Low Order Spectral Elements for M3D

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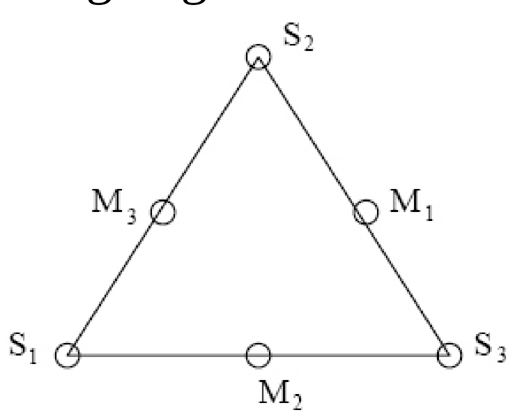
- Description of elements
- Accuracy of elements (Jin Chen)
- Tilt mode comparison
- ELM simulation example

2nd and 3rd order elements

- 2nd or 3rd order spectral finite elements in poloidal plane [Cohen et al, SIAM J. Numer. Anal. 38, 2047-2078 (2001)]
- Nodal points coincide with quadrature points
- Gives diagonal mass matrix
- Implemented in openmp, mpp versions of m3d
- Complementary to high order elements (B. Hientzsch)

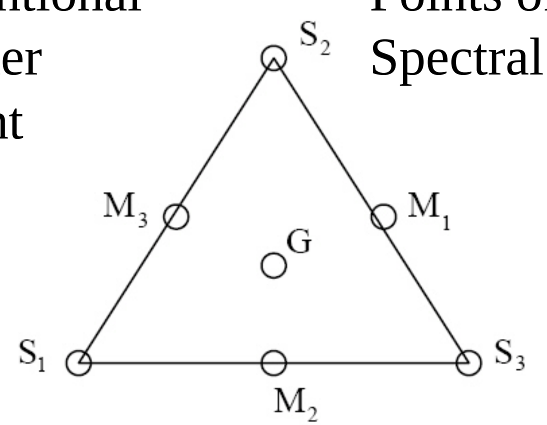
Nodal points of conventional 2nd order Lagrange element



Quadrature Points of Conventional 2nd order element

and

Nodal points And quadrature Points of 2nd order Spectral element



Nodal points and quadrature points

- Quadrature points integrate polynomials of degree 2k-1 exactly, where k is the polynomial order of the basis functions (k=1,2,or 3)
- Quadrature weights w must be positive

$$\int_{\Delta} f(\vec{x})d^2x = \sum_{a}^{quad.pts.} f(\overrightarrow{x_a})w_a$$

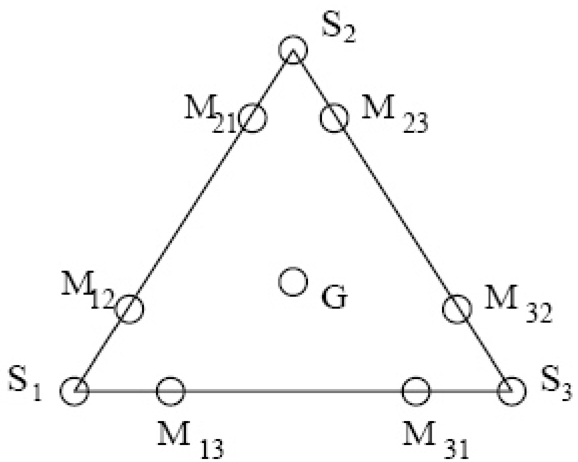
- Quadrature points = nodal points, basis functions N give diagonal mass matrix

$$N_i(\vec{x}_a) = \delta_{ia}$$

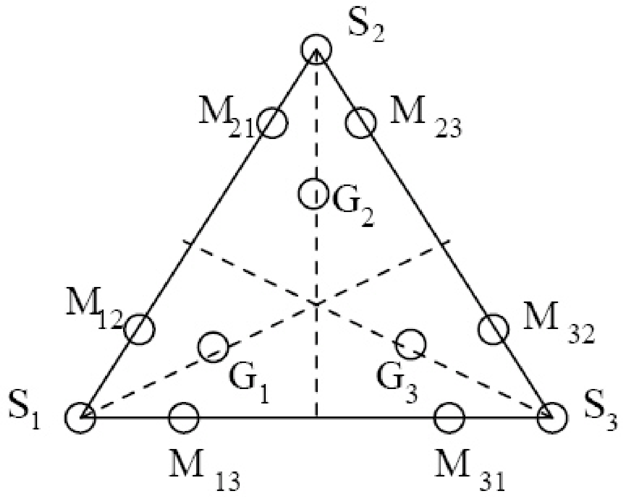
$$M_{ij} = \int N_i(\vec{x})N_j(\vec{x})d^2x = \delta_{ij}w_j$$

- diagonal mass matrix is important for efficiency of M3D

3^{rd} order elements



3rd order conventional Lagrange element



3rd order Cohen element

3^{rd} order is nontrivial. Add 2 bubble nodes. Side nodes are not equally spaced. The nodes are chosen to have positive quadrature weights. Quadrature is exact for 5^{th} order polynomials.

Comparisons of lumped HO elements by J. Chen in m3dp

$$poiss : \nabla^2 f = g$$

$$gcro : [f, g] = h$$

6800 nodes

f,g,h = sine, cosine
Error calculated at one point, but comparable to rms values

Table 1: Accuracy

operator	Linear	Lagrange HO	Lumped HO
pure poiss	.3133E-04	.1824E-10	.2778E-10
star poiss	.7480E-04	.1741E-07	.9668E-11
dagg poiss	.7689E-05	.1368E-07	.1316E-11
Helmholtz pure poiss	.8375E-04	.5808E-06	.5921E-11
Helmholtz star poiss	.2019E-03	.1122E-04	.1187E-10
Helmholtz dagg poiss	.3648E-04	.1582E-06	.1542E-10
pure poiss Neumann	.4887E-03	.7463E-03	.2022E-02
dxdr	.2424E-03	.7718E-11	.4413E-13
dxdz	.9665E-03	.1709E-09	.2709E-13
d2xdrdz	.9917E-03	.9688E-10	.5367E-11
d2xdzdr	.1301E-04	.7361E-10	.2437E-12
d2xdrdz - d2xdzdr	.9787E-03	.1705E-09	.5611E-11
grad	.4251E-03	.7760E-13	.4639E-13
gcro	.3830E-02	.6218E-09	.9326E-14
delsq	.6927E-03	.9690E-10	.1106E-09

— Note:
The HO elements is more accurate than the Linear elements.
But the lumped HO has more accuracy than the HO elements since it has a 3rd order term.

J. Chen – timings in m3dp

Table 2: Time

operator	Linear	Lagrange HO	Lumped HO
pure poiss	11.505164	17.993580	15.881487
star poiss	11.936641	17.842965	15.577935
dagg poiss	11.487363	17.065694	15.590550
Helmholtz pure poiss	11.593001	17.850698	15.764700
Helmholtz star poiss	11.827986	17.617935	15.462633
Helmholtz dagg poiss	11.127486	17.504207	15.329060
pure poiss Neumann	11.800331	17.994744	15.368874
dxdr	0.325041	2.822974	0.443981
dxdz	0.467021	2.539099	0.419528
d2xdrdz - d2xdzdr	0.560459	9.457784	2.098601
grad	0.680051	2.715444	0.961536
gcro	0.234130	2.418649	0.544330
delsq	0.355726	6.733015	0.554883

— Note:
The HO elements is more expensive than the linear ones. But The lumped HO elements overcomes this weak point due to the Lagrange property at the nodal points, which coincides with the collocation points in this case. The elliptic solver takes more time than the other derivative operators.

2D Incompressible MHD

$$\frac{\partial W}{\partial t} = -[W,\phi] + [C,\psi] + \mu\nabla^2 W$$

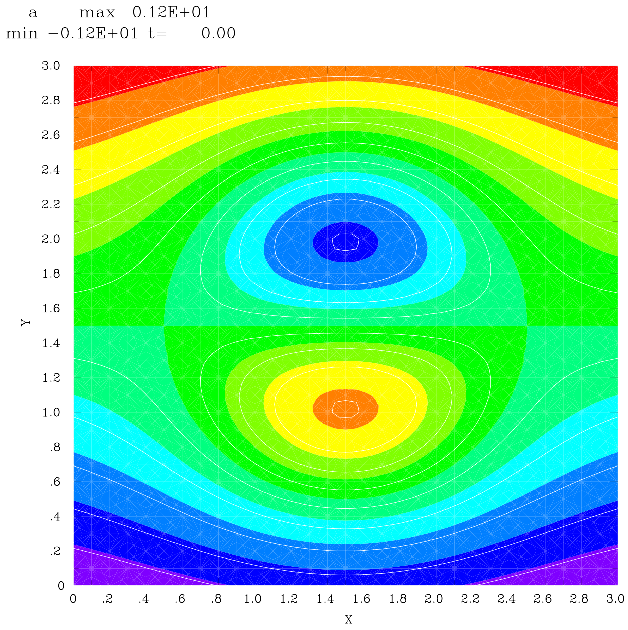
$$\frac{\partial C}{\partial t} = [\phi, C] + [W,\psi] + 2\left[\frac{\partial \phi}{\partial x}, \frac{\partial \psi}{\partial x}\right] + 2\left[\frac{\partial \phi}{\partial y}, \frac{\partial \psi}{\partial y}\right] + \eta\nabla^2 C$$

$$\nabla^2\phi = W$$

$$\nabla^2\psi = C$$

Strauss and Longcope, JCP 1998
Vorticity and current advance
Apply to tilt mode

Initial magnetic flux and current for tilt mode



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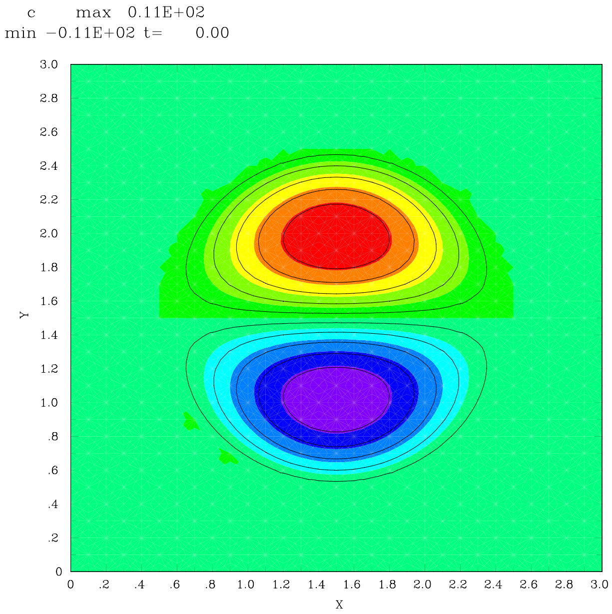
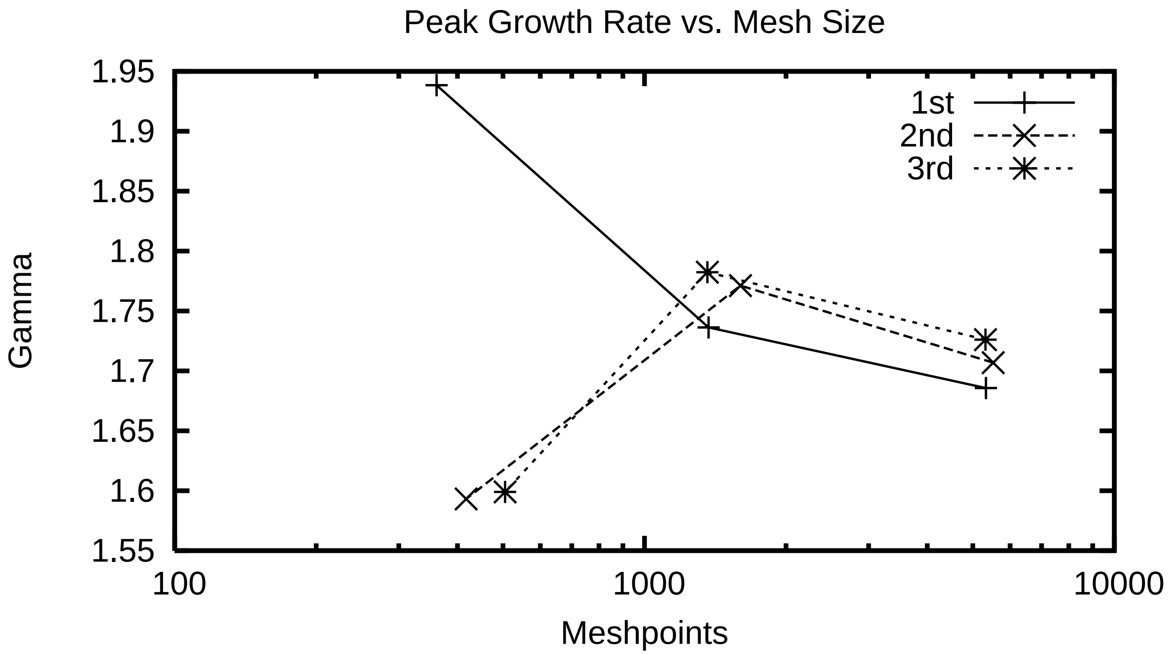


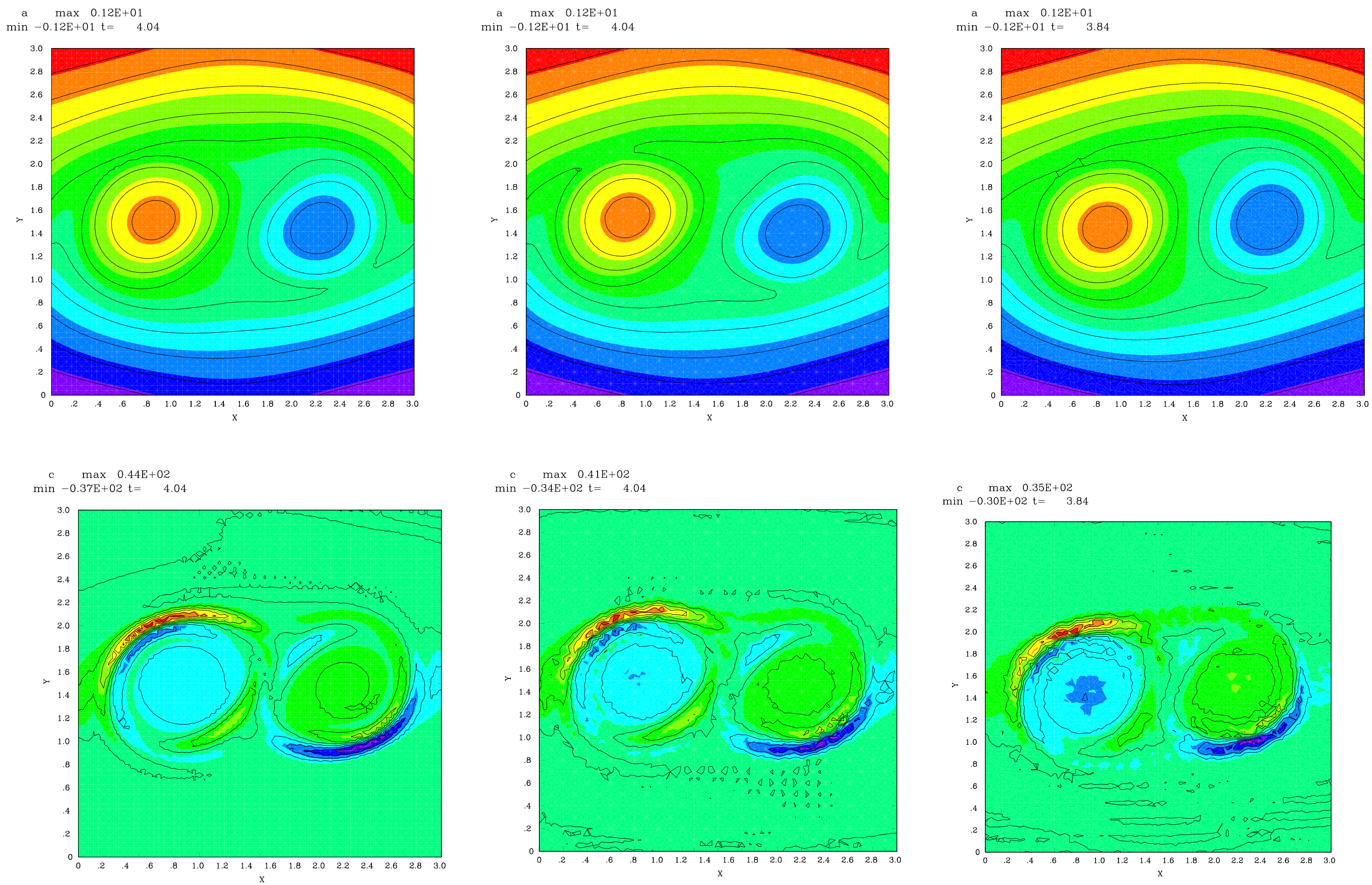
C

Convergence of peak tilt mode growth rate with order



Up to 5500 meshpoints

Magnetic flux and current in 1st, 2nd, 3rd order



$\eta = 10^{-4}, \mu = 5\times 10^{-3}$ 5500 meshpoints

2D Incompressible MHD

$$\frac{\partial W}{\partial t} = -[W, \phi] + [C, \psi] + \mu \nabla^2 W$$

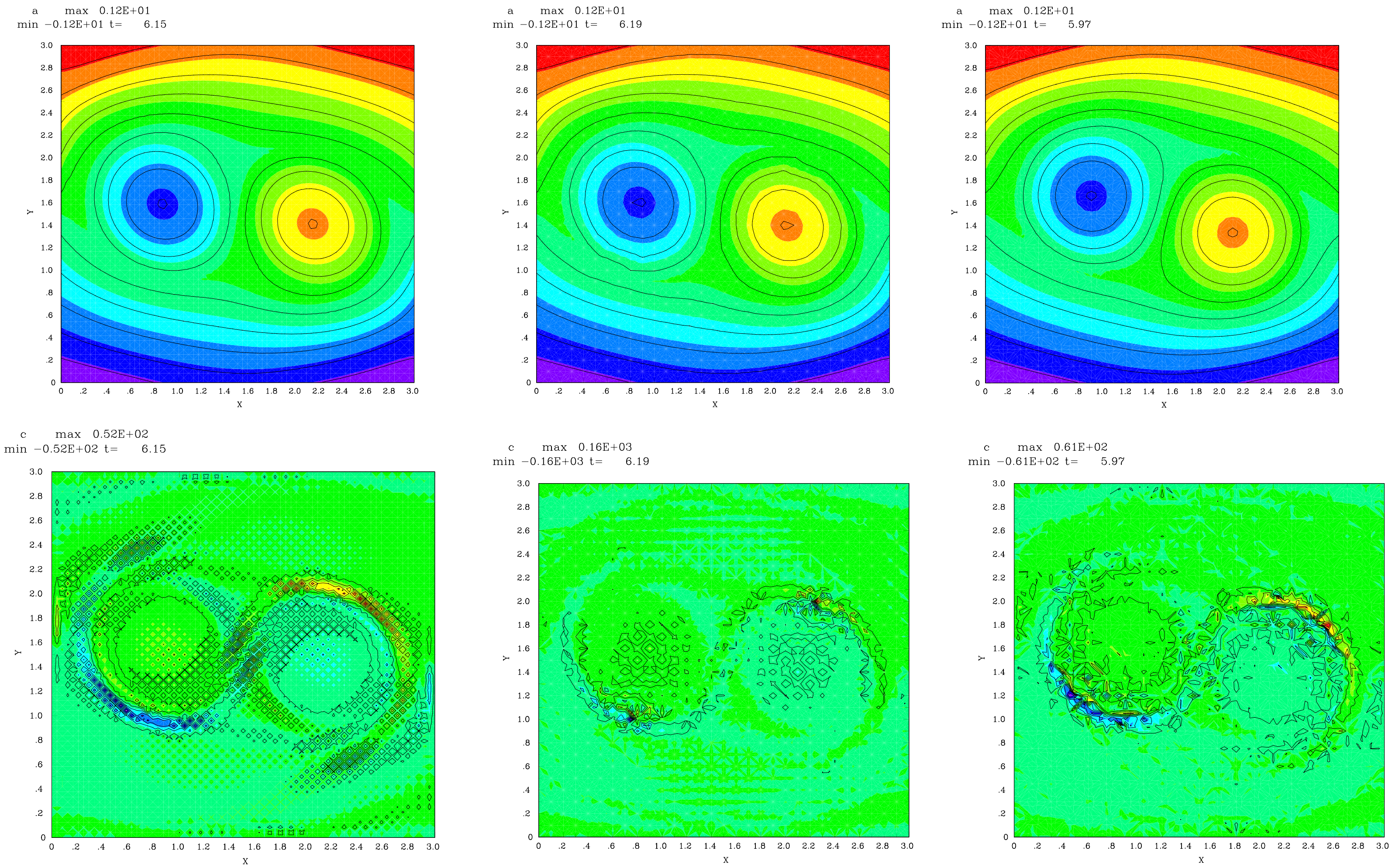
$$\frac{\partial \psi}{\partial t} = [\phi, \psi] + \eta \nabla^2 \psi$$

$$\nabla^2 \phi = W$$

$$C = \nabla^2 \psi$$

Standard formulation

Magnetic flux and current in 1st, 2nd, 3rd order

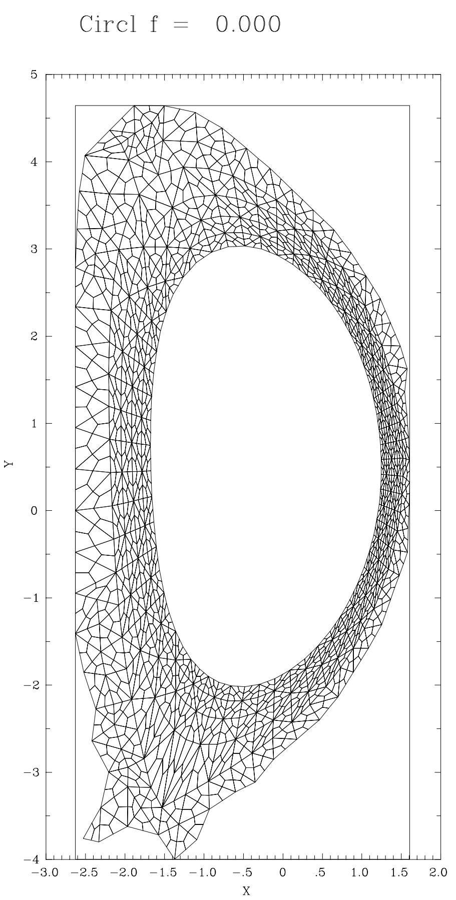


$\eta = 10^{-4}, \mu = 5\times10^{-3}$ 5500 meshpoints

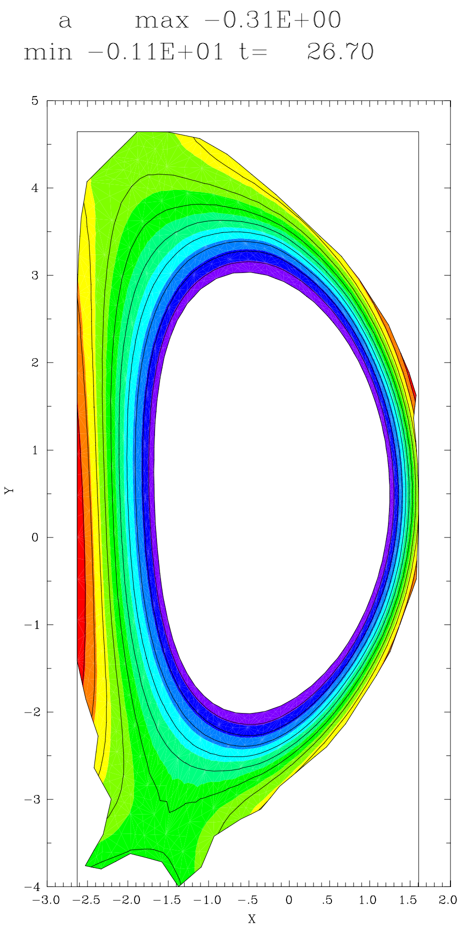
ELM simulations – M3D (3D full MHD)

- Preliminary nonlinear ELM simulations
  - ITER boundary and initial equilibrium
  - low resolution, n = 6, pseudo spectral

    in toroidal direction
  - Use 2nd order elements

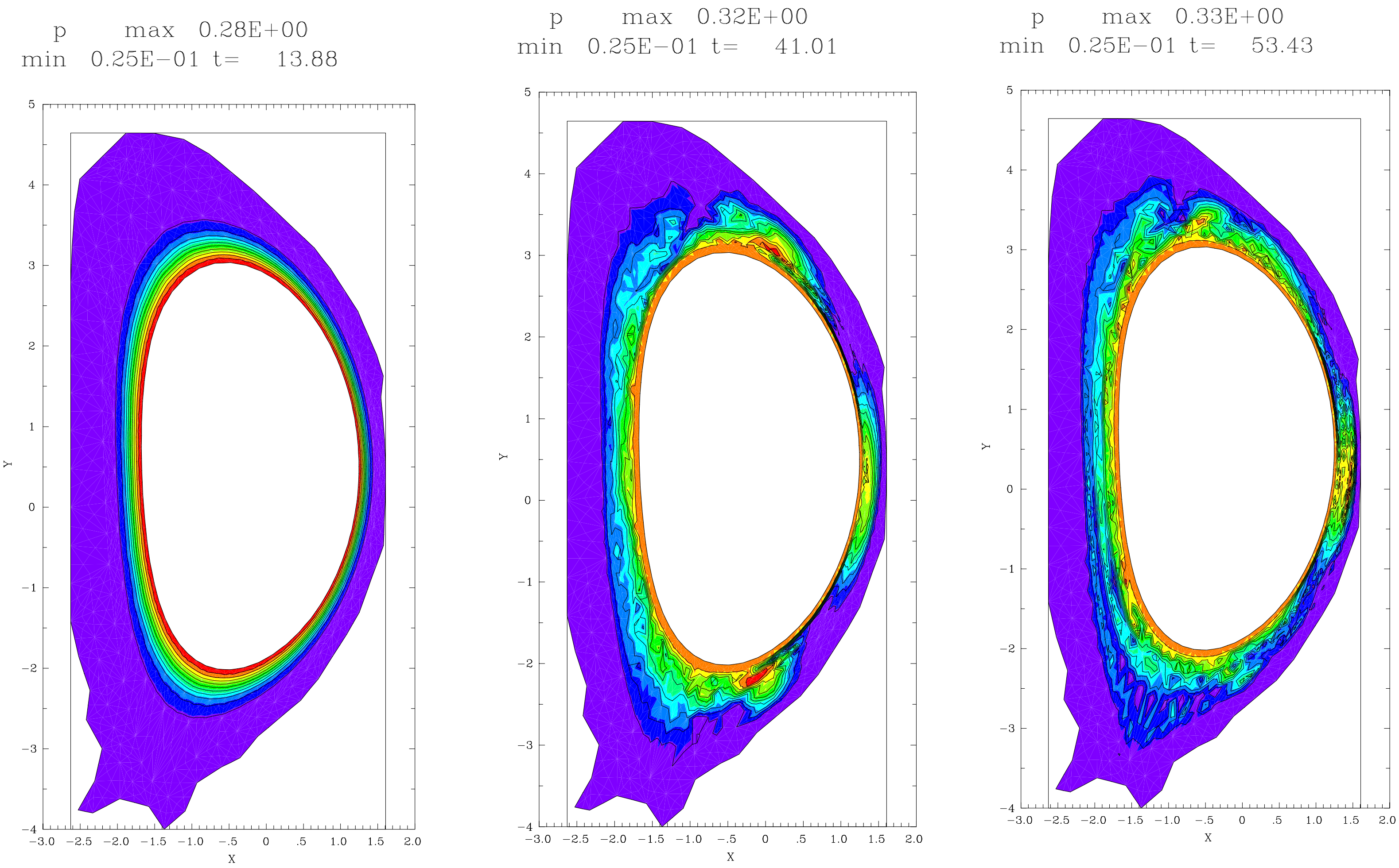


Mesh



Magnetic flux

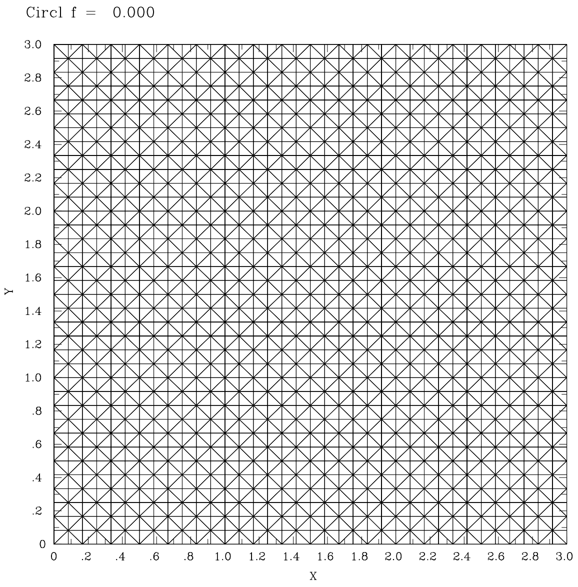
Nonlinear ELM pressure evolution

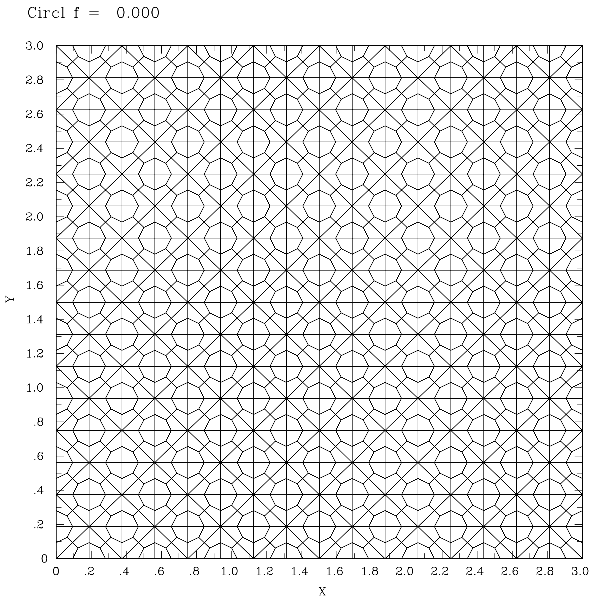


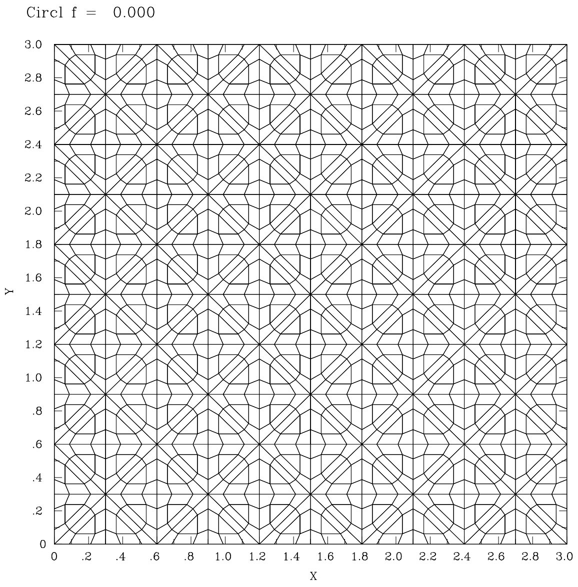
Summary & conclusions

- 2^{nd} and 3^{rd} order elements with diagonal mass matrix are implemented in M3D
- Accuracy is greatly improved compared to 1^{st} order
- Timing is much better than for standard Lagrange elements
- Tilt mode comparisons do not show clear advantage of higher order
- Can apply to realistic 3D ELM simulations
- Future work
  - More tests and applications
  - 4^{th} order?

Tilt mode – 1st, 2nd, 3rd order mesh (low resolution)





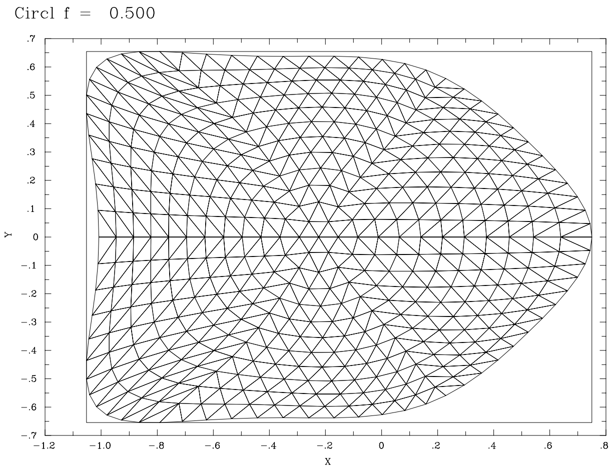


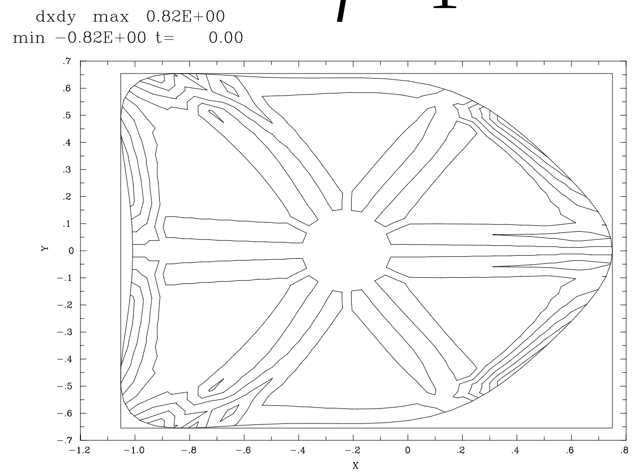
Commutation of derivatives

$$f = \frac{\partial}{\partial R}\left(\frac{\partial s}{\partial Z}\right) - \frac{\partial}{\partial Z}\left(\frac{\partial s}{\partial R}\right)$$

$$f \sim 1$$

1st order mesh





$$f \sim 10^{-11}$$

2nd order mesh