S. Jardin
10/25/01

1. Comments on the derivation of Eq. (2.20) in [Sugiyama, 2000] (SP)

Consider the momentum and gyroviscous terms in Eq. (2.2) of SP

$$n_i m_i \left(\frac{\partial \vec{V}_i}{\partial t} + \left(\vec{V}_i \bullet \nabla \right) \vec{V}_i \right) + \nabla \bullet \overleftrightarrow{\Pi}_i^{gv} \tag{1}$$

Substitute from (2.14) and (2.17) of SP

$$\begin{aligned} \vec{V}_i &= \vec{V} + \vec{V}_{di} \\ \vec{V}_{di} &= \vec{J}_\perp / en_e + \vec{V}_{*e} \\ \vec{V}_{*e} &\equiv -\vec{B} \times \nabla p_e / enB^2 \end{aligned}$$

Thus, Eq. (1) becomes

$$n_i m_i \left(\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \bullet \nabla \right) \vec{V} + \left(\vec{V}_{di} \bullet \nabla \right) \vec{V} \right) + n_i m_i \left(\frac{\partial \vec{V}_{di}}{\partial t} + \left(\vec{V}_i \bullet \nabla \right) \vec{V}_{di} \right) + \nabla \bullet \overleftrightarrow{\Pi}_i^{gv} \tag{2}$$

Now, from equations (26) and (35) in [Belova, 2001], we have the relation:

$$\begin{aligned} \nabla \bullet \overleftrightarrow{\Pi}_i^{gv} &= \left(\nabla \bullet \overleftrightarrow{\Pi}_i^{gv} \right)_\perp + \left(\nabla \bullet \overleftrightarrow{\Pi}_i^{gv} \right)_\parallel \\ &= -n_i m_i \left(\frac{\partial \vec{V}_*}{\partial t} + \left(\vec{V}_i \bullet \nabla \right) \vec{V}_* \right) - (n_i m_i \vec{V}_* \bullet \nabla) \vec{V}_\parallel + \nabla_\perp \tilde{\chi} + \upsilon_{gv\parallel} \end{aligned} \tag{3}$$

where we use the following definitions [Belova, 2001]:

$$\begin{aligned} \vec{V}_* &\equiv \vec{B} \times \nabla p_i / enB^2 \\ \upsilon_{gv\parallel} &\equiv \frac{1}{B^2} \nabla_\perp \left\{ \nabla_\perp \varphi, \left\| \mu U \right\| \right\} \\ \tilde{\chi} &\equiv -\frac{p_\perp}{2\Omega_i} \hat{b} \bullet \nabla \times V_\perp - \frac{1}{4\Omega_i} \nabla \times \vec{q}_\perp \end{aligned} \tag{4}$$

Thus, if $\vec{V}_* \equiv \vec{V}_{di}$ cancellations will occur in the inertial terms and we will get Equation (2.20) of SP (but with a different form for $\upsilon_{gv\parallel}$ and for χ). However, in general, $\vec{V}_* \neq \vec{V}_{di}$ as you can see from taking the cross product of Equation (2.2) in SP with the magnetic field:

$$\vec{V}_* = \vec{V}_{di} - \vec{B}\times\vec{T}/enB^2 \tag{5}$$

where the generalized polarization drift term is

$$\vec{T} = n_i m_i\left(\frac{\partial\vec{V}_i}{\partial t} + \left(\vec{V}_i\bullet\nabla\right)\vec{V}_i\right) + \nabla\bullet\overleftrightarrow{\Pi}_i^{gv} - n_i m_i \mu_{\perp i}\nabla^2 V_i \tag{6}$$

Thus, it is clear that Eq. (2.20) is not consistent with Eq. (2.2) in SP unless some additional approximations (involving the polarization drift term) are made. However, the second term in Eq. (5), involving the polarization drift, is of higher order in $\delta \sim \frac{\rho_i}{L}$. This is true both in the drift ordering, where $\frac{V_i}{V_{th}} \sim \delta, \quad \frac{\partial}{\partial t} \sim \omega \sim \delta^2\Omega_{ci}$, and in the Gyrokinetic ordering where $\frac{V_i}{V_{th}} \sim \delta, \quad \frac{\partial}{\partial t} \sim \omega \sim \delta\,\Omega_{ci}$. Thus to lowest order in the small parameter δ, Equations (2.20) and (2.2) in SP are consistent.

2. Alternative form keeping the effects of the polarization drift

Combine Eq. (1) and (3) to get for the inertial and gyroviscous terms in the ion momentum equation:

$$= n_i m_i\left(\frac{\partial\vec{V}_i}{\partial t} + \left(\vec{V}_i\bullet\nabla\right)\vec{V}_i\right) - n_i m_i\left(\frac{\partial\vec{V}_*}{\partial t} + \left(\vec{V}_i\bullet\nabla\right)\vec{V}_*\right) - (n_i m_i \vec{V}_*\bullet\nabla)\vec{V}_\parallel + \nabla_\perp\tilde{\chi} + \upsilon_{gv\parallel}$$

or

$$= n_i m_i\left(\frac{\partial(\vec{V}_i - \vec{V}_*)}{\partial t} + \left((\vec{V}_i - \vec{V}_*)\bullet\nabla\right)(\vec{V}_i - \vec{V}_*) + (\vec{V}_*\bullet\nabla)(\vec{V}_i - \vec{V}_*)_\perp\right) + \nabla_\perp\tilde{\chi} + \upsilon_{gv\parallel}$$

Now, define

$$\vec{V} \equiv \vec{V}_i - \vec{V}_* \tag{7}$$

And we have for the final form of the FLR corrected momentum equation:

$$n_i m_i\left(\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\bullet\nabla\right)\vec{V} + (\vec{V}_*\bullet\nabla)\vec{V}_\perp\right) + \nabla_\perp\tilde{\chi} + \upsilon_{gv\parallel} + \nabla P = \vec{J}\times\vec{B} \tag{8}$$

In terms of the velocity defined in Eq. (7), the electron velocity becomes:

$$\vec{V}_e = \vec{V}_i - \vec{J}/ne = \vec{V} + \vec{V}_* - \vec{J}/ne \tag{9}$$

Now, consider the Ohm's law, Eq. (2.3) in SP:

$$\vec{E}+\vec{V}_e\times\vec{B}=\eta\vec{J}^{*}-\frac{1}{ne}\nabla p_e-\frac{0.71}{e}\nabla_{\|}k_B T_e-N_e B$$
(10)

Substitute (9) into (10) to get:

$$\vec{E}+\vec{V}\times\vec{B}=\eta\vec{J}^{*}+\frac{1}{ne}\left(\vec{J}\times\vec{B}-\nabla p_e-\nabla_{\perp}p_i\right)-\frac{0.71}{e}\nabla_{\|}k_B T_e-N_e\vec{B}$$
(11)

Thus, Equations (2.20) and (2.22) in SP could be replaced with those from Equations (8) and (11) to retain the polarization term, and hence include the Hall term in Ohm's law.

3. Alternative form using the ion velocity:

Note further that [Belova,2001] has an identity in Eq. (39) and (40) , valid in the Gyrokinetic ordering, that can be used to write the ion inertial terms in another form. i.e., the identity:

$$\left(\frac{\partial\vec{V}_*}{\partial t}+\left(\vec{V}_i\bullet\nabla\right)\vec{V}_*\right)=(\vec{V}_*\bullet\nabla)\vec{V}_{\perp}-\frac{1}{n_i m_i}\hat{b}\times\nabla\left(\nabla_{\|}\left\|U\mu\right\|\right) \qquad (12)$$

allows one to rewrite the FLR corrected momentum equation as:

$$n_i m_i\left(\frac{\partial\vec{V}_i}{\partial t}+\left((\vec{V}_i-\vec{V}_*)\bullet\nabla\right)\vec{V}_i\right)+\hat{b}\times\nabla\left(\nabla_{\|}\left\|U\mu\right\|\right)+\nabla_{\perp}\tilde{\chi}+\upsilon_{gv\|}+\nabla P=\vec{J}\times\vec{B} \qquad (13)$$

where [Belova, 2001]

$$\left\|U\mu\right\|B=p_{\perp}V_{\|}+q_{\|}^{\perp}$$

If this form were used, the most natural form for the Ohm's law equation would be:

$$\vec{E}+\vec{V}_i\times\vec{B}=\eta\vec{J}^{*}+\frac{1}{ne}\left(\vec{J}\times\vec{B}-\nabla p_e\right)-\frac{0.71}{e}\nabla_{\|}k_B T_e-N_e\vec{B}$$
(14)

4. Relation to [Hazeltine, 1992] (HM)

We can make contact with the notation of [Hazeltine,1992] by making the variable transformation

$$\vec{V} \rightarrow \vec{V}_E + \vec{V}_\parallel$$
$$\vec{V}_i \rightarrow \vec{V} \qquad (15abc)$$
$$\vec{V}_* \rightarrow \vec{V}_{pi}$$

Thus, in their notation, Equation (8) can be written:

$$n_i m_i \left(\frac{\partial \vec{V}_E}{\partial t} + \left((\vec{V}_E + \vec{V}_\parallel + \vec{V}_{pi}) \bullet \nabla \right) \vec{V}_E + \frac{\partial \vec{V}_\parallel}{\partial t} + \left((\vec{V}_E + \vec{V}_\parallel) \bullet \nabla \right) \vec{V}_\parallel \right) + \nabla_\perp \tilde{\chi} + \upsilon_{gv\parallel} + \nabla P = \vec{J} \times \vec{B} \quad (16)$$

or, in HM notation, similar to their Equation (138):

$$n_i m_i \left(\frac{d\vec{V}_E}{dt} + \frac{d\vec{V}_\parallel}{dt} |_{MHD} \right) + \nabla_\perp \tilde{\chi} + \upsilon_{gv\parallel} + \nabla P = \vec{J} \times \vec{B} \qquad (17)$$

with

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\vec{V}_E + \vec{V}_\parallel + \vec{V}_{pi}) \bullet \nabla$$
$$\frac{d}{dt} |_{MHD} \equiv \frac{\partial}{\partial t} + (\vec{V}_E + \vec{V}_\parallel) \bullet \nabla$$

Similarly, in their notation, the Ohm's law, Eq (14) becomes:

$$\vec{E} + \vec{V} \times \vec{B} = \eta \vec{J}^* + \frac{1}{ne} \left(\vec{J} \times \vec{B} - \nabla p_e \right) - \frac{0.71}{e} \nabla_\parallel k_B T_e - N_e \vec{B}$$
(18)

Thus, our equations (17) and (18) correspond to HM Equations (137) and (138) . The additional terms in the momentum equation due to the gyroviscous stress are consistently dropped in HM where it is argued that since these terms are basically the perpendicular gradient of something, they are equivalent to the perpendicular pressure being slightly different and will not affect things in a fundamental way. The additional terms in the Ohm's law equation are associated with parallel electron temperature gradients and neoclassical effects.

5. Alternate formulation in terms of canonical momentum and generalized vorticity:

Consider now the FLR corrected momentum equations (8) and the generalized Ohm's law equation (11). The two terms $\vec{J} \times \vec{B} - \nabla P$ in Equation (11) almost cancel one another, and the first term leads to the whistler dynamics and the associated numerical problems. Equation (8) can be used to eliminate this term, to yield for the magnetic field evolution equation (without further approximation):

$$\frac{\partial \vec{B}^*}{\partial t} = \nabla\times\left[\vec{V}\times\vec{B}^* - \frac{m_i}{e}(\vec{V}_*\bullet\nabla)\vec{V}_\perp - \frac{1}{ne}\left(\nabla_\perp\tilde{\chi} + \upsilon_{gv\|}\right) - \frac{1}{ne}\nabla_\| p_i + \frac{0.71}{e}\nabla_\| k_B T_e + N_e\vec{B} - \eta\vec{J}^*\right]$$
(19)

where $\vec{B}^* \equiv \nabla\times(\vec{A}+\frac{m_i}{e}\vec{V})$ is the generalized vorticity, the curl of the canonical momentum [Steinhauer,1998]. Since $\vec{B} = \vec{B}^* - \frac{m_i}{e}\nabla\times\vec{V}$ and $\vec{J} = \nabla\times\vec{B}$, the FLR corrected momentum equation (8) can be written in terms of $\vec{B}^*$ as:

$$n_i m_i\left(\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\bullet\nabla\right)\vec{V} + (\vec{V}_*\bullet\nabla)\vec{V}_\perp\right) + \nabla_\perp\tilde{\chi} + \upsilon_{gv\|} + \nabla P = \left[\nabla\times\left(\vec{B}^* - \frac{m_i}{e}\nabla\times\vec{V}\right)\right]\times\left(\vec{B}^* - \frac{m_i}{e}\nabla\times\vec{V}\right) \qquad (20)$$

6.0 Dispersion Relation

Expanding around a homogeneous force-free equilibrium with $\nabla\times\vec{B} = \nabla P = \vec{V} = 0$, we can obtain the 2-fluid zero-pressure dispersion relation from either (8), (11) or (19), (20). Letting the background magnetic field and wave number have the forms (in Cartesian coordinates): $\vec{B}_0 = (0,0,B), \quad \vec{k} = (k_x, 0, k_z)$, we obtain:

$$\left[\frac{\omega^2}{V_A^2} - (k_x^2 + k_z^2)\right]\left[\frac{\omega^2}{V_A^2} - k_z^2\right] - \frac{\omega^2}{V_A^2}\left(\frac{V_A^2}{\Omega^2}\right)k_z^2(k_x^2 + k_z^2) = 0 \qquad (21)$$

where $V_A^2 = B^2/nm_i, \quad \Omega^2 = e^2B^2/m_i$. The two roots for ω^2/V_A^2, corresponding to the Hall modified fast wave (+) and shear Alfven wave (-) are given by:

$$\omega^2/V_A^2 = \frac{1}{2}\left[k_x^2 + 2k_z^2 + \frac{V_A^2}{\Omega^2}k_z^2\left(k_x^2 + k_z^2\right)\right] \pm \frac{1}{2}\left[k_x^4 + 2\frac{V_A^2}{\Omega^2}\left(k_x^2 + 2k_z^2\right)k_z^2\left(k_x^2 + k_z^2\right) + \frac{V_A^4}{\Omega^4}k_z^4\left(k_x^2 + k_z^2\right)^2\right]^{1/2}$$

The roots ω^2/V_A^2 are plotted in Figure 1 for values of the Hall parameter $V_A^2/\Omega^2 = 0$ and $V_A/\Omega = 0.2$ Note from the figure that when the Hall parameter vanishes, the two roots have their familiar form: $\frac{\omega^2}{V_A^2} = (k_x^2 + k_z^2)$ (fast wave) and $\frac{\omega^2}{V_A^2} = k_z^2$ (Alfven wave).

Further, when the Hall parameter is non-zero, and the wave number is large so that $k^2 \gg \left(\frac{V_A^2}{\Omega^2}\right)^{-2}$, then the roots take on the asymptotic forms:

$$\frac{\omega^2}{V_A^2} \sim \left(1+\frac{V_A^2}{\Omega^2}k_z^2\right)\left(k_x^2+k_z^2\right)+\ldots$$ (Hall modified fast wave) and

$$\frac{\omega^2}{V_A^2} \sim \left(\frac{V_A^2}{\Omega^2}\right)^{-1} - \left(\frac{V_A^2}{\Omega^2}\right)^{-2} \frac{\left(k_x^2+2k_z^2\right)}{k_z^2\left(k_x^2+k_z^2\right)}+\ldots$$ (Hall modified Alfven wave).

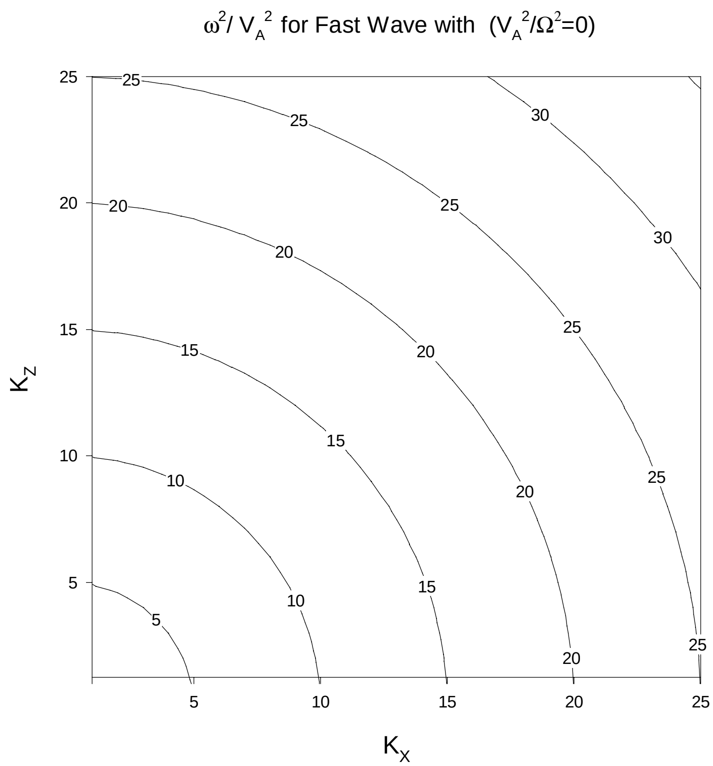

Figure 1a: Fast wave dispersion relation with no Hall term

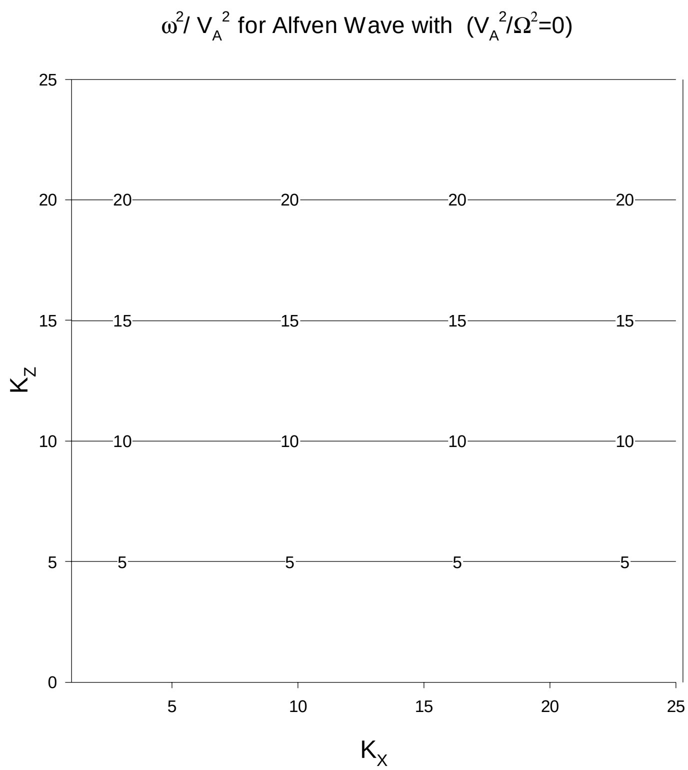

Figure 1b: Alfven wave dispersion relation with no Hall term

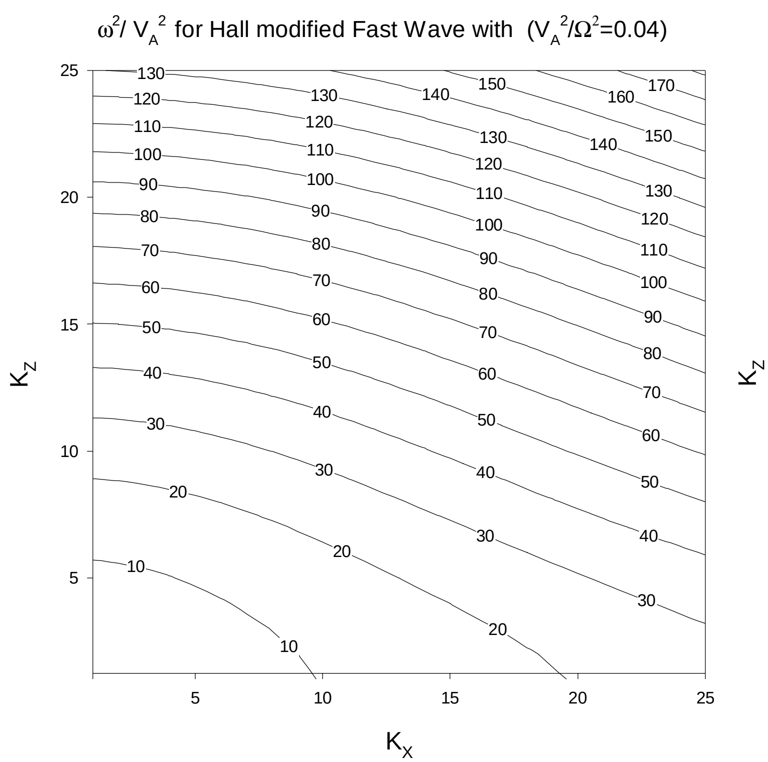

Figure 1c: Fast wave dispersion relation with Hall term

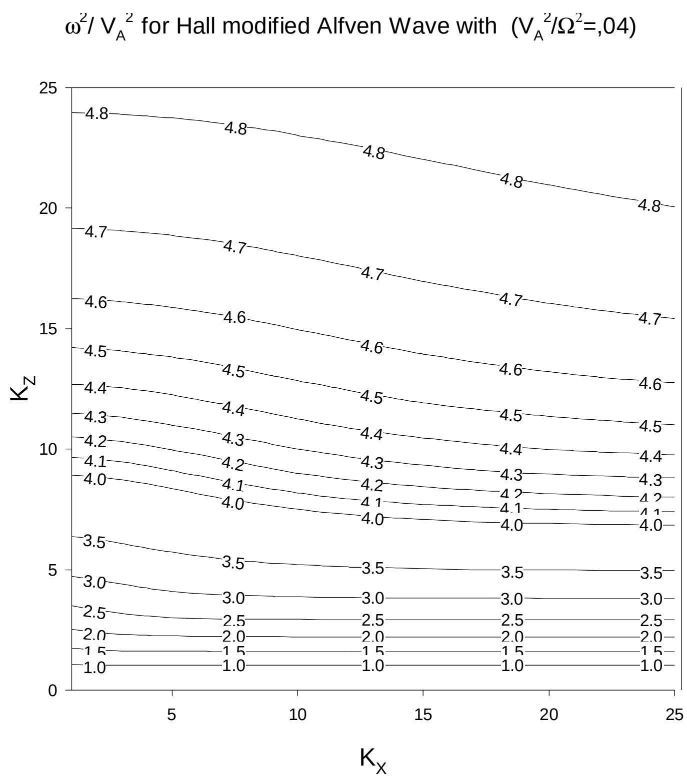

Figure 1d: Alfven wave dispersion relation with Hall term

References:

[Belova, 2001] E.V. Belova, “Nonlinear gyroviscous force in a collisionless plasma”, PPPL-3569, 2001, Phys. Plasmas, 2001

[Smolyakov,1998] A.I. Smolyakov, “Gyroviscous forces in a collisionless plasma with temperature gradients”, Can. J. Phys. **76**: 321-331 (1998)

[Hazeltine,1985] R.D. Hazeltine and J.D. Meiss, “Shear-Alfven Dynamics of Toroidally Confined Plasmas”, Physics Reports **121**: 1-164 (1985)

[Hazeltine,1991] R.D. Hazeltine and J.D. Meiss “Plasma Confinement”, Addison-Wesley, 1992

[Chang,1992] Z. Chang and J.D. Callen, “Generalized gyroviscous force and its effect on the momentum balance equation”, Phys. Fluids B **4**: 1766-1771 (1992)

[Sugiyama, 2000] L. E. Sugiyama and W. Park, “A nonlinear two-fluid model for toroidal plasmas”, Phys. Plasmas **7**: 4644-4658 (2000)

[Steinhauer, 1998] L. C. Steinhauer and A. Ishida, “Relaxation of a two-species magnetofluid and application to finite-β flowing plasmas”, Phys. Plasmas **5**: 2609-2622 (1998)