Progress on M3D-C1

an implicit 2-fluid code for high-order equations

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Extended MHD Models

Model	Momentum Equation	Ohm's law	Whist-lers[1]	KAW[2]	GV[3]	Slow dynamics[4]
General	$mn\frac{d\mathbf{V}}{dt} = -\nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i}) - \nabla\cdot\Pi_i^{gv}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J} + \frac{1}{ne}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e - \nabla\cdot\Pi_{\parallel e}\right)$	Yes	Yes	Yes	Either
Generalized Hall MHD[5]	$mn\frac{d\mathbf{V}}{dt} = -\nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i})$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J} + \frac{1}{ne}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e - \nabla\cdot\Pi_{\parallel e}\right)$	Yes	Yes	No	No
Neoclassical-MHD	$mn\frac{d\mathbf{V}}{dt} = -\nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i}) - \nabla\cdot\Pi_i^{gv}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J} - \frac{1}{ne}\nabla\cdot\Pi_{\parallel e}$	No	No	Yes	Yes
Generalized resistive MHD[5]	$mn\frac{d\mathbf{V}}{dt} = -\nabla p + \mathbf{J}\times\mathbf{B} - \nabla\cdot\Pi_{\parallel}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J}$	No	No	No	No
Generalized drift[6]	$mn\frac{d\mathbf{V}}{dt} = -mn\mathbf{V}_{di}\cdot\nabla\mathbf{V}_{\perp} + \upsilon_{gv} + nm\mu\nabla_{\perp}^2\mathbf{V} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i}) - \nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J}^{*} - \frac{1}{ne}\left[\nabla_{\parallel} p_e + \nabla\cdot\Pi_{\parallel e}\right]$	No	Yes	Yes	Yes

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Higher order modes present in Extended MHD models present new numerical challenges

Mode	Origin	Wave Equation	Dispersion	Comments
Whistler	in Ohm $\mathbf{J}\times\mathbf{B}$	$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\left(\frac{V_A^2}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2 \nabla^2 \mathbf{B}$	$\omega^2 = V_A^2 k^2 \left[1 + \frac{1}{\beta}(\rho_i k_\parallel)^2\right]$	•electron response •finite $k_\parallel$
KAW	in Ohm $\nabla_\parallel p_e$	$\frac{\partial^2 \mathbf{B}}{\partial t^2} = \left(\frac{V_A V_{th*}}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2 \nabla\times[\mathbf{bb}\cdot\nabla\times\mathbf{B}]$	$\omega^2 = V_A^2 k_\parallel^2 \left[1 + (\rho_s k_\perp)^2\right]$	•ion and e⁻ response •finite $k_\parallel$ $k_\perp$
Parallel ion GV	η_4 term in $\nabla\cdot\Pi^{GV}$	$\rho \frac{\partial^2 \mathbf{V}_\perp}{\partial t^2} = -\eta_4^2 \nabla_\parallel^4 \mathbf{V}_\perp$	$\omega_{\frac{L\pm}{R\pm}} = V_A k_\parallel \left[\pm 1 \pm \frac{1+\beta}{2\sqrt{\beta}}(\rho_i k_\parallel)\right]$	•ion response •finite $k_\parallel$
Perp. ion GV	η_3 term in $\nabla\cdot\Pi^{GV}$	$\rho \frac{\partial^2 \mathbf{V}_\perp}{\partial t^2} = -\eta_3^2 \nabla_\perp^4 \mathbf{V}_\perp$	$\omega^2 = V_A^2 k_\perp^2 \left[1 + \frac{\gamma\beta}{2} + \frac{\beta}{16}(\rho_i k_\perp)^2\right]$	•ion response • finite $k_\perp$

M3D-C^1

A parallel, implicit, extended MHD code using C^1 finite elements

Advantages of C^1 elements:

- more compact (fewer unknowns per variable)
- higher order continuity (fewer variables)
- block matrix patterns (more efficient solution)

Developmental Stages

2D	Reduced (2-variable)	NL	cylinder	done-published
2D	Reduced (4-variable)	NL	cylinder	done-periodic bc being added
2D	Extended (6-variable)	NL	cylinder	In progress
3D	Extended (6-variable)	linear	cylinder	
3D	Extended (6-variable)	linear	torus	
3D	Extended (6-variable)	NL	torus	

Coding principles:

- Extension of M3D
    - …same variables
- Fortran 90
- isolate communication routines (MPI/Shmem, OpenMP)
- CVS
- CCA-friendly
- X1-friendly

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Element Order

If an element with typical size h contains a complete polynomial of order M, then the error will be at most of order h^{M+1}

This follows directly from a local Taylor series expansion:

$$\phi(x,y)=\sum_{k=0}^{M}\sum_{l=0}^{k}\frac{1}{l!(k-l)!}\left[\frac{\partial^k\phi}{\partial x^l\partial z^{k-l}}\right]_{x_0,z_0}(x-x_0)^l(z-z_0)^{k-l}+O(h^{M+1})$$

Thus, linear elements will be $O(h^2)$
quadratic elements will be $O(h^3)$
cubic elements will be $O(h^4)$
quartic elements will be $O(h^5)$
complete quintic elements will be $O(h^6)$

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Element Continuity

Theorem: A finite element with continuity C^{k-1} belongs to Hilbert space H^k, and hence can be used for differential operators with order up to $2k$

H^k means that derivatives exist up to order k

Continuity	Hilbert Space	Applicability
C^0	H^1	second order equations
C^1	H^2	fourth order equations

This applicability is made possible by performing integration by parts in the Galerkin method, shifting derivatives from the unknown to the trial function

The vast majority of the literature concerns C^0 elements, (including Spectral Elements, NIMROD, Glasser's SEL.

Here we concentrate on C^1 elements

recall:

$$\iint_{domain} v_i \left[\nabla \bullet f(x,y) \nabla \phi\right] dxdy = -\iint_{domain} f(x,y) \nabla v_i \bullet \nabla \phi dxdy$$

$$\iint_{domain} v_i \left[\nabla^2 f(x,y) \nabla^2 \phi\right] dxdy = \iint_{domain} f(x,y) \nabla^2 v_i \nabla^2 \phi dxdy$$

NOTE: requires the trial function have appropriate boundary conditions

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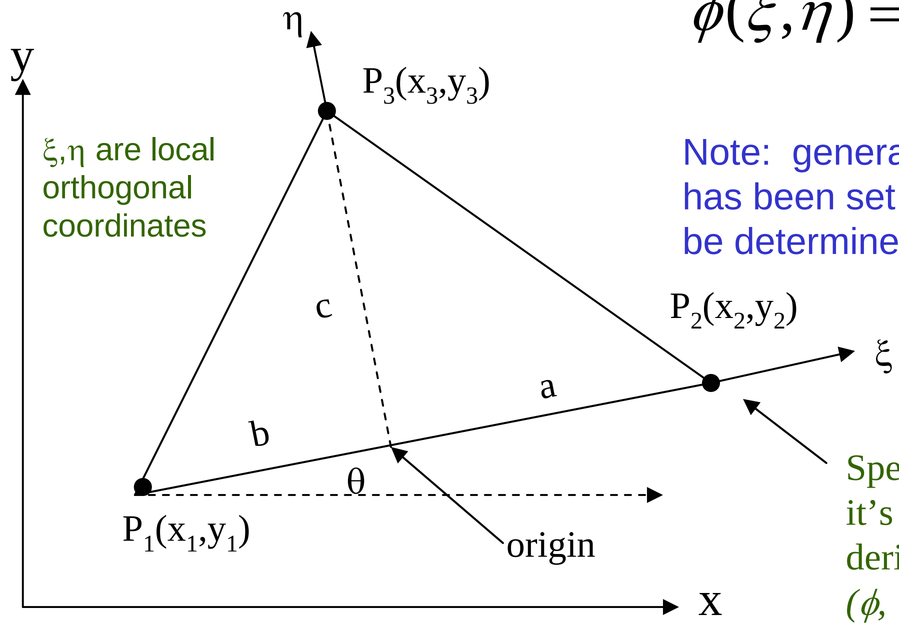
Reduced Quintic 2D Triangular Finite Element

$$\phi(\xi,\eta)=\sum_{k=1}^{20} a_k \xi^{m_k}\eta^{n_k}$$

Note: general quintic has 21 terms. 1 has been set to zero, and 2 others will be determined by constraints.



k	m_k	n_k
1	0	0
2	1	0
3	0	1
4	2	0
5	1	1
6	0	2
7	3	0
8	2	1
9	1	2
10	0	3
11	4	0
12	3	1
13	2	2
14	1	3
15	0	4
16	5	0
17	3	2
18	2	3
19	1	4
20	0	5

Specify the function and it's first and second derivatives at the 3 nodes $(\phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy})$

Guarantees C^0 continuity

For C^1, require that the normal slope along the edges ϕ_n have only cubic variation:

$$5b^4ca_{16} + (3b^2c^3 - 2b^4c)a_{17} + (2bc^4 - 3b^3c^2)\,a_{18} + (c^5 - 4b^2c^3)a_{19} - 5bc^4a_{20} = 0$$

$$5a^4ca_{16} + (3a^2c^3 - 2a^4c)a_{17} + (-2ac^4 - 3a^3c^2)\,a_{18} + (c^5 - 4a^2c^3)a_{19} - 5ac^4a_{20} = 0$$

20 – 2 = 18 unknowns:

These are determined in terms of [$\phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy}$] at P_1,P_2,P_3

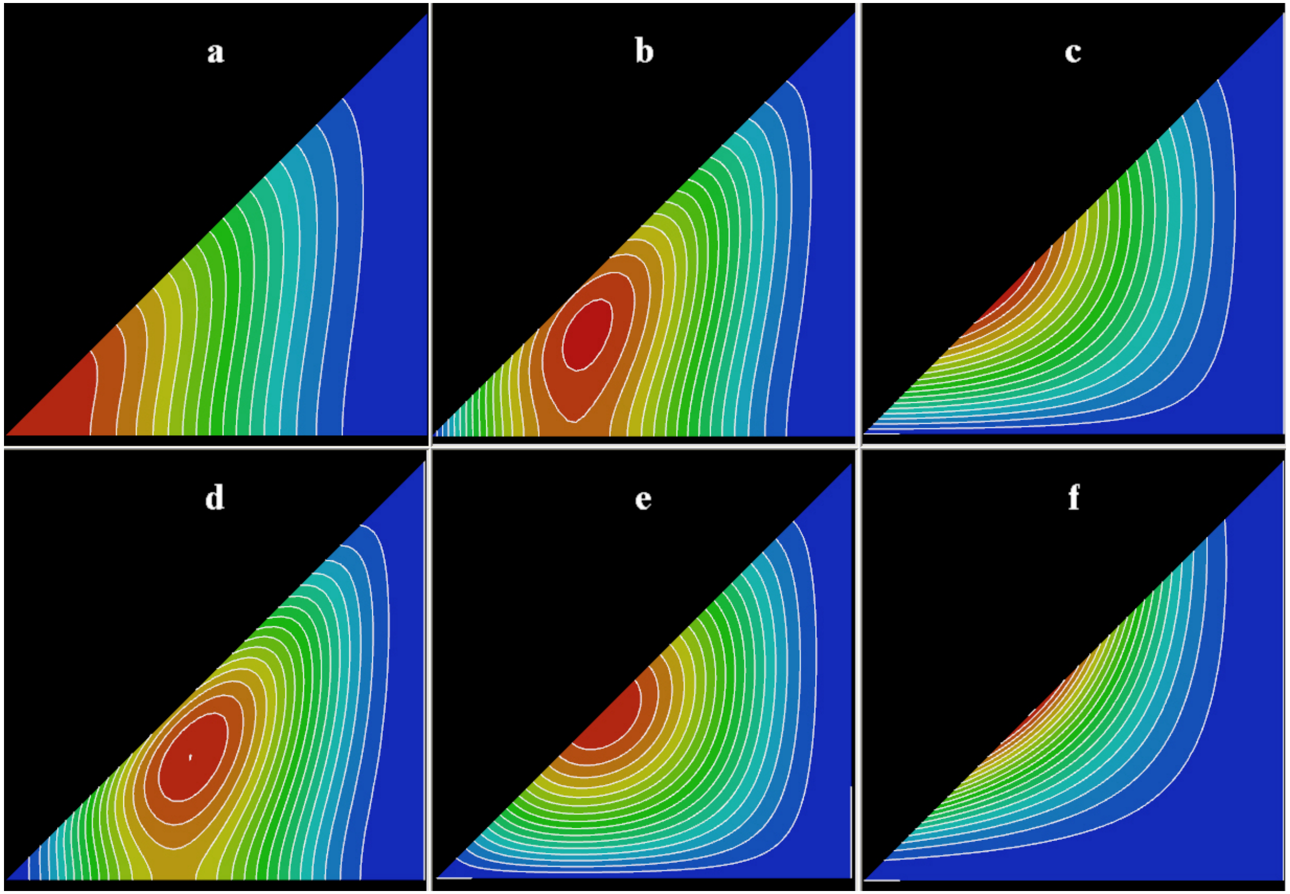
Implies C^1 continuity at edges and C^2 at nodes !

$a_i = g_{ij}\,\Phi_j$

The Trial Functions:

$$\phi = \sum_{i=1}^{20} a_i \,\xi^{m_i}\eta^{n_i} = \sum_{i=1}^{20}\sum_{j=1}^{18} g_{ij}\Phi_j \xi^{m_i}\eta^{n_i} = \sum_{j=1}^{18} v_j \Phi_j$$

$$v_j = \sum_{i=1}^{20} \xi^{m_i}\eta^{n_i} g_{ij}$$



These are the trial functions. There are 18 for each triangle.

The 6 shown here correspond to one node, and vanish at the other nodes, along with their derivatives

Each of the six has value 1 for the function or one of it's derivatives at the node, zero for the others.

Note that the function and it's derivatives (through 2nd) play the role of the amplitudes

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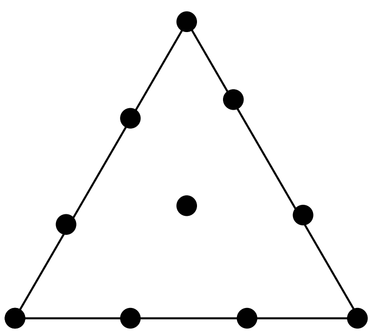
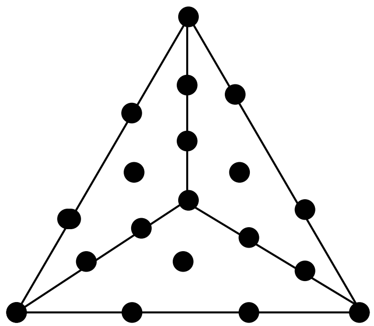
Comparison with a popular C^0 Element



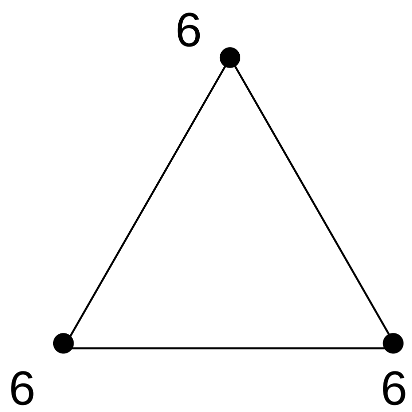
split



Lagrange Cubic: C^0, h^4

9 new unknowns: 2 new triangles

9/2 = $4^{1/2}$ unknowns/ triangle



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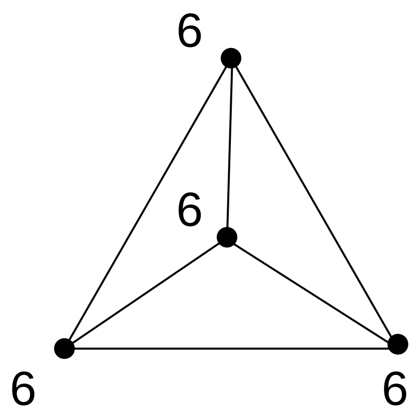

Reduced Quintic: C^1, h^5

6 new unknowns: 2 new triangles

6/2 = 3 unknowns/ triangle

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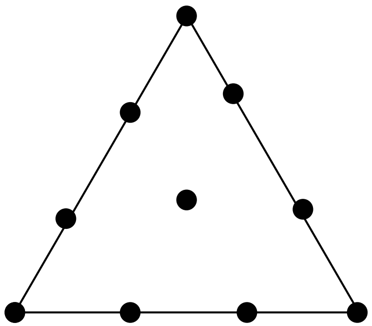
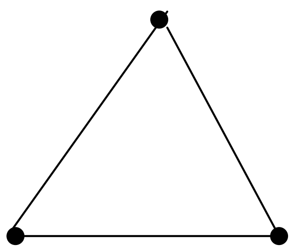
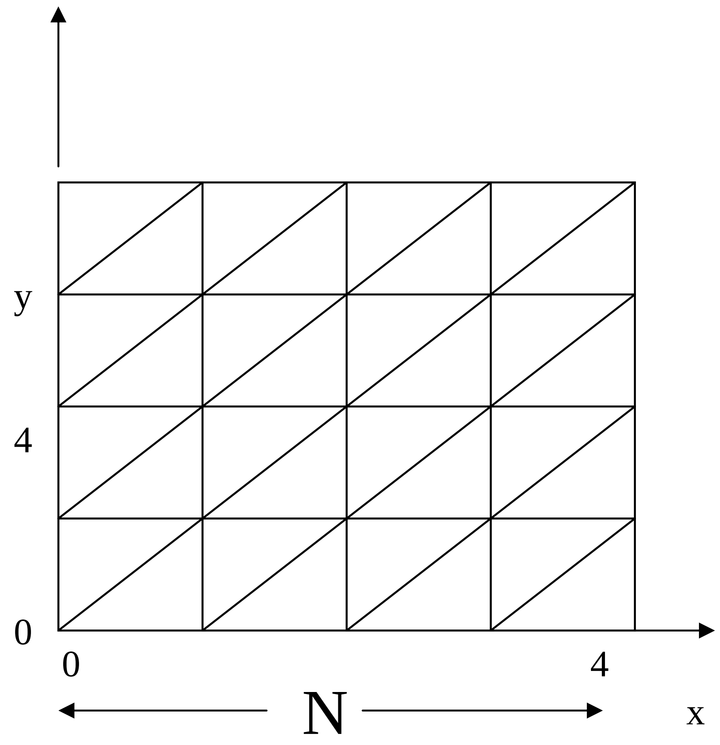
Comparison of reduced quintic to other popular triangular elements

	Vertex nodes	Line nodes	Interior nodes	accuracy order h^p	UK/T	continuity
linear element	3	0	0	2	½	C^0
Lagrange quadratic	3	3	0	3	2	C^0
Lagrange cubic	3	6	1	4	4½	C^0
Lagrange quartic	3	9	3	5	8	C^0
reduced quintic	18	0	0	5	3	C^1 *



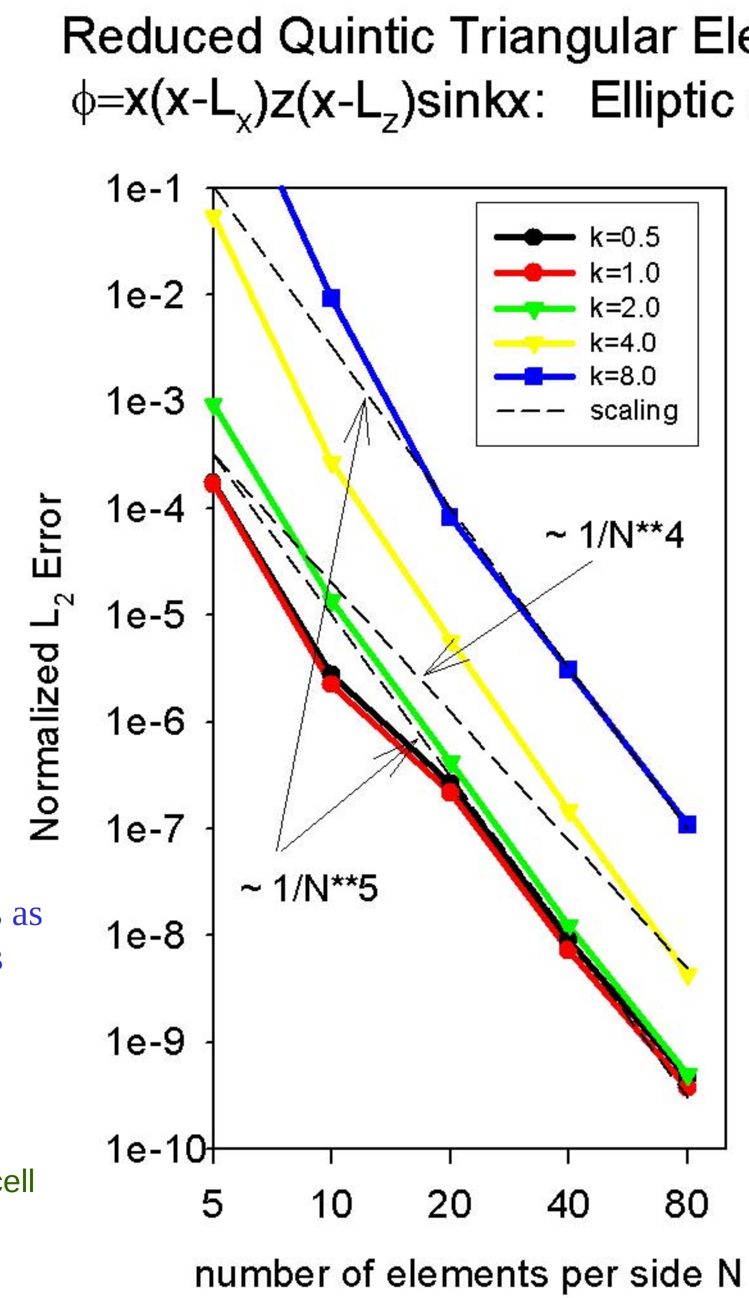


Results for Simple Problem



Verifies that error scales as N^{-5} for smooth functions

Note that there is 1 wavelength per cell when $k=\pi N/2$. For $k=8$, this is $N=5$

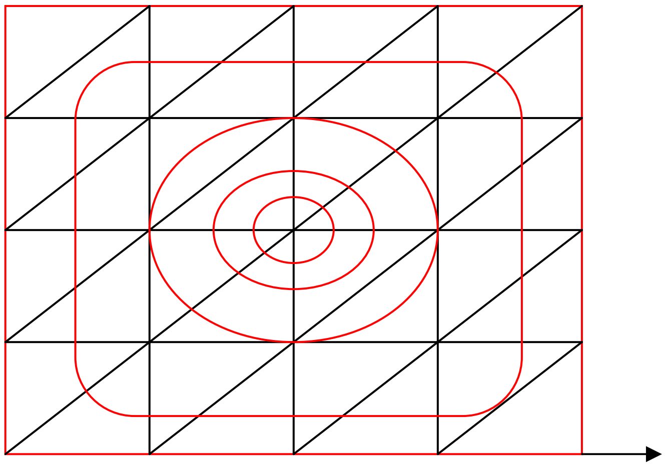


Anisotropic Diffusion

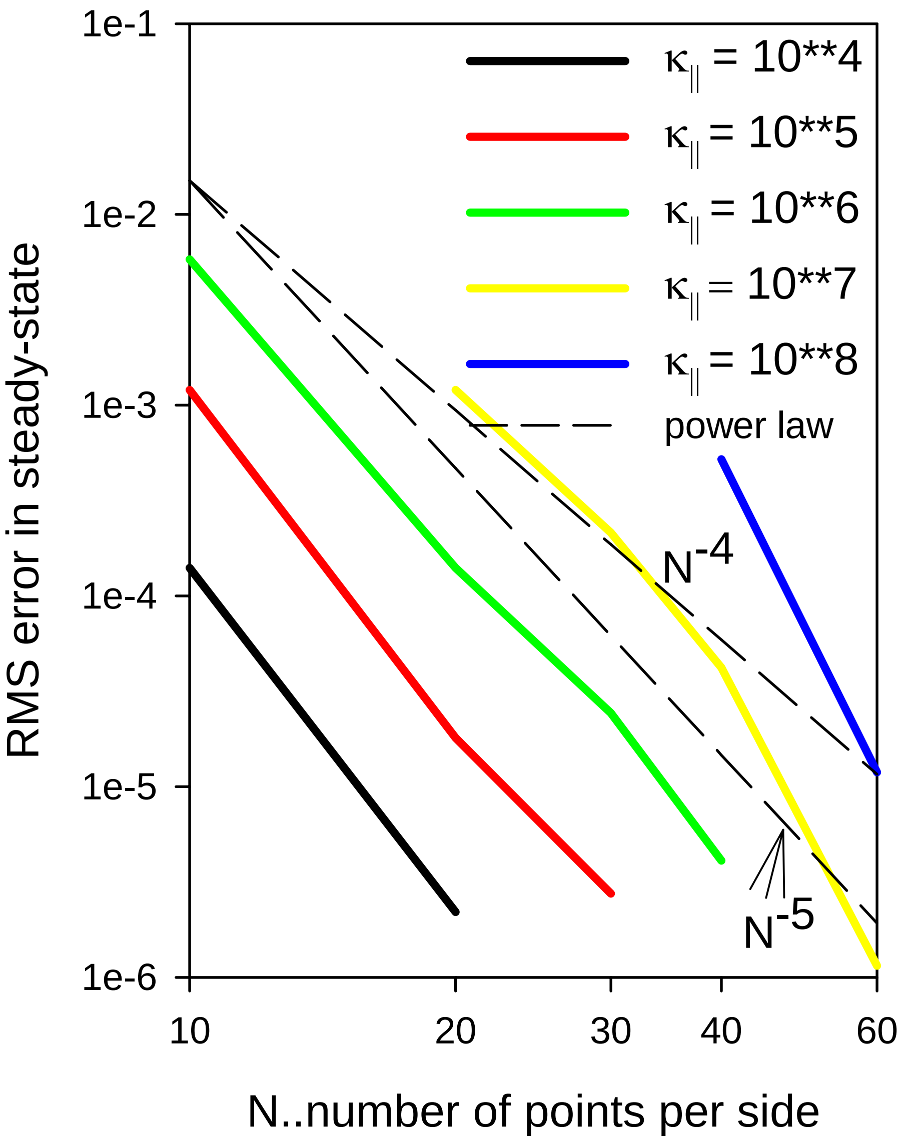
Shows greater than N^{-5} convergence

$$\frac{\partial \phi}{\partial t} = \nabla \bullet \left[\kappa_{\parallel} \frac{\vec{B}\vec{B}}{B^2} \bullet \nabla \phi \right] + \nabla \bullet \kappa \nabla \phi + S$$

Solve to steady state



$$S = \psi = \cos\frac{\pi x}{L} \cos\frac{\pi y}{L}$$



2D Incompressible MHD

$$\frac{\partial}{\partial t}\nabla^2\phi+\left[\nabla^2\phi,\phi\right]-\left[\nabla^2\psi,\psi\right]=\mu\nabla^4\phi$$

$$\frac{\partial\psi}{\partial t}+\left[\psi,\phi\right]=\eta\nabla^2\psi$$

note:

"reduced MHD"

ϕ is stream function

ψ is poloidal flux

θ-centering….time centered about n+1/2 for θ=0.5

$$\nabla^2\dot{\phi}+\left[\nabla^2\phi^n+\theta\delta t\nabla^2\dot{\phi},\phi^n+\theta\delta t\dot{\phi}\right]-\left[\nabla^2\psi^n+\theta\delta t\nabla^2\dot{\psi},\psi+\theta\delta t\dot{\psi}\right]$$
$$=\mu\left[\nabla^4\phi+\theta\delta t\nabla^4\dot{\phi}\right]$$

$$\dot{\psi}+\left[\psi^n+\theta\delta t\dot{\psi},\phi+\theta\delta t\dot{\phi}\right]=\eta\left[\nabla^2\psi^n+\theta\delta t\nabla^2\dot{\psi}\right]$$

$$\dot{\phi}=\frac{\phi^{n+1}-\phi^n}{\delta t},\qquad \dot{\psi}=\frac{\psi^{n+1}-\psi^n}{\delta t}\qquad \phi^n=\sum_{j=1}^{18}\nu_j\Phi_j^n\qquad \psi^n=\sum_{j=1}^{18}\nu_j\Psi_j^n$$

Multiply equations by each trial function
and integrate over space

$$\nu_j=\sum_{i=1}^{20}\xi^{m_i}\eta^{n_i}g_{ij}$$

Leads to the Matrix Implicit System

$$\begin{bmatrix} S_j^{11} & S_j^{12} \\ S_j^{21} & S_j^{22}\end{bmatrix}\begin{bmatrix}\Phi_j^{n+1}\\ \Psi_j^{n+1}\end{bmatrix} = \begin{bmatrix} D_j^{11} & D_j^{12}\\ D_j^{21} & D_j^{22}\end{bmatrix}\begin{bmatrix}\Phi_j^{n}\\ \Psi_j^{n}\end{bmatrix}$$

Solve each time step using SuperLU

$$\begin{bmatrix} S_j^{11} & S_j^{12} \\ S_j^{21} & S_j^{22}\end{bmatrix} = \begin{bmatrix} A_{i,j} + \theta\delta t[\bar{G}_{i,j,k}\Phi_k^* + \mu B_{i,j}] & -\theta\delta t\bar{G}_{i,j,k}\Psi_k^* \\ \theta\delta t K_{i,j,k}\Psi_k^* & M_{i,j} + \theta\delta t[K_{i,k,j}\Phi_k^* - \eta A_{i,j}] \end{bmatrix}$$

For linear problem, only need to form LU decomposition once and do a back-substitution each time step.

$$\begin{bmatrix} D_j^{11} & D_j^{12} \\ D_j^{21} & D_j^{22}\end{bmatrix} = \begin{bmatrix} \left\{\begin{matrix} A_{i,j} - \delta t[G_{i,j,k}\Phi_k^n - \theta\bar{G}_{i,j,k}\Phi_k^* \\ +(1-\theta)\mu B_{i,j}] \end{matrix}\right\} & \delta t(G_{i,j,k}\Psi_k^n - \theta\bar{G}_{i,j,k}\Psi_k^*) \\ \delta t K_{i,j,k}(-\tfrac{1}{2}\Psi_k^n + \theta\Psi_k^*) & \left\{\begin{matrix} M_{i,j} - \delta t[K_{i,k,j}(\tfrac{1}{2}\Phi_k^n - \theta\Phi_k^*) \\ -(1-\theta)\eta A_{i,j}] \end{matrix}\right\} \end{bmatrix}$$

$$\iint v_i(\xi,\eta)[\psi,\phi]\,d\xi d\eta = \sum_{j=1}^{18}\sum_{k=1}^{18} K_{i,j,k}\Psi_j\Phi_k$$

$$K_{i,j,k} = \sum_{p=1}^{20}\sum_{q=1}^{20}\sum_{r=1}^{20} g_{p,i}g_{q,j}g_{r,k}\left(m_q n_r - m_r n_q\right)F\left(m_p + m_q + m_r - 1, n_p + n_q + n_r - 1\right)$$

$$\iint v_i(\xi,\eta)\left[\nabla^2\psi,\psi\right]d\xi d\eta = \sum_{j=1}^{18}\sum_{k=1}^{18} G_{i,j,k}\Psi_j\Psi_k$$

$$G_{i,j,k} = \sum_{p=1}^{20}\sum_{q=1}^{20}\sum_{r=1}^{20} g_{p,i}g_{q,j}g_{r,k}\left(m_p n_r - m_r n_p\right)\begin{bmatrix} m_q(m_q-1)F\left(m_p + m_q + m_r - 3, n_p + n_q + n_r - 1\right) \\ +n_q(n_q-1)F\left(m_p + m_q + m_r - 1, n_p + n_q + n_r - 3\right)\end{bmatrix}$$

Note that stream function and vorticity are solved together

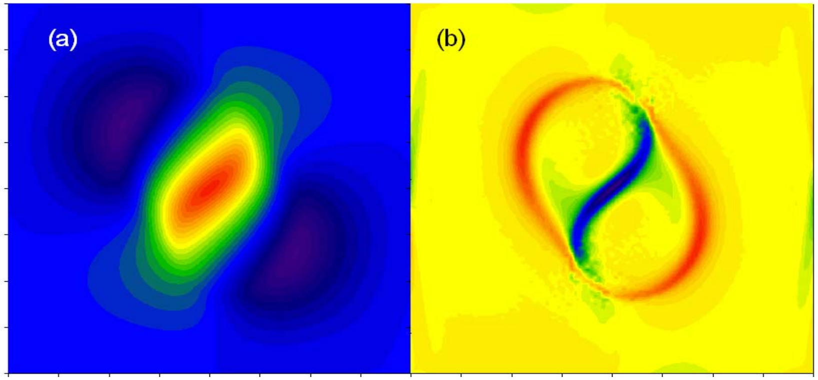
Tilting of a Plasma Column

Initial Condition:

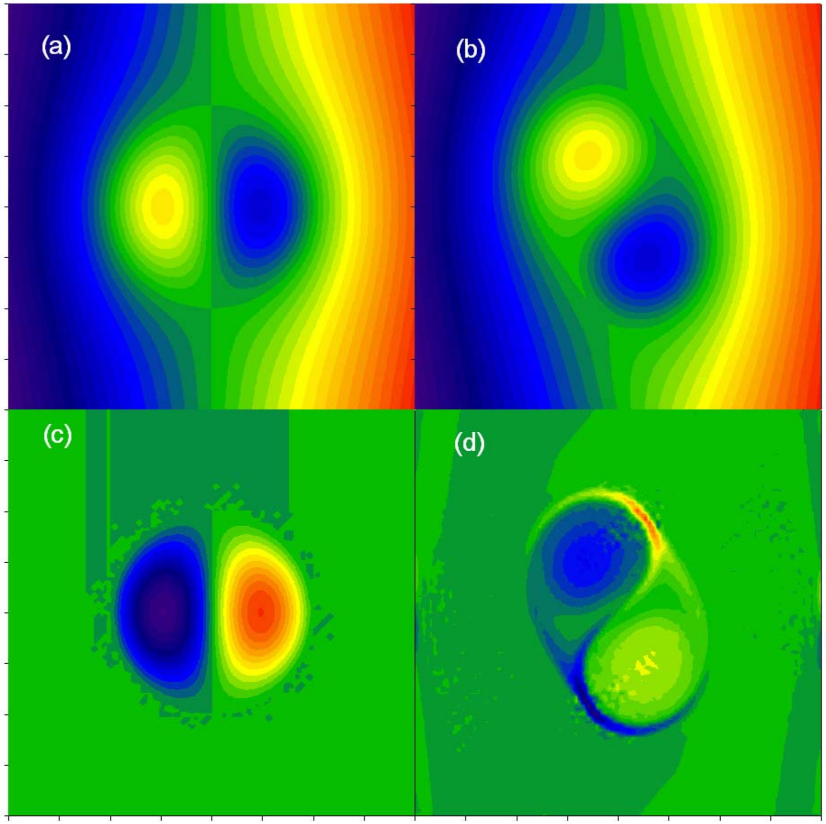
$$\psi = \begin{cases} [2/kJ_0(k)]J_1(kr)\cos\theta, & r<1 \\ (r-1/r)\cos\theta, & r>1 \end{cases}$$

$$J_1(k)=0$$

Give small perturbation and evolve in time

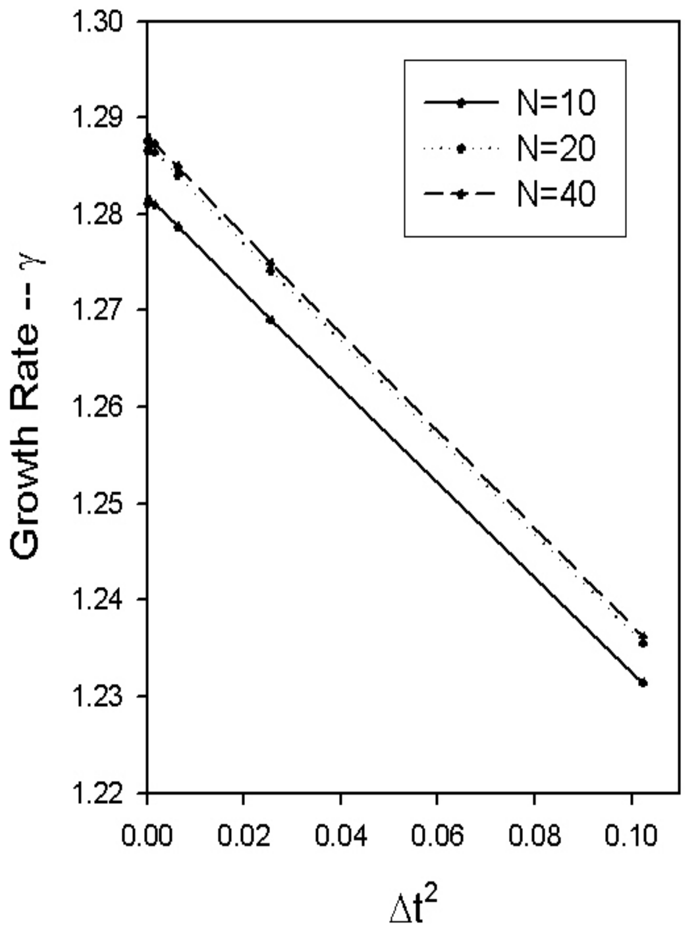


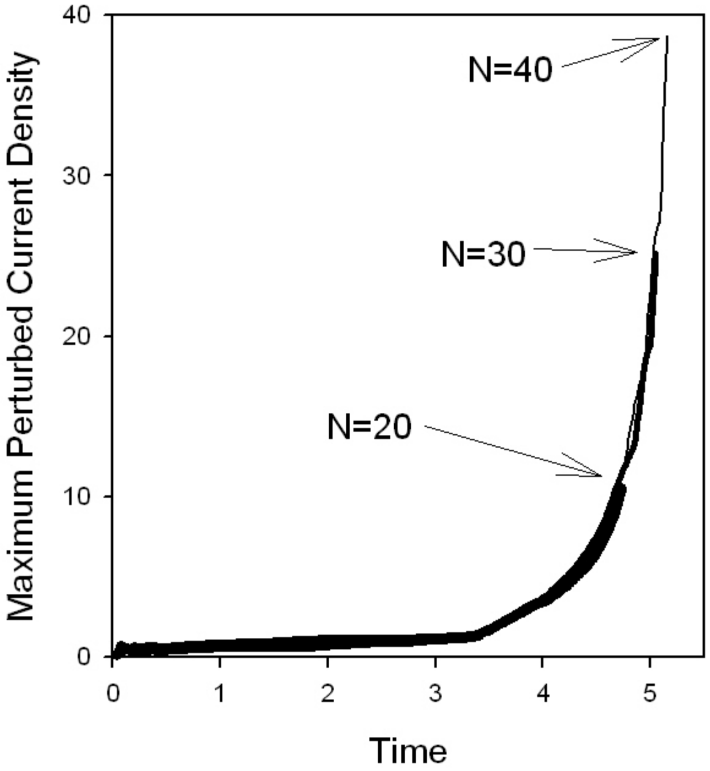
Stream function and vorticity at final time



Flux (top) and current (bottom) at initial and final times

Tilting of a Plasma Column-cont





Converged (in time) growth rate the same for N=30,40 out to 6 decimal places

Higher order formulation

By further manipulation, it is possible to get a 4th order PDE for Φ^{n+1} that is independent of Ψ^{n+1}…cuts matrix sizes down by 2

$$\left\{\nabla^2 + \theta\delta t L_1 + (\theta\delta t)^2 L_2\right\}\tilde{\Phi}^{n+1} = \left\{\nabla^2 + \theta\delta t L_1 + \theta(\theta-1)\delta t^2 L_2\right\}\tilde{\Phi}^{n}$$
$$-\theta\delta t^2 L_2 \Phi^0 + \delta t R$$

$$S'^{22}\tilde{\Psi}^{n+1} = D'^{22}\tilde{\Psi}^{n} - S'^{21}\tilde{\Phi}^{n+1} + D'^{21}\tilde{\Phi}^{n} +$$

$$L_1\tilde{\Phi}^{n+1} = \left[\nabla^2\tilde{\Phi}^{n+1},\tilde{\Phi}\right] + \left[\nabla^2\tilde{\Phi}^{n+1},\Phi^0\right] + \left[\nabla^2\tilde{\Phi},\tilde{\Phi}^{n+1}\right] + \left[\nabla^2\Phi^0,\tilde{\Phi}^{n+1}\right] - \mu\nabla^4\tilde{\Phi}^{n+1}$$

$$L_2\tilde{\Phi}^{n+1} = \left[\nabla^2\tilde{\Psi} + \nabla^2\Psi^0\left[\tilde{\Psi}+\Psi^0,\tilde{\Phi}^{n+1}\right]\right] - \left[\left[\tilde{\Phi}^{n+1},\nabla^2\tilde{\Psi}+\nabla^2\Psi^0\right],\tilde{\Psi}+\Psi^0\right]$$
$$-\left[\left[\nabla^2\tilde{\Phi}^{n+1},\tilde{\Psi}+\Psi^0\right],\tilde{\Psi}+\Psi^0\right] - 2\left[\left[\tilde{\Phi}_x^{n+1},\tilde{\Psi}_x+\Psi_x^0\right],\tilde{\Psi}+\Psi^0\right]$$
$$-2\left[\left[\tilde{\Phi}_y^{n+1},\tilde{\Psi}_y+\Psi_y^0\right],\tilde{\Psi}+\Psi^0\right]$$

$$R = -\left[\nabla^2\tilde{\Phi}^n,\tilde{\Phi}\right] - \left[\nabla^2\Phi^0,\tilde{\Phi}^n\right] - \left[\nabla^2\tilde{\Phi}^n,\Phi^0\right]$$
$$+\left[\nabla^2\tilde{\Psi}^n,\tilde{\Psi}^n\right] + \left[\nabla^2\Psi^0,\tilde{\Psi}^n\right] + \left[\nabla^2\tilde{\Psi}^n,\Psi^0\right] + \mu\nabla^4\tilde{\Phi}$$

Gives same results in 1/8th – 1/4th the time

Note: L_2 is 4th order

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M3D-C1 code has been set up in a general form, to allow non-trivial subsets of lower rank equations:

$$\begin{bmatrix} S_{11}^{v} & S_{12}^{v} & S_{13}^{v} \\ S_{21}^{v} & S_{22}^{v} & S_{23}^{v} \\ S_{31}^{v} & S_{32}^{v} & S_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^{v} & D_{12}^{v} & D_{13}^{v} \\ D_{21}^{v} & D_{22}^{v} & D_{23}^{v} \\ D_{31}^{v} & D_{32}^{v} & D_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^{v} & R_{12}^{v} & R_{13}^{v} \\ R_{21}^{v} & R_{22}^{v} & R_{23}^{v} \\ R_{31}^{v} & R_{32}^{v} & R_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n}$$

$$\begin{bmatrix} S_{11}^{p} & S_{12}^{p} & S_{13}^{p} \\ S_{21}^{p} & S_{22}^{p} & S_{23}^{p} \\ S_{31}^{p} & S_{32}^{p} & S_{33}^{p} \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^{p} & D_{12}^{p} & D_{13}^{p} \\ D_{21}^{p} & D_{22}^{p} & D_{23}^{p} \\ D_{31}^{p} & D_{32}^{p} & D_{33}^{p} \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^{p} & R_{12}^{p} & R_{13}^{p} \\ R_{21}^{p} & R_{22}^{p} & R_{23}^{p} \\ R_{31}^{p} & R_{32}^{p} & R_{33}^{p} \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n+1} + \begin{bmatrix} Q_{11}^{p} & Q_{12}^{p} & Q_{13}^{p} \\ Q_{21}^{p} & Q_{22}^{p} & Q_{23}^{p} \\ Q_{31}^{p} & Q_{32}^{p} & Q_{33}^{p} \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n}$$

Phase-I: Resistive MHD: done

$$\frac{\partial}{\partial t}\nabla^2\phi + \left[\nabla^2\phi, \phi\right] - \left[\nabla^2\psi, \psi\right] = \mu\nabla^4\phi$$

$$\frac{\partial \psi}{\partial t} + \left[\psi, \phi\right] = \eta\nabla^2\psi$$

Phase-II: Fitzpatrick-Porcelli model: now implemented- periodic bc being added

$$\frac{\partial}{\partial t}\nabla^2\phi = \left[\phi, \nabla^2\phi\right] + \left[\nabla^2\psi, \psi\right] + \mu\nabla^4\phi$$

$$\frac{\partial V_z}{\partial t} = \left[\phi, V_z\right] + c_\beta\left[I, \psi\right] + \mu\nabla^2 V_z$$

$$\frac{\partial \psi}{\partial t} = \left[\phi, \psi\right] + d_\beta\left[\psi, I\right] + \eta\nabla^2\psi$$

$$\frac{\partial I}{\partial t} = \left[\phi, I\right] + d_\beta\left[\nabla^2\psi, \psi\right] + c_\beta\left[V_z, \psi\right] + c_\beta^2\eta\nabla^2 I$$

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The 2D cylindrical two-fluid MHD equations and definition of the variables.

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\vec{E} + \vec{V} \times \vec{B} = \eta \vec{J} + \frac{1}{ne}\left(\vec{J} \times \vec{B} - \nabla p_e\right)$$

$$\mu_0 \vec{J} = \nabla \times \vec{B}$$

$$nM_i\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V}\right) + \nabla p = \vec{J} \times \vec{B} - \nabla \bullet \overleftrightarrow{\Pi}_i^{gv} + \mu n \nabla \bullet \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right]$$

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\vec{V}) = 0$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_e \vec{V}_i\right) = -p_e \nabla \bullet \vec{V}_i + \frac{\vec{J}}{ne} \bullet \left[\frac{3}{2}\nabla p_e - \frac{5}{2}\frac{p_e}{n}\nabla n + \vec{R}\right] - \nabla \bullet \vec{q}_e - Q_\Delta$$

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_i \vec{V}_i\right) = -p_i \nabla \bullet \vec{V}_i - \Pi_i : \nabla V_i + \nabla(\mu n \vec{V}) : \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right] - \nabla \bullet \vec{q}_i + Q_\Delta$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\frac{3}{2} p \vec{V}\right) = -p \nabla \bullet V - \Pi_i : \nabla V_i + \nabla(\mu n \vec{V}) : \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right] - \nabla \bullet (\vec{q}_i + \vec{q}_e)$$

$$+ \frac{\vec{J}}{ne} \bullet \left[\frac{3}{2}\nabla p_e - \frac{5}{2}\frac{p_e}{n}\nabla n + \vec{R}\right]$$

$$\vec{V} = \nabla \phi \times \hat{z} + \nabla_{\perp} \chi + v_{\varphi}$$

$$\vec{B} = \nabla \psi \times \hat{z} + I \hat{z}$$

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Numerical stability analysis for 2-fluid equations

$$\frac{\partial \vec{V}}{\partial t} = \vec{J} \times \vec{B}_0$$

$$\frac{\partial \vec{J}}{\partial t} = \nabla \times \frac{\partial \vec{B}}{\partial t} = \nabla \times \nabla \times \left[(\vec{V} - \vec{J}) \times \vec{B}_0 \right] = \vec{B}_0 \times \nabla^2 \vec{V} - \vec{B}_0 \times \nabla^2 \vec{J}$$

Whistler waves

Alfven Waves

$$\dot{V} = -\hat{z} \times \left\{ J + \theta \delta t \left[\hat{z} \times \nabla^2 (\vec{V} + \theta \delta t \dot{V}) - \hat{z} \times \nabla^2 \vec{J} \right] \right\}$$

$$\dot{J} = \hat{z} \times \nabla^2 \vec{V} - \hat{z} \times \nabla^2 (\vec{J} + \theta \delta t \dot{J})$$

$$\left[1 - (\theta \delta t)^2 \nabla^2 \right] (V^{n+1} - V^n) = \delta t \left\{ \theta \delta t \left[\nabla^2 V^n - \nabla^2 J^n \right] \right\} - \delta t \hat{z} \times J^n$$

$$\left[1 + \theta \delta t \hat{z} \times \nabla^2 \right] (J^{n+1} - J^n) = \delta t \hat{z} \times \nabla^2 [\theta V^{n+1} + (1-\theta) V^n] - \delta t \hat{z} \times \nabla^2 J^n$$

Note: these can be solved sequentially!

Note!

$$\begin{bmatrix} 1-(\theta\delta t)^2\nabla^2 & 0 & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \\ 0 & 1-(\theta\delta t)^2\nabla^2 & \\ 0 & \theta\delta t\nabla^2 & 1 \quad -\theta\delta t\nabla^2 \\ -\theta\delta t\nabla^2 & 0 & \theta\delta t\nabla^2 \quad 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ J_x \\ J_y \end{bmatrix}^{n+1} = \begin{bmatrix} 1-\theta(\theta-1)(\delta t)^2\nabla^2 & 0 & -\theta(\delta t)^2\nabla^2 & \delta t \\ 0 & 1-\theta(\theta-1)(\delta t)^2\nabla^2 & -\delta t & -\theta(\delta t)^2\nabla^2 \\ 0 & (\theta-1)\delta t\nabla^2 & 1 & -\delta t\nabla^2(\theta-1) \\ -(\theta-1)\delta t\nabla^2 & 0 & \delta t\nabla^2(\theta-1) & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ J_x \\ J_y \end{bmatrix}^{n}$$

Dispersion Relation

$$A = iB \qquad \delta t \nabla^2 \to E \quad \delta t^2 \nabla^2 \to \delta t E$$

$$A = A_0 + A_1 r + A_2 r^2 + A_3 r^3 = (r-1)\left[C_0 + C_1 r + C_2 r^2\right]$$

$$B = B_0 + B_1 r + B_2 r^2$$

$$A_0 = -1 + (1-\theta)^2 \left[\delta t E - E^2 + \theta^2 \delta t E^3\right]$$

$$A_1 = 3 - (1-\theta)(1-3\theta)\left[\delta t E - E^2 + \theta^2 \delta t E^3\right]$$

$$A_2 = -3 - \theta(2-3\theta)\left[\delta t E - E^2 + \theta^2 \delta t E^3\right]$$

$$A_3 = 1 - \theta^2 \left[\delta t E - E^2 + \theta^2 \delta t E^3\right]$$

$$B_0 = \delta t E^2 (1-\theta)(1-2\theta)$$

$$B_1 = \delta t E^2 \theta (3-4\theta)$$

$$B_2 = \delta t E^2 2\theta^2$$

$$C_0 = 1 - (1-\theta)^2 D$$

$$C_1 = -2[1 + \theta(1-\theta) D]$$

$$C_2 = 1 - \theta^2 D$$

$$D = \delta t E - E^2 + \theta^2 \delta t E^3$$

Wehave evaluated this dispersion relation numerically and find |r| ≤1 ; i.e. stability, for θ > ½

Summary

- Major upgrade to the M3D code is underway-based on C^1 finite elements
- Primary motivation is to allow efficient, high order, implicit solution of extended MHD equations with whister and KAW
- Staged implementation using reduced sets of equations with 2, 4, and then 6 variables
- Looks promising

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