Computation of Outer Region Matching Data for Resistive Instabilities with DCON

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Application of DCON to Resistive Instabilities

- DCON is widely used for computing the ideal MHD stability of axisymmetric toroidal plasmas.
- It is thoroughly verified and validated, robust, reliable, easy to use.
- It has also been applied to determination of ideal region matching conditions for resistive and related singular modes, Δ'–like quantities. But despite many efforts, the matching data are noisy and unreliable.
- The cause of this problem has recently been identified: the method used is similar to a shooting method, and is subject to a numerical instability.
- We are replacing this with the method of Pletzer & Dewar, using a Galerkin basis function expansion to discretize the ψ dimension and compute the matching data by linear system solution.
- The new method makes extensive use of the existing DCON code.

Ideal DCON

- Axisymmetric toroidal plasma, linear ideal MHD perturbations: 2D PDE.
  - Previous codes, *e.g*. PEST, ERATO, GATO:
    expand in 2D basis functions, convert to large matrix eigenvalue problem.
  - Problems: resolution near singular surfaces and separatrix, slow convergence, excessive human and computer time.
  - Newcomb, 1960: marginal ideal internal stability of cylindrical plasma reduced to numerical solution of 2nd-order ODE, initial value problem, crossing condition.
  - DCON: generalization of Newcomb to axisymmetric toroidal plasma, numerical solution of 2M-order ODE. 5 seconds on a fast workstation for toroidal mode number n=1, high β, tight aspect ratio, separatrix.
- Extension to free-boundary modes, plasma + vacuum, Morrell Chance.
- Interfaces to 28 Grad-Shafranov solvers, both direct and inverse, fit to bicubic splines.
- Reports equilibrium properties, Mercier & high-n ballooning stability, fixed and free boundary ideal MHD modes.
- Widely used around the world, extensively verified and validated.

Formulation of Ideal MHD DCON

Ideal MHD Energy Principle

$$\delta W = \frac{1}{2}\int_\Omega d\mathbf{x}\left[Q^2 + \mathbf{J}\cdot\xi\times\mathbf{Q} + (\xi\cdot\nabla P)(\nabla\cdot\xi) + \gamma P(\nabla\cdot\xi)^2\right]$$

Fourier Representation

$$\xi\cdot\nabla\psi(\psi,\theta,\zeta) = \sum_{m=m_{\rm low}}^{m_{\rm high}} \xi_m(\psi)e^{i(m\theta-n\zeta)}$$

$$\Xi(\psi) \equiv \{\xi_m(\psi) \mid m\in[m_{\rm min}, m_{\rm max}]\}$$

Euler-Lagrange Equation

$$\delta W = \frac{1}{2}\int_0^1 d\psi\left[\Xi'^\dagger\mathsf{F}\Xi' + \Xi'^\dagger\mathsf{K}\Xi + \Xi^\dagger\mathsf{K}^\dagger\Xi' + \Xi^\dagger\mathsf{G}\Xi\right]$$

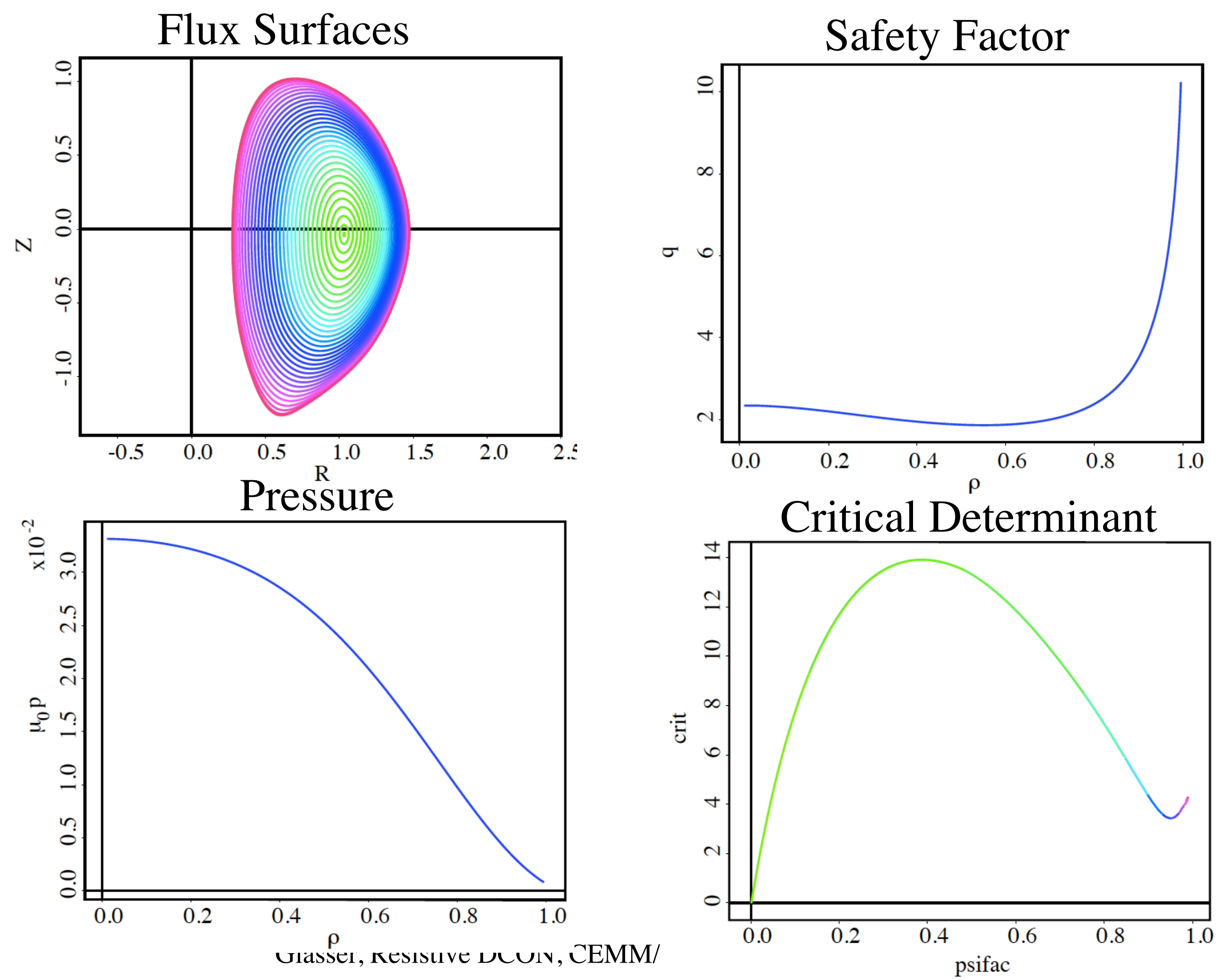
$$\mathsf{L}\Xi = -(\mathsf{F}\Xi' + \mathsf{K}\Xi)' + (\mathsf{K}^\dagger\Xi' + \mathsf{G}\Xi) = 0$$

Ordinary Differental Equation

$$\mathbf{u} = \begin{pmatrix}\Xi \\ \mathsf{F}\Xi' + \mathsf{K}\Xi\end{pmatrix}, \quad \mathsf{M} = \begin{pmatrix} -\mathsf{F}^{-1}\mathsf{K} & \mathsf{F}^{-1} \\ \mathsf{G}-\mathsf{K}^\dagger\mathsf{F}^{-1}\mathsf{K} & \mathsf{K}^\dagger\mathsf{F}^{-1}\end{pmatrix}, \quad \mathbf{u}' = \mathsf{M}\mathbf{u}$$

Ideal MHD DCON, NSTX Equilibrium

Aspect ratio = 1.46, β_n=5.6, mpert = 35, cpu = 6 seconds



Glasser, Resistive DCON, CEMM/

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Frobenius Expansion

Singular Surface Equations

$$\mathbf{u} \equiv \begin{pmatrix} \Xi \\ \mathsf{F}\Xi' + \mathsf{K}\Xi \end{pmatrix} \quad \mathsf{L} \equiv \begin{pmatrix} -\mathsf{F}^{-1}\mathsf{K} & \mathsf{F}^{-1} \\ \mathsf{G} - \mathsf{K}^\dagger \mathsf{F}^{-1}\mathsf{K} & \mathsf{K}^\dagger \mathsf{F}^{-1} \end{pmatrix} \quad \mathbf{u}' = \mathsf{L}\mathbf{u}$$

$$Q_{m,m'} = (m - nq)\delta_{m,m'}, \quad \mathsf{F} = \mathsf{Q}\bar{\mathsf{F}}\mathsf{Q}, \quad \mathsf{G} = \bar{\mathsf{G}}, \quad \mathsf{K} = \mathsf{Q}\bar{\mathsf{K}}$$

$$m_R - nq(\psi_R) = 0, \quad z \equiv \psi - \psi_R$$

$$\mathsf{F}^{-1} \sim z^{-2}, \quad \mathsf{F}^{-1}\mathsf{K} \sim \mathsf{K}^\dagger \mathsf{F}^{-1} \sim z^{-1}, \quad \mathsf{G} \sim \mathsf{K}^\dagger \mathsf{F}^{-1}\mathsf{K} \sim 1$$

Convergent Power Series Expansion

$$\Xi = z^p \sum_{n=0}^{N} \xi_n z^n$$

$$p = -\frac{1}{2} \pm \sqrt{-D_I}, \quad \text{large and small Mercier resonant powers}$$

$$p = 0, \quad \text{nonresonant powers}$$

Solved to arbitrarily high order N; automated using matrix formulation.
Essential for larger values of $|D_I|$; generally improves convergence.

Asymptotic Coefficients

Numerical Solution

$$\mathbf{U}' = \mathbf{L}\mathbf{U}$$

Power Series Solutions

$$\text{As } \psi \to \psi_R, \quad \mathbf{U}_A' \to \mathbf{L}\mathbf{U}_A$$

Asymptotic Coefficients

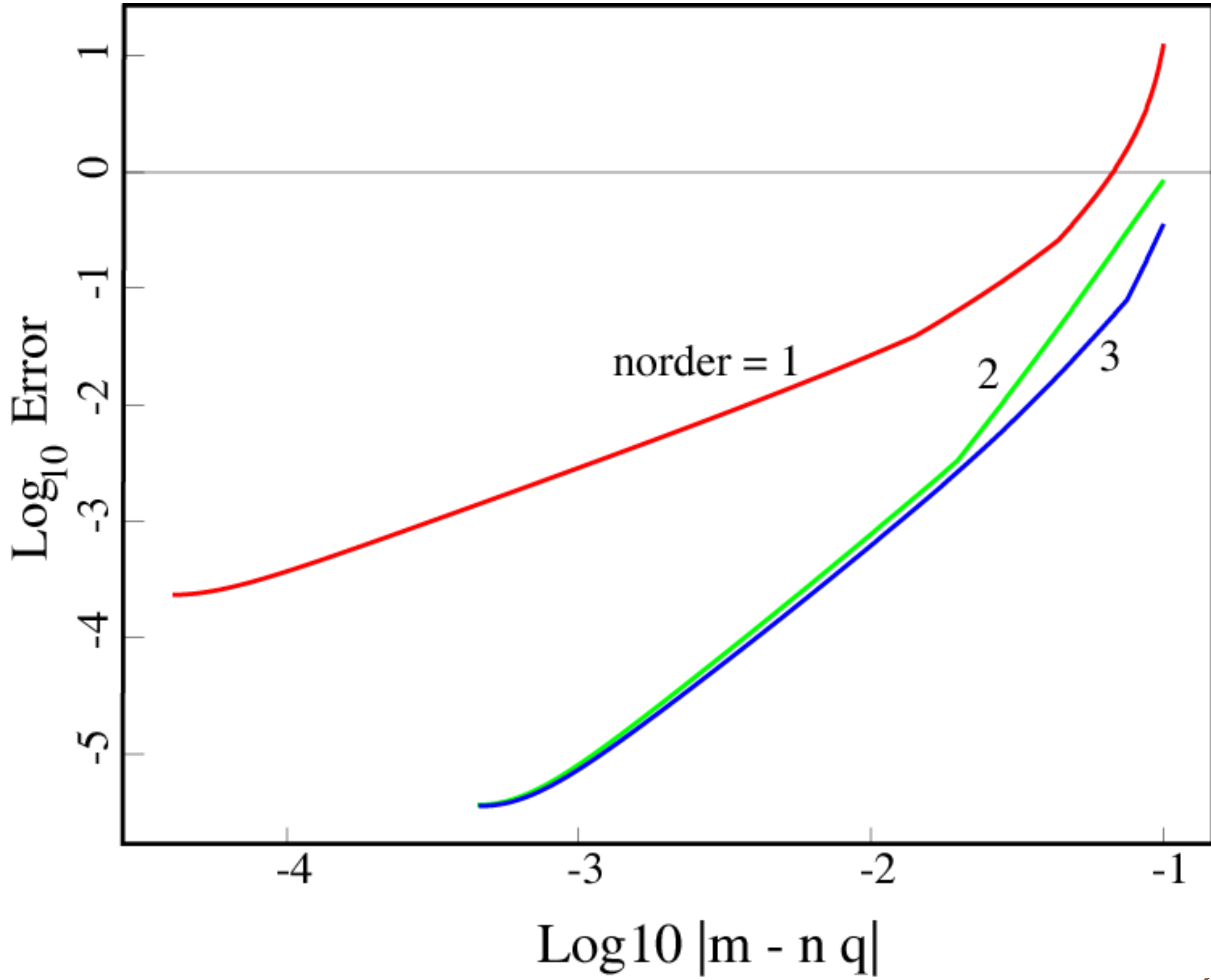
$$\mathbf{U} = \mathbf{U}_A \mathbf{C}_A, \quad \mathbf{C}_A = \mathbf{U}_A^{-1}\mathbf{U}$$

$$\text{As } \psi \to \psi_R, \quad \mathbf{C}_A' \to \mathbf{0}$$

Asymptotic coefficients are used
to construct matching conditions.

Effect of Increasing Power Series Order

Convergence of Asymptotic Coefficient Near Singular Surface



Initial Conditions at the Axis

- The equations have a regular singular point at the axis $\psi = 0$, just as they do at each resonant surface.
- DCON uses a numerical Grad-Shafranov solution fit to bicubic splines, with one edge of the bicubic splines near the origin. This introduces noise and inaccuracy into the ODEs.
- Newcomb proves a theorem that the small solution at the axis is the limit a simpler initial condition at a point near the axis, in the limit as that point approaches the axis.
- DCON uses Newcomb's procedure and gets clean, accurate results for ideal MHD stability.

Fixups

- Each independent solution grows from the axis as $r^m \sim \psi^{m/2}$.
- r^{20} grows much faster than r^1.
- If nothing is done about this, the higher m' s swamp the lower m' s. Instead of following M solutions, we follow 1 solution M times.
- Determinants approach zero, results turn to mush.
- Fixups
  - Keep track of the factor by which each independent solution grows.
  - Let unorm = ratio of the fastest to the slowest growth.
  - When unorm exceeds a specified threshold ucrit:
    - Subtract a multiple of the fastest-growing solution from each other solution
    - Repeat for each successively smaller solution
- Similar to Gaussian elimination. Triangularizes solution matrix, maintains linear independence.
- If ucrit is too large, noise develops. If it is small enough, the solutions are clean.

Crossing a Resonant Surface

- On approach to each resonant surface, each of the M independent solutions contains a linear combination of large, small, and non-resonant solutions.
- For ideal MHD, DCON eliminates the large resonant solution and restarts a new small resonant solution. This can be done with a fixup and without the need to evaluate asymptotic coefficients. After crossing, there are again M independent solutions.
- For resistive MHD, DCON evaluates the asymptotic coefficients. It imposes continuity on all non-resonant displacements and launches two new resonant solutions, large and small. After crossing, there are M+2 independent solutions.
- The resistive crossing procedure provides just enough additional degrees of freedom to match to arbitrary inner-region solutions *a posteriori*.

Convergence Problems

- DCON solves an initial value problem to compute ideal MHD stability, requiring only outward integration, and fixups to maintain linear independence of solutions.
- Resistive stability calculation requires matching across the whole domain, unwinding the fixups. This converts it from an initial-value problem into a 2-point boundary value problem, solved by a shooting method.
- Such methods are known to be numerically unstable because of excessive sensitivity to initial conditions.
- This explains the failure of resistive DCON to find reliable and accurate values of the matching data.
- Solution: replace the shooting method with a the method of Dewar & Pletzer, a Galerkin expansion in basis functions, solve matrix equation for matching data.

Dewar References

- A. D. Miller & R. L. Dewar, "Galerkin method for differential equations with singular points," *J. Comp. Phys.* **66**, 356-390 (1986). Introduces Galerkin method for singular ODEs, solves test problems.

- R. L. Dewar & A. Pletzer, "Two-dimensional generalization of the Newcomb equation," *J. Plasma. Phys.* **43**, 2, 291-310 (1990). Derives 2D Newcomb equations, equivalent to DCON equation.

- A. Pletzer & R. L. Dewar, "Non-ideal Variational method for determination of the outer-region matching data," J. Plasma Phys. 45, 3, 427-451 (1991).
  Solves cylindrical problem with non-monotonic q profile.

- A. Pletzer, A. Bondeson, and R. L. Dewar, "Linear stability of resistive MHD modes: axisymmetric toroidal computation of the outer region matching data," J. Comp. Phys. 115, 530-549 (1994). Solves toroidal problem, PEST 3, verified against MARS code.

Galerkin Expansion

Euler-Lagrange Equation

$$\mathbf{L}\Xi = -(\mathbf{F}\Xi' + \mathbf{K}\Xi)' + (\mathbf{K}^\dagger\Xi' + \mathbf{G}\Xi) = 0$$

Galerkin Expansion

$$(u, v) \equiv \int_0^1 u^\dagger(\psi)v(\psi)d\psi$$

$$\Xi(\psi) = \sum_{i=0}^{N} \Xi_i \alpha_l(\psi)$$

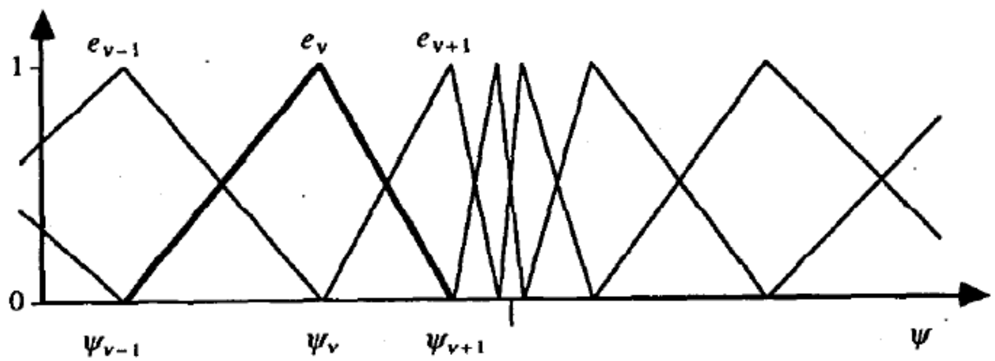
$$(\alpha_i, \mathbf{L}\Xi) = (\alpha_i, \mathbf{L}\alpha_j)\Xi_j = 0$$

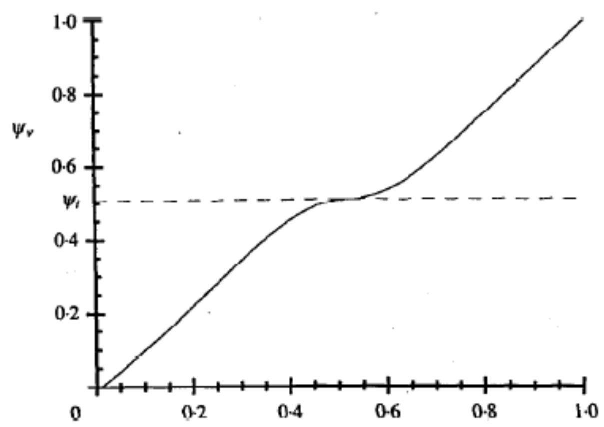
$$\mathbf{L}_{ij} = (\alpha_i', \mathbf{F}\alpha_j') + (\alpha_i', \mathbf{K}\alpha_j) + (\alpha_i, \mathbf{K}^\dagger\alpha_j') + (\alpha_i, \mathbf{G}\alpha_j)$$

Finite-Energy Response Driven by Large Solution

$$L_{ij}\check{\Xi}_j = -(\alpha_i, L\hat{\Xi})$$

Dewar and Pletzer: Linear Finite Elements on a Packed Grid

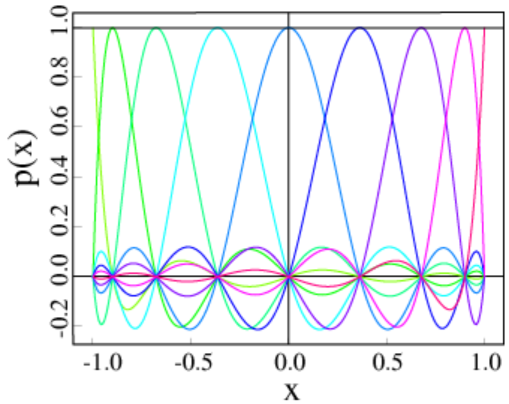




The choice of basis functions determines the rate of convergence.

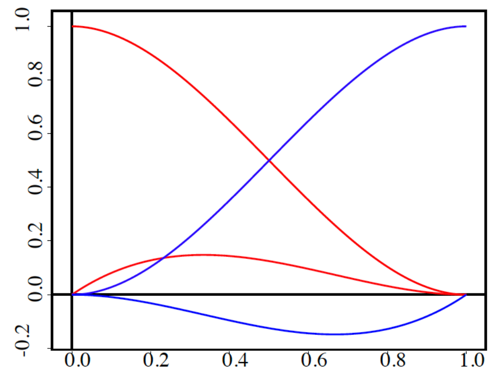
Higher-Order Basis Functions

C^0 Jacobi Nodal Basis



- Lagrange interpolatory polynomials
- Nodes at roots of $(1-x^2)\, P_n^{(0,0)\prime}(x)$
- Diagonally dominant

C^1 Hermite Cubics



- Cubic polynomials on (0,1).
- C^1 continuity: function values and first derivatives
- Useful for nonresonant solutions across the singular surface.

Adjustable Grid Packing: Equations

Grid Packing Function

$$\lambda(a) = \coth a = \frac{e^a + 1}{e^a - 1}, \quad a(\lambda) = \text{acoth}\ \lambda = \ln\left(\frac{1+\lambda}{1-\lambda}\right)$$

$$x(\xi, \lambda) = \frac{\tanh a\xi}{\lambda} = \frac{1}{\lambda}\left(\frac{e^{a\xi} - 1}{e^{a\xi} + 1}\right)$$

$$\lim_{\lambda \to 0} a(\lambda) = 2\lambda, \quad \lim_{\lambda \to 0} x(\xi, \lambda) = \xi$$

Center and Edge Grid Densities

$$\frac{\partial x}{\partial \xi} = \frac{1}{\lambda}\frac{2ae^{a\xi}}{(e^{a\xi} + 1)^2} = \frac{1}{\lambda}\frac{2ae^{-a\xi}}{(e^{-a\xi} + 1)^2}$$

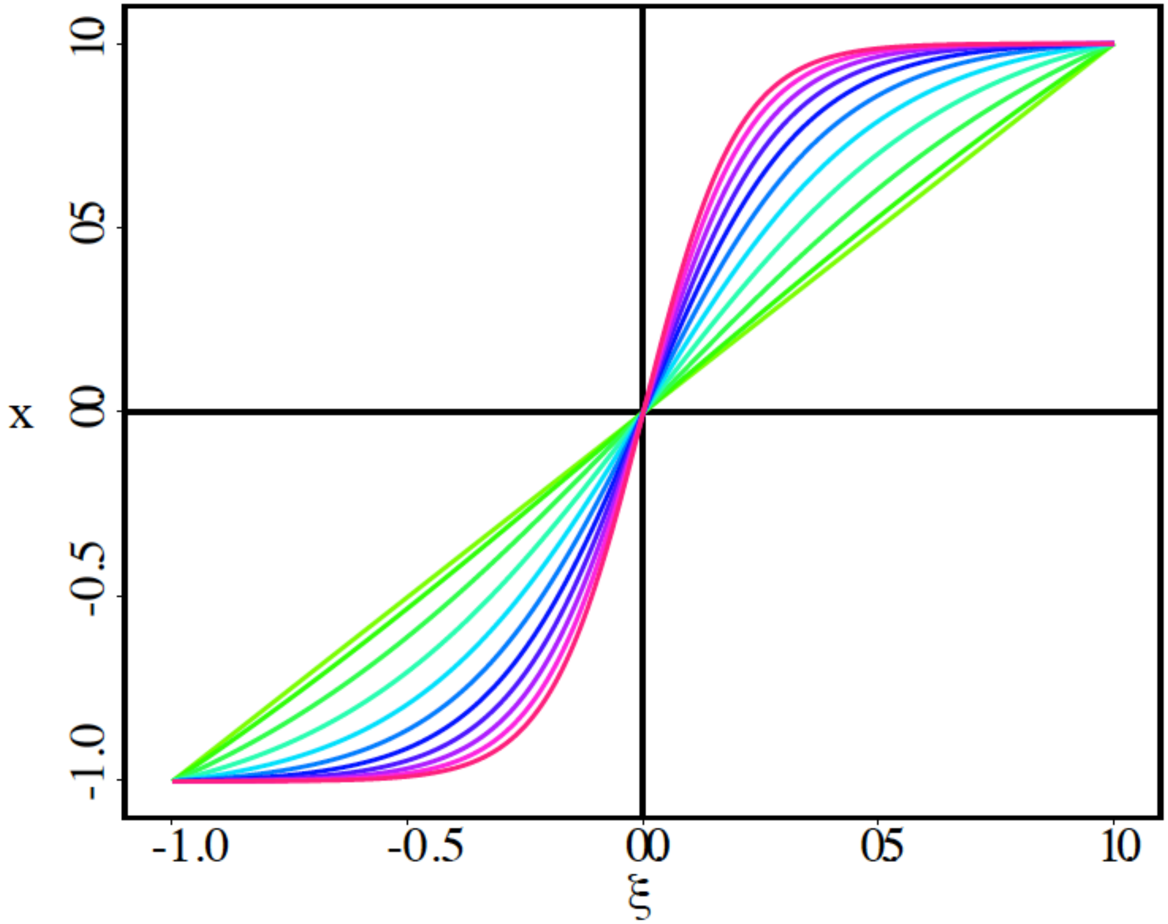
$$\left.\frac{\partial x}{\partial \xi}\right|_{\xi=0} = \frac{a}{2\lambda}$$

$$\left.\frac{\partial x}{\partial \xi}\right|_{\xi=\pm 1} = \frac{a}{2\lambda}\left(1 - \lambda^2\right)$$

Packing Ratio

$$P(\lambda) \equiv \frac{\partial x/\partial \xi|_{\xi=\pm 1}}{\partial x/\partial \xi|_{\xi=0}} = 1 - \lambda^2$$

Adjustable Grid Packing: Graphs



Singular Elements

- Weierstrass Convergence Theorem:
  Polynomial approximation uniformly convergent for analytic functions.
- Big and small resonant solutions are non-analytic near the singular surface.
- Supplement polynomial basis with small resonant solution near singular surface.
- DCON fits equilibrium data to Fourier series and cubic splines, computes resonant power series to arbitrarily high order.
- Convergence requires that the large solution be computed to at least n = 2*sqrt(-di) terms. PEST 3 is limited to n = 1.

Status and Plans

- DCON processes data from 28 different Grad-Shafranov solvers, diagnoses them, computes the **F**, **G**, and **K** matrices, and compute power series about singular surfaces. It accurately and reliably computes ideal Mercier, ballooning, and low-n free and fixed boundary modes.
- We have supplemented the existing DCON infrastructure with the method of Dewar & Pletzer, solving a set of inhomogeneous matrix equations for the outer region matching data, using the existing **F**, **G**, and **K** matrices, solved with a complex banded LAPACK routine.
- Improved choice of Galerkin basis functions and packing algorithm should result in faster, more reliable convergence, greater ease of use, compared to PEST 3.
- Solutions look good, but we have not yet achieved convergence with respect to the the rhs cutoff function. We have successfully reproduced the test cases in Miller & Dewar 1986.
- We are working with Dylan Brennan to validate resistive DCON against PEST 3.
- A separate code DELTAR implements the inner region resistive MHD equations of Glasser, Greene & Johnson, using the Fourier transform method of Glasser, Jardin & Tesauro, taking 35 microseconds of cpu time.
- Inner and out region solutions must be matched to form global eigenvalues and eigenfunctions.
- Future research will be devoted to improving the inner region model.

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