Development of Resistive DCON

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Presented at the
CEMM/SciDAC and Sherwood Meetings
San Diego, CA   March 23-26, 2014

Resistive DCON

- DCON computes the ideal MHD stability of axisymmetric toroidal plasmas. Thoroughly verified and validated, robust, reliable, easy to use, widely used.
- Ideal DCON integrates the Euler-Lagrange equation for Fourier components of the normal displacement from the magnetic axis to the plasma-vacuum interface. This is an initial value problem.
- Straightforward extension to compute the outer region matching data for resistive instabilities converts it to a shooting method, which is numerically unstable.
- Pletzer and Dewar introduced a singular Galerkin method, avoiding this problem.
- We have improved on their implementation with a better choice of basis functions and grid packing.
- This has been coupled to the inner region resistive MHD model of Glasser, Greene & Johnson, solved by DELTAR, and a vacuum region, solved by Chance's VACUUM.
- We have obtained good agreement with the straight-through linear MARS code.

Pletzer & Dewar References

- A. D. Miller & R. L. Dewar, "Galerkin method for differential equations with singular points," *J. Comp. Phys.* **66**, 356-390 (1986). Introduces Galerkin method for singular ODEs, solves test problems.

- R. L. Dewar & A. Pletzer, "Two-dimensional generalization of the Newcomb equation," *J. Plasma. Phys.* **43**, 2, 291-310 (1990). Derives 2D Newcomb equations, equivalent to DCON equation.

- A. Pletzer & R. L. Dewar, "Non-ideal Variational method for determination of the outer-region matching data," J. Plasma Phys. 45, 3, 427-451 (1991).
  Solves cylindrical problem with non-monotonic q profile.

- A. Pletzer, A. Bondeson, and R. L. Dewar, "Linear stability of resistive MHD modes: axisymmetric toroidal computation of the outer region matching data," J. Comp. Phys. 115, 530-549 (1994). Solves toroidal problem, PEST 3, verified against MARS code.

Galerkin Expansion

Euler-Lagrange Equation

$$\mathbf{L}\Xi = -(\mathbf{F}\Xi' + \mathbf{K}\Xi)' + (\mathbf{K}^\dagger\Xi' + \mathbf{G}\Xi) = 0$$

Galerkin Expansion

$$(u, v) \equiv \int_0^1 u^\dagger(\psi)v(\psi)d\psi$$

$$\Xi(\psi) = \sum_{i=0}^{N} \Xi_i \alpha_l(\psi)$$

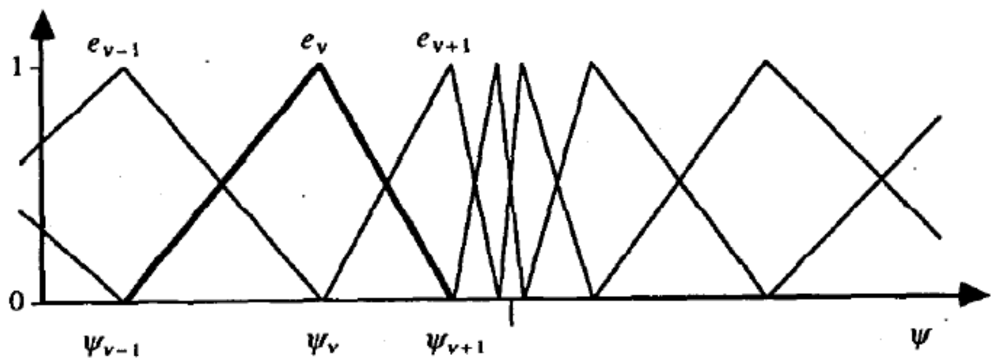
$$(\alpha_i, \mathbf{L}\Xi) = (\alpha_i, \mathbf{L}\alpha_j)\Xi_j = 0$$

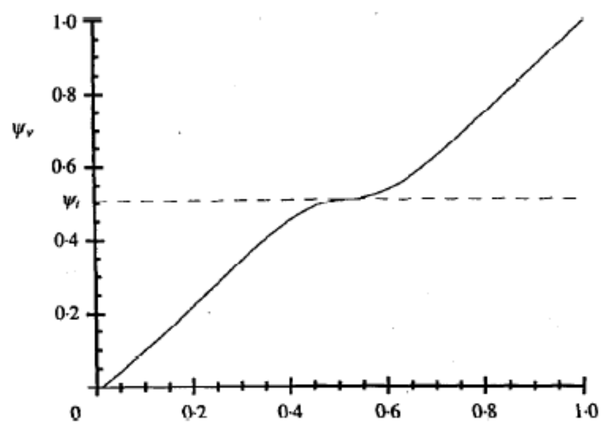
$$\mathbf{L}_{ij} = (\alpha_i', \mathbf{F}\alpha_j') + (\alpha_i', \mathbf{K}\alpha_j) + (\alpha_i, \mathbf{K}^\dagger\alpha_j') + (\alpha_i, \mathbf{G}\alpha_j)$$

Finite-Energy Response Driven by Large Solution

$$L_{ij}\check{\Xi}_j = -(\alpha_i, L\hat{\Xi})$$

Dewar and Pletzer:
Linear Finite Elements on a Packed Grid

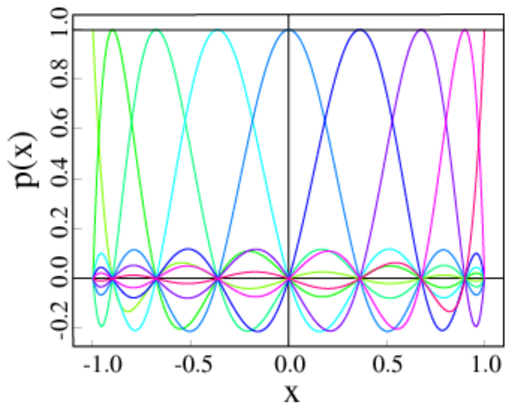




The choice of basis functions determines
the rate of convergence.

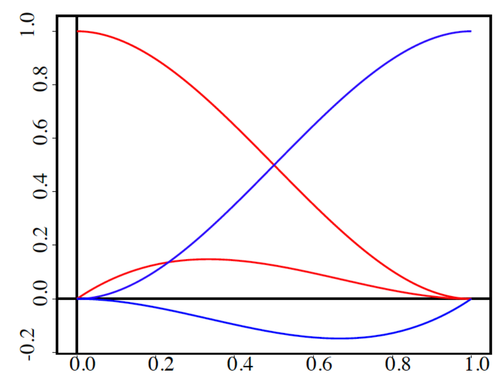
Higher-Order Basis Functions

C^0 Jacobi Nodal Basis



- Lagrange interpolatory polynomials
- Nodes at roots of $(1-x^2)\,P_n^{(0,0)\prime}(x)$
- Diagonally dominant

C^1 Hermite Cubics



- Cubic polynomials on (0,1).
- C^1 continuity: function values and first derivatives
- Useful for nonresonant solutions across the singular surface.

Singular Elements

- Weierstrass Convergence Theorem:
  Polynomial approximation uniformly convergent for analytic functions.

- Big and small resonant solutions are non-analytic near the singular surface.

- Supplement polynomial basis with small resonant solution near singular surface.

- DCON fits equilibrium data to Fourier series and cubic splines, computes resonant power series to arbitrarily high order.

- Convergence requires that the large solution be computed to at least n = 2*sqrt(-di) terms. PEST 3 is limited to n = 1.

Adjustable Grid Packing: Equations

Grid Packing Function

$$\lambda(a) = \coth a = \frac{e^a + 1}{e^a - 1}, \quad a(\lambda) = \text{acoth } \lambda = \ln\left(\frac{1+\lambda}{1-\lambda}\right)$$

$$x(\xi, \lambda) = \frac{\tanh a\xi}{\lambda} = \frac{1}{\lambda}\left(\frac{e^{a\xi} - 1}{e^{a\xi} + 1}\right)$$

$$\lim_{\lambda \to 0} a(\lambda) = 2\lambda, \quad \lim_{\lambda \to 0} x(\xi, \lambda) = \xi$$

Center and Edge Grid Densities

$$\frac{\partial x}{\partial \xi} = \frac{1}{\lambda}\frac{2ae^{a\xi}}{\left(e^{a\xi} + 1\right)^2} = \frac{1}{\lambda}\frac{2ae^{-a\xi}}{\left(e^{-a\xi} + 1\right)^2}$$

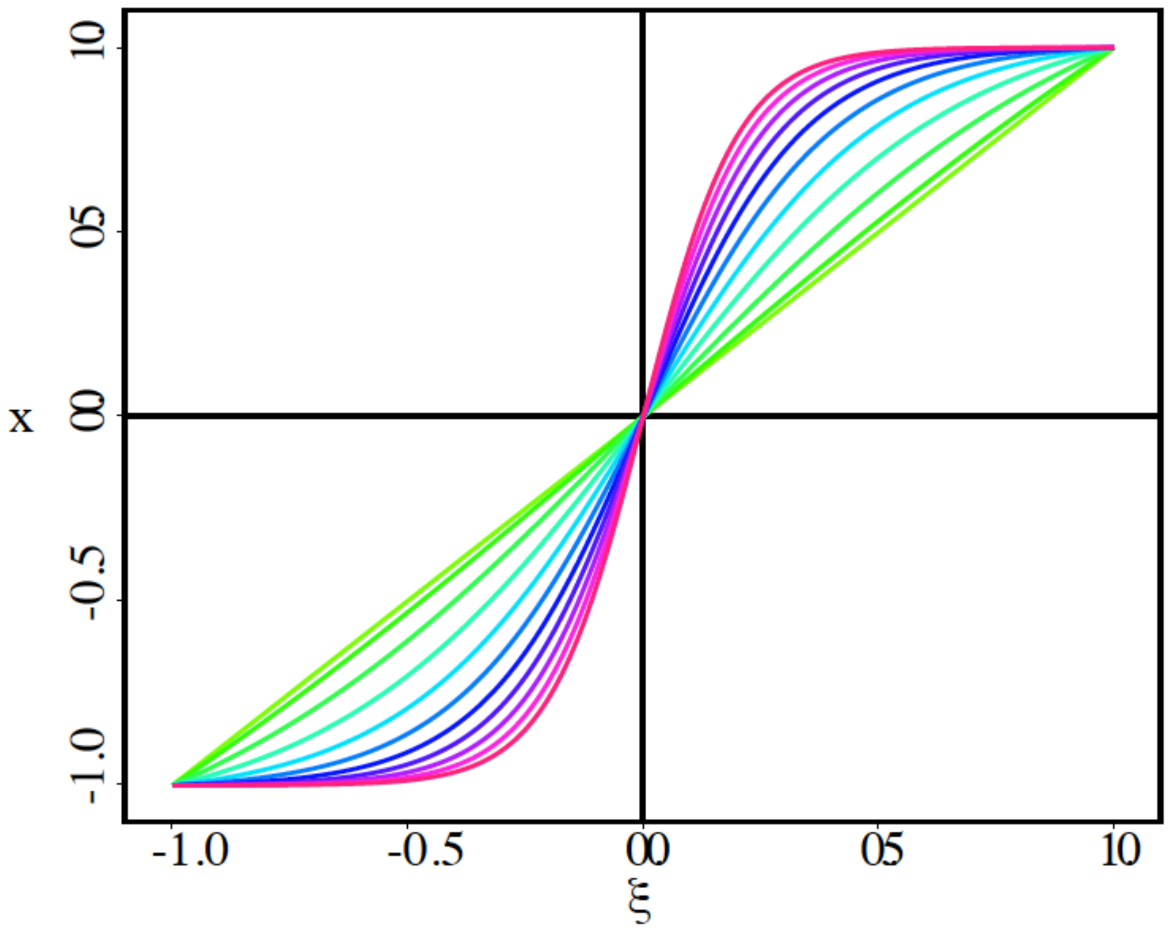
$$\left.\frac{\partial x}{\partial \xi}\right|_{\xi=0} = \frac{a}{2\lambda}$$

$$\left.\frac{\partial x}{\partial \xi}\right|_{\xi=\pm 1} = \frac{a}{2\lambda}\left(1 - \lambda^2\right)$$

Packing Ratio

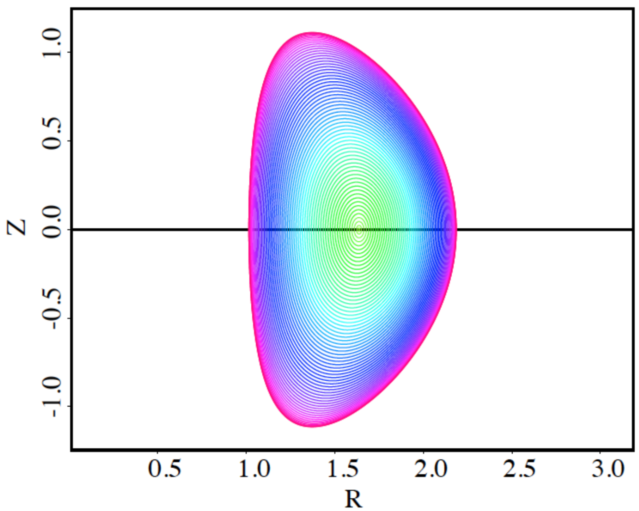
$$P(\lambda) \equiv \frac{\partial x / \partial \xi|_{\xi=\pm 1}}{\partial x / \partial \xi|_{\xi=0}} = 1 - \lambda^2$$

Adjustable Grid Packing: Graphs

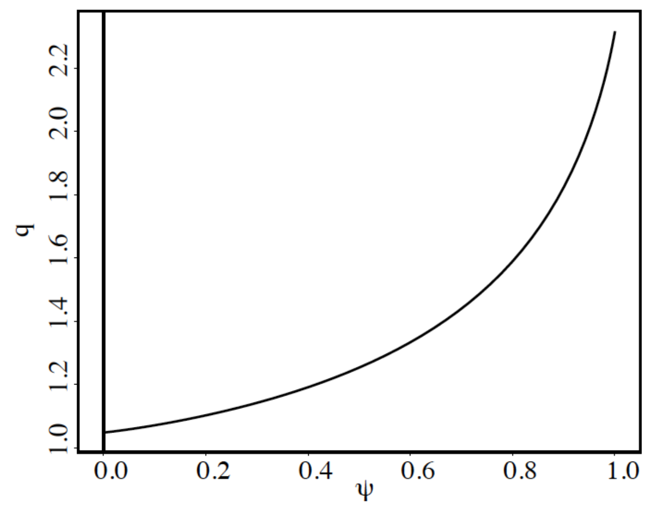


Chease Equilibrium, 1 Singular Surface, $\beta_N = 0.774$

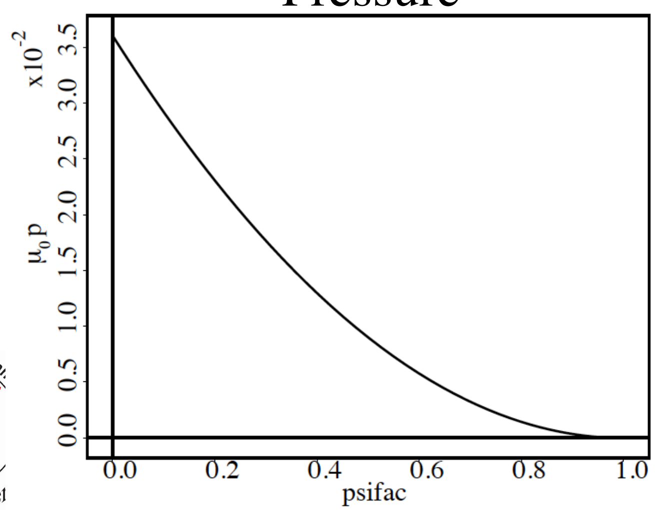
Flux Surfaces



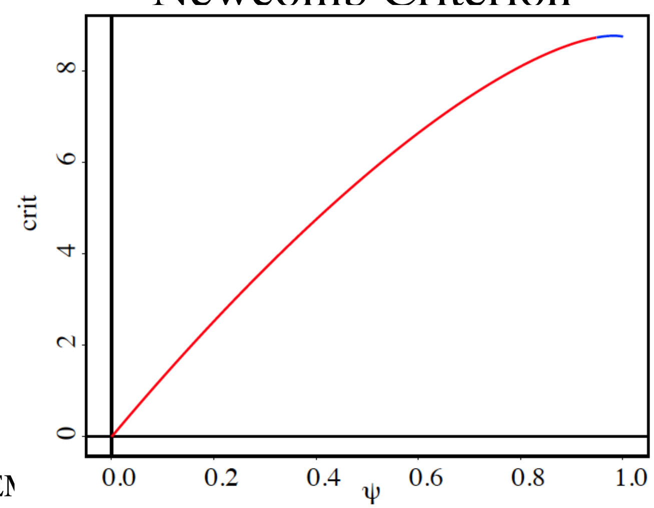
Safety Factor



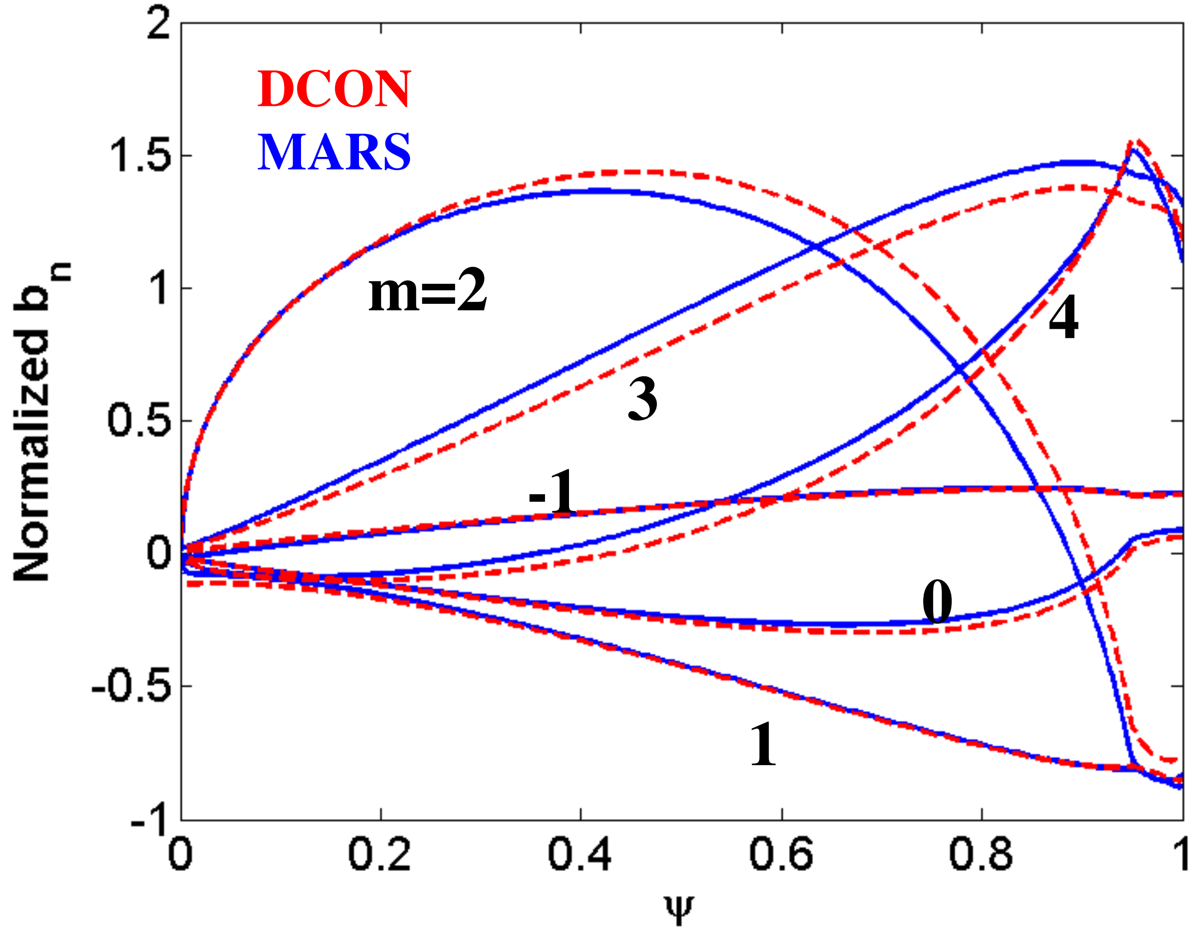
Pressure



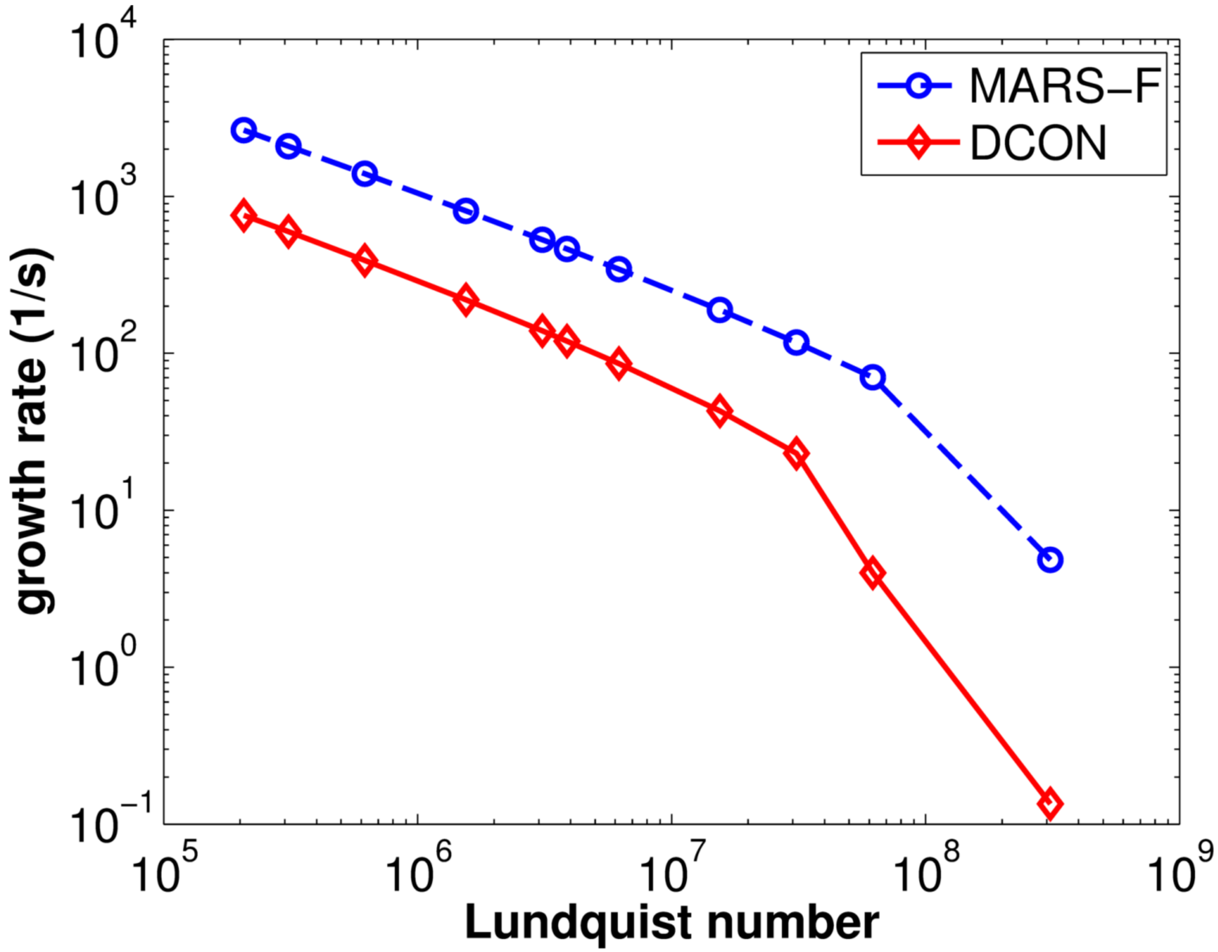
Newcomb Criterion



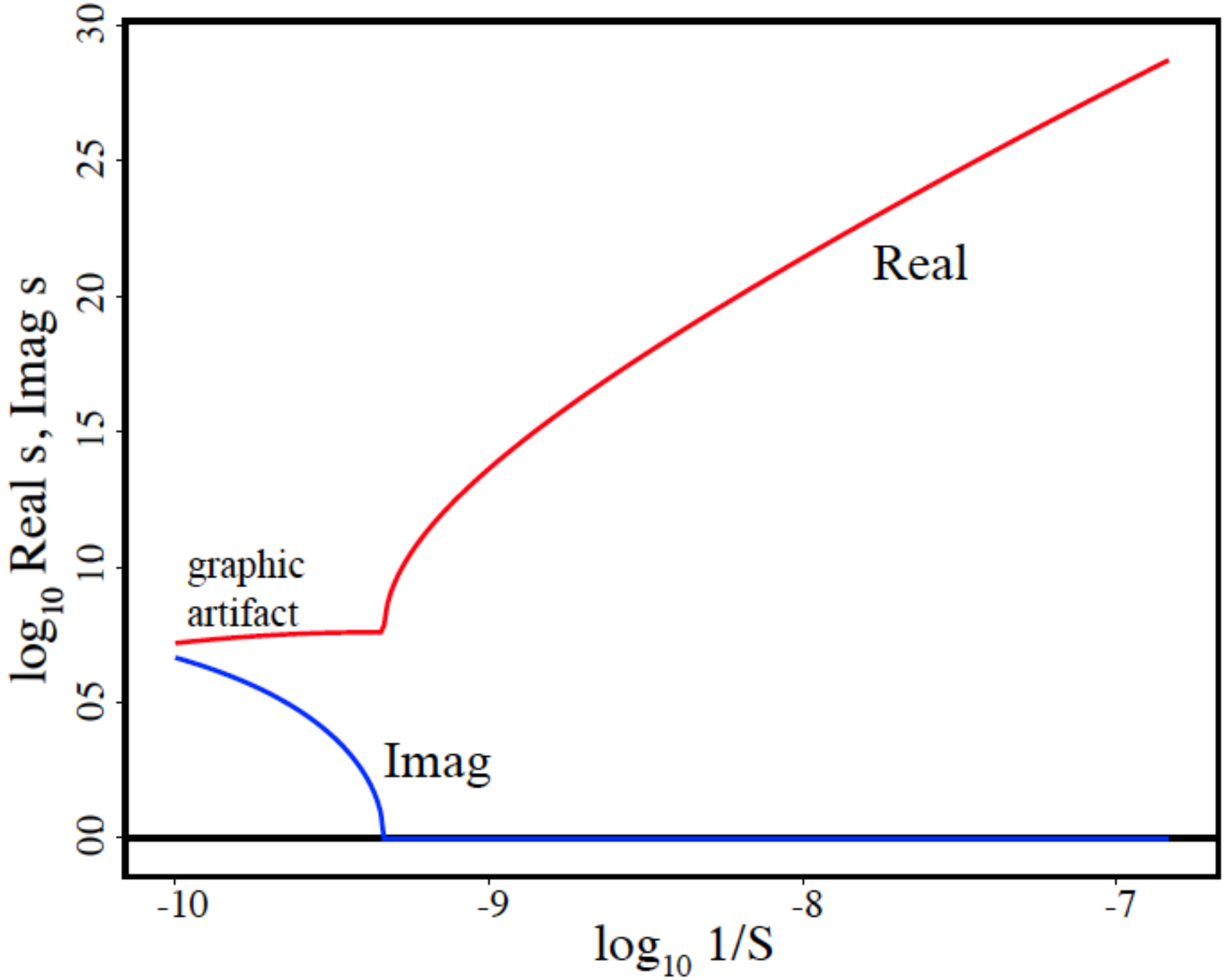
Comparison with MARS Code, 1 Singular Surface



Eigenvalue Benchmark with MARS Code

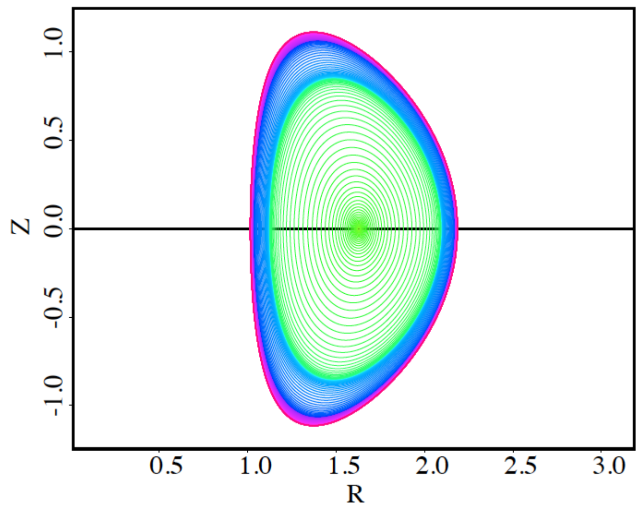


Glasser Effect: Complex Growth Rate Near $\Delta' = \Delta_C$

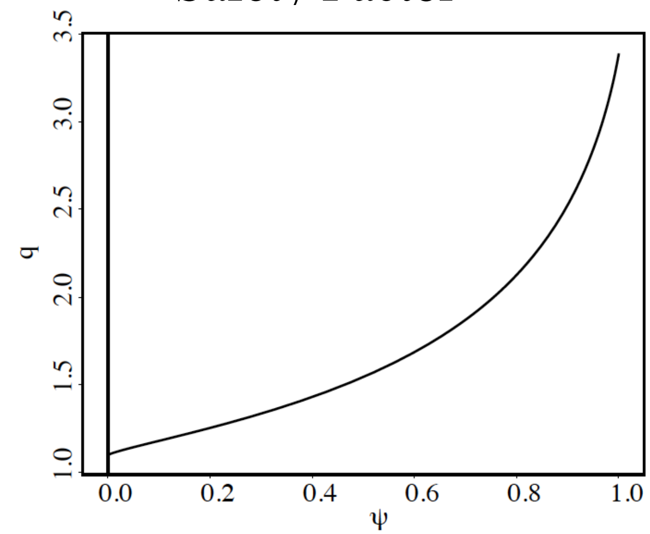


Chease Equilibrium, 2 Singular Surfaces, $\beta_N = 0.240$

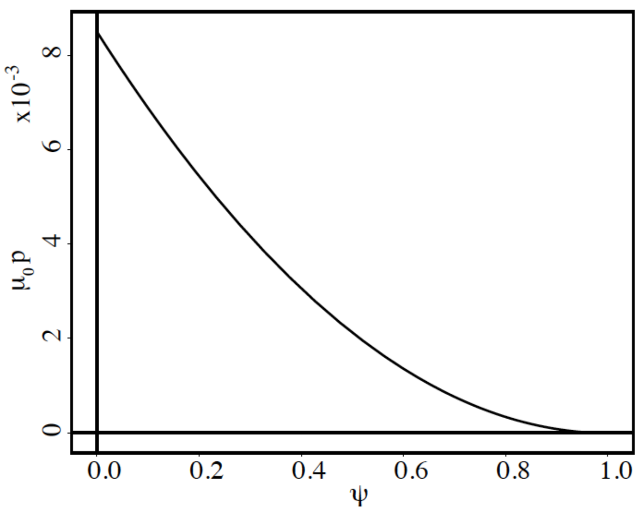
Flux Surfaces



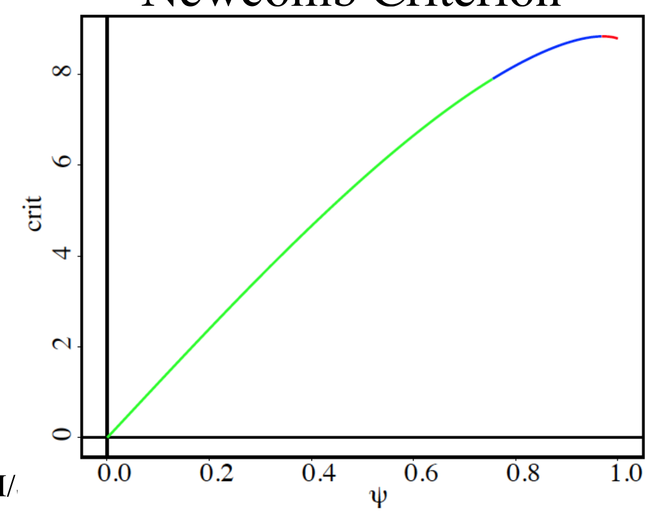
Safety Factor



Pressure



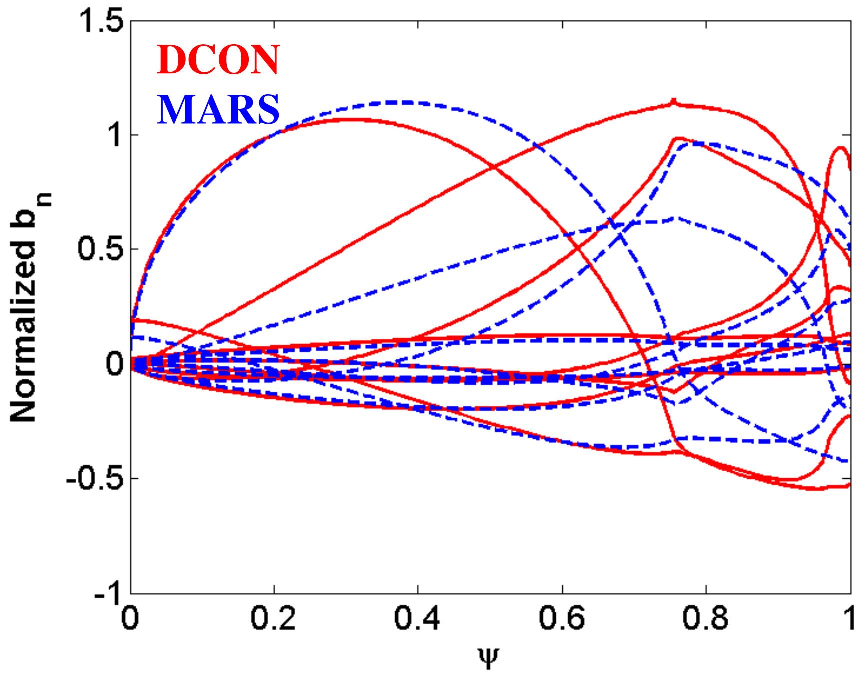
Newcomb Criterion



Comparison with MARS Code, 2 Singular Surfaces

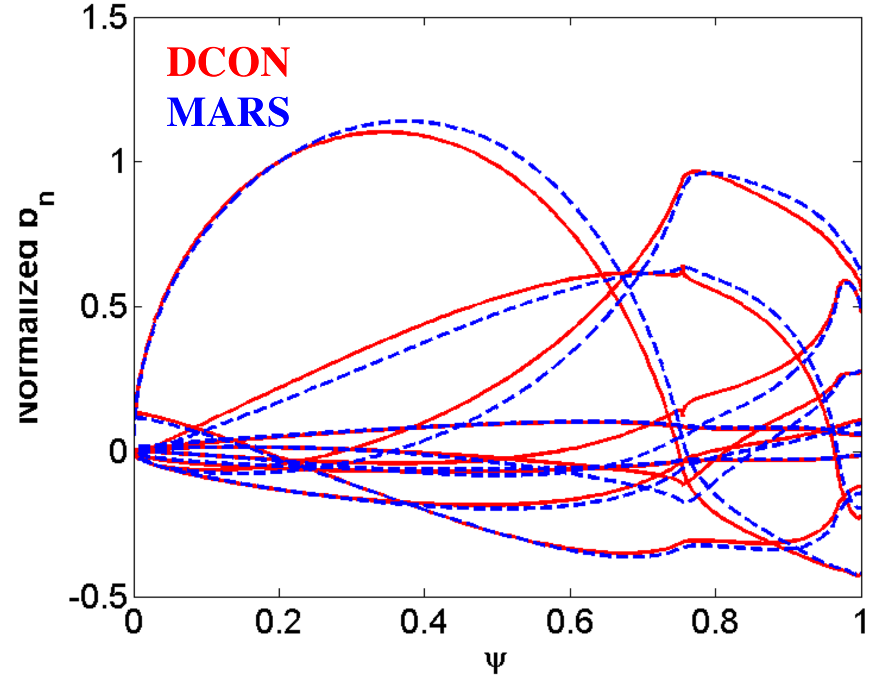
$$\xi_L = \xi_L^{(b)} + \Delta_{LR}\xi_R^{(s)} + \Delta_{LL}\xi_L^{(s)} + \xi_{reg,L} \qquad \xi_R = \xi_R^{(b)} + \Delta_{RR}\xi_R^{(s)} + \Delta_{RL}\xi_L^{(s)} + \xi_{reg,R} \qquad \xi = C_{R1}\xi_{R1} + C_{L1}\xi_{L1} + C_{R2}\xi_{R2} + C_{L2}\xi_{L2}$$

DCON solution from matching



CL1=1.0 CR1 =-1.005
CL2=0.769 CR2=-0.769

DCON solution with artificial coefficients



CL1=1.0 CR1 =-1.005
CL2=0.3 CR2=-0.3

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Matched Asymptotic Expansions

- The method of matched asymptotic expansions was introduced by Furth, Killeen, and Rutherford in order to obtain analytical results.

- Most recent work uses straight-through methods, such as M3D and NIMROD, using packed grids to resolve singular layers.

- Thermonuclear plasmas are in a regime where conditions for the validity of matched asymptotic expansion are very well satisfied.

- Resistive DCON and DELTAR provide numerical methods to do the full matching problem numerically and *very* efficiently.

- Inner region dynamics can be extended to include full fluid and kinetic treatments.

- Nonlinear effects are localized to the neighborhood of the singular layers and solved with the 2D HiFi code, exploiting helical symmetry, matched through ideal outer regions.

- Asymptotic matching and straight-through methods can complement and verify each other.

Future Work

- Improved benchmarks vs. MARS.
- Reconstruction of inner region eigenfunction by Fourier transformation.
- More complete fluid regime model of linear inner region; Braginskii.
- Neoclassical inner region model, drift kinetic equation; Ramos.
- Nonlinear model, NTM, with nonlinear effects localized to inner regions, coupled through ideal linear outer region. 2D HiFi code, helical symmetry.
- Verification with straight-through nonlinear codes: NIMROD, M3D-C1.
- Validation against experiments: NSTX, D-IIID.