# Interface Between Ideal MHD Theory and Transport Processes with Self Organization

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# Simplest Momentum Conservation Equation for a Magnetically Confined Plasma

$$\nabla p - \frac{1}{c}\mathbf{J} \times \mathbf{B} = 0$$

$$\mathbf{B} \cdot \nabla \psi = 0$$

$$\mathbf{B} \cdot \nabla p = 0 \Rightarrow p = p(\psi)$$

$$p = n(T_e + T_i)$$

$$\mathbf{B} \cdot \nabla T_e = 0 \Rightarrow T_e + T_e(\psi)$$

(large longitudinal electron thermal conductivity)

If $\qquad T_i \simeq T_e, \qquad T = \frac{1}{2}(T_e + T_i) = T(\psi),$

then $\qquad n = n(\psi).$

Thus surfaces of constant $n$ should overlap with surfaces of constant $T$. **Contradicted by the experiments** on axisymmetric plasmas with non-circular cross sections.

# Axisymmetric Configurations (Magnetically Confined Plasmas)

$$\mathbf{B} = \frac{1}{R}\left[\nabla\psi \times \mathbf{e}_\phi + I(\psi)\mathbf{e}_\phi\right]$$

$$\mathbf{F}_M = \frac{1}{c}\mathbf{J}\times\mathbf{B} = -\frac{1}{4\pi R^2}\left\{(\Delta_*\psi)\nabla\psi + I\nabla I\right\}$$

$$\mathbf{F}_{M\phi} = 0$$

and

$$J_\phi = -\frac{c}{4\pi R}\Delta_*\psi.$$

Then

$$\boxed{-\frac{1}{4\pi R^2}\left(\Delta_*\psi + I\frac{dI}{d\psi}\right) = \frac{dp}{d\psi}}$$

(so called G-S equation), where $\Delta_*\psi \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\psi\right) + \frac{\partial^2}{\partial z^2}\psi.$

On the other hand, under stationary conditions,

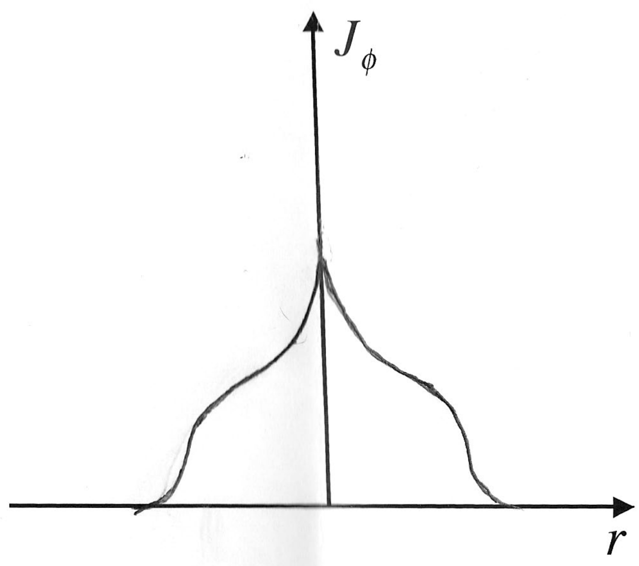
$$J_\phi = \sigma_\parallel E_\phi = \sigma_\parallel \frac{V_L^0}{2\pi R},$$

where $V_L^0$ is the “loop voltage”.

Considering toroidal configurations with large aspect ratios and circular cross sections, the plasma longitudinal conductivity in trapped particle regimes is, for $B_\phi^2 \gg B_p^2$,

$$\sigma_\parallel \simeq \sigma_\parallel^{cl}\left[1-1.95\left(\frac{r}{R_0}\right)^{1/2}\right], \text{ where } \sigma_\parallel^{cl} \propto T_e^{3/2}.$$

Then can $J_\phi \propto \frac{1}{R}\Delta_* \psi$ be proportional to $\frac{1}{R}\frac{T_e^{3/2}}{1-1.95\left(r/R_0\right)^{1/2}}$ ?




No evidence for this (as far as I know).

## Nonclassical Transport and the "Principle of Profile Consistency"

Starting from the experimental observation of temperature and density profiles in magnetically confined plasmas and analyzing the consistency conditions for the plasma-column equilibrium, analytical expressions for the nonclassical energy and particle flows are obtained, and an interpretation of existing experiments is provided.

The problem of understanding the nature of the particle and energy transport processes in magnetically confined plasmas has attracted considerable theoretical and experimental effort in recent years. In fact the theoretical effort has been mostly devoted to numerical simulation of the existing experimental observations, while a relatively simple analytical formulation of this problem would be highly desirable. In this spirit we present a set of criteria that appear to lead to a consistent description of both the electron thermal-energy transport and the particle transport. We label this set of criteria as the "principle of profile consistency." In fact, this is based on assuming that the observed flows of electron thermal energy and particles are those needed to reach a consistent set of radial profiles for the current density, the particle temperatures and the plasma density, while satisfying the equilibrium conditions for the considered plasma column. In addition, we start from the experimental observation that the electron temperature takes on a diffusion-like profile, in impurity-free plasmas, that is

$$T_e \simeq T_{e0} \exp\left(-\alpha_T \frac{r^2}{a^2}\right), \tag{1}$$

$a$ being the plasma column radius and $\alpha_T$ a weak function of $r/a$. Then, to the extent that the longitudinal resistivity $\eta_{\|}$ is classical, the current density profile is of the form

$$J_{\|} \simeq J_0 \exp\left(-\alpha_D \frac{q_s r^2}{q_0 a^2}\right), \tag{2}$$

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**This is where a self organization process has to come in, as the experimental evidence for the existence of a radial "profile consistency" (B.C., 1980) of the electron temperature suggests.**

# Brief Comments on Pulsar Models

In dealing with axisymmetric pulsar magnetospheres we have to take

$$\mathbf{B} \simeq \frac{1}{R}\left[\nabla\psi\times\mathbf{e}_\phi + I\left(\psi,\, z\right)\mathbf{e}_\phi\right]$$

As poloidal currents producing slowing down $\left[\omega_0 = \omega_0\left(t\right)\right]$ have to be present. That is, $I$ is not a function of $\psi$ only and is an odd function. The relevant magnetic configuration equation was derived originally in 1971 (published in Ap. J., 1973).

In this case the magnetic force $\mathbf{F}_M$ is given by

$$\mathbf{F}_M = \frac{1}{c}\mathbf{J}\times\mathbf{B} = -\frac{1}{4\pi R^2}\left\{\left(\Delta_*\psi\right)\nabla\psi + I\nabla I - \left(\nabla I\times\nabla\psi\right)\right\}$$

and has a **toroidal** component.

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# MAGNETIC CONFIGURATION IN THE NEIGHBORHOOD OF A COLLAPSED STAR

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ABSTRACT

It is shown that the magnetic configuration in the neighborhood of a collapsed star with parameters appropriate for models of X-ray stars or pulsars is nearly force-free, with $(\nabla \times \boldsymbol{B})/B$ nonconstant. In the case where the magnetic axis coincides with the rotation axis, a differential equation for the magnetic surfaces is derived. A proper double-expansion technique is used to obtain a significant asymptotic solution of this equation and to derive explicit expressions for the relevant magnetic-field components.

*Subject headings:* collapsed stars — magnetic fields — pulsars — X-ray sources

## I. INTRODUCTION

We consider a rotating collapsed (neutron) star with a magnetic-field configuration that is symmetric about its axis of rotation. We point out that, given the expected high value of the magnetic field, in the vicinity of the polar caps the current flow is nearly parallel to the magnetic field, $\boldsymbol{J} \simeq \alpha \boldsymbol{B}$; hence the field is approximately force-free. In addition, by considering the nature of the electromotive force driving this current and the resultant current flow, one must conclude that $\alpha$ is not constant. For this reason the treatment of force-free field configurations which are found in the literature (Lüst and Schlüter 1954; Chandrasekhar 1956; Chandrasekhar and Kendall 1957; Woltjer 1958; Morikawa 1969) cannot be utilized.

We therefore resort to an asymptotic solution of the general force-free field equations, valid in the vicinity of the rotation axis. In particular we refer, as in the analysis of the equilibrium of laboratory plasmas (Solovev and Shafranov 1970), to the magnetic surfaces of the considered configuration. These surfaces are labeled by the streaming function $\Psi$ which satisfies the equation $\boldsymbol{B}\cdot\nabla\Psi = 0$. Our solution enables us to give explicit expressions for the magnetic surfaces and field lines, and to formulate a precise criterion to establish limits for the current which flows through the star surface. Here we summarize our analysis, while a more detailed treatment of the same problem will be published elsewhere (Cohen, Coppi, and Treves 1972).

We assume that the collapsed star under consideration is surrounded by a plasmasphere and distinguish two regions: an active region near the symmetry axis, where poloidal currents can flow, and an inactive equatorial region where the plasma corotates with the star. The existence of poloidal currents in the active region results from slippage of the plasma with respect to the magnetic field; this slippage can be associated with the finite plasma resistivity and with relativistic effects and produces an electromotive force. The two regions are separated by the magnetic surface which intersects the star at colatitude $\theta_c$; if we assume that the poloidal magnetic field is dipolar, we find that $\theta_c$ is of order $(\omega_0 a/c)^{1/2}$ where $\omega_0$ is the angular velocity of the star and $a$ its radius.

We consider a region well inside the speed-of-light cylinder and write the equation of

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which is independent of $r$. The condition that the first, second, and fourth terms of equation (13) be dominated by $(4\chi/\hat{r}^3\Psi_c)\Psi_{\chi\chi}^{(1)}$ gives, respectively,

$$\sin^2\theta \ll 8\,, \qquad \sin^2\theta \ll 2\,, \qquad \sin^2\theta \ll \tfrac{4}{3}\,,$$

which together with expression (23) may be written

$$\sin^2\theta \ll \min\,[(\Psi_c/\hat{\alpha}_M)^{2/3}, \tfrac{4}{3}]\,. \tag{24}$$

Our solution is valid, then, as long as both the above condition and the force-free field approximation hold.

## IV. MAGNETIC SURFACES AND FIELD LINES

A point $(r, \theta, \phi)$ is related to a point $(a, \theta_0, \phi_0)$ at the surface of the star on the same magnetic surface by the equation $\Psi(r, \theta) = \Psi(a, \theta_0)$, or equivalently, by $\Psi(\hat{r}, \chi) = \Psi(\hat{r} = 1, \chi_0)$ where $\chi_0 = (\sin^2\theta_0)/\Psi_c$. One may use the latter form of this equation to obtain an expression for the magnetic surfaces in polar coordinates. Writing $\Psi(\hat{r}, \chi) = \Psi^{(0)} + \Psi^{(1)} \equiv \Psi_c\chi + \Psi^{(1)}(\hat{r}, \chi)$, the preceding equation gives

$$(\chi - \chi_0) = [\Psi^{(1)}(\hat{r} = 1, \chi_0) - \Psi^{(1)}(\hat{r}, \chi)]/\Psi_c$$

implying that $\chi - \chi_0$ is of order $(\Psi^{(1)}/\Psi_c) \lesssim (\Psi^{(1)}/\Psi^{(0)})$. We may obtain an expression for $(\chi - \chi_0)$ correct to first order in $(\Psi^{(1)}/\Psi_c)$ by simply replacing $\Psi^{(1)}(\hat{r}, \chi)$ by $\Psi^{(1)}(\hat{r}, \chi_0)$ in the above equation. Assuming that the separated form (16) is applicable, so that $\Psi^{(1)} = -(\hat{r}^3\Psi_c/4)\psi(\chi)$, the result may be expressed in the form

$$\frac{\sin^2\theta}{\hat{r}} = \sin^2\theta_0 + \tfrac{1}{4}\Psi_c\psi(\chi_0)(\hat{r}^3 - 1)\,. \tag{25}$$

For a current distribution corresponding to $\hat{\alpha}$ chosen as in equation (11), the coefficient of $(\hat{r}^3 - 1)$ is positive for all magnetic surfaces in the range of our approximations; $\theta$ increases with $r$ along a magnetic surface faster than for a dipole field. Since field

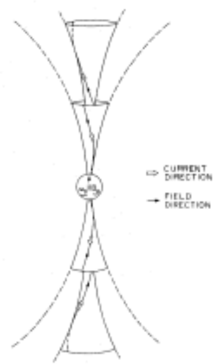


FIG. 1.—Sketch of magnetic surfaces and field lines

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# Magnetic Equation for a Rotating Neutron Star

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The magnetic configuration in the plasma-sphere surrounding a neutron star is described in terms of a model equation that is constructed to be valid from the surface of the star to distances of the order of the light speed cylinder and beyond. Significant asymptotic solutions of this equation, that are valid in limited regions around the star, are presented.

## 1. INTRODUCTION

One of the problems that have to be dealt with when considering possible models for pulsars concerns the macroscopic properties of the plasma that can surround a rotating neutron star. The high magnetic field that is associated with this type of star and the relatively high frequency of rotation that is appropriate for pulsar models are the most important parameters determining the nature of the plasma-sphere surrounding the star.

In the following sections we give an analytical procedure, for a fluid-like description of this plasma-sphere, that leads to solve an equation in the scalar labeling the (magnetic) surfaces of the relevant magnetic configuration. A number of important effects, that it is necessary to consider when analyzing the possible types of plasma flow and magnetic field configurations at distances of the order of or larger than the radius of the "light-speed-cylinder," are discussed and taken into account in the above mentioned magnetic equation. These include the fact that the plasma does not strictly corotate with the star and that its motion is not "frozen-in" everywhere with the magnetic field, the influence of the gravitational and of the centrifugal forces, and the effects of a "braking" force on the plasma resulting from the emission of radiation and particles. Moreover, the braking mechanism has to satisfy the condition that the star rate of energy loss be equal to the rate of angular momentum loss times the frequency of rotation. For the sake of simplicity, a two dimensional model is analyzed assuming that the axis of rotation coincides with that of the magnetic field configuration (Goldreich and Julian, 1969). This magnetic configuration depends upon two expansion parameters which are related to the portion of star surface from which poloidal currents are drawn, and to the ratio between the toroidal and the poloidal magnetic field at the star surface. An estimate of these parameters can be obtained

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respectively, where $\theta_c \simeq \psi_c^{1/2}$ as follows from Eq. (3.1). Therefore,

$$\Phi \neq 0 \text{ if } 0 < \psi < \psi_c \quad \text{and } \Phi = 0 \text{ if } \psi_c \leqslant \psi. \tag{3.4}$$

In addition, on each of the polar caps $\Phi$ is assumed to be a function of $\psi$ only, as there is no braking force close to the star. Two regions around the star are thus postulated, an active and an inactive one (see Fig. 1). The active region extends from the polar caps ($r = a$, $\psi < \psi_c$) to a "critical surface" beyond which the motion of the plasma is no longer tied to the star magnetic field. Current is drawn from the polar caps along constant $\Phi$-surfaces which, in the neighborhood of the star, coincide with magnetic surfaces. The inactive region is assumed to be bounded by the magnetic surface $\psi = \psi_c$ that is tangent to the critical surface. In addition, it is reasonable to assume that inside the inactive region there is no poloidal current so that the magnetic configuration, in a region relatively close to the star, is that of a dipole corotating with it. The braking of the plasma, which is connected with the rate of energy and

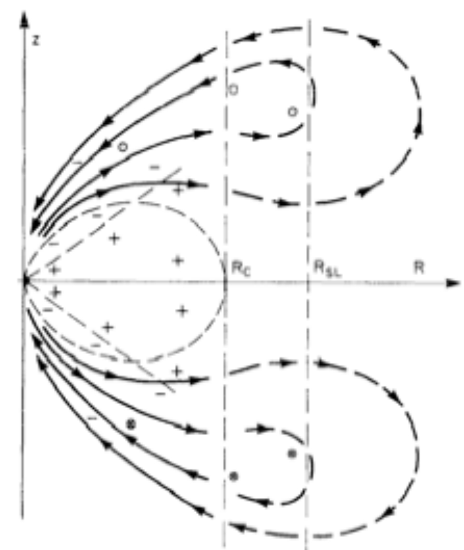



FIG. 1a. Current distribution in the rotating star magnetosphere. The star is indicated by the small circle around the origin. The light broken curve shows the boundary between the active and the inactive region while the straight broken line separates the region of positive charge (+) from the region of negative charge (−) close to the star. The current flow is indicated by heavy lines, solid for $R < R_c$ and broken for $R > R_c$ where the circuit closes. The orientation of the toroidal magnetic field in the active region is symbolized by crossed ⊗ and dotted ⊙ small circles. The dipole magnetic moment and the angular velocity of the star have been assumed to point in the same direction.

# Two-dimensional Plasma and Field Configurations Around Black Holes

General Relativity corrections are neglected at first. The plasma is rotating around a central object with a velocity

$$\mathrm{V}_\phi = R\Omega(R,\ z)$$

where

$$\Omega(R,\ z) \simeq \Omega_k(R) + \delta\Omega(R,\ z)\ ,$$

$\Omega_k \equiv \left(GM_*/R^3\right)^{1/2}$ is the Keplerian frequency for the central object of mass $M_*$ and whose gravity is prevalent (that is, the plasma self gravity can be neglected) and $|\delta\Omega|/\Omega_k < 1$. We assume, for simplicity that $I = I(\psi)$. Then

$$\mathbf{F}_{Mp} == -\frac{1}{4\pi R^2}\left\{(\Delta_*\psi) + I\frac{dI}{d\psi}\right\}\nabla\psi,$$

as in the case considered earlier of magnetically confined plasmas.

In the case that we consider, the total momentum conservation equation, that includes both the toroidal rotation velocity and the effect of the gravitational field of the a central object, is

$$-\rho\left(\Omega^2 R\mathbf{e}_R + \nabla\Phi_G\right) = -\nabla p + \frac{1}{c}\mathbf{J}\times\mathbf{B} \tag{I}$$

where

$$\Phi_G = \frac{GM_*}{\sqrt{R^2+z^2}}, \qquad \nabla\Phi_G \simeq -\frac{V_k^2}{R}\left(\mathbf{e}_R + \frac{z}{R}\mathbf{e}_z\right), \qquad V_k^2 \equiv \frac{GM_*}{R} \equiv \Omega_k^2 R^2.$$

Then we have

$$\mathbf{B}\cdot\nabla p = \rho R\left(\Omega^2 - \Omega_k^2\right)B_R - z\rho\Omega_k^2 B_z \neq 0$$

and if we apply the $\nabla\times$ operator on Eq. (I) we obtain

$$\nabla \times \left( \rho \nabla \Phi_G + \rho \Omega^2 R \mathbf{e}_R \right)$$

$$= \mathbf{e}_\phi \left\{ \frac{\partial \rho}{\partial z} \left( R\Omega^2 + \frac{\partial \Phi_G}{\partial R} \right) + \rho R 2\Omega \frac{\partial \Omega}{\partial z} - \frac{\partial \rho}{\partial R} \frac{\partial \Phi_G}{\partial z} \right\}$$

$$= \frac{1}{4\pi R^2} \left[ -\frac{2}{R} \left( \Delta_* \psi + I \frac{dI}{d\psi} \right) \mathbf{e}_R + \nabla \left( \Delta_* \psi \right) \right] \times \nabla \psi \, .$$

This can be rewritten as

$$2\Omega_k R \frac{\partial}{\partial z} \left( \rho \delta\Omega \right) + z\Omega_k^2 \left( \frac{\partial \rho}{\partial R} + \frac{3}{2} \frac{z}{R} \frac{\partial \rho}{\partial z} \right)$$

$$= \frac{1}{4\pi R^2} \left\{ \left[ \frac{2}{R} \left( \Delta_* \psi + I \frac{dI}{d\psi} \right) - \frac{\partial}{\partial R} \left( \Delta_* \psi \right) \right] \frac{\partial \psi}{\partial z} + \left[ \frac{\partial}{\partial z} \left( \Delta_* \psi \right) \right] \frac{\partial \psi}{\partial R} \right\} \qquad (*)$$

and we call it the "**Master Equation**".

We note that the vertical momentum conservation equation is, considering the expression for $F_{Mz}$ given earlier,

$$0 = -\frac{\partial p}{\partial z} - z\rho\Omega_k^2 - \frac{1}{4\pi R^2}\frac{\partial \psi}{\partial z}\left(\Delta_* \psi + I\frac{dI}{d\psi}\right). \qquad (**)$$

Clearly, we have two equations, (*) and (**), which give $\psi(R, z)$ and $p(R, z)$ for reasonable choices of the density $\rho(R, z)$, the poloidal current function $I(\psi)$, and $\delta\Omega(\psi_1)$.

Is it possible to find profiles that are consistent with known thermal energy balance equations?

In order to proceed further we consider a radial interval $\left|R-R_0\right|< R_0$ around a given radius $R_0$. Then

$$\Omega \simeq \Omega_k\left(R_0\right)+\left(R-R_0\right)\frac{d\Omega_k}{dR}\bigg|_{R=R_0}+\delta\Omega$$

and we comply with the isorotation condition $\Omega=\Omega\left(\psi\right)$ defining $\psi_z/\psi_{0k}$ by

$$\left(R-R_0\right)\frac{d\Omega_k}{dR}\bigg|_{R=R_0}=\Omega_k^0\,\frac{\psi_z}{\psi_{0k}}$$

and $\psi_1/B_0$ by

$$2\Omega R\delta\Omega \simeq 2\Omega_k^0\,\frac{d\Omega_k}{dR}\bigg|_{R=R_0}\frac{\psi_1}{B_0}=-\Omega_D^2\,\frac{\psi_1}{B_0R_0},$$

where $\Omega_D^2 \equiv -Rd\,\Omega^2/dR^2=3\Omega_k^2$ is considered to be the "driving factor" for the onset of the magnetic configurations that are analyzed and $\left|\psi_1/\left(B_0R_0^2\right)^2\right|<1$.

On the other hand, for the configurations we shall consider

$$\frac{\partial}{\partial R} \gtrsim \frac{\partial}{\partial z} \gg \frac{1}{R} \quad \text{and} \quad I\frac{dI}{d\psi} \sim \Delta_* \psi \, .$$

In this case $\nabla^2 \simeq \partial^2/\partial R^2 + \partial^2/\partial z^2$ and the Master Equation reduces to

$$\frac{\partial}{\partial z}\left[R\rho\left(\Omega^2 - \Omega_k^2\right)\right] + \Omega_k^2 z \frac{\partial}{\partial R}\rho + \frac{1}{4\pi}\left(B_z \nabla^2 B_R - B_R \nabla^2 B_z\right) \simeq 0\,,$$

that is independent of the toroidal field component.

In this connection we note that the derivation of the Master Equation is compatible with a pressure tensor of the form

$$\underline{\underline{P}} = p_{th}\underline{\underline{\mathbf{I}}} + p^F \mathbf{e}_\phi \mathbf{e}_\phi$$

# Ring Sequence (Analytic solution, with P. Rebusco)

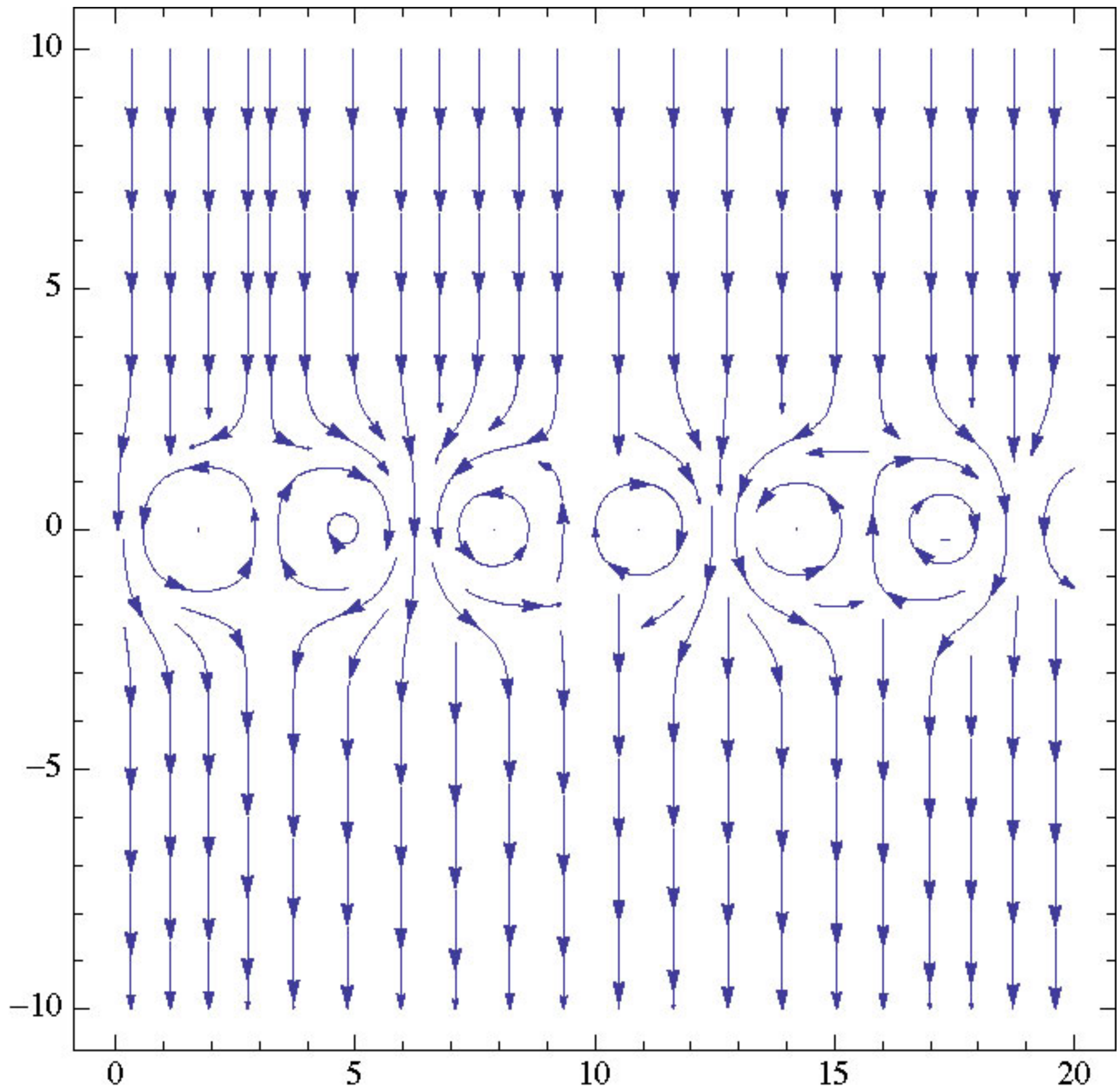
$$\delta\psi \propto N(R_*)\exp\left(-\frac{z^2}{2\Delta_z^2}\right)$$

$$\rho \propto D_*(R_*)\exp\left(-\frac{z^2}{2\Delta_z^2}\right)$$

$$C_0 \equiv \frac{\Delta_R^2}{\Delta_z^2} \qquad\qquad R_* \equiv \frac{R-R_0}{\Delta_R}$$

$$\boxed{N(R_*) = \alpha_0 C_0 R_* + \sin R_* + \frac{\varepsilon_*}{2}\sin 2R_*, \quad \varepsilon_* \lesssim \frac{1}{4}}$$

$$\boxed{D_*(R_*) \simeq C_0 \frac{\alpha_0 R\left[\alpha_0 C_0^2 + (C_0 - 1)\cos R_*\right] + \left[\alpha_0 (C_0 + 1) + \cos R_*\right]\sin R_*}{C_0\alpha_0 R_* + \sin R_* + \frac{\varepsilon_*}{2}\sin 2R_*} + \frac{3}{2}\varepsilon_* \frac{\sin^2 R_*}{1+\varepsilon_*\cos R_*}}$$



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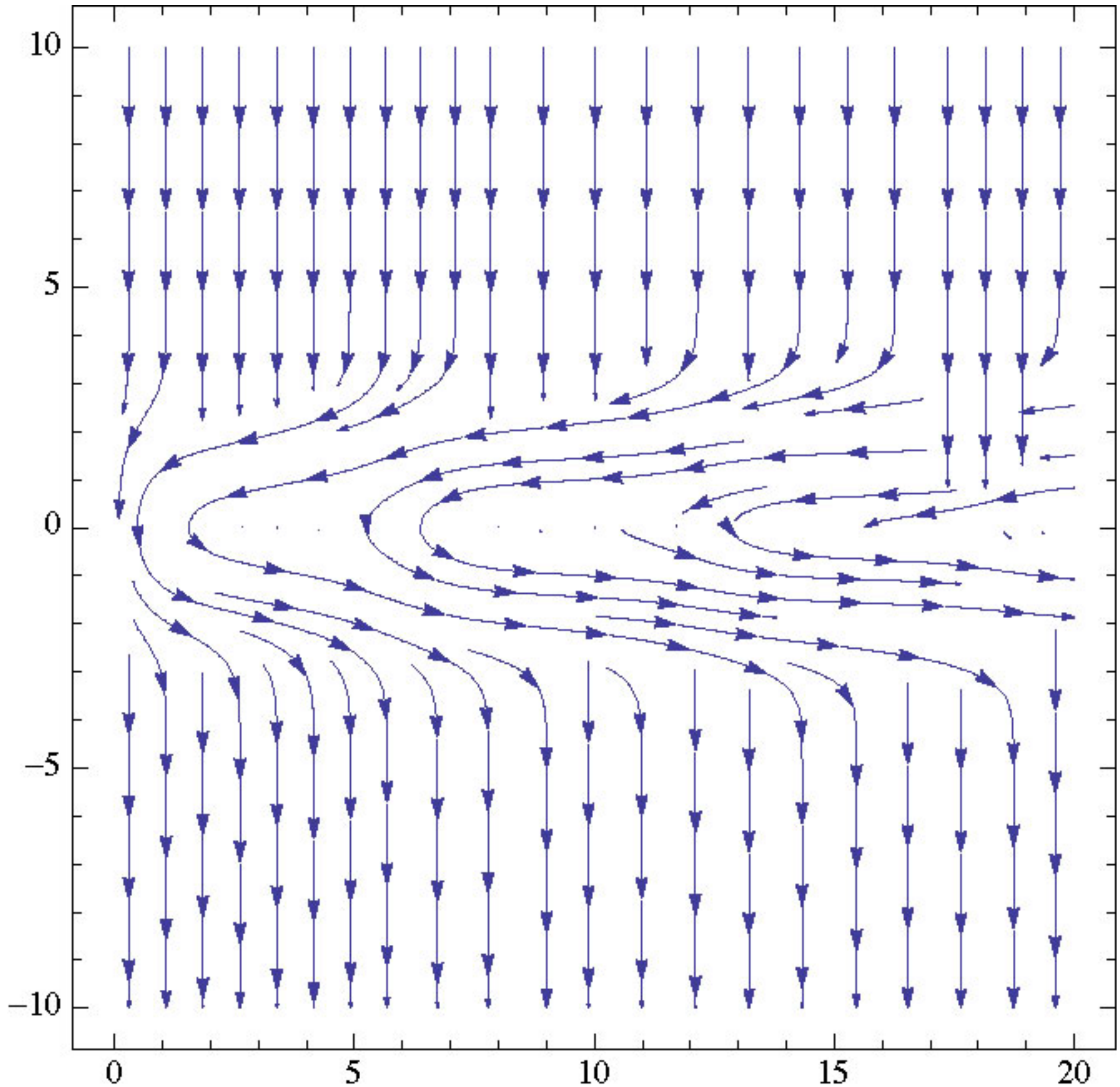
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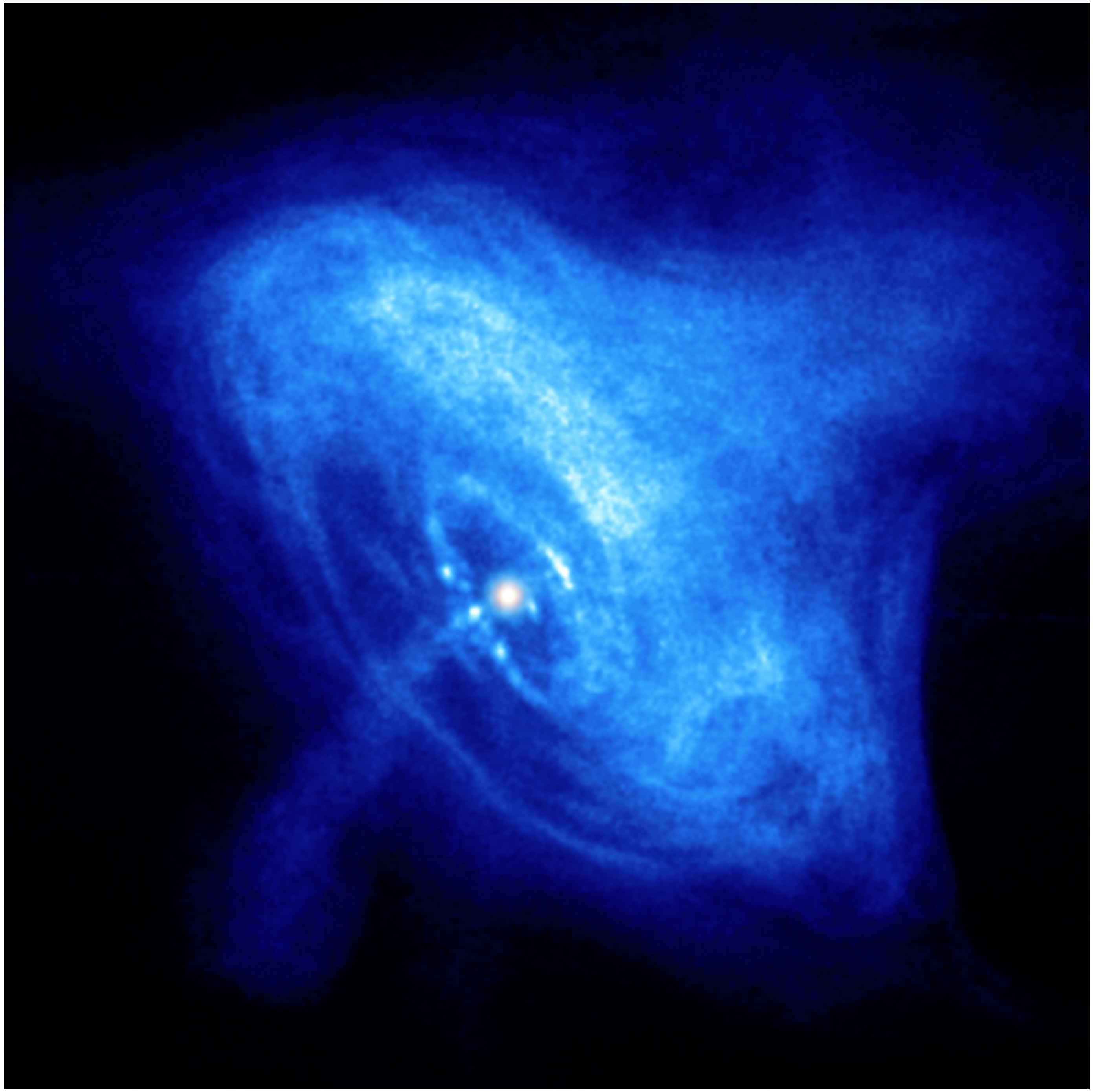

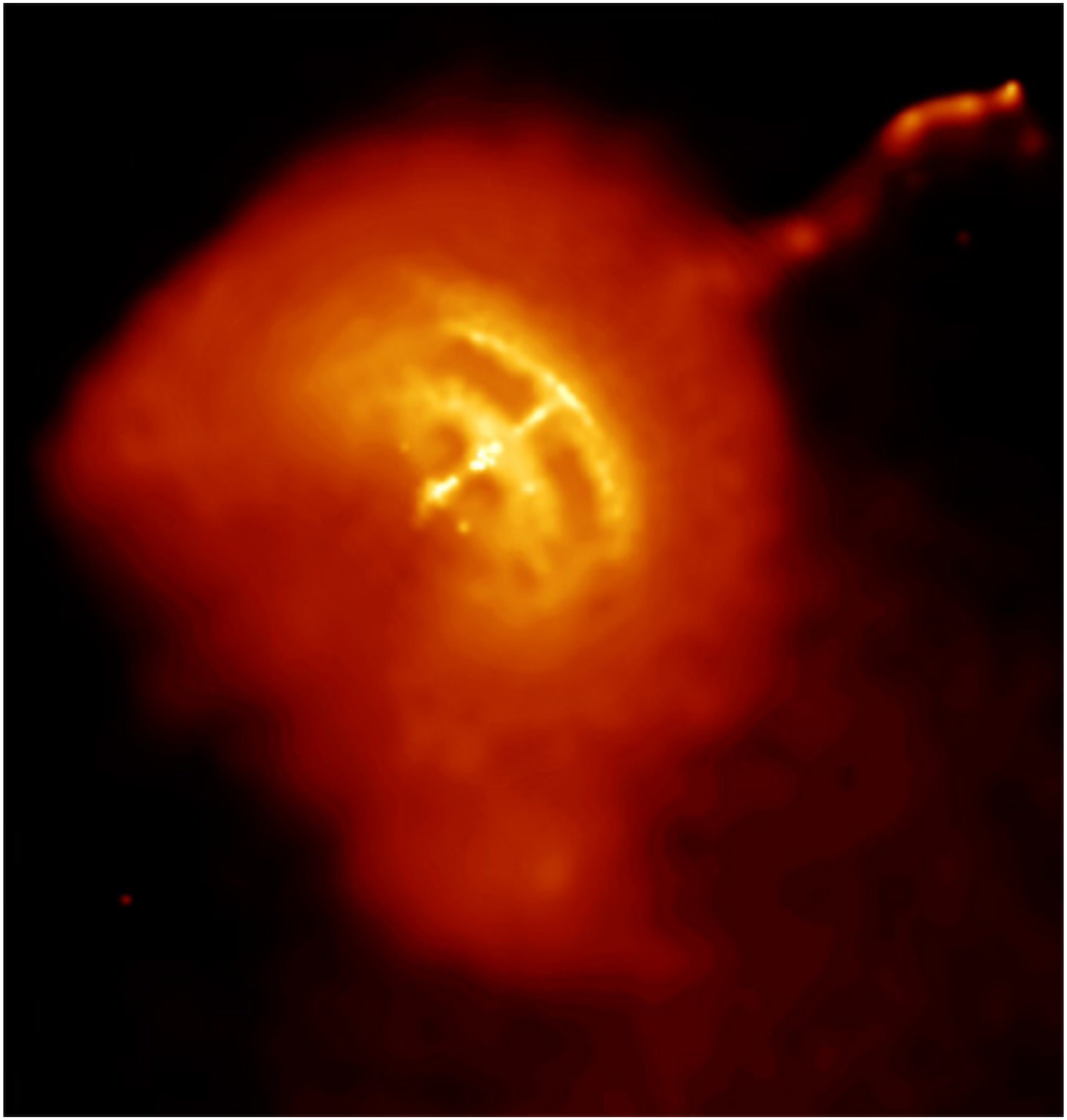

# Axisymmetric Configuration with Toroidal Rotation
# No gravity: Simplest Master "Equation"

$$0 = -\nabla p + \Omega^2 \rho R \mathbf{e}_R + \frac{1}{c}\mathbf{J} \times \mathbf{B} \tag{1}$$

$$B \cdot \nabla p = \Omega^2 \rho R B_R \tag{2}$$

$\mathbf{e}_\phi \cdot \nabla \times$ Eq. (1)

$$0 = \mathbf{e}_\phi \cdot \nabla \times \left(\Omega^2 \rho R \mathbf{e}_R\right) + \frac{1}{4\pi}\mathbf{e}_\phi \cdot \nabla \times \left(\mathbf{B} \cdot \nabla \mathbf{B}\right) \tag{3}$$

$$\Omega = \Omega\left(\psi\right) \tag{4}$$

$$\boxed{0 = +R\frac{\partial}{\partial z}\left(\Omega^2\rho\right) + \frac{1}{4\pi}\left[-\frac{\partial}{\partial R}\left(\mathbf{B}\cdot\nabla\mathbf{B}\right)_z + \frac{\partial}{\partial z}\left(\mathbf{B}\cdot\nabla\mathbf{B}\right)_R\right]} \tag{5}$$

$$\left(\mathbf{B}\cdot\nabla\mathbf{B}\right)_z = \left(B_R\frac{\partial}{\partial R} + B_z\frac{\partial}{\partial z}\right)B_z \tag{6}$$

$$\left(\mathbf{B}\cdot\nabla\mathbf{B}\right)_R = \left(B_R\frac{\partial}{\partial R} + B_z\frac{\partial}{\partial z}\right)B_R - \frac{B_\phi^2}{R} \tag{7}$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(RB_R\right) + \frac{\partial}{\partial z}B_z = 0 \tag{8}$$