CEMM Meeting and Sherwood Theory Conference. Lake Tahoe, April 2005

# THE GENERAL EXPRESSION OF THE GYROVISCOUS FORCE

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What **"the general expression"** means:

Lowest significant order in $\delta = \rho/L \ll 1$.

Applicable to both $\partial/\partial t \sim \delta\Omega_c,\ u \sim v_{th}$ (fast dynamics where $\Pi^{gyro} \sim \delta p$) and $\partial/\partial t \sim \delta^2\Omega_c,\ u \sim \delta v_{th}$ (slow dynamics where $\Pi^{gyro} \sim \delta^2 p$).

No simplifying assumptions on magnetic geometry, parallel gradients, $\beta$, parallel flow, compressibility etc.

Valid for arbitrary collisionality.

Does not require the distribution function to be close to a Maxwellian.

Divergence of the stress tensor (gyroviscous force) in exact, coordinate-free vector form.

**Define the gyroviscous stress** as the traceless and perpendicular (i.e. $\Pi^{gyro}_{ii} = \Pi^{gyro}_{ij} b_i b_j = 0$) part of the stress tensor in the fluid rest frame that does not depend explicitly on the collision frequencies:

$$m \int d^3\mathbf{v}\ [v_i - u_i(\mathbf{x},t)][v_j - u_j(\mathbf{x},t)]\ f(\mathbf{v},\mathbf{x},t) \ = \ p_\perp \delta_{ij} \ + \ (p_\parallel - p_\perp) b_i b_j \ + \ \Pi^{gyro}_{ij} \ + \ \Pi^{coll}_{\perp ij} \ .$$

Then,

$$\Pi^{gyro}_{ij} \ = \ \frac{1}{4}\Big[\epsilon_{ikl} b_k (\delta_{mj} + 3 b_m b_j) + \epsilon_{jkl} b_k (\delta_{mi} + 3 b_m b_i)\Big] K^{gyro}_{lm} \ \equiv \ \frac{1}{4}\epsilon_{[ikl} b_k (\delta_{mj]} + 3 b_m b_{j]}) K^{gyro}_{lm}$$

where, retaining the lowest-significant-order terms for both the fast dynamics and slow dynamics [1],

$$K^{gyro}_{ij} \ = \ \frac{m}{eB}\Big[\ p_\perp \frac{\partial u_{j]}}{\partial x_{[i}} \ + \ \frac{\partial (q_{T\parallel} b_{j]})}{\partial x_{[i}} \ + \ b_{[i} c_{j]} \ + \ \frac{\partial \Theta^{gyro}_{ijk}}{\partial x_k}\ \Big]$$

with

$$\mathbf{c} \ = \ (2q_{B\parallel} - 3q_{T\parallel})\kappa \ + \ \Big(\frac{p_\parallel - p_\perp}{B}\Big)\ \Big\{2(\mathbf{B}\cdot\nabla)\mathbf{u} - \nabla \times \Big[\frac{1}{en}\nabla p_\perp + \frac{1}{en}(\mathbf{B}\cdot\nabla)\Big(\frac{p_\parallel - p_\perp}{B}\mathbf{b}\Big)\Big]\Big\} \ ,$$

$$\kappa \ = \ (\mathbf{b}\cdot\nabla)\mathbf{b}$$

and

$$m \int d^3\mathbf{v}\ (v_i - u_i)(v_j - u_j)(v_k - u_k) f \ = \ q_{T\parallel}\delta_{[ij} b_{k]} \ + \ (2q_{B\parallel} - 3q_{T\parallel}) b_i b_j b_k \ + \ \Theta^{gyro}_{ijk} \ + \ \Theta^{coll}_{\perp ijk} \ .$$

The collision-independent perpendicular stress flux tensor $\Theta_{ijk}^{gyro}$ is needed only in the slow dynamics. For this case, with the required accuracy of $O(\delta p v_{th})$, $\Theta_{ijk}^{gyro}$ is also given in Ref.1. Its divergence is

$$\frac{\partial \Theta_{ijk}^{gyro}}{\partial x_k} \;=\; \frac{\partial}{\partial x_{[i}}\Big(\frac{1}{2}q_{T\perp j]} - \frac{\alpha j_\|}{B}b_{j]}\Big) \;+\; b_{[i}d_{j]} \;+\; \kappa_{[i}g_{\perp j]} \;+\; \epsilon_{[ilm}\Big\{\Big[\nabla\cdot(\alpha\mathbf{b})b_l + \alpha\kappa_l\Big]\frac{\partial b_{j]}}{\partial x_m} + \alpha b_l\frac{\partial \kappa_{j]}}{\partial x_m}\Big\},$$

where

$$\mathbf{q}_{T\perp} \;=\; \frac{1}{eB}\mathbf{b}\times\Big[2p_\perp\nabla\Big(\frac{p_\perp}{n}\Big) + \frac{4}{5}\nabla\Big(\tilde{r}_\perp - \frac{1}{4}\tilde{r}_\Delta\Big) + \tilde{r}_\Delta\kappa\Big],$$

$$\mathbf{g}_\perp \;=\; \frac{1}{eB}\mathbf{b}\times\Big[p_\perp\nabla\Big(\frac{p_\| - p_\perp}{n}\Big) + \frac{2p_\|(p_\| - p_\perp)}{n}\kappa + \frac{1}{2}\nabla\tilde{r}_\Delta + \Big(2\tilde{r}_\| - 2\tilde{r}_\perp - \frac{5}{2}\tilde{r}_\Delta\Big)\kappa\Big],$$

$$\alpha \;=\; \frac{1}{eB}\Big[\frac{p_\perp(p_\| - p_\perp)}{2n} + \tilde{r}_\Delta\Big],$$

and

$$\mathbf{d} \;=\; (3\alpha j_\|/B)\kappa \;+\; \nabla\times(\mathbf{g}_\perp\times\mathbf{b}) \;+\; \Big[(2\mathbf{g}_\perp + 2\alpha\nabla\times\mathbf{b} + \nabla\alpha\times\mathbf{b})\cdot\nabla\Big]\mathbf{b}$$

$$+\,\mathbf{b}\times\Big[\alpha\nabla(\nabla\cdot\mathbf{b}) + \nabla(\mathbf{b}\cdot\nabla\alpha) - (\mathbf{b}\cdot\nabla)\nabla\alpha\Big] \;-\; \alpha(\nabla\cdot\mathbf{b})\nabla\times\mathbf{b}.$$

The fourth rank moments $\tilde{r}_\perp$, $\tilde{r}_\parallel$, $\tilde{r}_\Delta$ are

$$\tilde{r}_\perp = \frac{m^2}{4}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \cos^2\lambda\ (f - f_{bimaxwell}),$$

$$\tilde{r}_\parallel = \frac{m^2}{2}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \sin^2\lambda\ (f - f_{bimaxwell}),$$

$$\tilde{r}_\Delta = \frac{m^2}{4}\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \cos^2\lambda\ (5\sin^2\lambda - 1)(f - f_{bimaxwell}),$$

where $\sin\lambda = (\mathbf{v}-\mathbf{u})\cdot\mathbf{b}/|\mathbf{v}-\mathbf{u}|$.

If the lowest-order distribution function is isotropic, $(p_\parallel - p_\perp)$, $(\tilde{r}_\parallel - \tilde{r}_\perp)$ and $\tilde{r}_\Delta$ vanish in lowest order. They can then be neglected in $\partial\Theta^{gyro}_{ijk}/\partial x_k$, which becomes:

$$\frac{\partial\Theta^{gyro}_{ijk}}{\partial x_k} = \frac{1}{2}\frac{\partial q_{T\perp j]}}{\partial x_{[i}} \qquad with \qquad \mathbf{q}_{T\perp} = \frac{2}{eB}\mathbf{b}\times\Big[p_\perp\nabla\Big(\frac{p_\perp}{n}\Big) + \frac{2}{5}\nabla\tilde{r}_\perp\Big].$$

If the lowest-order distribution function is a Maxwellian, then $\tilde{r}_\perp$ can be neglected too.

## SOME SPECIAL LIMITS

### 1) For fast dynamics and high collisionality (Braginskii [2]):

$$K_{ij}^{gyro1} \;=\; \frac{mp_\perp}{eB}\,\frac{\partial u_{j]}}{\partial x_{[i}},$$

in which case, $p_\perp = p$.

### 2) For slow dynamics and high collisionality (Mikhailowskii-Tsypin [3]):

$$K_{ij}^{gyro2} \;=\; \frac{m}{eB}\Big[\, p_\perp \frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial}{\partial x_{[i}}\Big(q_{T\parallel} b_{j]} + \frac{1}{2} q_{T\perp j]}\Big)\Big],$$

in which case, $p_\perp = p$ and $q_{T\parallel} b_j + q_{T\perp j}/2 = 2q_j/5$.

### 3) For fast dynamics and arbitrary collisionality (Macmahon [4]):

$$K_{ij}^{gyro3} \;=\; \frac{m}{eB}\Big\{\, p_\perp \frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial}{\partial x_{[i}}\Big(q_{T\parallel} b_{j]}\Big) \;+\; b_{[i}\Big[(2q_{B\parallel} - 3q_{T\parallel})\kappa_{j]} + 2(p_\parallel - p_\perp) b_k \frac{\partial u_{j]}}{\partial x_k}\Big]\Big\}.$$

**4) For slow dynamics under the assumption that, for arbitrary collisionality, the lowest-order distribution function could still be Maxwellian or just isotropic:**

$$K_{ij}^{gyro4} \;=\; \frac{m}{eB}\Big[\,p_\perp\frac{\partial u_{j]}}{\partial x_{[i}} \;+\; \frac{\partial}{\partial x_{[i}}\Big(q_{T\|}b_{j]} + \frac{1}{2}q_{T\perp j]}\Big) \;+\; b_{[i}(2q_{B\|} - 3q_{T\|})\kappa_{j]}\Big],$$

where $p_\perp = p$, and

$$\frac{1}{2}\mathbf{q}_{T\perp} \;=\; \frac{2}{5}\mathbf{q}_\perp \;=\; \frac{p}{eB}\mathbf{b}\times\nabla\Big(\frac{p}{n}\Big)$$

if the lowest-order distribution function is assumed to be Maxwellian (Simakov-Catto [5]), or

$$\frac{1}{2}\mathbf{q}_{T\perp} \;=\; \frac{2}{5}\mathbf{q}_\perp \;=\; \frac{1}{eB}\mathbf{b}\times\Big[p\nabla\Big(\frac{p}{n}\Big) + \frac{2}{5}\nabla\tilde{r}\Big]$$

if the lowest-order distribution function is assumed to be isotropic, with $\tilde{r} = \tilde{r}_\perp = \tilde{r}_\|$, but not necessarily Maxwellian (Ramos [1]).

## BRAGINSKII'S GYROVISCOUS FORCE

Braginskii's gyroviscosity term is:

$$\Pi_{ij}^{gyro1} \;=\; \frac{mp_\perp}{4eB}\,\epsilon_{[ikl}b_k\Big(\delta_{mj]} + 3b_m b_{j]}\Big)\,\frac{\partial u_{m]}}{\partial x_{[l}}\;.$$

Its divergence, in coordinate-free vector notation, is exactly:

$$\nabla\cdot\Pi^{gyro1} \;=\; \Big\{\Big[\nabla\times\Big(\frac{mp_\perp}{eB}\mathbf{b}\Big)\Big]\cdot\nabla\Big\}\mathbf{u} \;-\; \nabla\Big[\frac{mp_\perp}{2eB}\,(\mathbf{b}\cdot\omega)\Big]$$

$$-\,\nabla\times\Big\{\frac{mp_\perp}{eB}\,\Big[(\mathbf{b}\cdot\nabla)\mathbf{u} \;+\; \frac{1}{2}\Big(\nabla\cdot\mathbf{u} - 3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}]\Big)\,\mathbf{b}\,\Big]\Big\}$$

$$+\,(\mathbf{B}\cdot\nabla)\Big\{\frac{mp_\perp}{eB^2}\,\Big[\mathbf{b}\times[3(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\omega] \;+\; \frac{1}{2}(\mathbf{b}\cdot\omega)\,\mathbf{b}\,\Big]\Big\},$$

where $\omega \equiv \nabla\times\mathbf{u}$ is the vorticity.

**Using the explicit expression of Braginskii's gyroviscous force term, the momentum conservation equation can be written as:**

$$
mn\frac{\partial \mathbf{u}}{\partial t} \ + \ mn[(\mathbf{u}-\mathbf{u}_*)\cdot\nabla]\mathbf{u} \ + \ \nabla(p_\perp - \chi) \ + \ (\mathbf{B}\cdot\nabla)\Big(\frac{p_\parallel - p_\perp + \chi}{B}\,\mathbf{b}\Big)
$$

$$
-\ \nabla\times\Big\{\frac{mp_\perp}{eB}\Big[(\mathbf{b}\cdot\nabla)\mathbf{u} \ + \ \frac{1}{2}\Big(\nabla\cdot\mathbf{u} - 3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}]\Big)\,\mathbf{b}\,\Big]\Big\} \ + \ (\mathbf{B}\cdot\nabla)\Big\{\frac{mp_\perp}{eB^2}\,\mathbf{b}\times[3(\mathbf{b}\cdot\nabla)\mathbf{u} + \mathbf{b}\times\omega]\Big\}
$$

$$
+\ \nabla\cdot\Big(\Pi^{gyro} - \Pi^{gyro1} + \Pi_\perp^{coll}\Big) \ - \ en(\mathbf{E} + \mathbf{u}\times\mathbf{B}) \ - \ \mathbf{F}^{coll} \ = \ 0 \ ,
$$

where

$$
\mathbf{u}_* \equiv -\frac{1}{en}\nabla\times\Big(\frac{p_\perp}{B}\mathbf{b}\Big) \qquad and \qquad \chi \equiv \frac{mp_\perp}{2eB}(\mathbf{b}\cdot\omega) \ .
$$

Thus, $\mathbf{u}_*$ cancels only partially (i.e. except for derivatives of the magnetic field) the diamagnetic drift $\mathbf{u}_d = \mathbf{b}\times\nabla p_\perp/(enB)$ in the convective derivative $mn(\mathbf{u}\cdot\nabla)$. The parallel vorticity acts as an effective renormalization of the perpendicular pressure: $p_\perp \rightarrow p_\perp - \chi = p_\perp[1 - m(\mathbf{b}\cdot\omega)/(2eB)]$.

**Acknowledgment**

The author thanks the Center for Extended MHD Modeling and the U.S. Department of Energy for their support of this work.

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