Nonlinear Gyroviscous Force in a Collisionless Plasma

E. V. Belova

PPPL

Motivation:

- calculate FLR corrections to pressure tensor in terms of particle distribution function
- no closure relation is assumed, no truncation

using gyrokinetic formalism:

advantage: reduced phase-space (5D),
expanded in ρ/L already

Gyrokinetic ordering:

$$\epsilon = \frac{\rho_i}{L} \sim \frac{\omega}{\omega_{ci}} \sim \frac{k_\parallel}{k_\perp}, \qquad \epsilon_\delta = \frac{\delta F}{F} \sim \frac{e\varphi}{T},$$

$$\epsilon \sim \epsilon_\delta \ll 1$$

$$\epsilon_\perp = (k_\perp \rho)^2 \sim 1$$

Assumptions:

- electrostatic, straight $\mathbf{B}$
- second order in ϵ and ϵ_δ
- $\epsilon_\perp = (k_\perp \rho)^2 \ll 1$

Previous work:

- **Braginskii gyroviscosity**

  collisional ordering [Braginskii'65]
  neglects gradients of heat fluxes in $(\nabla\cdot\boldsymbol{\pi}_{\mathbf{g}})$

- **Fluid calculations**

  [Hazeltine and Meiss'85, Chang and Callen'92, Smolyakov'98], include higher order terms (in ϵ), heat fluxes

- **Gyrofluid calculations**

  [Brizard'92, Dorland and Hammet'92]
  gyrokinetic Vlasov equation is used to derive a set of reduced gyrofluid (and particle fluid) FLR equations

- **For $\nabla T \neq 0$ all give different answers!**

OUTLINE:

- **Gyrokinetic formalism**
- **Gyrofluid equations**
- **Particle-fluid equations**
- **Example: first four fluid equations**
- **Parallel momentum equation,** $(\nabla\cdot\boldsymbol{\pi}_{\mathbf{g}})_{\parallel}$
- **"Perpendicular momentum equation",** $(\nabla\cdot\boldsymbol{\pi}_{\mathbf{g}})_{\perp}$
- **Two forms of gyroviscous force**
- **FLR particle closure**
- **Summary**

Gyrokinetic formalism

Gyrokinetic Vlasov equation (up to $O(\epsilon^2)$):
[Dubin'83]

$$\frac{\partial F}{\partial t} + \left(\hat{\mathbf{b}} U + \frac{1}{B} \hat{\mathbf{b}} \times \nabla \langle \Phi \rangle \right) \cdot \nabla F - \hat{\mathbf{b}} \cdot \nabla \langle \Phi \rangle \frac{\partial F}{\partial U} = 0$$

$\bar{\mathbf{Z}} \equiv (\mathbf{X}, U, \mu, \theta)$ - gyrocenter coordinates,
$F = F(\mathbf{X}, U, \mu)$ - gyrocenter distribution function.

$$\Phi = \varphi - \frac{1}{2B} \frac{\partial \, \tilde{\varphi}^2}{\partial \mu} + \frac{1}{2B^2} \hat{\mathbf{b}} \cdot \nabla \tilde{\Psi} \times \nabla \, \tilde{\varphi},$$

$$\tilde{\varphi} = \varphi - \langle \varphi \rangle, \qquad \tilde{\Psi} = \int^{\theta} \tilde{\varphi} \, d\theta'$$

Relation between gyrocenter and guiding-center variables (I)

- First order transformation (in ϵ_δ):

$$\bar{Z}^j(\mathbf{Z}) = Z^j + G_1^j,$$

where components of generating vector are (n =1 or 2):

$$\begin{aligned}
&\mathbf{G_n} = -\hat{\mathbf{b}}\partial S_n/\partial U - \tfrac{1}{B}\hat{\mathbf{b}} \times \nabla S_n \\
&G_n^U = \nabla_\| S_n \\
&G_n^\mu = \partial S_n/\partial\theta \\
&G_n^\theta = -\partial S_n/\partial\mu
\end{aligned}$$

and S_1 is a gauge function for the first order transformation [Cary and Littlejohn'83]

$$\frac{\partial S_1}{\partial t} \ + \ U\nabla_\| S_1 + B\frac{\partial S_1}{\partial\theta} = \tilde{\varphi}$$

$$\begin{aligned}
S_1^{(1)} \ &= \ \frac{1}{B}\,\tilde{\Psi} + O(\epsilon^2) \\
S_1^{(2)} \ &= \ -\frac{1}{B^2}\int^\theta \left(\frac{\partial\,\tilde{\Psi}}{\partial t} + U\ \nabla_\|\tilde{\Psi}\right) d\theta\,' + O(\epsilon^3)
\end{aligned}$$

Relation between gyro-center and guiding-center variables (II)

• Second order transformation (in ϵ_δ):

$$\bar{Z}^j(\mathbf{Z}) = Z^j + G_1^j + \frac{1}{2} G_1^i \frac{\partial G_1^j}{\partial Z^i} + G_2^j + O(\epsilon^3),$$

S_2 is a gauge function for the second order transformation:

$$\begin{aligned} B\frac{\partial S_2}{\partial \theta} = -\mathbf{G}_1 \cdot \nabla \bar{\varphi} - G_1^\mu \frac{\partial \bar{\varphi}}{\partial \mu} \; & - \; \frac{1}{2}\left[(\mathbf{G}_1 \cdot \nabla \; \tilde{\varphi}) + G_1^\mu \frac{\partial \; \tilde{\varphi}}{\partial \mu} + G_1^\theta \frac{\partial \varphi}{\partial \theta} \right] \\ & + \; \frac{1}{2}\left[\langle \mathbf{G}_1 \cdot \nabla \; \tilde{\varphi} \rangle + \langle G_1^\mu \frac{\partial \; \tilde{\varphi}}{\partial \mu} \rangle + \langle G_1^\theta \frac{\partial \varphi}{\partial \theta} \rangle \right] \end{aligned}$$

• Guiding-center distribution function F_{gc} can be expressed in terms of the gyrocenter distribution function, F

$$F_{gc}(\mathbf{z}) = F(\mathbf{Z}(\mathbf{z}))$$

General gyro-fluid moment equation

Gyro-fluid (GF) moment is defined as:

$$M_{kl}(\mathbf{X},t) \equiv \|\mu^k U^l\|^{GF} = \int \mu^k U^l F d\mu dU$$

From gyrokinetic Vlasov equation up to $O(\epsilon^2)$, $O(\epsilon_\perp)$ order,

for l - even:

$$\frac{d' M_{kl}}{dt} + \nabla_\| M_{kl+1} + \frac{1}{2B^2}\left\{\nabla_\perp^2 \varphi, M_{k+1l}\right\} = 0$$

for l - odd:

$$\begin{aligned}\frac{d' M_{kl}}{dt} + \nabla_\| \left(M_{kl+1} + l M_{kl-1}\varphi\right) \; &+ \; \frac{1}{2B^2}\left\{\nabla_\perp^2 \varphi, M_{k+1l}\right\} \\ &+ \; \frac{1}{2B} l M_{k+1l-1} \nabla_\| \nabla_\perp^2 \varphi = 0\end{aligned}$$

where

$$\frac{d'}{dt} \equiv \frac{\partial}{\partial t} + (\mathbf{V}_E \cdot \nabla), \qquad \{f, g\} \equiv \hat{\mathbf{b}} \cdot \nabla f \times \nabla g$$

Relation between gyrofluid and particle-fluid moments

General particle-fluid (PF) moment:

$$m_{kl}(\mathbf{x},t) \equiv \|\mu^k U^l\|^{PF} = \int (v_\perp^2/2B)^k (v_\|)^l \, f \, d^3\mathbf{v}$$

m_{kl} in terms of guiding-center distribution function, F_{gc}:

$$m_{kl}(\mathbf{x},t) = \int \mu^k U^l \, F_{gc}(\mathbf{Z}) \, \delta(\mathbf{X} + \boldsymbol{\rho} - \mathbf{x}) \, d^6\mathbf{Z}$$

Gyrokinetic transformation (needed up to $O(\epsilon)$):

$$F_{gc}(\mathbf{Z}) = T_{gy}(F) = F + \mathbf{G}_1 \cdot \nabla F + G_1^\mu \partial F/\partial\mu + G_1^U \partial F/\partial U$$

Substituting F_{gc} and G_1^j, one can express PF moment in terms of GF moments ($O(\epsilon_\perp)$):

$$m_{kl}(\mathbf{x},t) = \left(M_{kl} + \frac{1}{2B}\nabla_\perp^2 M_{k+1l} + \frac{(k+1)}{B^2} M_{kl} \nabla_\perp^2 \varphi \right)(\mathbf{x},t)$$

with $O(\epsilon_\perp)$ accuracy, the inverse:

$$M_{kl}(\mathbf{X},t) = \left(m_{kl} - \frac{1}{2B}\nabla_\perp^2 m_{k+1l} - \frac{(k+1)}{B^2} m_{kl} \nabla_\perp^2 \varphi \right)(\mathbf{X},t)$$

General particle-fluid moment equation

Expressing GF moments in terms of the PF moments in the gyrofluid set of equations, one can obtain with $O(\epsilon^2)$ and $O(\epsilon_\perp)$ accuracy

for l - even:

$$\frac{d' m_{kl}}{dt} = -\nabla_\| m_{kl+1} + m_{kl}(k+1)\frac{d'}{dt}\nabla_\perp^2 \varphi/B^2 - \nabla_\perp \cdot \{\nabla_\perp \varphi, m_{k+1l}\}/B^2$$

for l - odd:

$$\frac{d' m_{kl}}{dt} = -\nabla_\| m_{kl+1} \; - \; l m_{kl-1}\nabla_\| \varphi - \nabla_\perp \cdot \{\nabla_\perp \varphi, m_{k+1l}\}/B^2$$

$$+ \; \Big[(k+1) m_{kl+1}/B^2 - l m_{k+1l-1}/B\Big] \nabla_\| \nabla_\perp^2 \varphi$$

where $d'/dt = \partial/\partial t + (\mathbf{v}_E \cdot \nabla)$.
Note that particle-fluid equations exhibit a generalized form of "gyroviscous cancellation" (a cancellation of $(\mathbf{v}_* \cdot \nabla)$ term from the total time derivative on LHS).

Example: nonlinear reduced fluid equations $O(\epsilon^2)$

$$n = \|1\|^{PF} = m_{00}, \quad nv_\| = \|U\|^{PF} = m_{01},$$

$$p_\perp = \|B\mu\|^{PF} = Bm_{10}, \quad p_\| = \|U^2\|^{PF} = m_{02}, \quad \ldots$$

$$\frac{d'n}{dt} \;=\; -\nabla_\| \, (nv_\|) + \frac{n}{B^2}\frac{d'}{dt}\nabla_\perp^2\varphi - \nabla_\perp \cdot\{\nabla_\perp \varphi, p_\perp\}/B^3$$

$$\frac{d'p_\perp}{dt} \;=\; -\nabla_\| \, \|U\mu B\| + \frac{2p_\perp}{B^2}\frac{d'}{dt}\nabla_\perp^2\varphi - \nabla_\perp \cdot\{\nabla_\perp \varphi, \|\mu^2\|\}/B$$

$$\frac{d'p_\|}{dt} \;=\; -\nabla_\| \, \|U^3\| + \frac{p_\|}{B^2}\frac{d'}{dt}\nabla_\perp^2\varphi - \nabla_\perp \cdot\{\nabla_\perp \varphi, \|U^2\mu\|\}/B^2$$

$$n\frac{d'v_\|}{dt} \;=\; -\nabla_\| \, (p_\| + n\varphi) - \nabla_\perp \cdot\{\nabla_\perp \varphi, \|U\mu\|\}/B^2 + \frac{p_\| - p_\perp}{B^2}\,\nabla_\|\,\nabla_\perp^2\varphi$$

where

$$\|U\mu\|B = p_\perp v_\| + q_\|^{(\perp)}, \quad \|U^3\| = 3p_\| v_\| + 2q_\|^{(\|)},$$

$$\|\mu^2\|B^2 = \frac{2p_\perp^2}{n} + 2R_\perp, \quad \|U^2\mu\|B = \frac{p_\perp p_\|}{n} + R_\times$$

Parallel momentum equation, $(\nabla\cdot\,\boldsymbol{\pi}_{\mathbf{g}})_{\|}$

From FLR parallel momentum equation, one can find parallel component of gyroviscous force (in terms of PF moments):

$$
\begin{aligned}
n\frac{d'v_{\|}}{dt} &= -\,\nabla_{\|}\,(p_{\|}+n\varphi)-\,\nabla_{\perp}\cdot\{\nabla_{\perp}\,\varphi,\|U\mu\|\}/B^2+\frac{p_{\|}-p_{\perp}}{B^2}\,\nabla_{\|}\,\nabla_{\perp}^2\varphi \\
n\frac{dv_{\|}}{dt} &= -\,\nabla_{\|}\,p_{\|}-\,(\nabla\cdot\,\boldsymbol{\pi}_{\mathbf{g}})_{\|}\,-n\,\nabla_{\|}\,\varphi,
\end{aligned}
$$

where $\boldsymbol{\pi}_{\mathbf{g}}$ is defined by: $\mathbf{P}=\mathbf{P}^{CGL}+\,\boldsymbol{\pi}_{\mathbf{g}}\,+\rho\mathbf{v}\mathbf{v}$

$$
\begin{aligned}
(\nabla\cdot\,\boldsymbol{\pi}_{\mathbf{g}})_{\|}=-(n\mathbf{v}_*\cdot\nabla)v_{\|}\;&+\;\frac{1}{B^2}\,\nabla_{\perp}\cdot\{\nabla_{\perp}\,\varphi,\|U\mu\|\} \\
&-\;\frac{p_{\|}-p_{\perp}}{B^2}\,\nabla_{\|}\,\nabla_{\perp}^2\varphi
\end{aligned}
$$

- Parallel momentum equation agrees with direct fluid calculations of [Smolyakov'98] for $p_{\perp}^{(0)}=p_{\|}^{(0)}$.

- extra terms $\sim\delta T$ and parallel vorticity terms [Brizard'92, Chang and Callen'92] are cancelled out, when parallel heat fluxes contribution is consistently retained.

“Perpendicular momentum equation”, $(\nabla\cdot\boldsymbol{\pi}_{\mathbf{g}})_\perp$

Nonlinear reduced fluid equations can be derived by taking moments of the gyrokinetic Vlasov equation, and expressing them in terms of the particle-fluid moments. Accuracy of $O(\epsilon)$ is required.
Perpendicular component of momentum equation can not be derived in the same way (moment of $\int \mathbf{V}_\perp F\, dU\, d\mu d\theta$ vanishes). It can be recovered, when $n\mathbf{v}_\perp(\mathbf{x},t)$ is expressed via moments of the gyrocenter distribution function F through the second order in ϵ and ϵ_δ.

From

$$n\frac{d\mathbf{v}_\perp}{dt} = -\,\nabla_\perp\, p_\perp -\,(\nabla\cdot\boldsymbol{\pi}_{\mathbf{g}})_\perp + n(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}),$$

follows:

$$\begin{aligned}
\mathbf{v}_\perp^{(0)} &= \mathbf{v}_* + \mathbf{v}_E \\
\mathbf{v}_\perp &= \mathbf{v}_* + \mathbf{v}_E + \underbrace{\frac{1}{nB}\hat{\mathbf{b}}\times(\nabla\cdot\boldsymbol{\pi}_{\mathbf{g}}) + \frac{1}{B}\hat{\mathbf{b}}\times\frac{d\mathbf{v}_\perp^{(0)}}{dt}}_{O(\epsilon^2)}
\end{aligned}$$

where $\mathbf{v}_* = \hat{\mathbf{b}}\times\nabla p_\perp/(nB)$, and $d/dt = \partial/\partial t + \mathbf{v}_\perp^{(0)}\cdot\nabla$.
If particle-fluid velocity $n\mathbf{v}_\perp = \int \mathbf{V}_\perp f\, d^3V$ is calculated up to the second order, the perpendicular component of gyroviscous force can be found.

Calculation of $n\mathbf{v}_\perp$ (I)

$$n\mathbf{v}_\perp(\mathbf{x},t) \equiv \int \mathbf{w}_\perp f(\mathbf{w},\mathbf{x},t)\, d^3\mathbf{w} = \int \mathbf{V}_\perp F_{gc}\, \delta(\mathbf{X}+\boldsymbol{\rho}-\mathbf{x})\, d^6\mathbf{Z}$$

where the guiding-center distribution function, F_{gc}, can be expressed via gyro-center distribution function, F, up to the second order:

$$F_{gc} = F + G_1^j \frac{\partial F}{\partial Z^j} + \frac{1}{2} G_1^j \frac{\partial}{\partial Z^j}\left(G_1^i \frac{\partial F}{\partial Z^i}\right) + G_2^j \frac{\partial F}{\partial Z^j}$$

Keeping all nonlinear terms $\sim O(\epsilon^2, \epsilon\epsilon_\delta, \epsilon_\delta^2)$ and $O(\epsilon_\perp)$, we obtain:

$$\begin{aligned} n\mathbf{v}_\perp = &\quad \frac{1}{B}\hat{\mathbf{b}} \times \nabla P_\perp + N\mathbf{V}_E + \frac{1}{4B}\hat{\mathbf{b}} \times \nabla(\nabla_\perp{}^2\, \|\mu^2\|) \\ &- \frac{3}{2B}\hat{\mathbf{b}} \times \nabla\chi_\perp - \frac{N}{B^2}\left(\frac{\partial}{\partial t} + \mathbf{V}_* \cdot \nabla\right) \nabla_\perp\, \varphi \\ &+ \frac{1}{B^3}\hat{\mathbf{b}} \times \nabla(\nabla_\perp\, P_\perp \cdot \nabla_\perp\, \varphi) + \frac{1}{2B^2}\mathbf{V}_E(\nabla_\perp{}^2\, P_\perp) \\ &+ \frac{N}{2B^3}\hat{\mathbf{b}} \times \nabla(\nabla_\perp\, \varphi)^2 \end{aligned}$$

where RHS is written in terms of the gyro-fluid moments, and $N\mathbf{V}_* = \hat{\mathbf{b}} \times \nabla P_\perp / B$, and $\chi_\perp = -(P_\perp/B)\, \hat{\mathbf{b}} \cdot \nabla \times \mathbf{V}_E$.

Calculation of $n\mathbf{v}_\perp$ (II)

In terms of particle-fluid moments $n\mathbf{v}_\perp$ becomes:

$$\begin{aligned} n\mathbf{v}_\perp = \quad & n(\mathbf{v}_* + \mathbf{v}_E) - \frac{1}{4B}\hat{\mathbf{b}} \times \nabla({\nabla_\perp}^2 \, \|\mu^2\|) + \frac{1}{2B}\hat{\mathbf{b}} \times \nabla\chi_\perp \\ - \; & \frac{n}{B^2}\frac{d}{dt} \, \nabla_\perp \, \varphi + \frac{1}{2nB^3}\hat{\mathbf{b}} \times \nabla(\nabla_\perp \, p_\perp)^2 \end{aligned}$$

Therefore,

$$(\nabla \cdot \boldsymbol{\pi}_\mathbf{g})_\perp = -n\frac{d\mathbf{v}_*}{dt} + \nabla_\perp \, \tilde{\chi}$$

From FLR fluid equation for $p_\perp$, it can be shown that

$$\frac{d\mathbf{v}_*}{dt} = (\mathbf{v}_* \cdot \nabla)\mathbf{v}_\perp - \frac{1}{n}\hat{\mathbf{b}} \times \nabla(\nabla_\parallel \, \|U\mu\|)$$

which allows to write the perpendicular gyroviscous force in [Chang and Callen'92] form, as:

$$(\nabla \cdot \boldsymbol{\pi}_\mathbf{g})_\perp = -n(\mathbf{v}_* \cdot \nabla)\mathbf{v}_\perp + \hat{\mathbf{b}} \times \nabla(\nabla_\parallel \, \|U\mu\|) + \nabla_\perp \, \tilde{\chi}$$

where

$$\tilde{\chi} \equiv \frac{1}{2}\chi_\perp - \frac{1}{4} \, {\nabla_\perp}^2 \, \|\mu^2\| + \frac{1}{2nB^2}(\nabla_\perp \, p_\perp)^2$$

RESULTS (PF)

For $\boldsymbol{\pi}_{\mathbf{g}}$ defined by: $\boldsymbol{\pi}_{\mathbf{g}} = \mathbf{P} - \mathbf{P}^{CGL} - \rho\mathbf{v}\mathbf{v}$, in terms of PF moments, we have calculated up to second order in ϵ and ϵ_δ, and up to $O((k_\perp\rho_i)^2)$:

$$
\begin{aligned}
(\nabla\cdot\,\boldsymbol{\pi}_{\mathbf{g}})_\parallel &= -(n\mathbf{v}_*\cdot\nabla)v_\parallel + \frac{1}{B^2}\,\nabla_\perp\cdot\{\nabla_\perp\,\varphi, \|U\mu\|\} \qquad (1)\\
&\quad -\frac{p_\parallel - p_\perp}{B^2}\,\nabla_\parallel\,\nabla_\perp^2\varphi
\end{aligned}
$$

$$
\begin{aligned}
(\nabla\cdot\,\boldsymbol{\pi}_{\mathbf{g}})_\perp &= -(n\mathbf{v}_*\cdot\nabla)\mathbf{v}_\perp + \hat{\mathbf{b}}\times\nabla(\nabla_\parallel\,\|U\mu\|) + \nabla_\perp\,\tilde{\chi} \qquad (2)\\
\text{or}&\\
&= -n\frac{d\mathbf{v}_*}{dt} + \nabla_\perp\,\tilde{\chi} \qquad (3)
\end{aligned}
$$

For Maxwellian f: $\tilde{\chi} = -\frac{p_\perp}{2B}\hat{\mathbf{b}}\cdot\nabla\times\mathbf{v}_\perp - \frac{1}{2B^2}\,\nabla_\perp\cdot(p_\perp\,\nabla_\perp\,T_\perp)$.

Note that two forms (2) and (3) of perpendicular gyroviscous force are obtained, which are related by the equation for $p_\perp$.
For $\nabla_\perp\;T_\perp = 0$, these two expressions are equivalent to that of [Chang and Callen'92] and [Hazeltine and Meiss, 1985], respectively.
For $\delta T_\perp = \delta T_\parallel$, our results agree with calculations by Smolyakov [1998].

Gyroviscous stress tensor in terms of F

(FLR particle closure)

GF gyroviscous stress tensor defined as:

$$\mathbf{\Pi}_g = \mathbf{P} - P_\perp(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - P_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}}$$

it is related to $\boldsymbol{\pi}_g$

$$\begin{aligned} \mathbf{\Pi}_g = \boldsymbol{\pi}_g + \rho \mathbf{V}\mathbf{V} \; &+ \; \left(p_\perp - P_\perp - \rho V_\perp^2/2\right)(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) \\ &+ \; \left(p_\parallel - P_\parallel - \rho V_\parallel^2\right)\hat{\mathbf{b}}\hat{\mathbf{b}} \end{aligned}$$

where $p_{\perp(\parallel)}$ and $P_{\perp(\parallel)}$ are PF and GF pressure, respectively.

Expressing particle-fluid moments in terms of gyrofluid moments:

$$\begin{aligned} (\nabla \cdot \mathbf{\Pi_g}) \; &= \; (n\mathbf{V}_E \cdot \nabla)\mathbf{V} + \hat{\mathbf{b}} \times \nabla \left(\nabla_\parallel \|\mu U\|\right) \\ &+ \; \nabla_\perp \left(\nabla_\perp^2 \|\mu^2\|/4 - 3\chi_\perp/2 + n\mathbf{V}_* \cdot \mathbf{V}_E\right) \\ &+ \; \nabla_\parallel \left(\nabla_\perp^2 \|U^2 \mu\|/2B - \chi_\perp\right) \\ &+ \; \hat{\mathbf{b}} \left(\nabla_\perp \cdot \{\nabla_\perp \varphi \,, \|U\mu\|\} + \{\varphi \,, \nabla_\perp^2 \|U\mu\|\}/2\right)/B^2 \end{aligned}$$

where

$$\chi_\perp = -\frac{P_\perp}{B}\hat{\mathbf{b}} \cdot \nabla \times \mathbf{V}_E, \quad \| * \| = \int (*) F \, dU d\mu,$$

Summary

FLR corrections to the ion stress tensor have been derived through $O(\epsilon_\perp)$ order, and can be used both in kinetic and fluid calculations.

• FLR corrections are calculated using gyrokinetic approach.

• Kinetic expression for $(\nabla \cdot \mathbf{\Pi_g})$ allows to include higher order FLR corrections into numerical models with so-called particle closure - it is more accurate than usual drift-kinetic and cheaper than the gyrokinetic schemes.

• A set of nonlinear FLR reduced fluid equations has been obtained in a general form.

• Previous calculations of $(\nabla\cdot \boldsymbol{\pi}_\mathbf{g})$ for $T \neq const$ have been corrected. Our results are in general agreement with the direct fluid calculations by Smolyakov [1998].

• Disagreement between the previous direct fluid and gyrofluid calculations of $(\nabla\cdot \boldsymbol{\pi}_\mathbf{g})_\parallel$ has been resolved.