Fully Implicit Solution of the full 2-fluid 2D MHD equations using high-order C^1 finite Elements

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We have developed a new numerical approach to solving the extended MHD equations using a compact representation that is specifically designed to yield efficient high-order-of-accuracy, implicit solutions of a the Extended-MHD equations.

The representation is based on a triangular finite element with fifth order accuracy that has continuous derivatives [C^1] across element boundaries, allowing its use with systems of equations containing spatial derivative operators of up to 4th order. The final set of equations is solved using the parallel sparse direct solver, Super-LU. The magnetic and velocity fields are decomposed in a potential, stream function form.

Subsets of the full set of 8 equations describing 2-fluid extended MHD yield (1) the 2-field reduced MHD equations, (2) the 4-field Fitzpatrick-Porcelli equations, and (3) the 6-field constant density, cold-ion model. Applications are presented showing the effect of "two-fluid" terms, compressibility, and a background "guide field" on 2D magnetic reconnection.

6-field model has now been implemented in 2D

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}, \qquad \vec{J} = \nabla \times \vec{B}$$

$$\vec{E} + \vec{V} \times \vec{B} = \frac{1}{ne}\left(\vec{R} + \vec{J} \times \vec{B} - \nabla p_e - \nabla \bullet \overleftrightarrow{\Pi}_e\right)$$

$$nM_i\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V}\right) + \nabla p_e = \vec{J} \times \vec{B} - \nabla \bullet \overleftrightarrow{\Pi}_i$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_e \vec{V}\right) = -p_e \nabla \bullet \vec{V} + \frac{\vec{J}}{ne} \bullet \left[\frac{3}{2}\nabla p_e + \vec{R} - \nabla \bullet \overleftrightarrow{\Pi}_e\right]$$

$$\vec{V} = \nabla U \times \hat{z} + \nabla_{\perp} \chi + V_z \hat{z},$$

$$\vec{B} = \nabla \psi \times \hat{z} + I \hat{z}$$

$$p_e$$

$$\vec{R} = \eta n e \vec{J}$$

$$\overleftrightarrow{\Pi}_i = -\mu n \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right] + h \mu n \nabla^2 \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right]$$

$$\overleftrightarrow{\Pi}_e = \lambda n e \left[\nabla \vec{J} + \nabla \vec{J}^{\dagger}\right]$$

Projections of the momentum equation:

$-\hat{z}\cdot\nabla\times$

$$n\nabla^2\dot{U}+n\left[\nabla^2U,U\right]+n\left(\nabla^2U,\chi\right)+n\nabla^2U\nabla^2\chi-\mu n\nabla^4U+\mu nh\nabla^4w+\left[\psi,\nabla^2\psi\right]=0$$

$\hat{z}\cdot$

$$n\dot{v}_\varphi+n\left[v_\varphi,U\right]+n\left(v_\varphi,\chi\right)-\mu n\nabla^2v_\varphi+\mu nh\nabla^4v_\varphi+\left[\psi,I\right]=0$$

$\nabla\bullet$

$$n\nabla^2\dot{\chi}+n\left[\nabla^2\chi,U\right]+\tfrac{1}{2}n\nabla^2\left|\nabla\chi\right|^2+2n\left(U_{xy}^2-U_{xx}U_{yy}+\chi_{xy}[U_{yy}-U_{xx}]+U_{xy}[\chi_{xx}-\chi_{yy}]\right)$$
$$-2\mu n\nabla^4\chi+2\mu nh\nabla^4\Delta+\left(\nabla^2\psi,\psi\right)+\left(\nabla^2\psi\right)^2+\nabla^2\left(\tfrac{1}{2}I^2+p\right)=0$$

$$w\equiv\nabla^2U,\quad \Delta\equiv\nabla^2\chi$$
$$\left[a,b\right]\equiv\hat{z}\bullet\nabla a\times\nabla b=a_xb_y-a_yb_x$$
$$\left(a,b\right)\equiv\nabla a\bullet\nabla b=a_xb_x+a_yb_y$$

Scalar Field Equations:

$$\dot{p}_e + [p_e, U] + \left(p_e, \chi\right) + \gamma p_e \nabla^2 \chi = \frac{1}{ne}\left[p_e, I\right] + S_e$$

$$\dot{\psi} + \left[\psi, U\right] + \left(\psi, \chi\right) = \eta \nabla^2 \psi - \lambda \nabla^4 \psi + \tfrac{1}{ne}\left[\psi, I\right]$$

$$\dot{I} + \left[I, U\right] + \left(I, \chi\right) + I \nabla^2 \chi + \left[\psi, v_\varphi\right] = \eta \nabla^2 I - \lambda \nabla^4 I + \tfrac{1}{ne}\left[\nabla^2 \psi, \psi\right]$$

$$S_e = \tfrac{2}{3}\left[\frac{1}{ne}\vec{J} \cdot (\vec{R} - \nabla \bullet \overleftrightarrow{\Pi}_e)\right]$$

Derivation of Implicit Equations

Taylor expand in time to get derivatives at advanced time. Use field equations to eliminate field time derivatives from momentum equation. For example:

$$
\begin{aligned}
& n\nabla^2\dot{U} + n\left[\nabla^2 U + \theta\delta t\nabla^2\dot{U}, U + \theta\delta t\dot{U}\right] + n\left(\nabla^2 U + \theta\delta t\nabla^2\dot{U}, \chi + \theta\delta t\dot{\chi}\right) \\
& \quad + n\left(\nabla^2 U + \theta\delta t\nabla^2\dot{U}\right)\left(\nabla^2\chi + \theta\delta t\nabla^2\dot{\chi}\right) + \left[\psi + \theta\delta t\dot{\psi}, \nabla^2\psi + \theta\delta t\nabla^2\dot{\psi}\right] \\
& \quad - \mu n(\nabla^4 U + \theta\delta t\nabla^4\dot{U}) = 0
\end{aligned}
$$

$$\dot{\psi} + \left[\psi, U + \theta\delta t\dot{U}\right] + \left(\psi, \chi + \theta\delta t\dot{\chi}\right) = S_\psi$$

Multiply by the time step, δt, and center the time derivatives about time *n+1/2,* so that , $\delta t\dot{U}_j = \left[U_j^{n+1} - U_j^n\right]$ etc.

Expand everything in C^1 finite elements: $U(x, y, t^n) = \sum_{j=1}^{18} \nu_j(x, y)U_j^n$

Multiply by each test function, integrate over domain, shift derivatives as needed, collect terms

Two systems of sparse matrix equations can be solved sequentially each time step

$$\begin{bmatrix} S_{11}^{v} & S_{12}^{v} & S_{13}^{v} \\ S_{21}^{v} & S_{22}^{v} & S_{23}^{v} \\ S_{31}^{v} & S_{32}^{v} & S_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^{v} & D_{12}^{v} & D_{13}^{v} \\ D_{21}^{v} & D_{22}^{v} & D_{23}^{v} \\ D_{31}^{v} & D_{32}^{v} & D_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^{v} & R_{12}^{v} & R_{13}^{v} & R_{14}^{v} \\ R_{21}^{v} & R_{22}^{v} & R_{23}^{v} & R_{24}^{v} \\ R_{31}^{v} & R_{32}^{v} & R_{33}^{v} & R_{34}^{v} \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ P \\ 1 \end{bmatrix}^{n}$$

Alfven wave physics

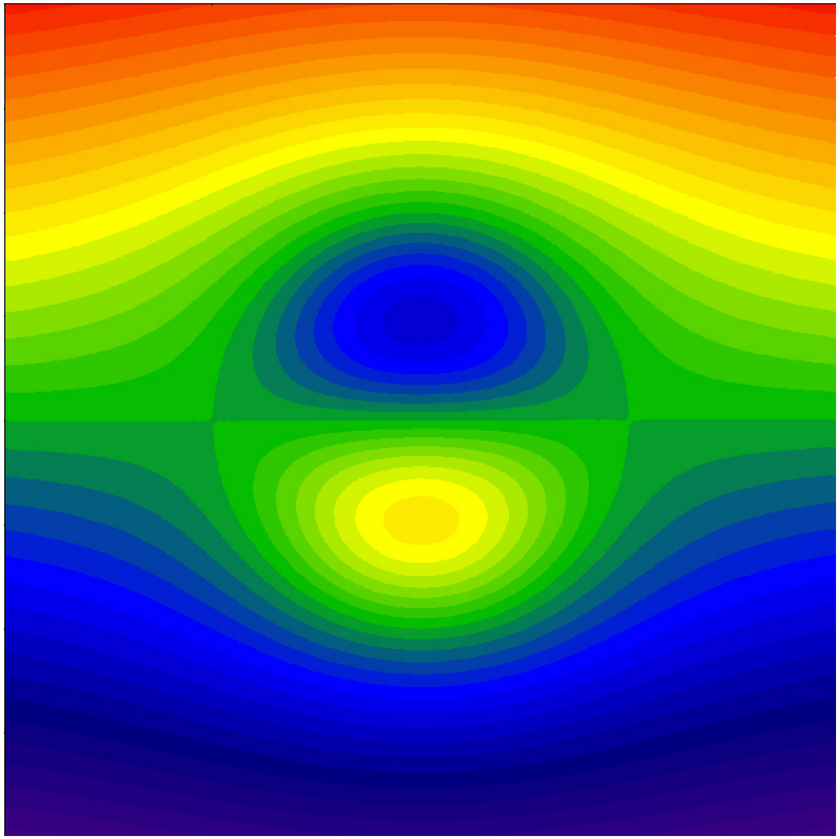
$$\begin{bmatrix} S_{11}^{b} & S_{12}^{b} & S_{13}^{b} \\ S_{21}^{b} & S_{22}^{b} & S_{23}^{b} \\ S_{31}^{b} & S_{32}^{b} & S_{33}^{b} \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ P_e \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^{b} & D_{12}^{b} & D_{13}^{b} \\ D_{21}^{b} & D_{22}^{b} & D_{23}^{b} \\ D_{31}^{b} & D_{32}^{b} & D_{33}^{b} \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ P_e \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^{b} & R_{12}^{b} & R_{13}^{b} \\ R_{21}^{b} & R_{22}^{b} & R_{23}^{b} \\ R_{31}^{b} & R_{32}^{b} & R_{33}^{b} \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n+1} + \begin{bmatrix} Q_{11}^{b} & Q_{12}^{b} & Q_{13}^{b} \\ Q_{21}^{b} & Q_{22}^{b} & Q_{23}^{b} \\ Q_{31}^{b} & Q_{32}^{b} & Q_{33}^{b} \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n}$$

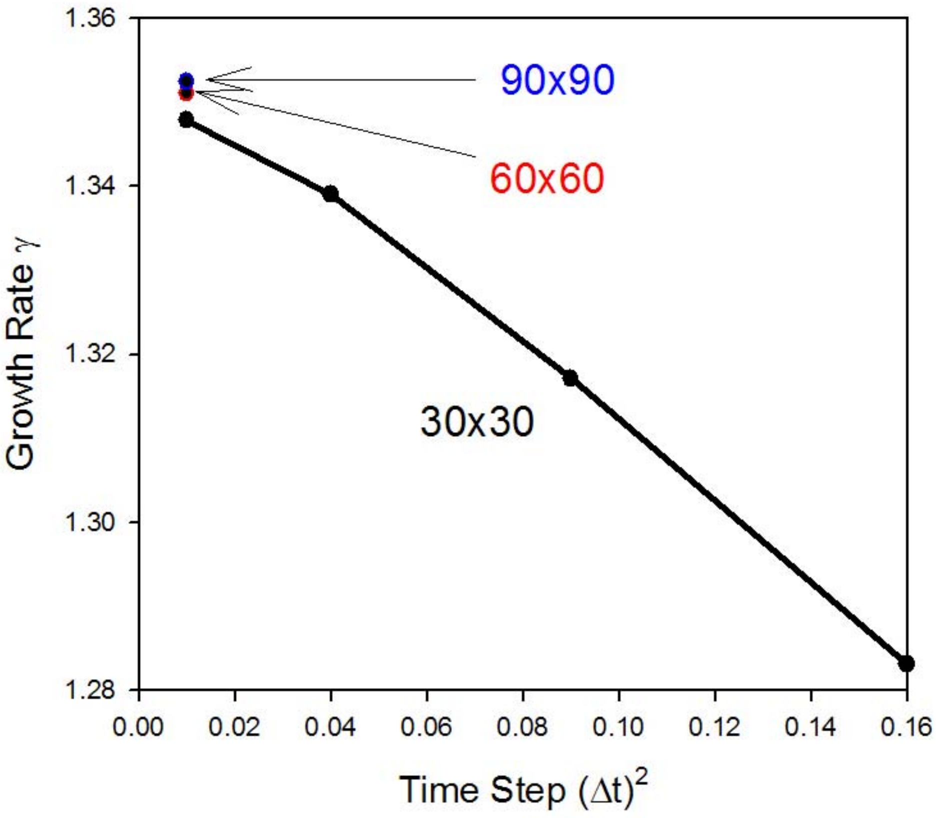
$$+ Q_{34}^{b} \begin{bmatrix} 0 \\ 0 \\ S_e \end{bmatrix}$$

Whistler wave physics

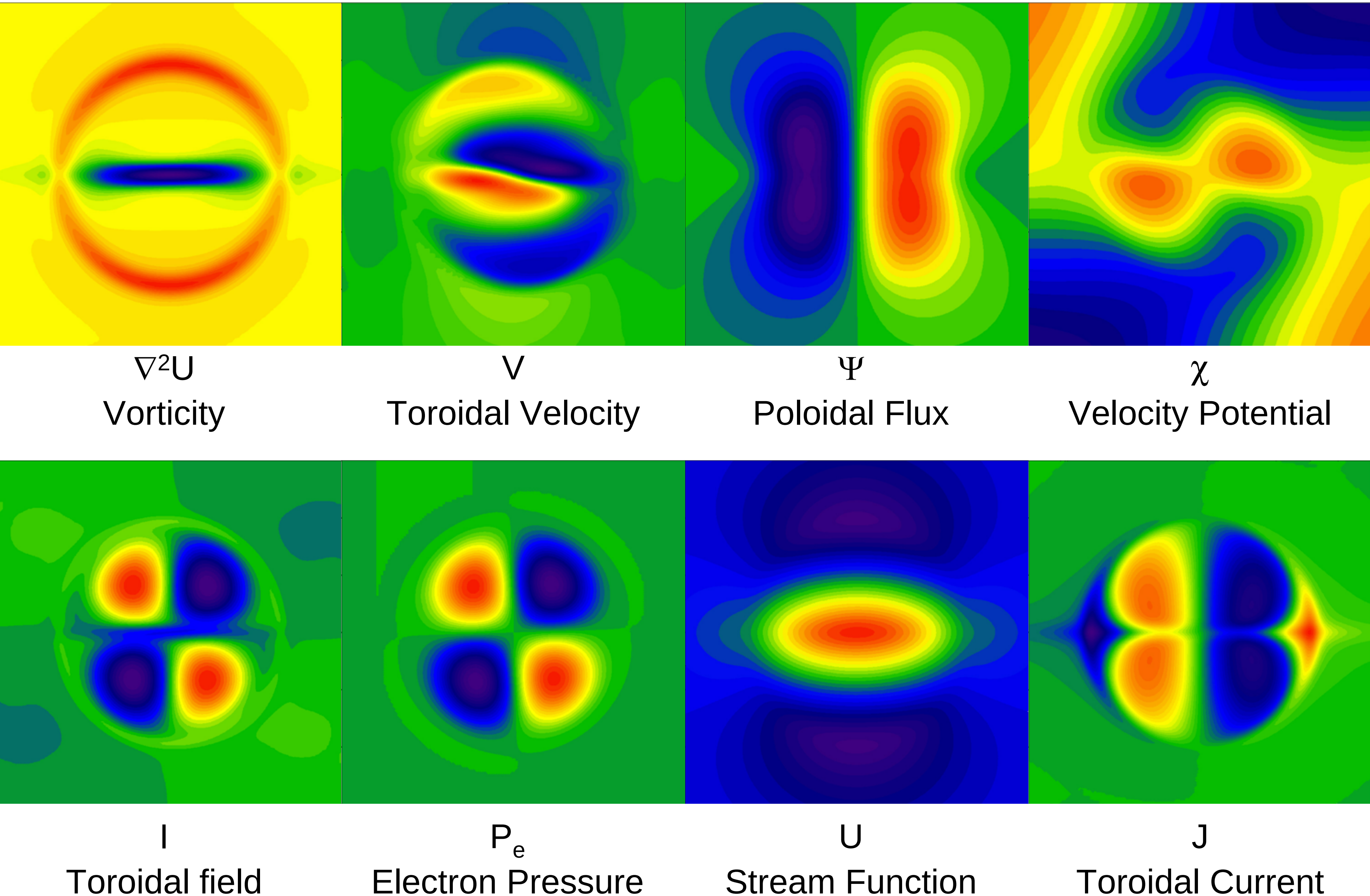
Linear analysis shows this to be stable for arbitrary time step

Tilting cylinder with 6-field 2-fluid model

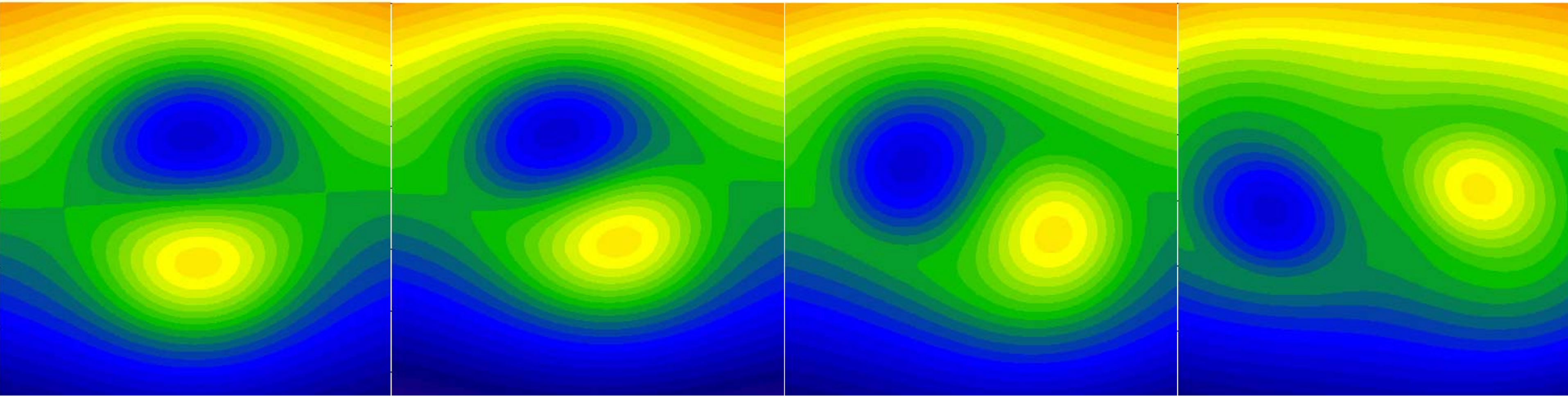




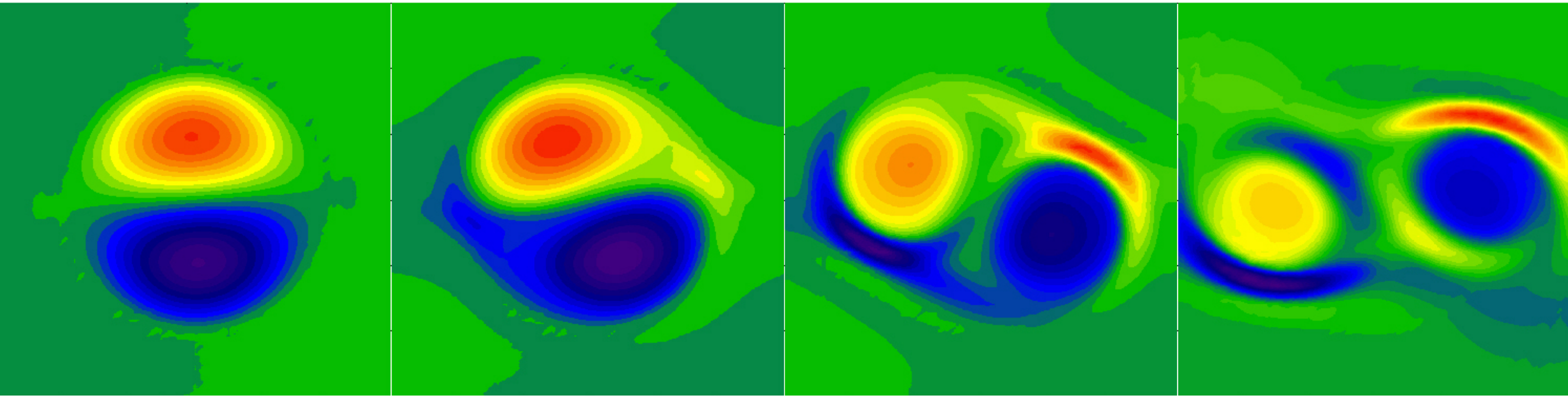
Linear eigenmode of tilting cylinder in 6-field 2-fluid model



Non-linear evolution of tilting cylinder in full 6-field 2-fluid model

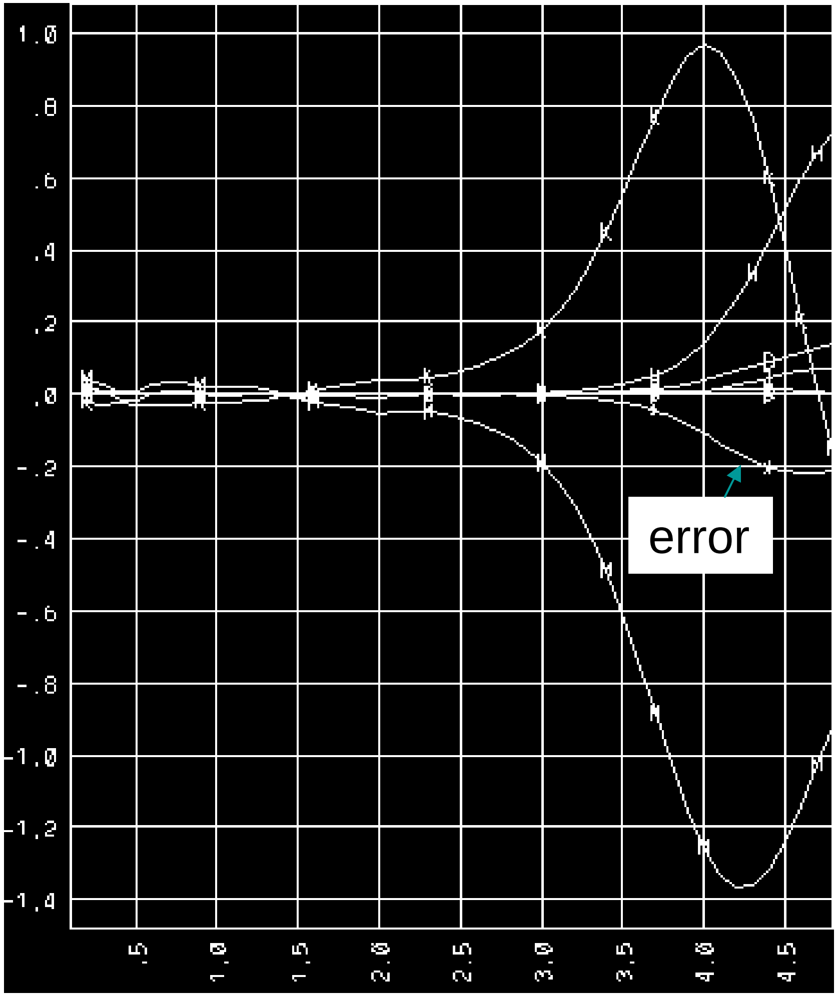


Ψ: t=0.8 Ψ: t=3.8 Ψ: t=4.0 Ψ: t=4.8

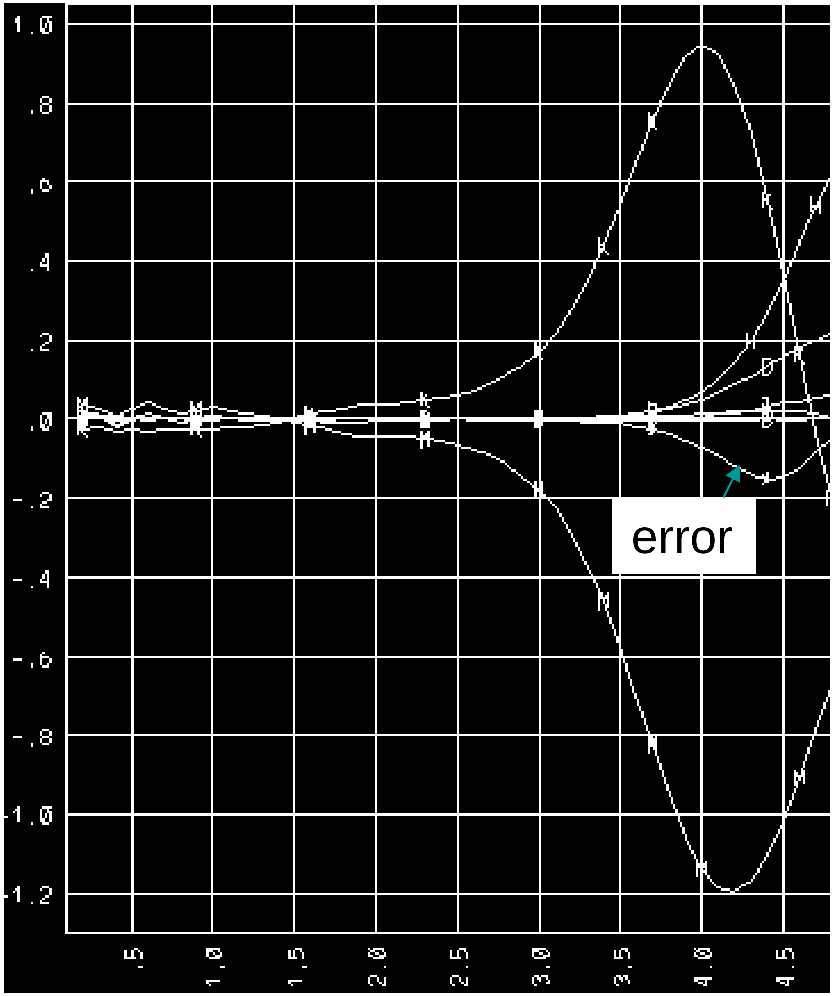


J: t=0.8 J: t=3.2 J: t=4.0 J: t=4.8

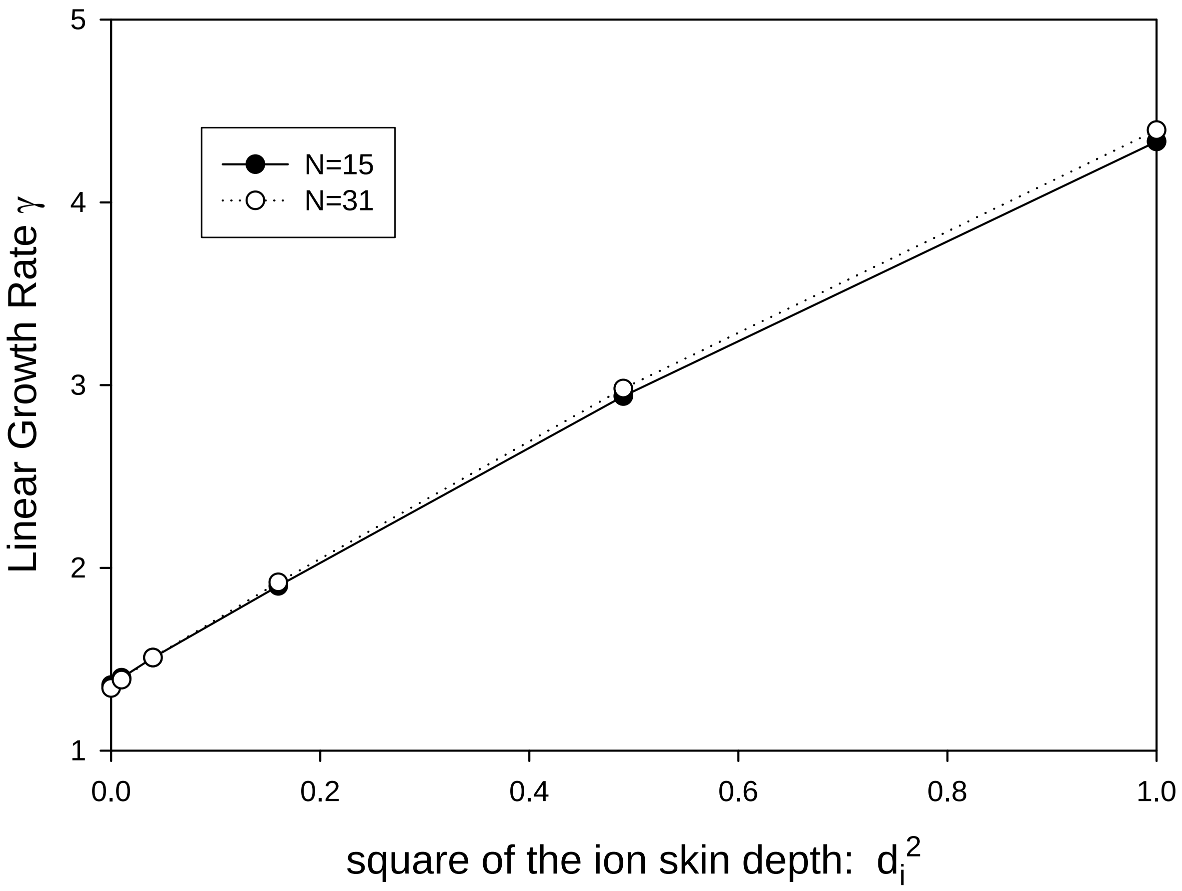
Energy error decreases with increasing number of nodes for sequence with hyper coef. H = C $(\Delta x)^2$



31 x 31 nodes



61 x 61 nodes



4-field (2-fluid) model predicts that growth rate of tilt mode increases linearly with the square of the ion skin depth d_i

Magnetic Reconnection

Modified GEM Challenge Problem:

$$\Psi(z) = \tfrac{1}{2}\log\left[\cosh\left(2z\right)\right] + \varepsilon\cos(k_x x)\cos\left(k_z z\right)$$

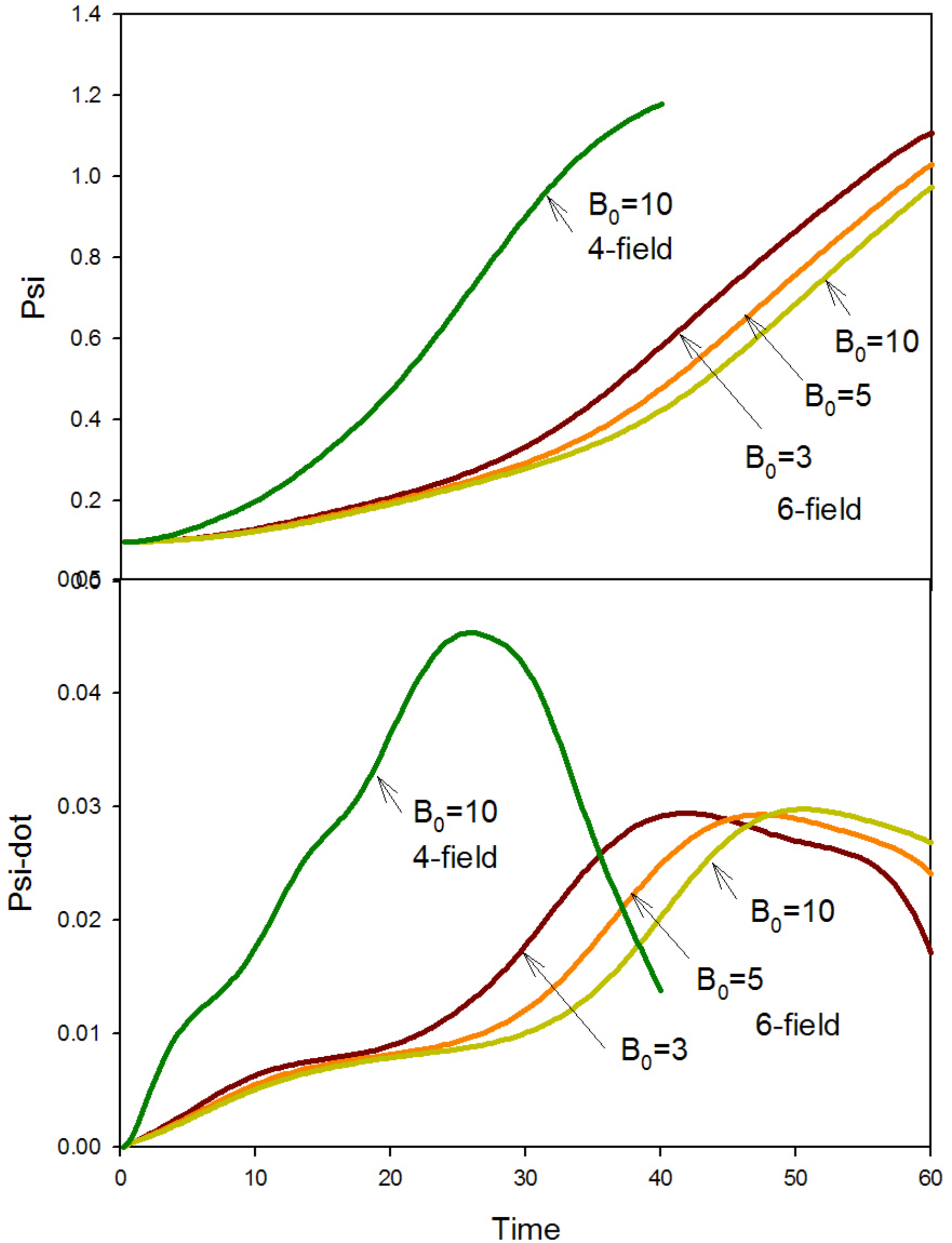
$$P_e(z) = 0.6 - \tfrac{1}{2}\tanh\left(2z\right)$$

$$P_i(z) = 0.$$

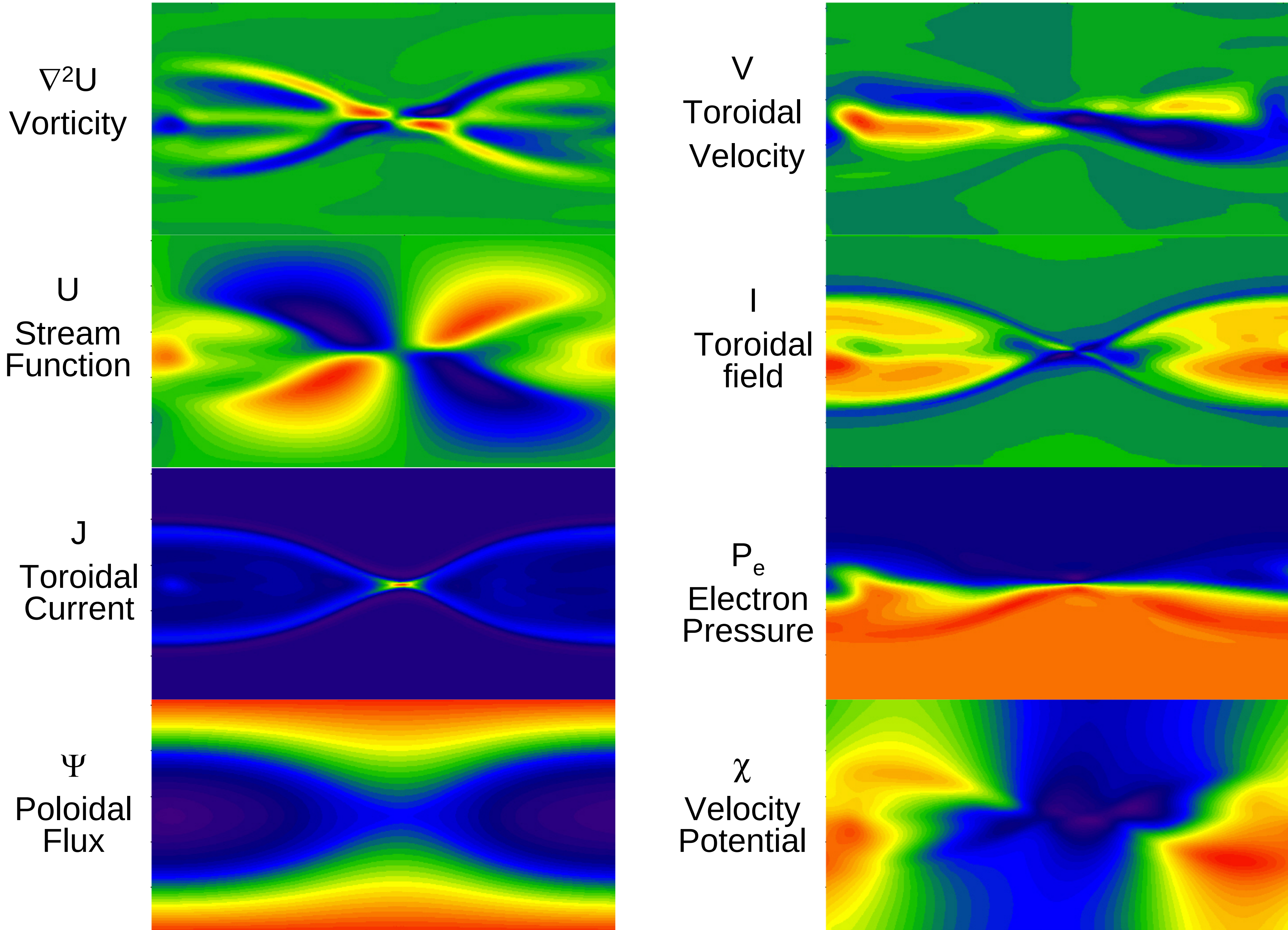
$$B_z = B_0$$

constant density n_0

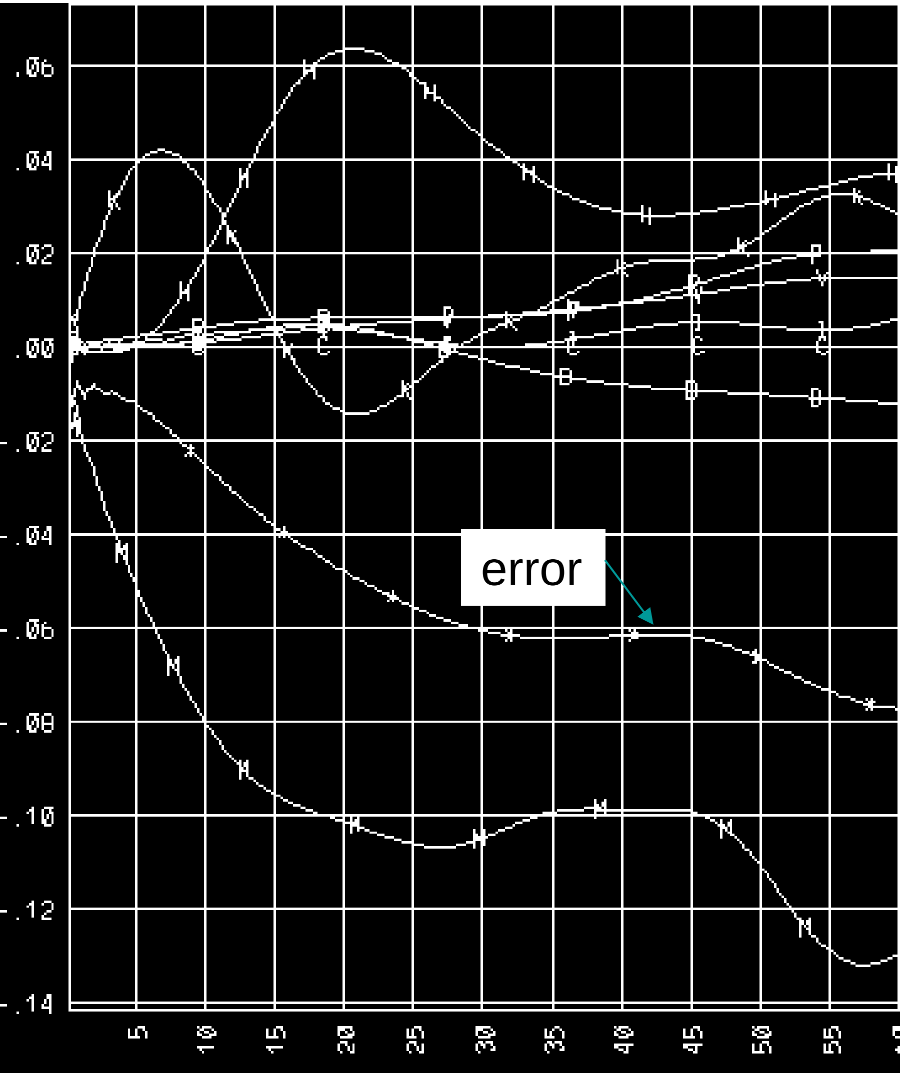
Full 6-field model gives lower reconnection rates than does reduced 4-field Fitzpatrick-Porcelli model



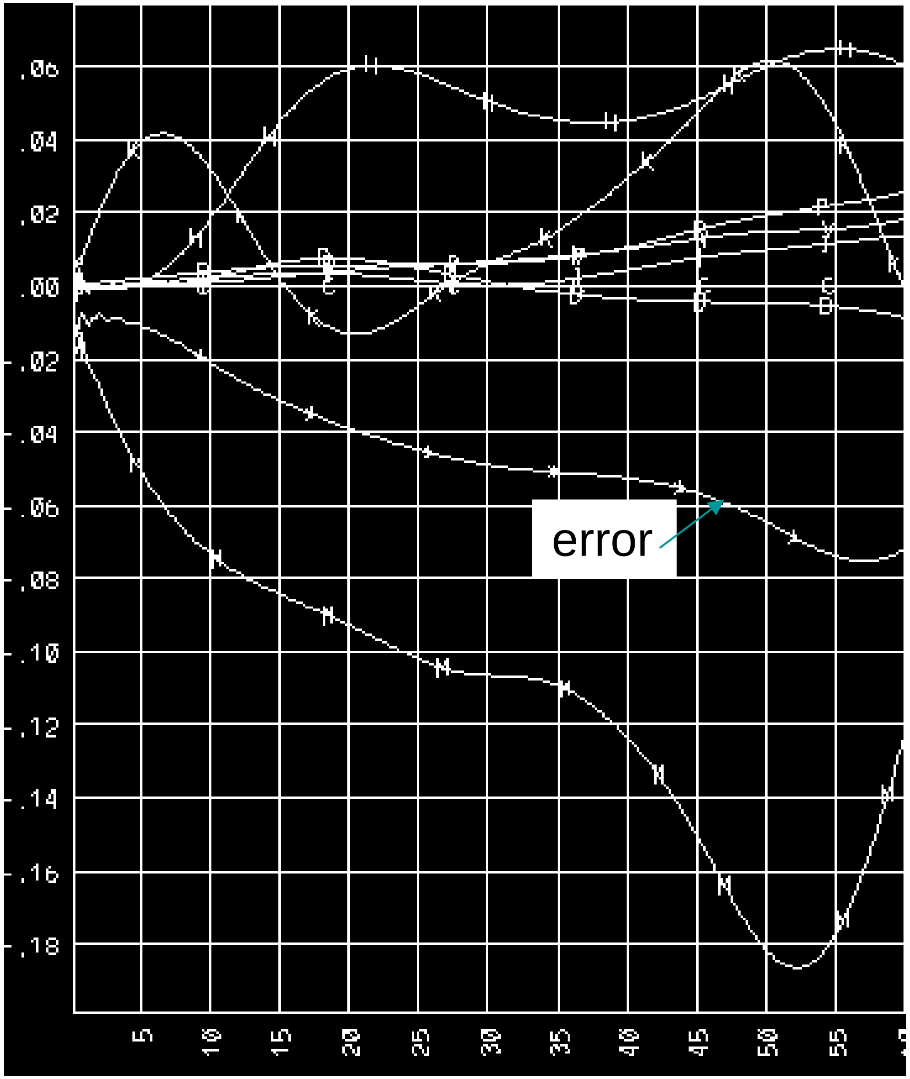
Magnetic Reconnection 6-field 2-fluid model



Relative energy error decreases with increasing number of nodes for sequence with hyper coef. $H = C (\Delta x)^2$



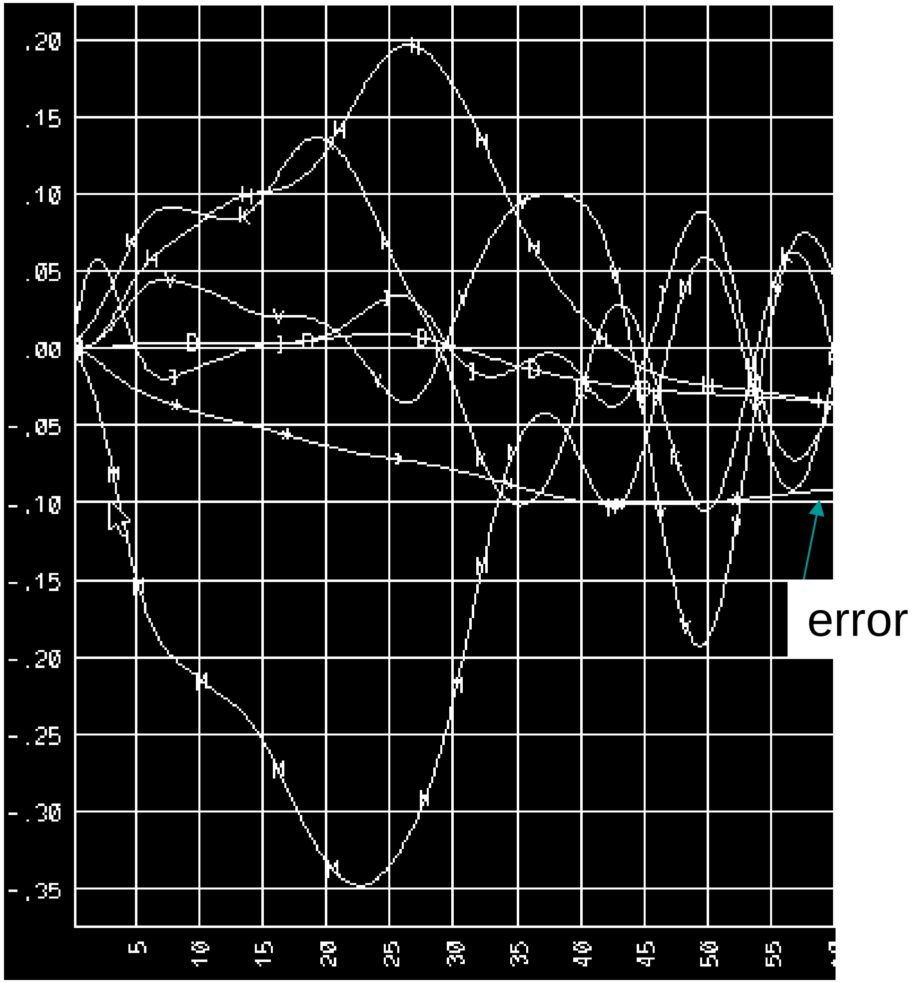
61 x 61



76 x 76

Relative error in 4-field model

61 x 61



76 x 76

Braginskii gyro-viscosity:

$$\nabla\cdot\overleftrightarrow{\Pi}=\left\{\left[\nabla\times\left(\frac{mp}{eB^2}\vec{B}\right)\right]\cdot\nabla\right\}\vec{V}-\nabla\left[\frac{mp}{2eB^2}\vec{B}\cdot\left(\nabla\times\vec{V}\right)\right]-\nabla\times\left\{\frac{mp}{eB^2}\left[(B\cdot\nabla)\vec{V}+\frac{1}{2}\left(\nabla\cdot\vec{V}-\frac{3}{B^2}\vec{B}\cdot\left[\left(\vec{B}\cdot\nabla\right)\vec{V}\right]\right)\vec{B}\right]\right\}$$
$$+(B\cdot\nabla)\left\{\frac{mp}{eB^2}\left(\frac{3}{B^2}\vec{B}\times\left[\left(\vec{B}\cdot\nabla\right)\vec{V}\right]+\frac{3}{2B^2}\left[\vec{B}\cdot\left(\nabla\times\vec{V}\right)\right]\vec{B}-\nabla\times\vec{V}\right)\right\}$$

Ramos

$$\hat{z}\cdot\quad\rightarrow\quad[V_z,\alpha I]-\alpha\left\{\left[\psi,\nabla_\perp^2U\right]+\left[\frac{\partial\psi}{\partial x},\frac{\partial U}{\partial x}-\frac{\partial\chi}{\partial y}\right]+\left[\frac{\partial\psi}{\partial y},\frac{\partial U}{\partial y}+\frac{\partial\chi}{\partial x}\right]\right\}-\frac{\partial\alpha}{\partial x}\left[\psi,\frac{\partial U}{\partial x}-\frac{\partial\chi}{\partial y}\right]-\frac{\partial\alpha}{\partial y}\left[\psi,\frac{\partial U}{\partial y}+\frac{\partial\chi}{\partial x}\right]$$
$$+\frac{1}{2}\alpha\nabla_\perp^2\chi\nabla_\perp^2\psi+\frac{1}{2}\left(\alpha\nabla_\perp^2\chi,\psi\right)-\frac{1}{2}\left[(\gamma\kappa,\psi)+\gamma\kappa\nabla_\perp^2\psi\right]+[\gamma\xi_z,\psi]+\frac{1}{2}[\lambda I,\psi]+\left[\alpha\nabla_\perp^2U,\psi\right]$$

$$-\hat{z}\cdot\nabla\times\quad\rightarrow\quad\left[\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y},\frac{\partial(\alpha I)}{\partial y}\right]-\left[\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x},\frac{\partial(\alpha I)}{\partial x}\right]+\left[\nabla_\perp^2U,\alpha I\right]+\nabla_\perp^2\left\{\alpha[\psi,V_z]\right\}-\frac{1}{2}\nabla_\perp^2\left(\alpha I\nabla_\perp^2\chi\right)+\frac{1}{2}\nabla_\perp^2(\gamma\kappa I)$$
$$+\frac{\partial}{\partial y}[\gamma\xi_x,\psi]-\frac{\partial}{\partial x}[\gamma\xi_y,\psi]+\frac{1}{2}\left\{\frac{\partial\lambda}{\partial x}\left[\frac{\partial\psi}{\partial x},\psi\right]+\frac{\partial\lambda}{\partial y}\left[\frac{\partial\psi}{\partial y},\psi\right]+([\lambda,\psi],\psi)+\left[\lambda\nabla_\perp^2\psi,\psi\right]\right\}+\frac{\partial}{\partial x}\left[\psi,\alpha\frac{\partial V_z}{\partial x}\right]+\frac{\partial}{\partial y}\left[\psi,\alpha\frac{\partial V_z}{\partial y}\right]$$

$$\nabla\cdot\quad\rightarrow\quad\left[\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y},\frac{\partial(\alpha I)}{\partial x}\right]+\left[\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x},\frac{\partial(\alpha I)}{\partial y}\right]+\left[\nabla_\perp^2\chi,\alpha I\right]+\frac{1}{2}\nabla_\perp^2\left\{\alpha\left[I\nabla_\perp^2U-(\psi,V_z)\right]\right\}+\frac{\partial}{\partial x}[\gamma\xi_x,\psi]+\frac{\partial}{\partial y}[\gamma\xi_y,\psi]$$
$$+\frac{1}{2}[[\lambda,\psi],\psi]+\lambda\left[\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial x}\right]+\frac{1}{2}\frac{\partial\lambda}{\partial x}\left[\frac{\partial\psi}{\partial y},\psi\right]-\frac{1}{2}\frac{\partial\lambda}{\partial y}\left[\frac{\partial\psi}{\partial x},\psi\right]+\frac{\partial}{\partial x}\left\{\alpha\left[\psi,\frac{\partial V_z}{\partial y}\right]\right\}-\frac{\partial}{\partial y}\left\{\alpha\left[\psi,\frac{\partial V_z}{\partial x}\right]\right\}+[V_z,[\alpha,\psi]]$$

$$\alpha\equiv\frac{ep}{mB^2}=\frac{ep/m}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+I^2}$$

$$\gamma=\frac{3ep/m}{\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+I^2\right]^2}$$

$$\vec{\xi}=\left\{\frac{\partial\psi}{\partial x}[\psi,V_z]+I\left[\psi,\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x}\right]\right\}\hat{x}+\left\{\frac{\partial\psi}{\partial y}[\psi,V_z]-I\left[\psi,\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y}\right]\right\}\hat{y}-\left\{\frac{\partial\psi}{\partial x}\left[\psi,\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y}\right]+\frac{\partial\psi}{\partial y}\left[\psi,\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x}\right]\right\}\hat{z}$$

$$\lambda=\gamma\left[(\psi,V_z)-I\nabla_\perp^2U\right]$$

$$\kappa\equiv\frac{\partial\psi}{\partial y}\left[\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y},\psi\right]-\frac{\partial\psi}{\partial x}\left[\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x},\psi\right]+I[V_z,\psi]$$

Breslau

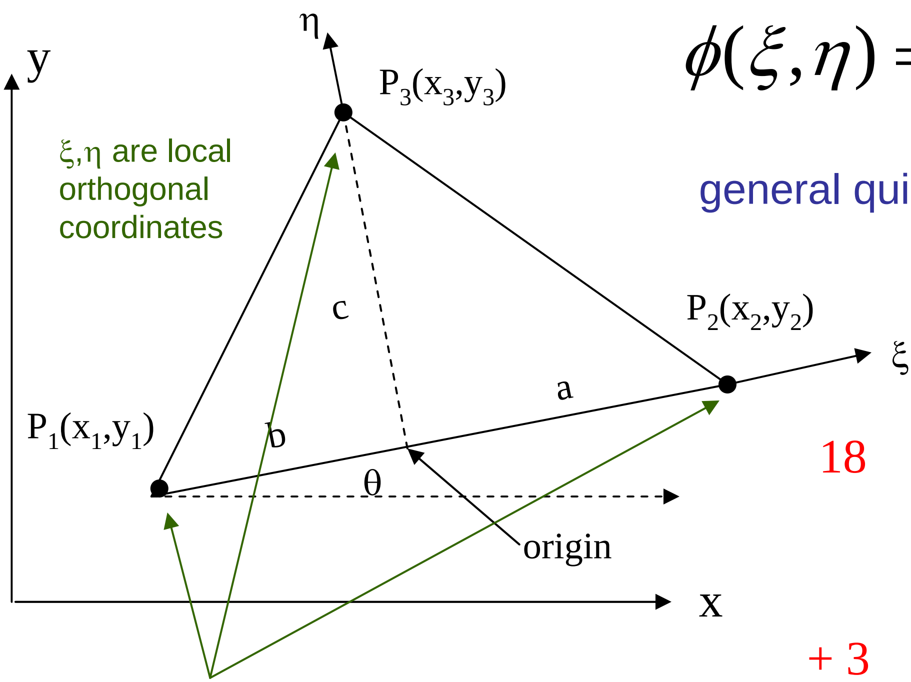
Near-Term Plans

- Additional error checking and benchmarking
- Add density and ion pressure equation
- Add electron thermal conductivity
- Add Braginskii ion-gyroviscosity
- Mesh adaptation
- Physics studies of the GEM reconnection with different 2-fluid models and quantify the effect of compressibility
- Extend to toroidal system, and non-axisymmetric modes

Summary and Conclusions

- *C^1* finite element method has been extended to 6-field 2-fluid MHD
- Uses a split semi-implicit time advance to give unconditional (linear) numerical stability
- Gives convergent results for tilt-mode problem: linear growth rate increases with d_i^2
- Applied to GEM reconnection, shows that compressibility reduces the reconnection rate

Divide domain into triangular regions: represent solution as a quintic polynomial within each region



$$\phi(\xi,\eta) = \sum_{k=1}^{21} a_k \xi^{m_k} \eta^{n_k}$$

general quintic has 21 terms.

k	m_k	n_k
1	0	0
2	1	0
3	0	1
4	2	0
5	1	1
6	0	2
7	3	0
8	2	1
9	1	2
10	0	3
11	4	0
12	3	1
13	2	2
14	1	3
15	0	4
16	5	0
17	3	2
18	2	3
19	1	4
20	0	5
21	4	1

18 constraints to match the function and derivatives at nodes

\+ 3 constraints on quintic coefficients to enforce C^1 continuity at edges

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21 coefficients of the quintic polynomial

The function and it's first and second derivatives at the 3 nodes are the global unknowns (6 per node) $(\phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy})$

Error ~ h^5 *(since complete Taylor series through h^4)*

C^1 continuity allows treatment of 4th spatial derivatives *(Galerkin Method)*

Most compact representation for this accuracy "reduced quintic"