CEMM Meeting. Santa Fe NM, April 2013.

TWO-FLUID TEARING MODES IN CYLINDRICAL GEOMETRY*

J. J. Ramos

M.I.T. Plasma Science and Fusion Center

*Work supported by the U.S. Department of Energy

- RECENTLY DERIVED ANALYTIC DISPERSION RELATIONS FOR THE HALL-MHD RESISTIVE TEARING INSTABILITY IN SLAB GEOMETRY [1-3] APPLY TO RATHER GENERAL PARAMETER REGIMES, MAKING THEM USEFUL FOR TESTING THE EXTENDED-MHD CODES. THEY WERE USED IN A SUCCESSFUL BENCHMARK OF NIMROD [4].

- A GENERALIZATION TO CYLINDRICAL GEOMETRY HAS BEEN SUGGESTED IN ORDER TO FACILITATE COMPARISONS WITH M3D-C1 SIMULATIONS.

- SUCH CYLINDRICAL GEOMETRY GENERALIZATION IS CONCEPTUALLY TRIVIAL BUT ALGEBRAICALLY CUMBERSOME. IT IS NOT IMMEDIATELY OBVIOUS THAT THE NEW GEOMETRY WOULD NOT BRING ABOUT QUALITATIVE CHANGES.

TWO-FLUID, HALL-MHD MODEL

$$\frac{\partial n}{\partial t} \ + \ \nabla \cdot (n\mathbf{u}) \ = \ 0$$

$$\mathbf{j} \ = \ \nabla \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} \ + \ \nabla \times \mathbf{E} \ = \ 0$$

$$\mathbf{E} \ = \ -\mathbf{u} \times \mathbf{B} \ + \ \frac{1}{en}(\mathbf{j} \times \mathbf{B} - \nabla p_e) \ + \ \eta \mathbf{j}$$

$$m_\iota n \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] \ + \ \nabla(p_\iota + p_e) \ - \ \mathbf{j} \times \mathbf{B} \ = \ 0$$

$$p_s n^{-\Gamma_s} = \text{constant} \quad (s = \iota, e)$$

FORCE-FREE CYLINDRICAL EQUILIBRIUM

$$n_0 \ , \quad p_{\iota 0} \ , \quad p_{e0} \ = \ \text{constants}$$

$$\mathbf{u}_0 = 0$$

$$\mathbf{B}_0 \ = \ B_{0\theta}(r) \ \mathbf{e}_\theta \ + \ B_{0z}(r) \ \mathbf{e}_z$$

$$\mathbf{j}_0 \times \mathbf{B}_0 \ = \ 0 \quad \Rightarrow \quad \mathbf{j}_0 \ = \ \lambda_0 \mathbf{B}_0 \ , \qquad \lambda_0 \ = \ \frac{\mathbf{j}_0 \cdot \mathbf{B}_0}{B_0^2} \ = \ \frac{(rB_{0\theta})'}{rB_{0z}} \ = \ - \ \frac{B'_{0z}}{B_{0\theta}} \qquad \left[' \equiv \frac{d}{dr} \right]$$

NORMAL MODE LINEARIZED SYSTEM

Consider normal modes of the form

$$f(\mathbf{x}, t) \;=\; f_0(r) \;+\; f_1(r)\;\exp(im\theta + ik_z z + \gamma t)$$

Define

$$\mathbf{k}(r) \;\equiv\; \frac{m}{r}\mathbf{e}_\theta \;+\; k_z\mathbf{e}_z$$

$$F(r) \;\equiv \mathbf{k}\cdot\mathbf{B}_0 \;=\; \frac{m}{r}B_{0\theta} + k_z B_{0z}\;, \qquad\qquad F(r_s) \;=\; 0$$

$$G(r) \;\equiv \mathbf{e}_r\cdot(\mathbf{k}\times\mathbf{B}_0) \;=\; \frac{m}{r}B_{0z} - k_z B_{0\theta}$$

$$c_S^2 \equiv (m_\iota n_0)^{-1}\sum_s \Gamma_s p_{s0}\;, \qquad \xi \equiv \frac{u_{1r}}{\gamma}\;, \qquad Q \equiv B_{1z} + \frac{i}{rk^2}\Big[m\lambda_0 B_{1r} - k_z(rB_{1r})'\Big]$$

Eliminate algebraically $\mathbf{E}$, j, $B_{1\theta}$, $u_{1\theta}$, u_{1z} $p_{\iota 1}$, p_{e1} and n_1, to obtain an exact linearized system for ξ, B_{1r} and Q

$$\gamma^2 m_\iota n_0 \left\{ \xi - \left[\frac{c_S^2}{\gamma^2 + c_S^2 k^2} \; \frac{(r\xi)'}{r} \right]' \right\} \; + \; \frac{\gamma}{\eta k^2} F^2 \xi \;\; = \;\; \frac{i}{k^2} \left[\lambda_0' G - \left(\lambda_0^2 + \frac{\gamma}{\eta} \right) F \right] B_{1r} \; -$$

$$- \; \left(\frac{\gamma^2}{\gamma^2 + c_S^2 k^2} \; \frac{r}{m} \; GQ \right)' \; + \; \left(\frac{ir}{\eta e n_0 m} F^2 - \frac{2k_z}{mk^2} G \right) Q$$

$$(\gamma + \eta k^2) B_{1r} \; - \; \eta k^2 \left[\frac{1}{rk^2} (rB_{1r})' \right]' \; - \; \frac{2\eta m k_z}{r^2 k^2} \lambda_0 B_{1r} \;\; = \;\; F \left(i\gamma\xi + \frac{rk^2}{e n_0 m} Q \right) \; + \; \frac{2i\eta k_z}{r} Q$$

$$\left[\gamma + \frac{1}{\gamma m_\iota n_0} \left(F^2 + \frac{\gamma^2 G^2}{\gamma^2 + c_S^2 k^2} \right) + \frac{F^2}{\eta e^2 n_0^2} \right] Q \; + \; \eta \left[k^2 Q - \frac{1}{r} (rQ')' \right] \; + \; \frac{ik_z}{e n_0 r} \left[\frac{2k_z}{k^2} GQ - \frac{1}{m} (r^2 F Q)' \right] \; =$$

$$= \; \frac{i\eta}{r} \left\{ r \left(\frac{1}{rk^2} \Big[m\lambda_0 B_{1r} - k_z (rB_{1r})' \Big] \right)' \right\}' \; - \; \frac{i}{rk^2} (\gamma + \eta k^2) \Big[m\lambda_0 B_{1r} - k_z (rB_{1r})' \Big] \; -$$

$$- \; \frac{m}{e n_0 r k^2} \left[\lambda_0' G - \left(\lambda_0^2 + \frac{\gamma}{\eta} \right) F \right] B_{1r} \; + \; \frac{\gamma}{r} \left[\frac{c_S^2 m / r}{\gamma^2 + c_S^2 k^2} G (r\xi)' - (B_{0z} r\xi)' \right] \; - \; \frac{i\gamma m}{\eta e n_0 r k^2} F^2 \xi$$

MARGINALLY STABLE IDEAL-MHD SOLUTION

$$\eta = 0 \ , \qquad \gamma = 0$$

$$Q = 0 \ , \qquad B_{1r} = iF\xi$$

$$\left[\frac{1}{rk^2}(rB_{1r})'\right]' - \left(1 + \frac{\lambda_0' G}{k^2 F} - \frac{\lambda_0^2}{k^2} - \frac{2mk_z\lambda_0}{r^2k^4}\right) B_{1r} \ = \ 0$$

applicable to the "outer" region away from $r = r_s$ and whose discontinuity at $r = r_s$ defines the tearing stability index:

$$\Delta' = \frac{B_{1r}(r_{s+}) - B_{1r}(r_{s-})}{B_{1r}(r_s)}$$

ASYMPTOTIC EXPANSIONS IN THE MODE SINGULAR LAYER

IN A LAYER WHERE $x \equiv (r - r_s) \ll r_s$, USE THE LOWEST-ORDER TAYLOR EXPANSION OF EQUILIBRIUM QUANTITIES:

$$r = r_s \ , \qquad k = k(r_s) \ , \qquad B_0 = B_0(r_s)$$

$$B_{0\theta} = - \frac{k_z}{k(r_s)} B_0(r_s) \ , \qquad B_{0z} = \frac{m}{r_s k(r_s)} B_0(r_s) \ , \qquad G = k(r_s) \ B_0(r_s)$$

$$F = \left[k(r_s) \ \lambda_0(r_s) + \frac{m k_z}{r_s^2 k(r_s)} \right] B_0(r_s) \ (r - r_s) \ \equiv \ \frac{k(r_s) \ B_0(r_s)}{L_B} \ x$$

ALSO, NEGLECT ηk^2 COMPARED TO γ

$$\gamma^2 m_\iota n_0 \left(\xi - \frac{c_S^2}{\gamma^2 + c_S^2 k^2}\ \xi''\right) \ + \ \frac{\gamma B_0^2}{\eta L_B^2}\ x^2\ \xi \ = \ \frac{iB_0}{k}\left(\lambda_0' - \frac{\gamma}{\eta L_B}\ x\right) B_{1r} \ -$$

$$- \ B_0\ \frac{\gamma^2}{\gamma^2 + c_S^2 k^2}\ \frac{rk}{m}\ Q' \ + \ B_0 \left(\frac{iB_0 r k^2}{\eta e n_0 m L_B^2}\ x^2 - \frac{2k_z}{mk}\right) Q$$

$$\gamma B_{1r} \ - \ \eta B_{1r}'' \ - \ \frac{2\eta m k_z}{r^2 k^2}\lambda_0 B_{1r} \ = \ \frac{kB_0}{L_B}\ x \left(i\gamma\xi + \frac{rk^2}{en_0 m}Q\right) \ + \ \frac{2i\eta k_z}{r}Q$$

$$\left[\gamma + \frac{k^2 B_0^2}{\gamma m_\iota n_0}\left(\frac{x^2}{L_B^2} + \frac{\gamma^2}{\gamma^2 + c_S^2 k^2}\right) + \frac{k^2 B_0^2 x^2}{\eta e^2 n_0^2 L_B^2}\right] Q \ - \ \eta Q'' \ + \ \frac{ik_z B_0}{en_0}\left[\frac{2k_z}{rk}Q - \frac{rk}{mL_B}(xQ)'\right] \ =$$

$$= \ \frac{i\eta}{rk^2}\Big(m\lambda_0 B_{1r}'' - rk_z B_{1r}'''\Big) \ - \ \frac{i\gamma}{rk^2}\Big(m\lambda_0 B_{1r} - rk_z B_{1r}'\Big) \ -$$

$$- \ \frac{mB_0}{en_0 rk}\left(\lambda_0' - \frac{\gamma}{\eta L_B}\ x\right) B_{1r} \ - \ \frac{\gamma^3 m B_0}{rk(\gamma^2 + c_S^2 k^2)}\ \xi' \ - \ \frac{i\gamma m B_0^2}{\eta e n_0 r L_B^2}\ x^2\ \xi$$

INFINITE ASPECT RATIO LIMIT

$$\frac{rk_z}{m} = -\frac{nr}{mR} \rightarrow 0$$

$$\gamma^2 m_\iota n_0 \left(\xi - \frac{c_S^2}{\gamma^2 + c_S^2 k^2}\xi''\right) + \frac{\gamma B_0^2}{\eta L_B^2} x^2 \xi = \frac{iB_0}{k}\left(\lambda_0' - \frac{\gamma}{\eta L_B} x\right) B_{1r} - \frac{B_0 \gamma^2}{\gamma^2 + c_S^2 k^2} Q' + \frac{iB_0^2 k}{\eta e n_0 L_B^2} x^2 Q$$

$$\gamma B_{1r} - \eta B_{1r}'' = \frac{kB_0}{L_B} x \left(i\gamma\xi + \frac{k}{en_0}Q\right)$$

$$\left[\gamma + \frac{k^2 B_0^2}{\gamma m_\iota n_0}\left(\frac{x^2}{L_B^2} + \frac{\gamma^2}{\gamma^2 + c_S^2 k^2}\right) + \frac{k^2 B_0^2 x^2}{\eta e^2 n_0^2 L_B^2}\right] Q - \eta Q'' =$$

$$= \frac{i\lambda_0}{k}(\eta B_{1r}'' - \gamma B_{1r}) - \frac{B_0}{en_0}\left(\lambda_0' - \frac{\gamma}{\eta L_B} x\right) B_{1r} - \frac{\gamma^3 B_0}{\gamma^2 + c_S^2 k^2}\xi' - \frac{i\gamma k B_0^2}{\eta e n_0 L_B^2} x^2 \xi$$

WHICH IS THE SAME AS THE SLAB GEOMETRY SYSTEM

CONCLUSIONS

IN THE INFINITE ASPECT RATIO LIMIT, THE RESULTS IN CYLINDRICAL GEOMETRY SHOULD BE THE SAME AS THOSE IN SLAB GEOMETRY WITH THE IDENTIFICATIONS

$$k \;\rightarrow\; \frac{m}{r_s}\,, \qquad\qquad \frac{1}{L_B} \;\rightarrow\; \lambda_0(r_s) \;=\; -\,\frac{B'_{0z}(r_s)}{B_{0\theta}(r_s)}$$

MORE WORK IS NEEDED TO ASCERTAIN THE FINITE ASPECT RATIO CASE

OVERALL SUMMARY INCLUDING SUPERSONIC RESULTS

1: $\epsilon_\gamma = \epsilon_\eta^{3/5}\left(\frac{\Delta'^2}{C^2k^3L_B}\right)^{2/5}$ **[5]**

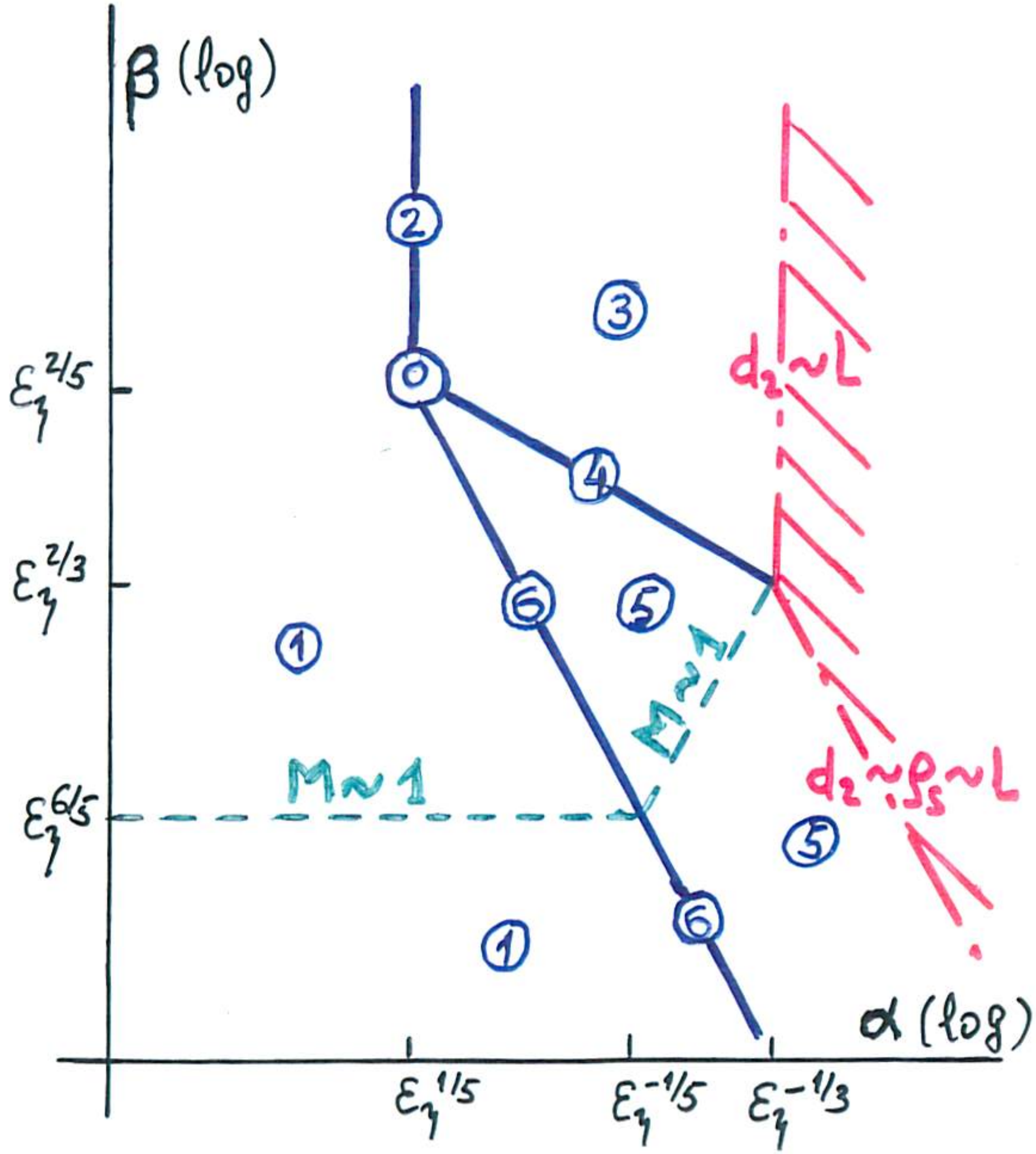
2: $\epsilon_\gamma = \epsilon_\eta^{3/5}\left(\frac{\Delta'^2}{C^2k^3L_B}\right)^{2/5} f_2^{-4/5}\left(\frac{\epsilon_\gamma^{1/2}\alpha}{\epsilon_\eta^{1/2}}\right)$ **[1]**

3: $\epsilon_\gamma = \epsilon_\eta^{1/2}\alpha^{1/2}\left(\frac{\Delta'^2}{C^2k^3L_B}\right)^{1/2}$ **[6]**

4: $\epsilon_\gamma = \epsilon_\eta^{1/2}\alpha^{1/2}\left(\frac{\Delta'^2}{C^2k^3L_B}\right)^{1/2} f_4^{-1}\left(\frac{\epsilon_\gamma k L_B}{\alpha\beta}\right)$ **[2]**

5: $\epsilon_\gamma = \epsilon_\eta^{1/3}\alpha^{2/3}\beta^{1/3}\left(\frac{\Delta'}{\pi k^2 L_B}\right)^{2/3}$ **[7]**

6: $\epsilon_\gamma = \epsilon_\eta^{3/5}\left(\frac{\Delta'^2}{C^2k^3L_B}\right)^{2/5} f_6^{-4/5}\left(\frac{\alpha^2\beta}{\epsilon_\eta^{1/2}\epsilon_\gamma^{1/2}kL_B}\right)$ **[1]**



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