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REPRESENTATION OF THE COLLISIONAL MOMENTS*

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For a quasineutral plasma with one ion species of unit charge, and to leading order in the small mass ratio limit $m_e/m_\iota \to 0$, the general system of fluid moment equations is:

$$\mathbf{u}_e \;=\; \mathbf{u}_\iota \;-\; \frac{1}{en}\mathbf{j} \;,$$

$$\mathbf{j} \;=\; \nabla\times\mathbf{B} \;,$$

$$\frac{\partial\mathbf{B}}{\partial t} \;+\; \nabla\times\mathbf{E} \;=\; 0 \;,$$

$$\mathbf{E} \;=\; -\mathbf{u}_\iota\times\mathbf{B} \;+\; \frac{1}{en}\Big(\mathbf{j}\times\mathbf{B} - \nabla\cdot\mathsf{P}_e^{CGL} + \mathbf{F}_e^{coll}\Big) \;,$$

$$\frac{\partial n}{\partial t} \;+\; \nabla\cdot(n\mathbf{u}_\iota) \;=\; 0 \;,$$

$$m_\iota n\Big[\frac{\partial\mathbf{u}_\iota}{\partial t} + (\mathbf{u}_\iota\cdot\nabla)\mathbf{u}_\iota\Big] \;+\; \nabla\cdot\Big(\mathsf{P}_e^{CGL} + \mathsf{P}_\iota^{CGL} + \hat{\mathsf{P}}_\iota^{gyr} + \hat{\mathsf{P}}_\iota^{coll}\Big) \;-\; \mathbf{j}\times\mathbf{B} \;=\; 0 \;,$$

where

$$\mathsf{P}_\alpha^{CGL} = p_{\alpha\perp}\mathsf{I} + (p_{\alpha\parallel} - p_{\alpha\perp})\mathbf{b}\mathbf{b} \;, \qquad \hat{\mathsf{P}}_\alpha^{gyr} : \mathsf{I} = \hat{\mathsf{P}}_\alpha^{gyr} : (\mathbf{b}\mathbf{b}) = \hat{\mathsf{P}}_\alpha^{coll} : \mathsf{I} = \hat{\mathsf{P}}_\alpha^{coll} : (\mathbf{b}\mathbf{b}) = 0 \;,$$

and the evolution equations for the CGL stress tensors, with $p_\alpha = (2p_{\alpha\perp} + p_{\alpha\parallel})/3$, are:

$$\frac{3}{2}\Big[\frac{\partial p_\iota}{\partial t} + \nabla\cdot(p_\iota \mathbf{u}_\iota)\Big] \;+\; \Big(\mathbf{P}_\iota^{CGL} + \hat{\mathbf{P}}_\iota^{gyr} + \hat{\mathbf{P}}_\iota^{coll}\Big) : (\nabla \mathbf{u}_\iota) \;+\; \nabla\cdot\mathbf{q}_\iota \;-\; g_\iota^{coll} \;=\; 0 \;,$$

$$\frac{1}{2}\Big[\frac{\partial p_{\iota\parallel}}{\partial t} + \nabla\cdot(p_{\iota\parallel}\mathbf{u}_\iota)\Big] \;-\; \mathbf{b}\cdot\Big(\mathbf{P}_\iota^{CGL} + \hat{\mathbf{P}}_\iota^{gyr} + \hat{\mathbf{P}}_\iota^{coll}\Big)\cdot\Big[\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}_\iota\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u}_\iota - \mathbf{b}\times(\nabla\times\mathbf{u}_\iota)\Big] -$$

$$-\;\mathbf{b}\cdot\Big(\mathbf{Q}_\iota^{CGL} + \hat{\mathbf{Q}}_\iota^{gyr} + \hat{\mathbf{Q}}_\iota^{coll}\Big) : (\nabla\mathbf{b}) \;+\; \nabla\cdot\mathbf{q}_{\iota B} \;-\; g_{\iota B}^{coll} \;=\; 0 \;,$$

$$\frac{3}{2}\Big[\frac{\partial p_e}{\partial t} + \nabla\cdot\Big(p_e\mathbf{u}_e\Big)\Big] \;+\; p_e\nabla\cdot\mathbf{u}_e \;+\; (p_{e\parallel} - p_{e\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\Big] - \nabla\cdot\mathbf{u}_e/3\Big\} \;+\; \nabla\cdot\mathbf{q}_e \;-\; g_e^{coll} \;=\; 0 \;,$$

$$\frac{1}{2}\Big[\frac{\partial p_{e\parallel}}{\partial t} + \nabla\cdot\Big(p_{e\parallel}\mathbf{u}_e\Big)\Big] \;+\; p_{e\parallel}\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\Big] \;+\; q_{eT\parallel}\mathbf{b}\cdot\nabla(\ln\; B) \;+\; \nabla\cdot\mathbf{q}_{eB} \;-\; g_{eB}^{coll} \;=\; 0 \;.$$

The above system involves the following collisional moments:

$$\mathbf{F}_e^{coll} \; = \; m_e \int d^3\mathbf{v} \; (\mathbf{v} - \mathbf{u_e}) \; C_e(\mathbf{v}, \mathbf{x}, t),$$

$$\hat{\mathsf{P}}_{\iota,jk}^{coll} \; = \; \frac{m_\iota}{4eB} \, \epsilon_{[jlm} b_l (\delta_{nk]} + 3 b_n b_{k]}) \left(-\mathsf{G}_{\iota,mn}^{coll} + \frac{\partial \hat{\mathsf{Q}}_{\iota,mnp}^{coll}}{\partial x_p} \right) ,$$

$$g_\alpha^{coll} \; = \; \mathbf{G}_\alpha^{coll} : \mathbf{I}/2 \; ,$$

$$g_{\alpha B}^{coll} \; = \; \mathbf{G}^{coll} : (\mathbf{bb})/2 \; ,$$

where

$$\mathbf{G}_\alpha^{coll} \; = \; m_\alpha \int d^3\mathbf{v} \; (\mathbf{v} - \mathbf{u}_\alpha)(\mathbf{v} - \mathbf{u}_\alpha) \; C_\alpha(\mathbf{v}, \mathbf{x}, t)$$

and the contributions from the collisional parts of the stress-flux tensors $\hat{\mathbf{Q}}_\alpha^{coll}$ (in particular the collisional perpendicular heat fluxes) can be neglected in the low collisionality regimes of interest, characterized by $v_{th\alpha}/\nu_\alpha^{coll} \gtrsim L$.

The collision operators are taken in their complete, quadratic Fokker-Plank form:

$$C_\alpha(\mathbf{v},\mathbf{x},t) \;=\; -\sum_\beta \;\frac{c^4 e_\alpha^2 e_\beta^2 \ln\Lambda_{\alpha\beta}}{8\pi m_\alpha}\;\Gamma_{\alpha\beta}(\mathbf{v},\mathbf{x},t)\;,$$

$$\Gamma_{\alpha\beta}(\mathbf{v},\mathbf{x},t) \;=\; \frac{\partial}{\partial\mathbf{v}}\cdot\int d^3\mathbf{w}\;\mathsf{U}(\mathbf{v},\mathbf{w})\cdot\Big[\frac{f_\alpha(\mathbf{v},\mathbf{x},t)}{m_\beta}\;\frac{\partial f_\beta(\mathbf{w},\mathbf{x},t)}{\partial\mathbf{w}} \;-\; \frac{f_\beta(\mathbf{w},\mathbf{x},t)}{m_\alpha}\;\frac{\partial f_\alpha(\mathbf{v},\mathbf{x},t)}{\partial\mathbf{v}}\Big]\;,$$

$$\mathsf{U}(\mathbf{v},\mathbf{w}) \;=\; \frac{|\mathbf{v}-\mathbf{w}|^2\;\mathsf{I}\;-\;(\mathbf{v}-\mathbf{w})(\mathbf{v}-\mathbf{w})}{|\mathbf{v}-\mathbf{w}|^3}\;.$$

Formal manipulations on their velocity moments (integrations by parts and expansions in m_e/m_ι and ρ_ι/L) yield simplified expressions which are still applicable to any collisionality regime and do not require the distribution functions to be close to Maxwellians.

These simplified expressions are well suited for Chapman-Enskog, neoclassical or other kinds of approximations.

For each species, define the thermal speed $v_{th\alpha} \equiv \sqrt{p_\alpha/(m_\alpha n)}$, the dimensionless phase space coordinate

$$\boldsymbol{\xi} \equiv \frac{\mathbf{v} - \mathbf{u}_\alpha(\mathbf{x}, t)}{v_{th\alpha}(\mathbf{x}, t)} \ ,$$

and the dimensionless distribution function

$$\hat{f}_\alpha(\boldsymbol{\xi}, \mathbf{x}, t) \ \equiv \ \frac{v_{th\alpha}^3}{n} \, f_\alpha(\mathbf{u}_\alpha + v_{th\alpha}\boldsymbol{\xi}, \ \mathbf{x}, \ t) \ ,$$

so that

$$\int d^3\boldsymbol{\xi} \ \hat{f}_\alpha(\boldsymbol{\xi}, \mathbf{x}, t) \ = \ 1 \qquad and \qquad \int d^3\boldsymbol{\xi} \ \boldsymbol{\xi} \ \hat{f}_\alpha(\boldsymbol{\xi}, \mathbf{x}, t) \ = \ 0 \ .$$

Define also the collision frequencies:

$$\nu_e^{coll} \ \equiv \ \frac{c^4 e^4 n \ln \Lambda_{e\iota}}{m_e^2 v_{the}^3} \ = \ \frac{c^4 e^4 n \ln \Lambda_{ee}}{m_e^2 v_{the}^3} \qquad and \qquad \nu_\iota^{coll} \ \equiv \ \frac{c^4 e^4 n \ln \Lambda_{\iota\iota}}{m_\iota^2 v_{th\iota}^3}$$

Expanding the distribution functions in powers of $\delta \sim \rho_\iota/L$ (and dropping the $\mathbf{x}, t$ arguments) we have:

$$\hat{f}_\alpha(\boldsymbol{\xi}) \;=\; \hat{f}_\alpha^{(0)}(\xi,\ \xi_\|) \;+\; \hat{f}_\alpha^{(1)}(\boldsymbol{\xi}) \;+\; O(\delta^2)\ ,$$

where $\xi = |\boldsymbol{\xi}|$**,** $\xi_\| = \mathbf{b}\cdot\boldsymbol{\xi}$**,** $\hat{f}_\alpha^{(0)} = O(1)$ **and** $\hat{f}_\alpha^{(1)} = O(\delta)$**.**

It is also useful to define the following two-point convolutions:

$$\Im_\alpha^{(0)}(\xi,\ \xi_\|) \;\equiv\; \int d^3\boldsymbol{\zeta}\ \ \hat{f}_\alpha^{(0)}\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\| + \zeta_\|\Big)\ \ \hat{f}_\alpha^{(0)}(\zeta,\ \zeta_\|) \;=\; O(1)$$

and

$$\Im_\alpha^{(1)}(\boldsymbol{\xi}) \;\equiv\; \int d^3\boldsymbol{\zeta}\ \ \hat{f}_\alpha^{(0)}\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\| + \zeta_\|\Big)\ \ \hat{f}_\alpha^{(1)}(\boldsymbol{\zeta}) \;=\; O(\delta)\ .$$

COLLISIONAL FRICTION FORCE

$$\mathbf{F}_e^{coll} \ = \ \frac{\nu_e^{coll}\ p_e}{4\pi v_{the}} \left\{ \Big[- \int d^3\boldsymbol{\xi}\ \frac{\xi_\|}{\xi^3}\ \hat{f}_e^{(0)}(\xi,\xi_\|) \ + \ \frac{\mathbf{b}\cdot\mathbf{j}}{2env_{the}} \int d^3\boldsymbol{\xi}\ \Big(\frac{\boldsymbol{\xi} - 3\xi_\|\mathbf{b}}{\xi^3}\Big)\cdot\frac{\partial \hat{f}_e^{(0)}(\xi,\xi_\|)}{\partial\boldsymbol{\xi}} \Big]\ \mathbf{b} \ - \right.$$

$$\left. - \ \frac{1}{2env_{the}} \Big[\int d^3\boldsymbol{\xi}\ \Big(\frac{\boldsymbol{\xi} - \xi_\|\mathbf{b}}{\xi^3}\Big)\cdot\frac{\partial \hat{f}_e^{(0)}(\xi,\xi_\|)}{\partial\boldsymbol{\xi}} \Big]\ \mathbf{j} \ - \ \int d^3\boldsymbol{\xi}\ \frac{\boldsymbol{\xi}}{\xi^3}\ \hat{f}_e^{(1)}(\boldsymbol{\xi}) \ + \ O\Big(\frac{\delta^2\ m_e^{1/2}}{m_\iota^{1/2}}\Big) \ + \ O\Big(\frac{m_e}{m_\iota}\Big) \right\} .$$

In the generalized Ohm's law, $\mathbf{F}_e^{coll}$ is to be compared with $\nabla p_e \sim p_e/L$. The order of magnitude of the different terms that contribute to $\mathbf{F}_e^{coll}$ is:

$$\int d^3\boldsymbol{\xi}\ \frac{\xi_\|}{\xi^3}\ \hat{f}_e^{(0)} \ \sim \ \frac{q_{e\|}}{p_e v_{the}} \ \sim \ \frac{q_{e\|}}{p_e v_{th\iota}}\ \Big(\frac{m_e}{m_\iota}\Big)^{1/2} \quad \textit{but vanishes for isotropic}\ \ \hat{f}_e^{(0)} = \hat{f}_e^{(0)}(\xi) \ ,$$

$$\frac{\mathbf{b}\cdot\mathbf{j}}{env_{the}} \int d^3\boldsymbol{\xi}\ \Big(\frac{\boldsymbol{\xi} - 3\xi_\|\mathbf{b}}{\xi^3}\Big)\cdot\frac{\partial \hat{f}_e^{(0)}}{\partial\boldsymbol{\xi}} \ \sim \ \frac{\delta\ v_{th\iota}}{v_{the}} \ \sim \ \delta\ \Big(\frac{m_e}{m_\iota}\Big)^{1/2} \ \textit{but vanishes for isotropic}\ \ \hat{f}_e^{(0)} = \hat{f}_e^{(0)}(\xi) \ ,$$

$$\frac{1}{env_{the}} \Big[\int d^3\boldsymbol{\xi}\ \Big(\frac{\boldsymbol{\xi} - \xi_\|\mathbf{b}}{\xi^3}\Big)\cdot\frac{\partial \hat{f}_e^{(0)}}{\partial\boldsymbol{\xi}} \Big]\ \mathbf{j} \ \sim \ \frac{\delta\ v_{th\iota}}{v_{the}} \ \sim \ \delta\ \Big(\frac{m_e}{m_\iota}\Big)^{1/2} \ ,$$

$$\int d^3\boldsymbol{\xi}\ \frac{\boldsymbol{\xi}}{\xi^3}\ \hat{f}_e^{(1)} \ \sim \ \frac{q_e^{(1)}}{p_e v_{the}} \ \sim \ \frac{q_e^{(1)}}{p_e v_{th\iota}}\ \Big(\frac{m_e}{m_\iota}\Big)^{1/2} \ \sim \ \delta\ \Big(\frac{m_e}{m_\iota}\Big)^{1/2} \ .$$

ION COLLISIONAL PERPENDICULAR VISCOSITY

In its leading order:

$$\hat{\mathsf{P}}_{\iota}^{coll} \; = \; \frac{3\nu_{\iota}^{coll} m_{\iota} p_{\iota}}{8\pi e B} \int d^3\boldsymbol{\xi} \; \frac{\Im_{\iota}^{(1)}(\boldsymbol{\xi})}{\xi^3} \left[(\boldsymbol{\xi} + 3\xi_{\parallel}\mathbf{b}) \; (\mathbf{b} \times \boldsymbol{\xi}) \; + \; (\mathbf{b} \times \boldsymbol{\xi}) \; (\boldsymbol{\xi} + 3\xi_{\parallel}\mathbf{b}) \right] \; \sim \; \delta^2 \, \frac{\nu_{\iota}^{coll} L}{v_{th\iota}} \, p_{\iota} \; .$$

In the momentum conservation equation, $\hat{\mathsf{P}}_{\iota}^{coll}$ is to be compared with $\hat{\mathsf{P}}_{\iota}^{gyr} \sim \delta p_{\iota}$ in the fast dynamics (MHD) ordering, or $\hat{\mathsf{P}}_{\iota}^{gyr} \sim \delta^2 p_{\iota}$ in the slow dynamics (drift) ordering.

ION COLLISIONAL HEAT EXCHANGES

In their leading order:

$$g_\iota^{coll} \;=\; \frac{\nu_e^{coll} m_e}{8\pi m_\iota}\,\Big[2p_e \int d^3\boldsymbol{\xi}\;\frac{\hat f_e^{(0)}(\xi,\xi_\|)}{\xi} \;+\; p_\iota \int d^3\boldsymbol{\xi}\;\frac{\boldsymbol{\xi}}{\xi^3}\cdot\frac{\partial \hat f_e^{(0)}(\xi,\xi_\|)}{\partial\boldsymbol{\xi}}\;\int d^3\boldsymbol{\zeta}\;(\zeta^2-\zeta_\|^2)\;\hat f_\iota^{(0)}(\zeta,\zeta_\|) \;-$$

$$-\;p_\iota \int d^3\boldsymbol{\xi}\;\frac{\xi_\|\mathbf{b}}{\xi^3}\cdot\frac{\partial \hat f_e^{(0)}(\xi,\xi_\|)}{\partial\boldsymbol{\xi}}\;\int d^3\boldsymbol{\zeta}\;(\zeta^2-3\zeta_\|^2)\;\hat f_\iota^{(0)}(\zeta,\zeta_\|)\Big] \;\sim\; \nu_e^{coll}\,\frac{m_e}{m_\iota}\,(p_e-p_\iota) \;\sim\; \nu_\iota^{coll}\,\Big(\frac{m_e}{m_\iota}\Big)^{1/2}\,(p_e-p_\iota)$$

and

$$g_{\iota B}^{coll} \;=\; \frac{\nu_\iota^{coll} p_\iota}{8\pi}\;\int d^3\boldsymbol{\xi}\;\Big(\frac{\xi^2-3\xi_\|^2}{\xi^3}\Big)\,\Im_\iota^{(0)}(\xi,\;\xi_\|) \;\sim\; \nu_\iota^{coll}\;(p_{\iota\|}-p_{\iota\perp})\;,$$

with $g_{\iota B}^{coll} = O\Big(\nu_\iota^{coll} p_\iota \delta\Big) + O\Big(\nu_\iota^{coll} p_\iota \sqrt{m_e/m_\iota}\Big) + O\Big(\nu_\iota^{coll} p_e \sqrt{m_e/m_\iota}\Big)$ **for isotropic** $\hat f_\iota^{(0)} = \hat f_\iota^{(0)}(\xi)$ **.**

In the ion pressure equations, g_ι^{coll} **is to be compared with** $\partial p_\iota/\partial t$ **and** $g_{\iota B}^{coll}$ **is to be compared with** $\partial(p_{\iota\|}-p_{\iota\perp})/\partial t$ **.**

ELECTRON COLLISIONAL HEAT EXCHANGES

$$g_e^{coll} \;=\; \frac{1}{en}\mathbf{j}\cdot\mathbf{F}_e^{coll} \;-\; g_\iota^{coll}$$

where

$$\frac{1}{en}\mathbf{j}\cdot\mathbf{F}_e^{coll} \;\lesssim \delta\,\nu_\iota^{coll}\,\Big(\frac{m_e}{m_\iota}\Big)^{1/2}\,p_e \qquad and \qquad g_\iota^{coll} \;\sim\; \nu_\iota^{coll}\,\Big(\frac{m_e}{m_\iota}\Big)^{1/2}\,(p_e - p_\iota)\ .$$

In its leading order:

$$g_{eB}^{coll} \;=\; \frac{\nu_e^{coll} p_e}{8\pi}\;\int d^3\boldsymbol{\xi}\,\Big(\frac{\xi^2 - 3\xi_\parallel^2}{\xi^3}\Big)\,\Big[\Im_e^{(0)}(\xi,\ \xi_\parallel) + \hat{f}_e^{(0)}(\xi,\ \xi_\parallel)\Big] \;\sim\; \nu_e^{coll}\,(p_{e\parallel} - p_{e\perp})\ ,$$

with $g_{eB}^{coll} = O\Big(\nu_e^{coll} p_e \delta\Big) + O\Big(\nu_e^{coll} p_\iota m_e/m_\iota\Big) + O\Big(\nu_e^{coll} p_e m_e/m_\iota\Big)$ **for isotropic** $\hat{f}_e^{(0)} = \hat{f}_e^{(0)}(\xi)$ **.**

In the electron pressure equations, g_e^{coll} **is to be compared with** $\partial p_e/\partial t$ **and** g_{eB}^{coll} **is to be compared with** $\partial(p_{e\parallel} - p_{e\perp})/\partial t$ **.**

LOOKING AHEAD

To proceed to the next step in the analysis, a decission has to be made on the relative orderings among several independent small parameters in the theory.

The three independent parameters that should be considered small but finite are:

$$\delta \sim \rho_\iota / L \ , \qquad \nu_\iota^{coll} / \Omega_{c\iota} \quad and \quad \sqrt{m_e/m_\iota} \ ,$$

and the relative orderings among them are crucial to determine the appropriate collisional terms to be included in the model.

In addition, the decission has to be made on the orderings of $\omega/\Omega_{c\iota}$ and $u_\iota/v_{th\iota}$ relative to δ (fast or slow dynamics).

It would be desirable to agree on a set of ”canonical orderings for low-collisionality extended-MHD”, presumably based on the CEMM time scales of interest in ITER.