Extended MHD Analysis of the Gravitational Interchange.

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The gravitational interchange is analyzed using three different extended MHD models to illustrate the different behavior present in each of these models.

- We consider a single temperature extended MHD model that includes ion gyroviscosity and a two-fluid Ohm's Law.
- Two partial models are also considered.
  - The gyroviscous model includes gyroviscosity with a MHD Ohm's law.
  - The two-fluid model uses a two-fluid Ohm' law but neglects gyroviscosity.
- The differences between these models highlights the importance of using a self-consistent model.

Outline

The gravitational interchange (g-mode) is useful for studying interchange dynamics in simplified geometry.

- A fictitious gravity is introduced to represent magnetic curvature.
  - Interchange force balance: $\nabla\left(\mu_0 P + \frac{B^2}{2}\right) = \vec{B}\cdot\nabla\vec{B}$
  - g-mode force balance : $\nabla\left(\mu_0 P + \frac{B^2}{2}\right) = \rho\vec{g}$
  - $\rho' g < 0$ corresponds to "bad curvature."
- Extended MHD introduces drifts which can stabilize interchange modes [Roberts and Taylor PRL 1962].
  - The g-mode is a simple model used to study the dynamics of this stabilization.
  - The g-mode is a useful benchmark for extended MHD codes [Schnack et al. PoP 2006].
- We analyze the extended MHD g-mode dispersion relation of Zhu et al. [Zhu et al. PRL 2008].
  - Zhu analyzes the relation in the simplified model that only includes gyroviscosity.
  - Our analysis includes two-fluid effects with and without gyroviscosity.

The extended MHD model includes ion gyroviscosity and two-fluid corrections to Ohm's law.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = \vec{J} \times \vec{B} - \nabla p - \delta \nabla \cdot \vec{\vec{\pi}}_{gv}$$

$$\vec{\vec{\pi}}_{gv} = \frac{p_i}{4\Omega_{ci}} \left[\hat{b} \times \vec{\vec{W}} \cdot \left(\vec{\vec{I}} + 3\hat{b}\hat{b} \right) - \left(\vec{\vec{I}} + 3\hat{b}\hat{b} \right) \cdot \vec{\vec{W}} \times \hat{b} \right]$$

$$\vec{\vec{W}} = \nabla \vec{v} + \nabla \vec{v}^{T} - \frac{2}{3} \vec{\vec{I}} \nabla \cdot \vec{v}$$

$$\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = -\gamma p \nabla \cdot v$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left[-\vec{v} \times \vec{B} + \frac{\lambda}{ne} \left(\vec{J} \times \vec{B} - \nabla p_e \right) \right]$$

- Separate electron and ion temperature evolution is not considered.
- δ and λ are markers used to track gyroviscous and two-fluid effects.

Zhu et al. derive an extended MHD dispersion relation for the local g-mode assuming a static equilibrium.

$$
\begin{aligned}
\vec{u}_0 &= 0 \\
\vec{B}_0 &= B_z \hat{e}_z \\
\frac{d}{dx}\left(p + \frac{B^2}{2\mu_0}\right) &= \rho \vec{g} \cdot \hat{e}_x \\
ne\vec{E} &= \nabla p_i - \rho \vec{g}
\end{aligned}
$$

- Equilibrium quantities vary in the x direction.
- Perturbed quantities vary in x and y.
- Local interchange ordering is used with $k_y L_n \gg 1$ and $\tilde{u}_y \ll \tilde{u}_x$.

The dispersion relation is normalized such that only one parameter, G, depends on k_y/Ω.

Symbol	Normalized Quantity	Physical Quantity	Normalization
$X = \frac{\omega}{\Gamma_{MHD}}$	Mode frequency	ω	Γ_{MHD}
$G = \frac{\omega_g}{\Gamma_{MHD}}$	Gravitational drift	$\omega_g = -\frac{k_y}{\Omega} g$	Γ_{MHD}
$P = \frac{\omega_p}{\omega_g}$	Ion diamagnetic drift	$\omega_p = \frac{k_y}{\Omega} \frac{p_i'}{m_i n}$	ω_g
$N = \frac{\omega_n}{\omega_g}$	Ion density drift	$\omega_n = \frac{k_y}{\Omega} \frac{p_i n'}{m_i n^2}$	ω_g
$R^2 G^2 = k_y^2 r_i^2$	Ion gyroradius	$k_y r_i = \frac{k_y}{\Omega} \sqrt{\frac{p_i}{n m_i}}$	G

- $\Gamma_{MHD} = \sqrt{-\frac{n'}{n} g}$ is the $0 - \beta$ MHD growth rate.
- Scaling G is equivalent to scaling k_y for a fixed equilibrium.
- The normalized gyroradius is not a free parameter: $R^2 = N$.
- The extended MHD model is physically valid in the limit $RG \ll 1$.

The extended MHD g-mode is characterized by a cubic dispersion relation.

$$(A_0 + A_2) X^3 + (X_{*1} + X_{*3}) X^2 + \left(\Gamma_0^2 + \Gamma_2^2\right) X + D_1 = 0$$

$$(A_0 + A_2) = 1 + \gamma\beta + \delta^2 \frac{\tau G^2 R^2}{4} \beta$$

$$X_{*1} = G \left(\delta \left[(1+\gamma\beta)(1+\beta) P + (2+\gamma\beta)\tau\beta\right] + \lambda \left[1 + P - \gamma N\right]\right)$$

$$X_{*3} = -\delta^2 \lambda G^3 \frac{R^2}{4} N$$

$$\Gamma_0^2 = 1 + \gamma\beta + \frac{\tau\beta}{N}$$

$$\Gamma_2^2 = \delta\lambda G^2 \left[(1+\beta)(P+1) P - ((1+\gamma\beta\tau) + (1+\beta)\gamma P) N + (1+P)\tau\beta\right]$$

$$D_1 = \lambda G (P - \gamma N)$$

- $\tau p = p_i$ and $\beta = \mu_0 p / B^2$
- The red terms arise due to gyroviscosity, the blue terms arise due to two-fluid effects, and the magenta terms arise due to an interaction between the two effects.

The gyroviscous dispersion relation reduces to a quadratic.

$$\left(AX^2 + X_* X + \Gamma^2\right) X = 0$$

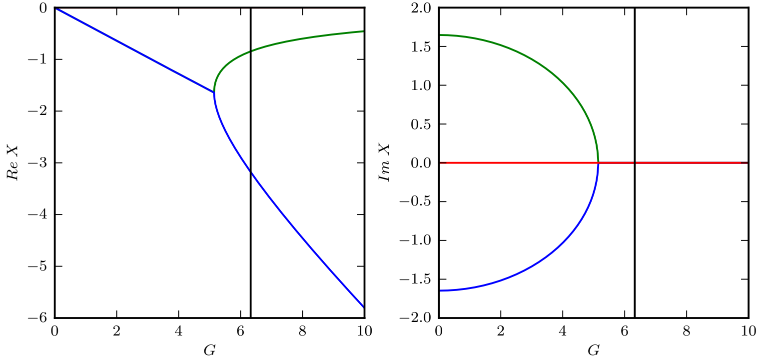
$$A = 1 + \gamma\beta + \frac{\tau G^2 R^2}{4}\beta$$

$$X_* = G\left[\left(1+\gamma\beta\right)\left(1+\beta\right)P + \left(2+\gamma\beta\right)\tau\beta\right]$$

$$\Gamma^2 = 1 + \gamma\beta + \frac{\tau\beta}{N}$$

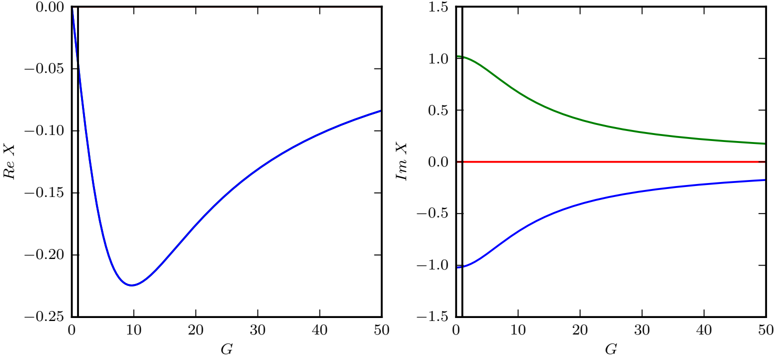
- This dispersion relation is obtained by setting $\delta = 1$ and $\lambda = 0$ in the full extended MHD dispersion relation.
- The two nontrivial solutions are $X = \frac{-X_*}{2A}\left(1 \mp \sqrt{1 - \frac{4A\Gamma^2}{X_*^2}}\right)$.
- Complete stabilization is not always possible when $\beta \neq 0$ and $P < N$.
- X scales as G^{-1} in the limit $|G| \gg 1$.

$P \geq N$ is a sufficient condition for complete gyroviscous stabilization.



- The unstable g-mode (green) and its damped counter part (blue) have the same real frequency.
- The two branches have distinct real frequencies after stabilization.
- Both waves asymptote to 0 for $G > 60$ (not shown).

Other regimes are not stabilized by FLR effects in the gyroviscous model!



- Both branches of the g-mode (blue and green) have the same real frequency for all G.
- The real and imaginary frequencies asymptote to zero at large G.

The two-fluid dispersion relation is represented by a two parameter model.

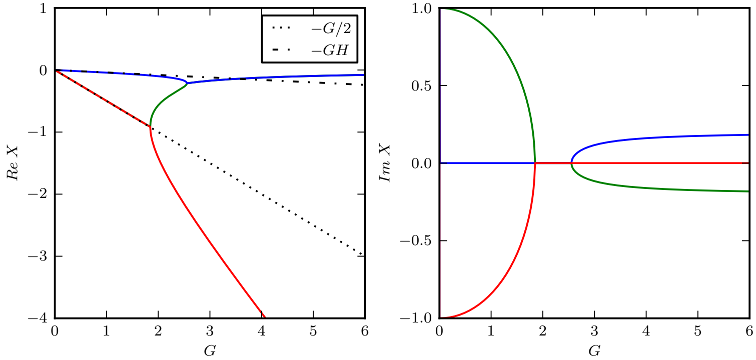
$$X^3 + G\,(1+H)\,X^2 + X + GH = 0$$
$$H = P - \gamma N$$

- The two-fluid model is obtained by setting $\delta = 0$, $\lambda = 1$, and $\beta = 0$ in the full extended MHD dispersion relation.
  - Setting $\beta = 0$ does not change the qualitative behavior of the model, since none of the two-fluid terms containing β depend on G.
- The modified ion diamagnetic frequency H is related to the ITG stability parameter $\eta_i = \frac{nT'}{n'T}$.
  - $H = N\,(\eta_i - (\gamma - 1))$
  - $H > 0$ implies that $\eta_i > \frac{2}{3}$ for $\gamma = \frac{5}{3}$.
- The term GH introduces a third nontrivial mode.

Analysis of the two-fluid model identifies the asymptotic stability boundaries.

- Stability requires that $G^2 (1 + H)^2 \geq 3$.
  - $H = -1$ is always unstable.
- Two-fluid effects are stabilizing in the limit $|G| \ll 1$ when $H > -\frac{1}{4}$.
  - $X_1 = -GH$ is an ion drift wave.
  - $X_{2,3} = -\frac{G}{2} \pm i \left(1 - \frac{G^2}{8} (1 + 4H)\right)$ are the branches of the g-mode.
  - The drift wave and the branches of the g-mode propagate in the same direction for $H > 0$.
- The g-mode is stable in the limit $|G| \gg 1$ when $-1 < H \leq 0$.
  - Two of the modes are independent of G: $X_{1,2} = \pm\sqrt{\frac{-H}{1+H}}$.
  - The third mode is an ion drift wave: $X_3 = -G (1 + H)$.

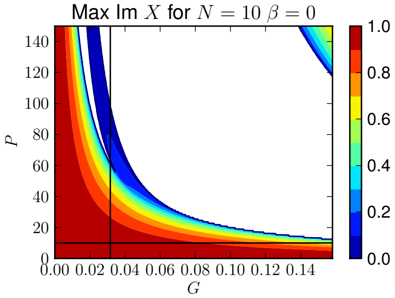
The two-fluid model yields a second instability for $H > 0$.



- Instability occurs when the ion drift wave (blue) intersects the low frequency branch of the stabilized g-mode (green).
- The drift wave and the g-mode drift in opposite directions for $H < 0$.
- There region of stability disappears for $H > \frac{1}{8}$.

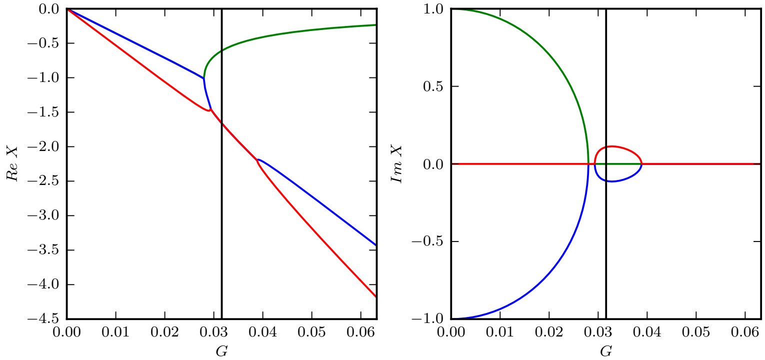
The full model yields multiple unstable regions.

- The g-mode exists at low G.
  - Stabilization of the g-mode is weak when $P < N$.
- A second instability exists at intermediate G when $P \gg \gamma N$.
  - Here the ion drift wave and the g-mode propagate in the same direction.
- A third instability exists for $RG \gtrsim 1$ (top right corner).
  - Here the extended MHD model is not valid.
  - The maximum growth rate of this third mode exceeds the MHD growth rate.



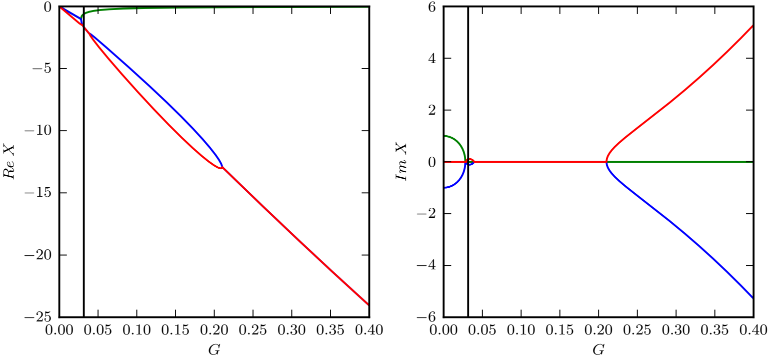
- White regions are stable.
- Vertical line indicates $GR = 0.1$.
- Similar behavior is observed for finite β.

The second instability is due to an interaction between the ion drift wave and the g-mode.



- The second instability occurs when the ion drift wave (red) intersects the high frequency branch of the stabilized g-mode (blue).
- The vertical line indicates $RG = 0.1$.

A third unphysical instability occurs at $G^2R^2 \gtrsim 1$, where extended MHD is not valid.



- This instability is due to an interaction between the two branches of the previously stabilized mode.
- It exists in codes that use the extended MHD model and may be problematic due to its large growth rate.

The analysis of the extended MHD g-mode reveals new behavior.

- The gyroviscous model is always more stable than MHD.
  - Zhu showed the complete stabilization is not always possible.
  - The growth rate asymptotes to zero even without complete stabilization.
- A 2-parameter model is used to analyze the two-fluid model.
  - Two-fluid effects introduce an ion drift wave.
  - The ion drift wave can interact with the low frequency branch of the g-mode to produce a new instability.
  - Its growth rate is comparable to that of the MHD g-mode.
- Multiple instabilities exist in the full model.
  - One instability is driven unstable by the interaction between the ion drift wave and the high frequency branch of the g-mode.
  - A second instability exists in a regime where extended MHD is not physically valid.
    - This instability may be a concern for extended MHD codes due to its large growth rate.

Extra Slides

Extras

Zhu et al. show that complete gyroviscous stabilization is not always possible.

- Complete stabilization occurs when

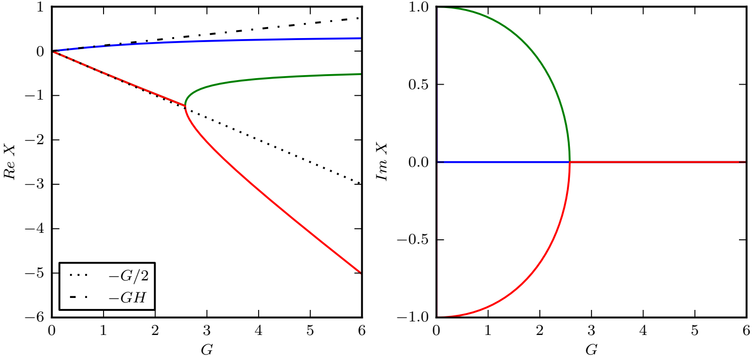
$$G^2\left[\frac{\left[\left(1+\gamma_s\beta\right)\left(1+\beta\right)P+\left(2+\gamma_s\beta\right)\tau\beta\right]^2}{1+\gamma\beta+\frac{\tau\beta}{N}}-\tau R^2\beta\right]\geq 4\left(1+\gamma\beta\right).$$

- Stabilization is impossible when the bracketed term is negative.
- The bracketed term is positive for $\beta=0$ or when the temperature and density gradients are parallel ($P\geq N$).
- The growth rate asymptotes to zero even in cases where complete stabilization is absent.

A second instability occurs in the two-fluid model when the g-mode and the original ion drift wave propagate in the same direction.

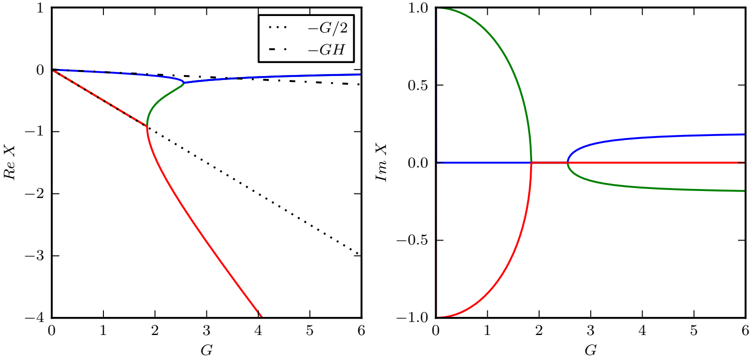
- The second instability occurs for $H > 0$ or equivalently $\eta_i > 2/3$.
  - The mode grows at a rate comparable to the MHD growth rate.
  - The mode is destabilized by an interaction between an ion drift wave and the low frequency branch of the g-mode.
- This mode is not an ITG mode despite having many similarities.
  - ITG requires finite $k_{\|}$, but here $k_{\|} = 0$.
  - ITG is a modified sound wave, but this mode is a modified g-mode.
  - ITG requires gyroviscosity, but this mode lacks gyroviscosity.
- Two-fluid effects are destabilizing at small G when $H < -1/4$.
- The two-fluid model is stable in the large G limit for $-1 < H \leq 0$.

Two-fluid effects lead to complete stability for $-1/4 < H \leq 0$.



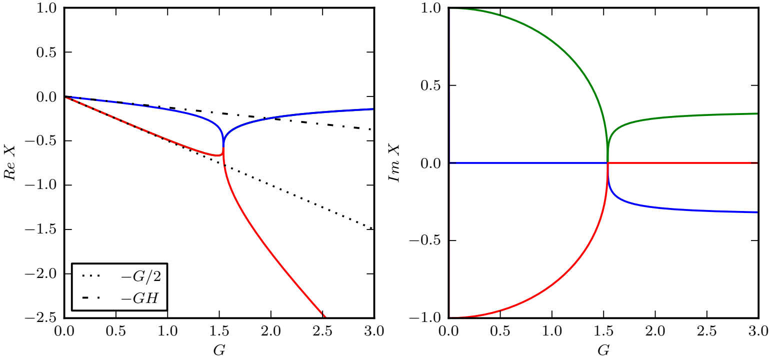
- The ion drift wave (blue) and the branches of the g-mode (red and green) propagate in opposite directions.
- The branches of the g-mode become real waves after stabilization.

For $0 < H < 1/8$ there is only a window of stability.



- Two-fluid effects stabilize the g-mode at small G.
- A region of stability occurs at intermediate G.
- A second instability is present at large G.
  - Instability occurs when the ion drift wave (blue) intersects the low frequency branch of the stabilized g-mode (green).

The window of stability disappears for $H = 1/8$.



- Complete stabilization only occurs at $G = \pm\frac{8}{9}\sqrt{3}$.
- The necessary stability condition $G^2(1+H)^2 \geq 3$ is satisfied exactly.
- The drift wave intercepts both branches of the g-mode at the point of stabilization.
- Complete stabilization never occurs for $H > \frac{1}{8}$.

The asymptotic analysis is also applied to the full extended MHD model.

- The three modes in the limit $|G| \ll 1$ are:
  - an ion drift wave $X_1 = \frac{-D_1}{\Gamma_0^2}$
  - the g-mode: $X_{2,3} = Y_1 \pm i \frac{\Gamma_0^2}{A_0} \sqrt{1 + \frac{1}{2\Gamma_0^2} \left(3Y_1^2 A_0 + 2X_{*1} Y_1 + \Gamma_2^2\right)}$
    - $Y_1 = \frac{A_0 D_1 - X_{*1}\Gamma_0^2}{2\Gamma_0^2 A_0}$
- The limit $|G| \gg 1$ is dominated by a new instability.
  - $X_1 = \frac{GN}{\tau\beta}$ is an ion drift wave.
  - $X_{2,3} = -\frac{\Gamma_2^2}{2X_{*3}} \left(1 \pm i\sqrt{1 - \frac{4D_1 X_{*3}}{\Gamma_2^4}}\right)$ is a new instability driven by Γ_2^2.
  - Stability is possible when $G \gg 1$.
  - Even if unstable, the growth rate decreases as G^{-1}.
- The asymptotic behavior can change if A_2, Γ_2^2, or X_{*3} are 0.
  - In these cases complete stability is not always possible.