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PROGRESS IN THE TWO-FLUID THEORY OF THE TEARING MODE*

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PURPOSE: DERIVE ANALYTIC RESULTS FOR CEMM VERIFICATION TASKS

EXTENDED-MHD EFFECTS ON THE TEARING MODE:

TWO-FLUID HALL (DOES NOT REQUIRE DENSITY OR TEMPERATURE EQUILIBRIUM GRADIENTS).

TWO-FLUID DIAMAGNETIC (WITH INDEPENDENT DENSITY, ION TEMPERATURE AND ELECTRON TEMPERATURE EQUILIBRIUM GRADIENTS).

PARALLEL CLOSURES ($p_{\parallel} - p_{\perp}$ AND $q_{\parallel}$).

BASIC TWO-FLUID SYSTEM, NEGLECTING PARALLEL CLOSURES:

$$\frac{\partial n}{\partial t} \;+\; \nabla\cdot(n\mathbf{u}) \;=\; 0\;,$$

$$\mathbf{j} \;=\; \nabla\times\mathbf{B}\;,$$

$$\frac{\partial \mathbf{B}}{\partial t} \;+\; \nabla\times\mathbf{E} \;=\; 0\;,$$

$$\mathbf{E} \;=\; -\mathbf{u}\times\mathbf{B} \;+\; \frac{1}{en}(\mathbf{j}\times\mathbf{B}-\nabla p_e) \;+\; \eta\mathbf{j}\;,$$

$$m_\iota n[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}] \;+\; \nabla(p_\iota+p_e) \;+\; \nabla\cdot\mathsf{P}_\iota^{GV} \;-\; \mathbf{j}\times\mathbf{B} \;=\; 0\;,$$

$$\frac{3}{2}(\frac{\partial p_\iota}{\partial t}+\mathbf{u}\cdot\nabla p_\iota) \;+\; \frac{5}{2}p_\iota\nabla\cdot\mathbf{u} \;+\; \nabla\cdot\mathbf{q}_{\iota\perp} \;=\; 0\;,$$

$$\frac{3}{2}[\frac{\partial p_e}{\partial t}+(\mathbf{u}-\frac{1}{en}\mathbf{j})\cdot\nabla p_e] \;+\; \frac{5}{2}p_e\nabla\cdot(\mathbf{u}-\frac{1}{en}\mathbf{j}) \;+\; \nabla\cdot\mathbf{q}_{e\perp} \;=\; 0\;.$$

The resistivity, η, will be taken as constant and the ion gyroviscous stress tensor and the diamagnetic perpendicular heat fluxes are:

$$
\begin{aligned}
\mathsf{P}^{GV}_{\iota,jk} &= \frac{m_\iota p_\iota}{4eB}\,\epsilon_{[jlm}b_l\,(\frac{\partial u_n}{\partial x_m}+\frac{\partial u_m}{\partial x_n})(\delta_{nk]}+3b_n b_{k]})\;, \\
\mathbf{q}_{\iota\perp} &= \frac{5p_\iota}{2eB}\mathbf{b}\times\nabla(\frac{p_\iota}{n})\;, \\
\mathbf{q}_{e\perp} &= -\frac{5p_e}{2eB}\mathbf{b}\times\nabla(\frac{p_e}{n})\;.
\end{aligned}
$$

EQUILIBRIUM WITH FLOW IN ONE-DIMENSIONAL SLAB GEOMETRY:

$$n_0 = n_0(x)$$

$$p_{\iota 0} = p_{\iota 0}(x)$$

$$p_{e0} = p_{e0}(x)$$

$$\mathbf{u}_0 = \mathbf{u}_0(x) = u_{0y}(x)\mathbf{e}_y + u_{0z}(x)\mathbf{e}_z,$$

$$\mathbf{B}_0 = \mathbf{B}_0(x) = B_{0y}(x)\mathbf{e}_y + B_{0z}(x)\mathbf{e}_z,$$

$$\mathbf{j}_0 = \mathbf{j}_0(x) = -B'_{0z}(x)\mathbf{e}_y + B'_{0y}(x)\mathbf{e}_z.$$

$$\frac{1}{2}B_0^2(x) + p_{e0}(x) + p_{\iota 0}(x) - \frac{m_\iota p_{\iota 0}(x)}{2eB_0^2(x)}[u'_{0y}(x)B_{0z}(x) - u'_{0z}(x)B_{0y}(x)] = constant\ .$$

$$E_{0x}(x) = -u_{0y}(x)B_{0z}(x) + u_{0z}(x)B_{0y}(x) - \frac{1}{en_0(x)}[\frac{1}{2}B_0^2(x) + p_{e0}(x)]' \ ,$$

$$E_{0y}(x) = \eta j_{0y}(x) \ ,$$

$$E_{0z}(x) = \eta j_{0z}(x) \ ,$$

$$\frac{\partial \mathbf{B}_0(x)}{\partial t} = -\eta \mathbf{B}_0''(x) \simeq 0 \ .$$

TWO-FLUID TEARING MODES IN THE ABSENCE OF EQUILIBRIUM FLOW AND DENSITY OR TEMPERATURE GRADIENTS:

Consider the simplest case of a static equilibrium ($\mathbf{u}_0 = 0$) with constant n_0, p_{e0} and $p_{\iota 0}$. From equilibrium force balance, the magnitude of the magnetic field B_0 is constant too. No diamagnetic effects, only the two-fluid Hall effect retained.

Sheet pinch equilibrium magnetic field profile:

$$B_{0y}(x) = B_{0y}^{\infty} \tanh(\frac{x}{L}) \ ,$$

$$B_{0z}(x) = [B_0^2 - B_{0y}^2(x)]^{1/2} \ .$$

Linear stability analysis for perturbed quantities independent of z and periodic in y:

$$Q(\mathbf{x}, t) = Q_0(x) + Q_1(x) \exp(iky + \gamma t) \ .$$

DIMENSIONLESS PARAMETERS OF THE PROBLEM:

$$\epsilon_B = \frac{B_{0y}^{\infty}}{B_0} \ ,$$

to be treated as arbitrary.

$$\epsilon_\eta = \frac{\eta k (m_\iota n_0)^{1/2}}{B_{0y}^{\infty}} = \frac{\eta k (m_\iota n_0)^{1/2}}{\epsilon_B B_0} \ll 1 \ ,$$

which is the basic expansion parameter.

$$\epsilon_\gamma = \frac{\gamma (m_\iota n_0)^{1/2}}{k B_{0y}^{\infty}} = \frac{\gamma (m_\iota n_0)^{1/2}}{k \epsilon_B B_0} \ ,$$

which is the eigenvalue of the problem.

$$k d_\iota = \frac{k m_\iota^{1/2}}{e n_0^{1/2}} \ ,$$

to be treated as arbitrary, ranging from the single-fluid limit $kd_\iota \rightarrow 0$ to $kd_\iota = O(1)$.

$$kL,$$

whose allowed range will be such that the normalized instability index,

$$\tilde{\Delta}'(kL) = \frac{\Gamma(1/4)}{2\pi\Gamma(3/4)\, k^{3/2}L^{1/2}}\, \frac{B'_{1x}(0+) - B'_{1x}(0-)}{B_{1x}(0)} = \frac{\Gamma(1/4)}{\pi\Gamma(3/4)}[(kL)^{-5/2} - (kL)^{-1/2}]\ ,$$

is positive and comparable to or less than unity. Only the limit $\tilde{\Delta}'(kL) \gg 1$ is excluded.

$$\beta = \frac{2(p_{e0} + p_{\iota 0})}{B_0^2}\ ,$$

which will be required to satisfy $\epsilon_\eta^{2/5} \ll \beta \leq 1$. Thus the present analysis excludes the very low or zero beta limits, but covers the regime of interest for magnetic fusion plasmas. Within this range of applicability, and for $\tilde{\Delta}'(kL) \lesssim 1$, our dispersion relation is independent of β.

HALL-TEARING DISPERSION RELATION:

Under the above applicability conditions:

$$f\Big(\frac{kd_\iota \epsilon_\gamma^{1/2}}{\epsilon_\eta^{1/2}}\Big)\,\frac{\epsilon_\gamma^{5/4}}{\epsilon_\eta^{3/4}} \;=\; \tilde{\Delta}'(kL)\;,$$

or, calling $\tilde{\epsilon}_\eta = \epsilon_\eta \tilde{\Delta}'^{-2}$ **and** $\tilde{\epsilon}_\eta = \epsilon_\eta \tilde{\Delta}'^{-2}$**:**

$$f\Big(\frac{kd_\iota \tilde{\epsilon}_\gamma^{1/2}}{\tilde{\epsilon}_\eta^{1/2}}\Big)\,\tilde{\epsilon}_\gamma^{5/4} \;=\; \tilde{\epsilon}_\eta^{3/4},$$

where f **is the function of a single variable**

$$f(w) \;=\; \Big[1+\frac{w^2}{2}+\Big(w^2+\frac{w^4}{4}\Big)^{1/2}\Big]^{-1/4}\Big[\frac{1}{2}+\frac{1}{4}\Big(\frac{1}{w^2}+\frac{1}{4}\Big)^{-1/2}\Big]+$$

$$+\Big[1+\frac{w^2}{2}-\Big(w^2+\frac{w^4}{4}\Big)^{1/2}\Big]^{-1/4}\Big[\frac{1}{2}-\frac{1}{4}\Big(\frac{1}{w^2}+\frac{1}{4}\Big)^{-1/2}\Big]\;,$$

having the asymptotic behaviors $f(w)\rightarrow 1\ for\ w \ll 1$ **and** $f(w)\rightarrow w^{-1/2}\ for\ w \gg 1$**.**

This expression defines implicitly a normalized growth rate of the form:

$$\tilde{\epsilon}_\gamma = F(\tilde{\epsilon}_\eta, kd_\iota)$$

or

$$\epsilon_\gamma = \tilde{\Delta}'^2 \; F(\epsilon_\eta \tilde{\Delta}'^{-2}, kd_\iota) \; ,$$

where F is a function of two variables with the asymptotic behaviors $F(v,w) \rightarrow v^{3/5} \; for \; w \ll v^{1/5}$ and $F(v,w) \rightarrow v^{1/2} w^{1/2} \; for \; w \gg v^{1/5}$.

Thus,

$$\epsilon_\gamma = \epsilon_\eta^{3/5} \tilde{\Delta}'^{4/5} \qquad for \qquad kd_\iota \ll \epsilon_\eta^{1/5} \tilde{\Delta}'^{-2/5}$$

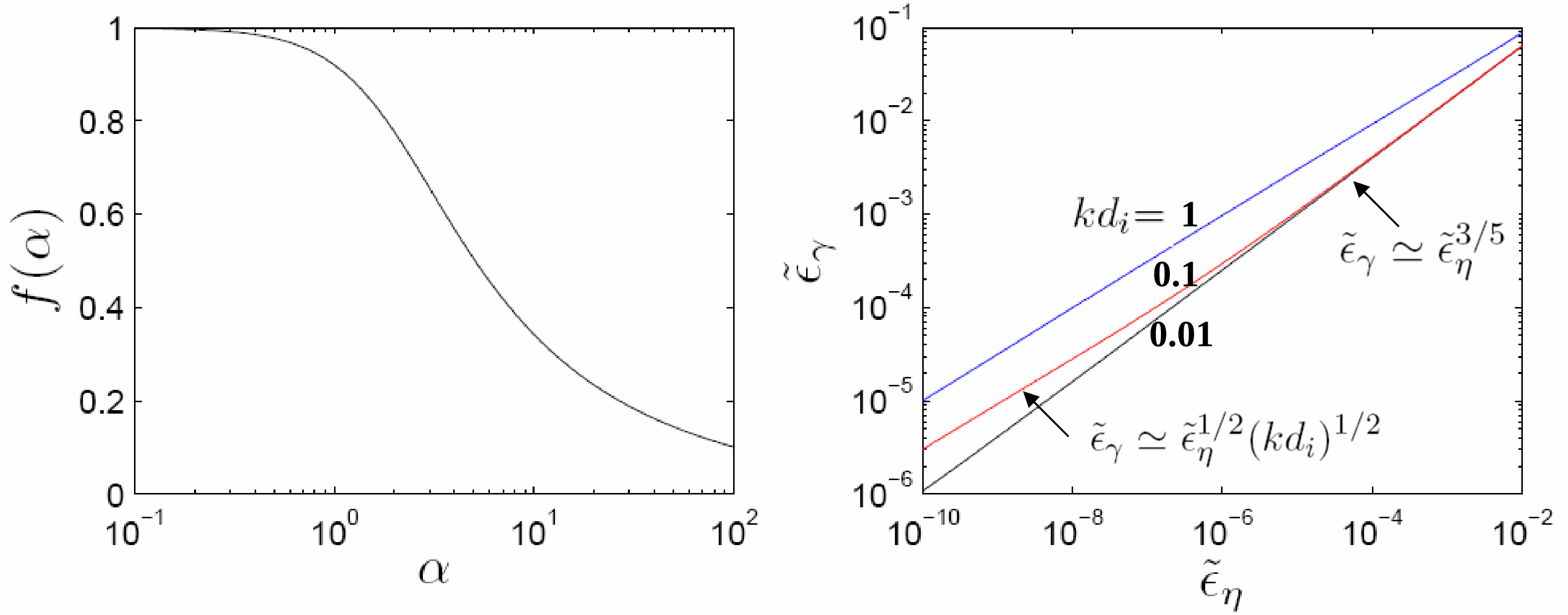
and

$$\epsilon_\gamma = \epsilon_\eta^{1/2} (kd_\iota)^{1/2} \tilde{\Delta}' \qquad for \qquad kd_\iota \gg \epsilon_\eta^{1/5} \tilde{\Delta}'^{-2/5} \; .$$

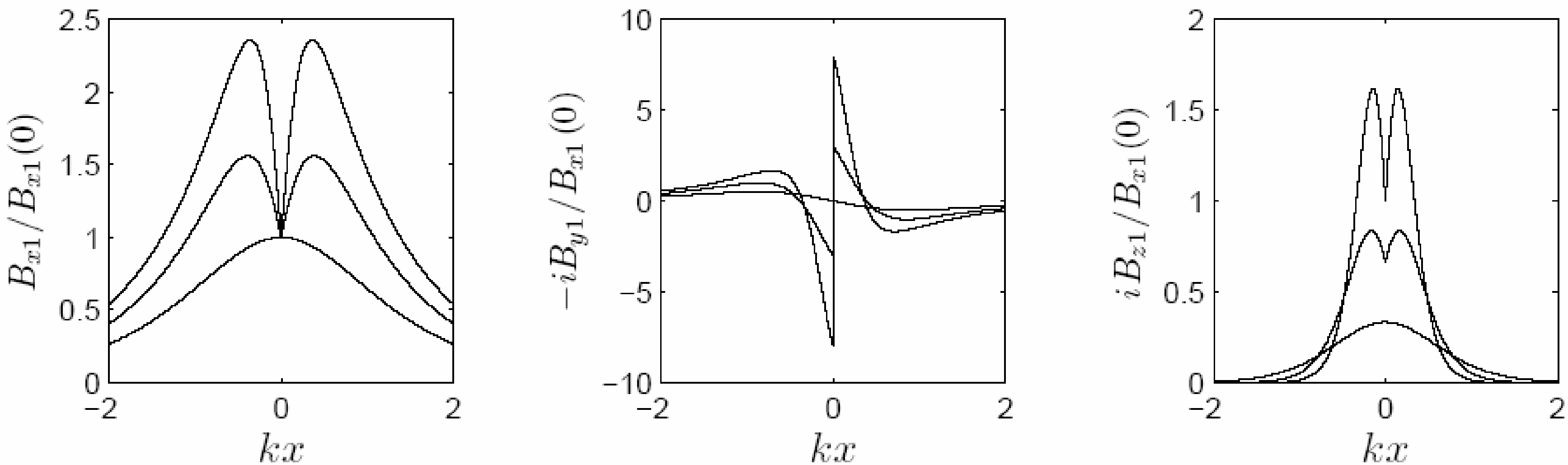
ANALYTICAL DISPERSION RELATION

$$\boxed{\tilde{\epsilon}_\gamma(\tilde{\epsilon}_\eta, kd_i): \quad \tilde{\epsilon}_\gamma^{5/4} f\left(kd_i \frac{\tilde{\epsilon}_\gamma^{1/2}}{\tilde{\epsilon}_\eta^{1/2}}\right) = \tilde{\epsilon}_\eta^{3/4}}$$

$$\tilde{\epsilon}_\eta = \frac{\eta k}{\mu_0 c_A \epsilon_B \tilde{\Delta}'^2} \qquad \tilde{\epsilon}_\gamma = \frac{\gamma}{k c_A \epsilon_B \tilde{\Delta}'^2} \qquad \tilde{\Delta}' \equiv \frac{k^{-1}\Delta'}{C(kL)^{1/2}} \simeq 0.94 \frac{1-(kL)^2}{(kL)^{5/2}}$$



OUTER REGION NORMAL MODES

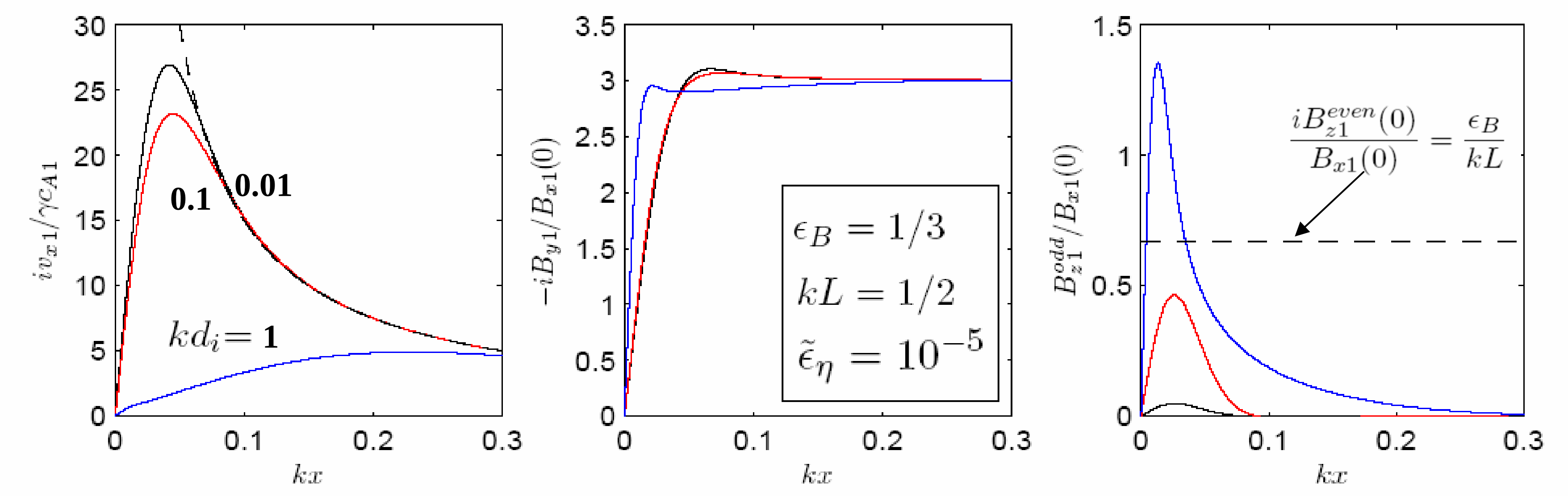


$$\epsilon_B = 1/3$$

$$kL = 1, 1/2, 1/3 \quad \rightarrow \quad \tilde{\Delta}' \simeq 0, 4, 13$$

independent of kd_i

INNER REGION NORNAL MODES

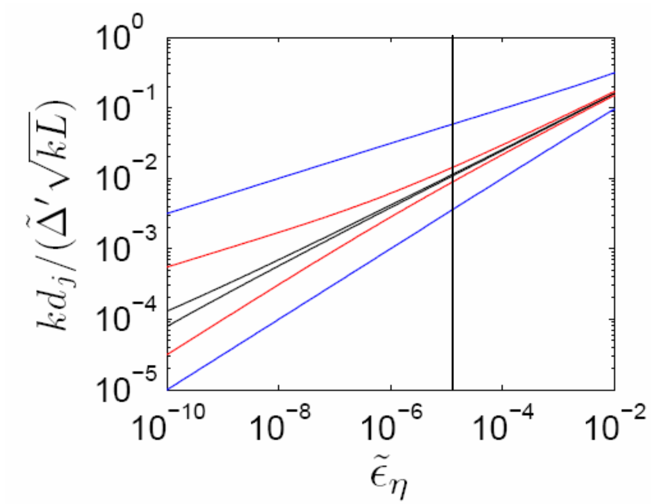


Two well-separated inner regions

for $kd_i/\tilde{\epsilon}_\eta^{1/5} = 10$ (blue case):

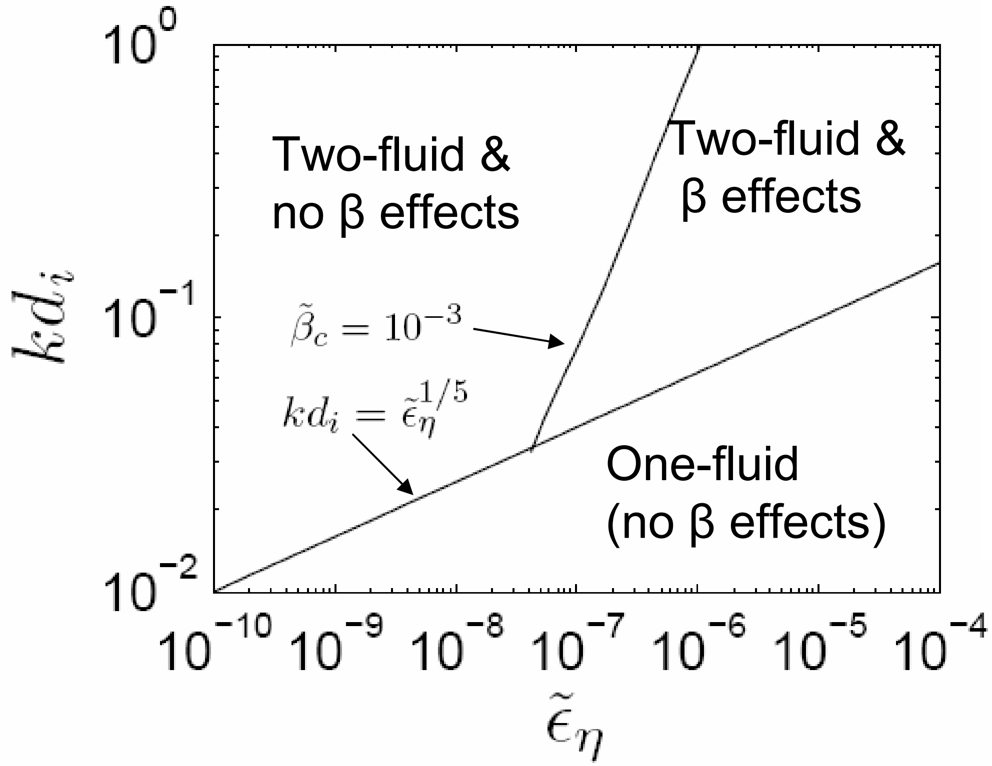
1) Innermost resistive layer, kd_1

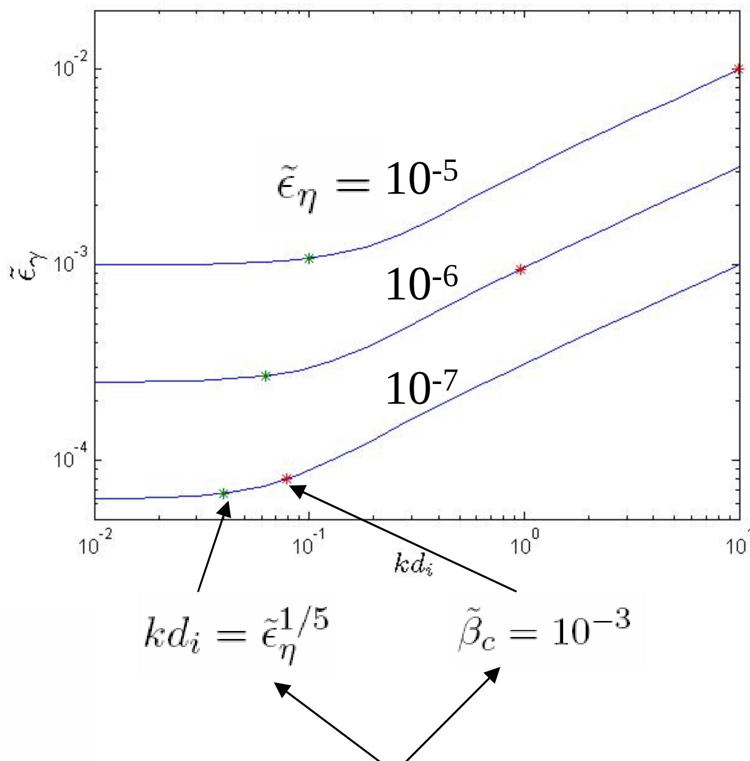
2) Intermediate layer, kd_2



BETA EFFECTS

- ϵ_γ not affected by β for $\dfrac{\beta}{kL\tilde{\Delta}'^2} \gtrsim \tilde{\beta}_c(\tilde{\epsilon}_\eta, kd_i)$ or $kd_i \lesssim \tilde{\epsilon}_\eta^{1/5}$.
- β effects are expected mild.





β affects only the arc between red and green points