that retains only the terms that are lowest order d. The result is

$$\frac{\partial n}{\partial t} = -\nabla\cdot n\mathbf{V}_i \quad , \tag{7.1a}$$

$$\nabla p = \mathbf{J}\times\mathbf{B} \quad , \tag{7.1b}$$

$$\mathbf{E} = -\mathbf{V}_i\times\mathbf{B} + \frac{1}{n}\nabla_\perp p_i \quad , \tag{7.1c}$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \quad . \tag{7.1d}$$

Inertia has been ordered out of the system, and with it all waves. In an axisymmetric equilibrium the flow will be only perpendicular to the flux surfaces. Equation (7.1b) becomes the Grad-Shafranov equation, while, from (7.1c), the outward flux of particles is

$$n\mathbf{V}_{\perp i} = n(\mathbf{V}_E + \mathbf{V}_{*i}) \quad . \tag{7.2}$$

When this expression is used with an equation of state, such as $p = nT$, Equation (7.1a) takes the form of a diffusion equation for the density. Field diffusion results from (7.1d) if the friction force (resistivity) is retained in the electron equation (7.1c).

8. The Standard Drift Model

While the drift ordering introduced in Section 6 produced the lowest order FLR corrections to the fluid equations, it resulted in more complicated equations than in the MHD ordering and provided no special insights. In contrast to the drift *ordering*, the *drift model* makes use of the velocity decomposition given by Equations (6.4a-d), along with a remarkable result called the *gyro-viscous cancellation*, to produce a simplified set of equations that yield significant physical insight.

Essentially, the standard drift model makes a velocity transformation to a frame moving with the MHD velocity $\mathbf{V}_E$:

$$\mathbf{V}_i = \mathbf{V}_{\parallel i} + \mathbf{V}_E + \mathbf{V}_{*i} + O(d^2) \quad , \tag{8.1a}$$

$$\mathbf{V}_e = \mathbf{V}_i - \frac{1}{n}\mathbf{J} = \mathbf{V}_{\parallel i} + \mathbf{V}_E + \mathbf{V}_{*i} - \frac{1}{n}\mathbf{J} + O(d^2) \quad . \tag{8.1b}$$

The motivation is to arrive at a set of equations that look like the MHD equations with corrections. Using this last result in the electron equation, Equation (6.2b), we have

$$\begin{aligned}
\mathbf{E} &= -\left(\mathbf{V}_E + \mathbf{V}_{*i} - \frac{1}{n}\mathbf{J}_\perp\right)\times\mathbf{B} - \frac{1}{n}\nabla p_e + O(d^2) \quad , \\
&= -\mathbf{V}_E \times \mathbf{B} - \frac{1}{n}\nabla_\parallel p_e + \frac{1}{n}\underbrace{\left(-\nabla_\perp p + \mathbf{J}\times\mathbf{B}\right)}_{O(d^2)} + O(d^2) \quad , \\
&= -\mathbf{V}_E \times \mathbf{B} - \frac{1}{n}\nabla_\parallel p_e \quad .
\end{aligned} \tag{8.2}$$

This is the generalized Ohm's law in the drift model. Notice that the high frequency whistler response, which comes from $\mathbf{J}\times\mathbf{B}$, has been dropped as a higher order ($O(d^2)$) correction. (A similar argument led to the elimination of whistler waves from the MHD model; see Section 5.) This is consistent with the low frequency ordering $w \sim d^2\Omega$; the whistler branch merges with the Alfvén branch at low frequency. The ion equation becomes

$$\begin{aligned}
d^2\left(n\frac{d}{dt}\left(\mathbf{V}_{\parallel i} + \mathbf{V}_E\right) + n\frac{d\mathbf{V}_{*i}}{dt} + \nabla\cdot\Pi_i^{gv}\left(\mathbf{V}_i\right)\right) &= -\nabla p + \mathbf{J}\times\mathbf{B} - d^2\mathbf{b}\cdot\nabla\cdot\Pi_i^{nc} \\
&\quad + d^2 n m_A \nabla^2\mathbf{V}_i + O(d^4) \quad ,
\end{aligned} \tag{8.3}$$

where we have retained the notation $d/dt = \P/\P t + \mathbf{V}_i\cdot\nabla$.

One reason for the utility of the drift model is an enormous simplification of the equation of motion that occurs because, in the proper reference frame, the parallel ion gyro-viscous force $\nabla\cdot\Pi_{gv}$ algebraically cancels a significant part of the advective acceleration $n\mathbf{V}_i\cdot\nabla\mathbf{V}_i$. The remaining terms primarily introduce a slight modification to the total pressure.

The *gyro-viscous cancellation* is usually written as[14]

$$n\left(\frac{\P\mathbf{V}_{*i}}{\P t} + \mathbf{V}_i\cdot\nabla\mathbf{V}_{*i}\right) + \nabla\cdot\Pi_i^{gv}\left(\mathbf{V}_i\right) \approx \nabla c - \mathbf{b}n\mathbf{V}_{*i}\cdot\nabla V_{\parallel i} \quad , \tag{8.4}$$

where

$$c = -p_i\mathbf{b}\cdot\left(\nabla\times\mathbf{V}_{\perp i}\right) \tag{8.5}$$

is a scalar related to the parallel component of the ion vorticity. Both terms involved in the cancellation result from the transport by advection at the streaming velocity ($\mathbf{V}_i$) of momentum inherent in the gyro-motion ($\mathbf{V}_{*i}$). These fluxes almost cancel because the diamagnetic drift does not correspond to any real drift of the guiding centers[22]. Both the gyro-viscous force and the advective term $\mathbf{V}_{*i}\cdot\nabla\mathbf{V}_i$ explicitly contain the pressure, the velocity, and two gradients, so one can surmise (at least mathematically) how this cancellation might come about.[14] Since, from Equations (3.2b) and (3.2d),

$$
\begin{aligned}
\nabla \cdot \Pi_i^{gv} &\sim \nabla \cdot \left[p(\mathbf{b} \times \nabla \mathbf{V}_i) \right] \quad , \\
&\sim \nabla p \cdot (\mathbf{b} \times \nabla) \mathbf{V}_i \quad , \\
&= -(\mathbf{b} \times \nabla p) \cdot \nabla \mathbf{V}_i \quad , \\
&\sim -n \mathbf{V}_{*i} \cdot \nabla \mathbf{V}_i \quad ,
\end{aligned}
\tag{8.6}
$$

it is at least plausible that such a cancellation might take place. However, the actual calculation is extremely complex and tedious[6,8,14], and seems to have been carried out only under restricted conditions (i.e., uniform magnetic field, sheared slab geometry, uniform temperature, etc.). Further, there is not universal agreement on the exact form of the cancellation. Some authors[6,8] find extra terms on the right hand side of Equation (8.4).

The gyro-viscous cancellation cannot occur in either Hall or ideal MHD, since the advective acceleration and the gyro-viscous force enter at different orders in each of those models.

Using the gyro-viscous cancellation, Equation (8.4), in the momentum equation, Equation (8.3), and assuming that the magnetic field is approximately constant (so that the unit vector $\mathbf{b}$ can be moved freely through derivative operators), we find

$$
\begin{aligned}
nd^2 \Bigg[\left(\frac{\partial \mathbf{V}_E}{\partial t} + \mathbf{V}_E \cdot \nabla \mathbf{V}_E \right) &+ \left(\frac{\partial \mathbf{V}_{\parallel i}}{\partial t} + \mathbf{V}_E \cdot \nabla \mathbf{V}_{\parallel i} \right) \\
&+ \mathbf{V}_{*i} \cdot \nabla \mathbf{V}_E + \mathbf{V}_{\parallel i} \cdot \nabla (\mathbf{V}_E + \mathbf{V}_{\parallel i}) \Bigg] = -\nabla \left[p \left(1 + d^2 c \right) \right] + \mathbf{J} \times \mathbf{B} \\
&- d^2 \mathbf{b} \cdot \nabla \cdot \Pi_i^{nc} + d^2 n m_A \nabla^2 \mathbf{V}_i + O(d^4) \quad .
\end{aligned}
\tag{8.7}
$$

This is usually expressed in terms of the perpendicular and parallel momentum balance:

$$
\begin{aligned}
nd^2 \left. \frac{d\mathbf{V}_E}{dt} \right|_{MHD} &= -d^2 n \mathbf{V}_{*i} \cdot \nabla \mathbf{V}_E - d^2 n \mathbf{V}_{\parallel i} \cdot \nabla \mathbf{V}_E + d^2 n m_A \nabla^2 \mathbf{V}_E \\
&- d^2 (\mathbf{I} - \mathbf{bb}) \cdot (\mathbf{b} \cdot \nabla \cdot \Pi_i^{nc}) - \nabla_\perp \left[p \left(1 + d^2 c \right) \right] + \mathbf{J} \times \mathbf{B} \quad ,
\end{aligned}
\tag{8.8a}
$$

$$
\begin{aligned}
nd^2 \left. \frac{d\mathbf{V}_{\parallel i}}{dt} \right|_{MHD} &= -d^2 n \mathbf{V}_{\parallel i} \cdot \nabla \mathbf{V}_{\parallel i} + d^2 n m_A \nabla^2 \mathbf{V}_{\parallel i} \\
&- d^2 \mathbf{b} \cdot (\mathbf{b} \cdot \nabla \cdot \Pi_i^{nc}) - \mathbf{b} \cdot \nabla \left[p \left(1 + d^2 c \right) \right] \quad ,
\end{aligned}
\tag{8.8b}
$$

where

$$
\left. \frac{d}{dt} \right|_{MHD} \equiv \frac{\partial}{\partial t} + \mathbf{V}_E \cdot \nabla \quad .
\tag{8.9}
$$

These equations differ from the MHD model primarily through a modification to the scalar pressure and the appearance of a source term proportional to $\mathbf{V}_{*i}$. This last term explicitly introduces the diamagnetic drift frequency ($\omega_{*i} \sim \mathbf{V}_{*i} \cdot \nabla$).

The drift model thus naturally produces a set of equations that a) explicitly contain the lowest order FLR corrections to the MHD model, b) look very much like the MHD equations when cast in terms of the velocity $\mathbf{V} = \mathbf{V}_E + \mathbf{V}_{\|i}$, c) separate easily into perpendicular and parallel parts, d) remove most of the complications of the gyro-viscous stress (see Equation (3.2b)), and, e) eliminate the whistler branch. It is no wonder that these equations have proven to be powerful for the analysis of hot, confined plasmas.

There are, however, some caveats in the drift model. In the first place, the derivation formally admits only slow flows, which is consistent with the result of force balance through first order in δ. Since this ordering refers to the total ion velocity $\mathbf{V}_i$ (see Equations (6.1) and (6.2a)), it is applicable both the MHD velocity $\mathbf{V}_E$ and the drift velocity $\mathbf{V}_{*i}$. (A relaxation of this restriction will be discussed in Section 11.) Second, the assumption of very low frequency may limit the validity of the model to time scales much longer than the Alfvén transit time L/V_A. Acceptable frequencies are on the order of the diamagnetic drift frequency $\omega_{*i} \sim V_{*i}/L << \omega_A$. Third, the form of the gyro-viscous cancellation used here assumes a uniform magnetic field, or at least a sheared slab. There is no generally accepted form that is much less restrictive. Therefore, the specific form of the equations should be considered only approximate. Finally, the artificial viscosity in the momentum equation is unphysical. In non-linear numerical simulations it may be required to be so large as to obscure the delicate cancellations that have occurred.

The continuity equation must also be modified by the velocity transformation. The drift model is then summarized as:

continuity:

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{V}_E = -\nabla \cdot n\left(\mathbf{V}_{*i} + \mathbf{V}_{\|i}\right) \quad , \tag{8.10a}$$

perpendicular momentum:

$$\begin{aligned} n\delta^2 \left.\frac{d\mathbf{V}_E}{dt}\right|_{MHD} &= -\delta^2 n\mathbf{V}_{*i} \cdot \nabla\mathbf{V}_E - \delta^2 n\mathbf{V}_{\|i} \cdot \nabla\mathbf{V}_E + \delta^2 nm_A \nabla^2 \mathbf{V} \\ &\quad - \delta^2 (\mathbf{I} - \mathbf{bb}) \cdot (\mathbf{b} \cdot \nabla \cdot \Pi_i^{nc}) - \nabla_\perp \left[p\left(1 + \delta^2 \chi\right)\right] + \mathbf{J} \times \mathbf{B} \quad , \end{aligned} \tag{8.10b}$$

parallel momentum:

$$\begin{aligned} n\delta^2 \left.\frac{d\mathbf{V}_{\|i}}{dt}\right|_{MHD} &= -\delta^2 n\mathbf{V}_{\|i} \cdot \nabla\mathbf{V}_{\|i} + \delta^2 nm_A \nabla^2 \mathbf{V}_{\|i} \\ &\quad - \delta^2 \mathbf{b} \cdot (\mathbf{b} \cdot \nabla \cdot \Pi_i^{nc}) - \mathbf{b} \cdot \nabla \left[p\left(1 + \delta^2 \chi\right)\right] \quad , \end{aligned} \tag{8.10c}$$

generalized Ohm's law:

$$\mathbf{E} = -\mathbf{V}_E \times \mathbf{B} - \frac{1}{n}\nabla_{\|} p_e \quad , \tag{8.10d}$$

Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad , \tag{8.10e}$$

Ampere's law:

$$\mathbf{J} = d\nabla \times \mathbf{B} \quad . \tag{8.10f}$$

Of course, these must be supplemented by an appropriate drift-ordered energy equation to determine the scalar pressure.

9. A Generalized Drift Model

A generalized drift model has recently been introduced[17], which attempts to some of the restrictions of the standard drift model. This model retains the drift ordering of Equation (6.1), but introduces the more general velocity transformation

$$\mathbf{V}_i = \mathbf{V}_E + \mathbf{V}_{di} + \mathbf{V}_{\|i} \quad , \tag{9.1}$$

where the *generalized drift velocity* $\mathbf{V}_{di}$ contains corrections to the perpendicular ion velocity to *all* orders in d. The electron velocity can be written in two independent ways: from Equations (6.4d) and (9.2);

$$\begin{aligned} \mathbf{V}_e &= \mathbf{V}_i - \frac{1}{n}\mathbf{J} \quad , \\ &= \mathbf{V}_{\|i} - \frac{1}{n}\mathbf{J}_{\|} + \mathbf{V}_E + \mathbf{V}_{di} - \frac{1}{n}\mathbf{J}_{\perp} \quad , \end{aligned} \tag{9.2a}$$

and from Equation (6.2b);

$$\mathbf{V}_e = \mathbf{V}_{\|e} + \mathbf{V}_E + \mathbf{V}_{*e} \quad , \tag{9.2b}$$

where

$$\mathbf{V}_{*e} = -\frac{1}{nB^2}\mathbf{B} \times \nabla p_e \quad . \tag{9.3}$$

From Equations (9.2a,b), we conclude that

$$\mathbf{V}_{\|e} = \mathbf{V}_{\|i} - \frac{1}{n}\mathbf{J}_{\|} \quad , \tag{9.4}$$

and

$$\mathbf{V}_{di} = \frac{1}{n}\mathbf{J}_{\perp} + \mathbf{V}_{*e} \quad . \tag{9.5}$$

Note that, from Equation (6.2a),

$$\mathbf{V}_{di} = \mathbf{V}_{*i} + O(d^2) \quad . \tag{9.6}$$

This is a result of the drift ordering, i.e., force balance to second order in d.

Using this *ansatz* in the generalized Ohm's law, Equation (6.2b), we find again Equation (8.10d), except it is now exact to all orders of δ. In light of Equation (9.6), we also can replace $\mathbf{V}_{*i}$ with $\mathbf{V}_{di}$ everywhere in Equations (8.10a-c). This becomes the generalized drift model.

The advantage of this model is that the electron dynamics become exact (within the context of $m_e = 0$). In principle, the whistler branch should then reappear in the ion dynamics, but it is again removed by the drift ordering. It appears at $O(\delta^4)$ and is dropped. The entire model remains accurate to $O(\delta^2)$. It seems that rapid flows and relatively high frequencies remain disallowed by the fundamental ordering.

10. Dispersive Modes in the Drift Model

The finite Larmor radius effects of the drift model introduce modifications to the normal modes of the usual MHD equations. The FLR corrections to the electromagnetic Alfvén wave are especially interesting and troublesome because they introduce high frequency dispersive modes with $\omega \sim k^2$. The corresponding Courant condition $\Delta t < \Delta x^2$ is too restrictive for long time scale computations using explicit methods, and the non-Hermitian property of the wave operators dictates that implicit methods use non-symmetric linear algebra solvers.

To examine these waves, it is instructive to consider the generalized Ohm's law without assuming force balance to $O(\delta^2)$. From the derivation preceding Equation (8.2),

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} - \frac{1}{ne}\nabla_{\|} p_e + \frac{1}{ne}\left[-\nabla_{\perp} p + \mathbf{J}\times\mathbf{B}\right] \quad . \tag{10.1}$$

(Here, and for the remainder of this section, we revert to dimensional equations.) Recall that the term in brackets on the right hand side of Equation (10.1) is ordered out of the drift model (see Equation (8.2)). The first term on the right hand side yields the usual MHD Alfvén waves. The remaining terms are FLR corrections. The second term ($\sim \nabla_{\|} p_e$) is introduces *kinetic Alfvén waves* (KAW), and the last term ($\sim \mathbf{J}\times\mathbf{B}$) introduces *whistler waves*. The third term ($\sim \nabla_{\perp} p$) yields no new modes in an incompressible plasma.

In a uniform, homogenous plasma with straight field lines, the linearized whistler wave equation comes about from combining Equation (10.1) with Faraday's law and Ampere's law,

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{B_0}{ne}(\mathbf{b}\cdot\nabla)\mathbf{J} = -\frac{B_0}{ne\mu_0}(\mathbf{b}\cdot\nabla)\nabla\times\mathbf{B} \quad ,$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\frac{B_0}{ne\mu_0}\nabla\times\frac{\partial \mathbf{B}}{\partial t} \quad ,$$

$$= \left(\frac{V_A^2}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2 \nabla\times\nabla\times\mathbf{B} \quad . \tag{10.2}$$

These waves have the dispersion relation

$$\omega^2 = \left(\frac{V_A^2}{\Omega} \right)^2 k^2 k_\|^2 \quad , \tag{10.3}$$

so that shorter wavelengths have higher frequencies. This is the source of the computational problem. The semi-implicit operator that stabilizes the whistler waves[18] is based on Equation (10.2).

Under similar conditions, but assuming overall incompressibility, i.e.,

$$\nabla \cdot \mathbf{V} \equiv \nabla \cdot \left(\mathbf{V}_E + \mathbf{V}_{\|i} \right) = 0 \quad , \tag{10.4}$$

the KAW arises from a combination of the second term in Equation (10.3), Faraday's law, Ampere's law, the perpendicular momentum equation, and the electron energy equation

$$\frac{\partial p_e}{\partial t} = -\gamma p_{e0} \nabla \cdot \mathbf{V}_e \quad . \tag{10.5}$$

Note that Equation (10.4) does not imply that the electrons are themselves incompressible. In fact

$$\nabla \cdot \mathbf{V}_e = \nabla \cdot \left(\mathbf{V}_i - \frac{1}{ne} \mathbf{J}_\perp \right) - \frac{1}{ne} \nabla \cdot \mathbf{J}_\| \quad . \tag{10.6}$$

Now $\mathbf{J}_\perp = ne(\mathbf{V}_{*i} - \mathbf{V}_{*e})$, so that, using Equations (8.1a,b), we have

$$\begin{aligned} \nabla \cdot \mathbf{V}_e &= \nabla \cdot \left(\mathbf{V}_E + \mathbf{V}_{\|i} \right) + \nabla \cdot \mathbf{V}_{*e} - \frac{1}{ne} \nabla \cdot \mathbf{J}_\| \quad , \\ &= -\frac{1}{ne} \nabla \cdot \mathbf{J}_\| \quad , \end{aligned} \tag{10.7}$$

since $\nabla \cdot \mathbf{V}_{*a} = 0$. Then Equation (10.5) becomes

$$\frac{\partial p_e}{\partial t} = \frac{\gamma p_{e0}}{ne} \nabla \cdot \mathbf{J}_\| \quad . \tag{10.8}$$

Putting all this together, we find the KAW wave equation to be

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = \left(\frac{V_A V_{th*}}{\Omega} \right)^2 (\mathbf{b} \cdot \nabla)^2 \nabla \times \left[\mathbf{bb} \cdot \nabla \times \mathbf{B} \right] \quad , \tag{10.9}$$

where V_{th*} is the thermal speed evaluated with the electron temperature and the ion mass. The KAW dispersion relation is

$$\omega^2 = \left(\frac{V_A V_{th*}}{\Omega} \right)^2 k_\perp^2 k_\|^2 \quad . \tag{10.10}$$

It can be shown that the KAW survives the assumption of overall incompressibility and remains dispersive[23].

Whistler waves come from a combination of the generalized Ohm's law and Faraday's law. They are completely related to the electrons. The KAW involves both electron and ion dynamics, and their frequency is low enough to be captured in the drift ordering. Nonetheless, the similarity between the whistler wave equation (Equation (10.2)) and the KAW wave equation (Equation (10.9)) strongly suggests that the same semi-implicit operator that has been developed for treating the whistler waves[18,24] may also be successful in stabilizing the KAW. (The difference is that whistler equation involves the curl of the total current density, while the KAW involves the curl of the parallel current density.) This is for now a matter of speculation, but even if a separate operator based on Equation (10.9) is required, it will only be a small modification of the existing whistler operator[24].

11. Fast Flows and Very Low Frequencies

The drift ordering only admits flows that are $O(dV_{thi})$. This restriction can be effectively removed by explicitly writing the ion velocity as

$$\mathbf{V}_i = V_{thi}\left(x_E \mathbf{V}_E + x_* \mathbf{V}_{*i} + x_{\|i} \mathbf{V}_{\|i}\right) \quad . \tag{11.1}$$

Allowing for fast flows and low frequencies, we set $x_E \sim x_\| \sim 1$, $x_* \sim d$, and $e \sim d^2$. Then the equation of motion (including the gyro-viscous cancellation) and Ohm's law become

$$n\left(d^2 \frac{¶\mathbf{V}}{¶t} + d\mathbf{V}\cdot\nabla\mathbf{V}\right) = -d^2 n\mathbf{V}_{*i}\cdot\nabla\mathbf{V}_\perp - d\nabla\left(p + dc\right) + \mathbf{J}\times\mathbf{B} \quad , \tag{11.2a}$$

and

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} - d\frac{1}{n}\nabla_\| p_e \quad , \tag{11.2b}$$

where $\mathbf{V} = \mathbf{V}_E + \mathbf{V}_{\|i}$. Except for the relative ordering of the individual terms, these equations have the same form as in the drift model, Equations (8.10b-d), and have the same overall accuracy. Thus equations of this form allow fast flows. The relevant restriction is now $\mathbf{V}_i - \mathbf{V}_E \sim O(d)$.

12. Discussion

The role of small parameters in two-fluid modeling has been emphasized by writing the fluid equations for the ion and electron species (with $m_e = 0$) in non-dimensional form. The parameters are $e = w/\Omega_i$, $x = V_0/V_{thi}$, $d = V_{thi}/\Omega_i L = r_i/L$, and n/Ω_i. In terms of these parameters the normalized plasma pressure is not arbitrary, but is given by $b = d/x$. For a highly magnetized plasma $d \ll 1$, and important physical effects can be highlighted by the relative ordering of the remaining parameters with respect to d. In all cases the electrons and ions flow as separate fluids.

Properties of the various models is summarized in Table I.

Hall MHD appears in the ordering $x \sim 1/d$ and $e \sim 1$, and applies to very low-b plasmas, $b \sim d^2$ (see Section 3). In this ordering the consistent Ohm's law contains the whistler (Hall) terms, but ignores the diamagnetic contributions. Unbalanced forces appear at $O(1)$, so this model is appropriate for describing situations that are far from equilibrium, such as fast Z-pinches, gun plasma formation, and coronal mass ejections.

Table I

Properties of Fluid Models

Model	V_i	w	b	$\mathbf{J}\times\mathbf{B}$	Whistlers[†]	KAW[††]
Hall MHD	V_{thi}/d	Ω_{ci}	$O(d^2)$	$mn\frac{d\mathbf{V}_i}{dt}+O(d)$	Yes	No
Ideal MHD	V_{thi}	$d\Omega_{ci}$	$O(d)$	$O(d)$	No	No
Drift	dV_{thi}	$d^2\Omega_{ci}$	$O(1)$	$\nabla p+O(d^2)$	No	Yes

[†]Whistler waves are high frequency phenomena that disappear as the frequency is ordered successively lower.
[††]Kinetic Alfvén waves are finite pressure phenomena that appear as b becomes successively larger.

MHD appears in the ordering $x \sim 1$ and $e \sim d$ (see Section 4). MHD admits the possibility of fast flows ($V_0 \sim V_{thi}$), but restricts the frequency to be low ($w \sim d\Omega_i$). It is applicable when $b \sim d$. Unbalanced forces can appear at $O(d)$, so MHD is appropriate for describing situations that deviate moderately equilibrium, such as spheromaks, RFPs, and possibly sawtooth crashes. FLR effects appear at $O(d^2)$ and are ignored in this model.

In MHD, the parallel Braginskii viscous force is divergent when $n/\Omega_i < O(d)$. This is the regime is of interest in modern tokamaks. Therefore, the viscous forces do not formally appear in the MHD model. However, a force of the form $nmm\nabla^2\mathbf{V}$ is often used for numerical purposes. This force is non-physical. In fact, it does not vanish under rigid rotations as is required of the actual viscous force. (A more correct form would be $\nabla\cdot(\nabla\mathbf{V}+\nabla\mathbf{V}^T)$.)

The drift ordering (see Section 6) is restricted to slow flows ($V_0 \sim dV_{thi}$) and very low frequencies ($w \sim d^2\Omega_i$). It is applicable to situations where $b \sim O(1)$. Unbalanced forces appear only at $O(d^2)$ (see Equations 6.2a,b). This model is appropriate for systems that deviate only slightly and slowly from equilibrium, such as a hot plasma confined in a tokamak. (It reflects favorably on the progress in tokamak confinement that such a model is required to accurately describe the dynamics.) If all FLR corrections are ignored this ordering produces the transport model, in which all inertial effects are removed and the slow motions across the field are diffusive in nature (see Equations (7.1-2)). Retaining lowest order corrections ($O(d^2)$ in the ion dynamics and $O(d)$ in the electron dynamics) yields dynamical equations that describe the ions and electrons as separate fluids.

In the drift ordering the gyro-viscous force appears at the same order as the acceleration, and must be retained in the model[2]. The neo-classical parallel viscous force also enters at the same order, and should be retained. An isotropic "collisional" viscosity is also included. The same remarks apply here as in the case of MHD.

The power of the drift ordering is evident when the velocity is decomposed into parallel, $\mathbf{E}\times\mathbf{B}$ drift, and diamagnetic drift components. With the aid of the gyro-viscous cancellation (see Equation (8.4-5)), the ion and electron equations can be put into a form that looks like the MHD equations with only slight corrections (see Equations (8.10a-f)). The major correction to the ion equation is a term of the form $nm\mathbf{V}_{*i}\cdot\nabla\mathbf{V}$, which introduces the ion diamagnetic drift frequency. The major correction to the electrons is the elimination due to high order force balance of the whistler branch (which comes from the Hall term $\mathbf{J}\times\mathbf{B}$). The frequency of the whistler waves is too high to be captured in the drift ordering. However, the remaining term, $-\nabla_{\|}p_e/n$, introduces kinetic Alfvén waves (see Section 10), which, like the whistlers, are dispersive ($w^2 \sim k^4$).

A generalized drift ordering has been presented[17] in which the Ohms law is exact to all orders in d, and the perpendicular velocity contains the polarization drift as well as the diamagnetic and $\mathbf{E}\times\mathbf{B}$ drifts. This model also eliminates the whistler branch, but still allows dispersive kinetic Alfvén waves.

While the drift ordering (as well as the generalized drift ordering[17]) formally requires small flows, it is possible to introduce an ordering $x \sim 1$, $e \sim d^2$ that allows fast flows but retains the form of the drift equations. The only difference is the relative size of the various terms in the model. Thus the form of the drift model equations can also accommodate fast flows ($V_0 \sim V_{thi}$).

A goal of two-fluid modeling is to produce a set of equations that contains the essential corrections to MHD (such as the ion drifts) while eliminating any high frequency parasitic waves (such as whistlers and kinetic Alfvén waves) that are not essential for the physics but must be dealt with in a numerical algorithm. While we have succeeded in eliminating analytically the fastest of these (the whistler branch), the slightly lower frequency KAW remains. Their dispersion relation makes them unacceptable for explicit time advancement and requires implicit (or semi-implicit) methods. Options for dealing with these modes are discussed below.

One option is to retain the full two-fluid Ohm's law,

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \frac{1}{ne}\left(-\nabla p_e + \mathbf{J}\times\mathbf{B}\right) \quad , \tag{12.1}$$

and treat the Hall term with a well-known semi-implicit operator[18,23]. Work in this regard is underway[23]. As speculated in Section 10, it seems likely that this operator may also stabilize the KAW, but this remains to be demonstrated. (In this case it is only consistent to use $\mathbf{V}_{*i}$ instead of $\mathbf{V}_{di}$ in the equation of motion.)

A second option is to use the Ohm's law of the generalized drift model,

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} - \frac{1}{ne}\nabla_{\|}p_e \quad , \tag{12.2}$$

where $\mathbf{V} = \mathbf{V}_i - \mathbf{V}_{di}$ and $\mathbf{V}_{di} = \dfrac{1}{ne}\mathbf{J}_\perp + \mathbf{V}_{*e}$, and use a semi-implicit advance based on Equation (10.9) to stabilize the KAW branch. This seems to be more accurate (exact Ohm's law and all ion drifts) than the approach based on Equation (12.1), while requiring no significant extra computational effort (the new semi-implicit term is just a slight modification of the Hall semi-implicit operator).

In any case, the FLR corrections to the ion dynamics are captured by the drift modification to the equation of motion, i.e.,

$$
\begin{aligned}
nm\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) &= -\nabla p + \mathbf{J}\times\mathbf{B} + nm\mu\nabla^2\mathbf{V} - \mathbf{b}\cdot\nabla\cdot\Pi_i^{nc} \\
&\quad \underbrace{-nm\mathbf{V}_{di}\cdot\nabla\mathbf{V}_\perp - \nabla\chi}_{FLR\ \ corrections} \quad ,
\end{aligned}
\tag{12.3}
$$

where $\chi = -(p/2\Omega)\mathbf{b}\cdot\nabla\times\mathbf{V}_\perp$ appears as a correction to the pressure. The extra advective term can be readily included in a predictor-corrector algorithm.

We note that a third option would be to use the full Ohm's law, Equation (12.1), and retain the full gyro-viscous stress (Equation (3.2b)) in the momentum equation (i.e., perform the gyro-viscous cancellation numerically). This approach is appropriate for dealing with more general situations that may significantly depart from equilibrium (and where the assumptions involved in the gyro-viscous cancellation do not apply). However, the drift model is well adapted to the special case of tokamak dynamics and is recommended for those applications.

The equations to be implemented in any model will be written in dimensional form. The relative ordering introduced in Section 2 will now appear as relative sizes of the dependent variables.

From the beginning we have neglected the friction force in the equations of motion. The primary effect is to modify Ohm's law to be[17]

$$
\mathbf{E} = \eta\mathbf{J}^* - \mathbf{V}\times\mathbf{B} - \frac{1}{ne}\nabla_\| p_e \quad , \tag{12.4}
$$

where η is the resistivity and

$$
\mathbf{J}^* = \mathbf{J} - \frac{3}{2}\frac{n}{B^2}\mathbf{B}\times\nabla T_e \quad . \tag{12.5}
$$

The MHD velocity then becomes $\mathbf{V}_E = (\mathbf{E} - \eta\mathbf{J}^*)\times\mathbf{B}/B^2$. This can be implemented with little extra effort.

The proposed two-fluid computational model thus consists of (12.2-5), along with the modified continuity equation (Equation (8.10a)), Ampere's law and Faraday's law, and a semi-implicit operator based on Equation (10.9). Appropriate forms of the energy equations for p_e and p_i must also be derived. We remark as before that the specific form of the equations depends on assumptions (such as the gyro-viscous cancellation in a sheared slab magnetic field) that are only approximate in toroidal geometry. However, such uncertainties in the details of the formulation may be masked by the presence of the isotropic artificial viscosity that is introduced for numerical purposes. Further, the model has the virtues that both the ion drifts and Ohm's law are exact to all orders of d, that the modifications to the momentum equation are simple and their physical meaning is clear, and that the semi-implicit operator is only a slight modification of one that is presently being implemented[24].

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