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GYROVISCOSITY EFFECTS ON NON-DIFFUSIVE EQUILIBRIA WITH FLOW*

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TWO-FLUID EQUILIBRIUM MODEL THAT CAN BE IMPLEMENTED WITH CURRENT VERSIONS OF EXTENDED-MHD CODES

$$\nabla \cdot \mathbf{B} = 0 \ , \qquad \mathbf{j} = \nabla \times \mathbf{B} \ , \qquad \mathbf{E} = -\nabla \Phi$$

$$\nabla \cdot (n\mathbf{u}_\iota) = \nabla \cdot (n\mathbf{u}_e) = 0 \ , \qquad \mathbf{u}_e = \mathbf{u}_\iota - \frac{1}{en}\mathbf{j}$$

$$m_\iota n(\mathbf{u}_\iota \cdot \nabla)\mathbf{u}_\iota \ + \ \nabla(nT_\iota) \ + \ \nabla \cdot \mathsf{P}_\iota^{GV} \ - \ en(\mathbf{u}_\iota \times \mathbf{B} - \nabla\Phi) \ = \ 0$$

$$\nabla(nT_e) \ + \ en(\mathbf{u}_e \times \mathbf{B} - \nabla\Phi) \ = \ 0$$

$$\mathbf{B} \cdot \nabla T_\iota \ = \ \mathbf{B} \cdot \nabla T_e = \ 0$$

$$\mathsf{P}_{\iota,jk}^{GV} \ = \ \frac{m_\iota n T_\iota}{4eB^2}\epsilon_{jlm}B_l \left(\frac{\partial u_{\iota,m}}{\partial x_n} + \frac{\partial u_{\iota,n}}{\partial x_m}\right)\left(\delta_{nk} + \frac{3}{B^2}B_nB_k\right) \ + \ (j \leftrightarrow k)$$

The ion gyroviscosity can be switched off by setting $T_\iota = 0$, which results in a Hall-MHD model.

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AXISYMMETRIC SOLUTION ($\partial/\partial\varphi = 0$)

$$\mathbf{B} \; = \; \nabla\psi \times \nabla\varphi \; + \; RB_\varphi \nabla\varphi$$

$$\mathbf{j} \; = \; \nabla(RB_\varphi) \times \nabla\varphi \; - \; \Delta^*\psi \nabla\varphi$$

$$\mathbf{u}_\iota \; = \; \frac{1}{n}\nabla\Psi_\iota \times \nabla\varphi \; + \; Ru_{\iota\varphi}\nabla\varphi$$

$$\mathbf{u}_e \; = \; \frac{1}{n}\nabla\Psi_e \times \nabla\varphi \; + \; Ru_{e\varphi}\nabla\varphi$$

$$RB_\varphi \; = \; e(\Psi_\iota - \Psi_e)$$

$$-\Delta^*\psi \; = \; enR(u_{\iota\varphi} - u_{e\varphi})$$

$$T_\iota = T_\iota(\psi) \; , \qquad T_e = T_e(\psi)$$

The three components of the generalized Ohm's law yield:

$$\Psi_e = \Psi_e(\psi)$$

$$e\Phi = T_e(\psi)\ln\left[\frac{n}{N_*(\psi)}\right]$$

$$\frac{u_{e\varphi}}{R} = \frac{B_\varphi}{nR}\frac{d\Psi_e}{d\psi} - \frac{T_e}{eN_*}\frac{dN_*}{d\psi} + \frac{1}{e}\frac{dT_e}{d\psi}\left[\ln\left(\frac{n}{N_*}\right) - 1\right]$$

The three components of the ion momentum equation yield equations for Ψ_ι, n and $u_{\iota\varphi}$.

1. COLD ION SOLUTION WITH PURELY TOROIDAL FLOW

Setting $T_\iota = 0$ and $\Psi_\iota = 0$, the three components of the ion momentum equation yield:

$$0 \;=\; 0$$

$$T_e \;\mathbf{B}\cdot\nabla \ln n \;-\; m_\iota u_{\iota\varphi}^2 \;\mathbf{B}\cdot\nabla \ln R \;=\; 0$$

$$\left[e\frac{u_{\iota\varphi}}{R} \;+\; \frac{T_e}{N_*}\frac{dN_*}{d\psi} \;-\; \frac{dT_e}{d\psi}\ln\left(\frac{n}{N_*}\right)\right]|\nabla\psi|^2 \;-\; T_e\;\nabla\psi\cdot\nabla\ln n \;+\; m_\iota u_{\iota\varphi}^2\;\nabla\psi\cdot\nabla\ln R \;=\; 0\;.$$

This system admits the integral

$$u_{\iota\varphi}^2 \;=\; \frac{R}{2}\;\frac{\partial[R^2\Omega^2(\psi,R)]}{\partial R}\;, \qquad\qquad n \;=\; N(\psi)\exp\left[\frac{m_\iota R^2\Omega^2(\psi,R)}{2T_e(\psi)}\right]$$

where $\Omega^2(\psi,R)$ satisfies

$$\Omega^2(\psi,R) \;+\; \frac{R}{2}\;\frac{\partial\Omega^2(\psi,R)}{\partial R} \;=\; \left[\Omega_*(\psi) \;+\; \frac{m_\iota R^2}{2e}\;\frac{\partial\Omega^2(\psi,R)}{\partial\psi}\right]^2 \quad \textbf{with} \quad \Omega_*(\psi)\equiv\frac{d}{d\psi}\left[\frac{T_e}{e}\ln\left(\frac{N}{N_*}\right)\right]$$

or, solving perturbatively,

$$\Omega^2(\psi,R) \;=\; \Omega_*(\psi)^2\;\left[1 \;+\; \frac{m_\iota R^2}{e}\;\frac{d\Omega_*(\psi)}{d\psi} \;+\; \ldots\right]$$

. ## 2. FINITE ION TEMPERATURE SOLUTION WITH PURELY TOROIDAL FLOW

Setting $\Psi_\iota = 0$ but $T_\iota \neq 0$, the three components of the ion momentum equation yield:

$$\nabla \cdot \left(R^2 \nabla\varphi \cdot \mathbf{P}_\iota^{GV} \right) \ = \ 0$$

$$(T_\iota + T_e)\ \mathbf{B}\cdot\nabla\ln n \ - \ m_\iota u_{\iota\varphi}^2\ \mathbf{B}\cdot\nabla\ln R \ + \ \frac{1}{n}\ \mathbf{B}\cdot\left(\nabla\cdot\mathbf{P}_\iota^{GV}\right) \ = \ 0$$

$$\left[e\frac{u_{\iota\varphi}}{R} - \frac{dT_\iota}{d\psi} + \frac{T_e}{N_*}\frac{dN_*}{d\psi} - \frac{dT_e}{d\psi}\ln\left(\frac{n}{N_*}\right)\right]|\nabla\psi|^2 + \nabla\psi\cdot\left[m_\iota u_{\iota\varphi}^2\nabla\ln R - (T_\iota + T_e)\nabla\ln n - \frac{1}{n}\nabla\cdot\mathbf{P}_\iota^{GV}\right] = 0.$$

This is a system of three equations for the two unknowns $u_{\iota\varphi}$ and n, which has a special rigid rotation solution only if T_ι is constant:

$$T_\iota \ = \ T_{\iota 0} \ = \ constant\ , \qquad \frac{u_{\iota\varphi}}{R} \ = \ \Omega_0 \ = \ constant\ , \qquad n \ = \ N(\psi)\exp\left\{\frac{m_\iota \Omega_0 R^2}{2[T_{\iota 0} + T_e(\psi)]}\right\}$$

with the additional constraint

$$\frac{d}{d\psi}\left[T_e \ln\left(\frac{N}{N_*}\right)\right] \ = \ e\Omega_0 \ - \ \frac{T_{\iota 0}}{N}\frac{dN}{d\psi}\ .$$

3. COLD ION SOLUTION WITH FINITE POLOIDAL FLOW

Setting $T_\iota = 0$ and $\Psi_\iota \neq 0$, the components of the ion momentum equation in the directions of $\nabla\varphi$ and $\mathbf{u}_\iota$ are

$$\nabla\varphi \cdot \left[\nabla\Psi_\iota \times \nabla\left(\psi + \frac{m_\iota}{e} R u_{\iota\varphi}\right)\right] = 0 \ ,$$
$$\nabla\varphi \cdot \left[\nabla\Psi_\iota \times \nabla\left(e\Phi + \frac{m_\iota}{2} u_\iota^2\right)\right] = 0 \ ,$$

which have the general integrals

$$\psi + \frac{m_\iota}{e} R u_{\iota\varphi} = \chi(\Psi_\iota) \ , \qquad e\Phi + \frac{m_\iota}{2} u_\iota^2 = W(\Psi_\iota) \ .$$

Combining this result with the solution for $e\Phi$ and the component of the ion momentum equation in the direction of $\nabla\Psi_\iota$, one gets the following equations for n and Ψ_ι:

$$\frac{m_\iota}{2R^2}\left\{\frac{1}{n^2}|\nabla\Psi_\iota|^2 + \frac{e^2}{m_\iota^2}\left[\chi(\Psi_\iota) - \psi\right]^2\right\} + T_e(\psi)\ln\left[\frac{n}{N_*(\psi)}\right] - W(\Psi_\iota) = 0 \ ,$$

$$R^2\nabla\cdot\left(\frac{1}{nR^2}\nabla\Psi_\iota\right) - \frac{e^2}{m_\iota^2}\left[\Psi_\iota - \Psi_e(\psi)\right] + \frac{ne^2}{m_\iota^2}\left[\chi(\Psi_\iota) - \psi\right]\frac{d\chi(\Psi_\iota)}{d\Psi_\iota} - \frac{nR^2}{m_\iota}\frac{dW(\Psi_\iota)}{d\Psi_\iota} = 0 \ .$$

4. FINITE ION TEMPERATURE AND FINITE POLOIDAL FLOW

In the general case, $T_\iota \neq 0$ and $\Psi_\iota \neq 0$, the components of the ion momentum equation in the directions of $\nabla\varphi$ and $\mathbf{u}_\iota$ are

$$\nabla\varphi \cdot \left[\nabla\Psi_\iota \times \nabla\left(e\psi + m_\iota R u_{\iota\varphi}\right)\right] \;=\; \nabla \cdot \left(R^2 \nabla\varphi \cdot \mathbf{P}_\iota^{GV}\right) \;,$$

$$\nabla\varphi \cdot \left[\nabla\Psi_\iota \times \nabla\left(e\Phi + \frac{m_\iota}{2} u_\iota^2 + T_\iota\right)\right] \;+\; T_\iota \, \nabla\varphi \cdot \left(\nabla\Psi_\iota \times \nabla \ln n\right) \;=\; \nabla \cdot \left(\mathbf{u}_\iota \cdot \mathbf{P}_\iota^{GV}\right) \;.$$

These have the integrability constraints

$$\oint_{\Psi_\iota} dl \, \frac{R}{|\nabla\Psi_\iota|} \, \nabla \cdot \left(R^2 \nabla\varphi \cdot \mathbf{P}_\iota^{GV}\right) \;=\; 0 \;,$$

$$\oint_{\Psi_\iota} dl \, \frac{R}{|\nabla\Psi_\iota|} \, \left[\nabla \cdot \left(\mathbf{u}_\iota \cdot \mathbf{P}_\iota^{GV}\right) \;-\; T_\iota \, \nabla\varphi \cdot \left(\nabla\Psi_\iota \times \nabla \ln n\right)\right] \;=\; 0 \;.$$

PERTURBATION NEAR SINGLE-FLUID SOLUTION

In the limit $\rho_\iota/L \to 0$ and $d_\iota/L \to 0$ our system recovers the single-fluid solution:

$$\Psi_\iota^{(0)} \; = \; \Psi_{\iota *}(\psi) \; , \qquad\qquad \mathbf{u}_\iota^{(0)} \; = \; \frac{1}{n} \, \frac{d\Psi_{\iota *}(\psi)}{d\psi} \, \mathbf{B} \; + \; R^2 \; \Omega_*(\psi) \; \nabla\varphi$$

Since $\mathbf{P}_\iota^{GV} \leq O(\rho_\iota/L \; nT_\iota)$, in a perturbative scheme we may evaluate the leading order gyroviscosity based on this single-fluid flow solution.

Neither of the two integrability conditions imposes any restriction on the $R^2 \; \Omega_*(\psi) \; \nabla\varphi$ part of the single-fluid flow.

Constraints imposed by the integrability conditions on the $n^{-1} \; d\Psi_{\iota *}(\psi)/d\psi \; \mathrm{B}$ part of the flow are being investigated.