Role of conservation laws in Drift-MHD closures

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Outline

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Analytic investigations of the NTM threshold require an FLR fluid model

- FLR kinetic calculations show that the polarization drift is stabilizing for islands rotating between the ion diamagnetic and electric drift frequencies.

- In the nonlinear regime torque balance determines the electric drift frequency but not the island rotation frequency (Kuvshinov 98). Determination of the island rotation frequency requires the calculation of the flattening of the density and electric field across the island (transport problem). A fluid model is highly desirable for this calculation.

- Most available fluid models give qualitatively incorrect results for islands propagating in the stabilizing band of frequency.

- What do analytic solutions tell us about the ingredients for a successful closure?

- What are the numerical requirements on closures?

Analytic closures rely heavily on reduction

- Reduction refers to the use of time-scale orderings to simplify the equations of motion. MHD itself follows from the moment equations by reduction. Tokamak reduced MHD (Strauss) simplifies the dynamics further by eliminating the fast wave.

- Flute-reduced MHD (Drake and Antonsen 1983, Callen et al. 1986) avoids the aspect-ratio ordering underlying RMHD by using instead $k_\parallel \ll L^{-1} \ll k_\perp$. There results the vorticity equation

  $$\rho_m \frac{DU}{Dt} = \frac{B_0^2}{4\pi} \mathbf{B} \cdot \boldsymbol{\nabla} \left(\frac{J_\parallel}{B_0} \right) + 2(\mathbf{B}_0 \times \boldsymbol{\kappa}_0) \cdot \boldsymbol{\nabla} p;$$

  and Ohm's law

  $$\frac{B_0^2}{c} \frac{\partial \psi}{\partial t} + \mathbf{B}_0 \cdot \boldsymbol{\nabla} \phi = \frac{J_\parallel}{\sigma_\parallel};$$

- Reduction enables the calculation of cross-moments, thus reducing the burden of closure. For example,

  $$q_{e\perp} = -\frac{5}{2} \frac{p_e}{eB} \mathbf{b} \times \boldsymbol{\nabla} T_e.$$

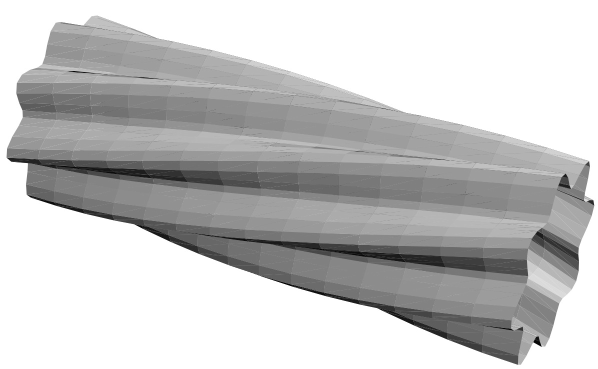
  Of course, reduction does not specify the pressure anisotropy $p_\parallel - p_\perp$

Flute-reduction is groovy, baby!

- Flute-reduction was recently used by Hegna to calculate the effect of Pfirsch-Schlüter currents on island stability in spherical tori.
- Reduction is physically compelling: it is unlikely that a fluid closure adequate in different frequency regimes exists!

However,

- Reduction is only locally applicable. For finite aspect-ratio, $k_\parallel \sim k_\perp$ away from mode-rational surfaces. The solutions of the reduced equations must be matched to perturbed equilibrium solutions outside the resonant layer(s).



Implicit algorithms are an alternative to reduction

- By its local nature, flute-reduction is poorly suited to numerical computation (especially in an RFP!).
- Implicit algorithms, like reduction, simplify the calculation by exploiting time-scale separation to eliminate fast oscillations. Implicit and reduced models should, in principle, be equivalent.
- It appears unlikely that flute-reduction can suggest improved implicit algorithms.
- At first sight, there are no apparent obstacles to using reduced closures in unreduced equations.

Part II:

Role of conservation laws in drift-MHD

Conservation laws and the integration miracle

- For MHD, the "miracle" is that $\boldsymbol{\nabla} \cdot J = 0$ can be integrated to yield Grad-Shafranov:

$$J_{\|} = -BI'(\psi) - 4\pi I(\psi)P'(\psi)/B.$$

- In some drift-MHD models, as many first-integrals can be found as there are equilibrium equations (one of the first-integrals is the GS Eq.).

- This can be seen to follow from the existence of "Casimir" invariants:

$$\begin{aligned} C_1 &= \int d^2x\, A(\psi) \qquad \text{(Frozen-in flux);} \\ C_2 &= \int d^2x\, UB(\psi) \qquad \text{(Vorticity consvn.);} \end{aligned}$$

- The equilibria are extremals of the energy + Casimirs:

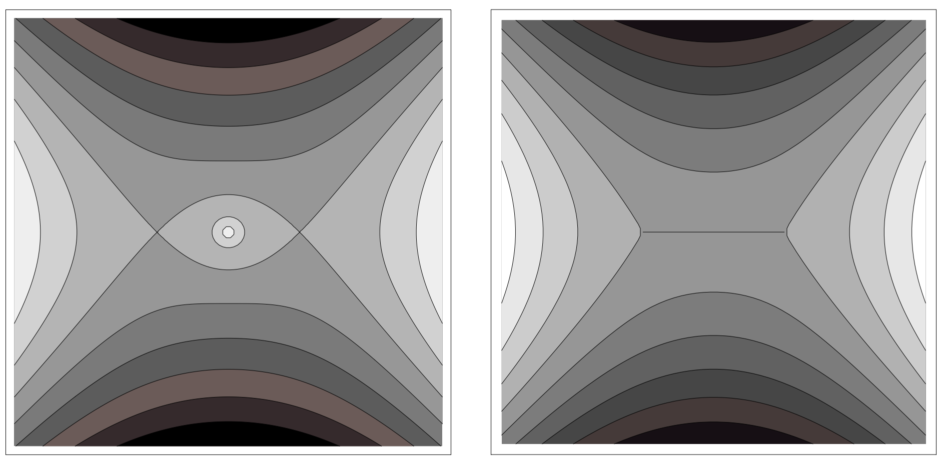
$$F = \int d^2x \left(|\boldsymbol{\nabla}\psi|^2 + |\boldsymbol{\nabla}\phi|^2 + A(\psi) + UB(\psi) \right).$$

- We conjecture that physical closures have integrable equilibrium equations.

- Conservation of the Casimirs determines all the profile functions during ideal and quasi-ideal processes, such as fast reconnection.

Role of conservation properties in MHD reconnection

- In MHD, rapid reconnection leads to the formation of current ribbons. (Park et al. 1986). Generalizing the nonlinear kink solutions of Rosenbluth, Dagazian and Rutherford shows that these current ribbons are a consequence of the frozen-in law (Waelbroeck 1989).



- Do conservation laws remain relevant in drift-MHD?

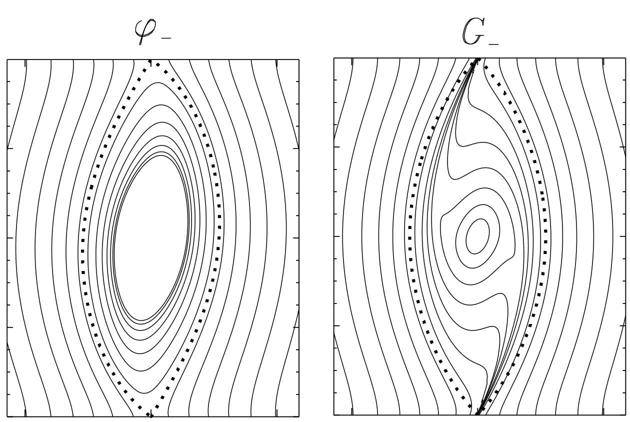
The role of conservation properties in drift-MHD reconnection

- In drift-MHD, the current ribbon persists, but its length is reduced to $\sim d_e$ (Rogers and Drake).
- An effective dissipation results from the mixing of the conserved quantities (Grasso, Califano, Pegoraro and Porcelli 2001):

$$\frac{\partial G_\pm}{\partial t} + \mathbf{v}_\pm \cdot \boldsymbol{\nabla} G_\pm = 0,$$

where

$$\begin{aligned} G_\pm &= \psi - d_e^2 J \pm d_e \rho_s U \\ \phi_\pm &= \varphi \pm (\rho_s/d_e)\psi. \end{aligned}$$



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Hamiltonian methods yield simple closures with guaranteed conservation laws!

- Finite ion Larmor-radius models are unwieldy affairs that generally do not yield integrable equilibrium equations and have fewer conservation laws than their $T_i = 0$ brethrens.

- Relatively simple FLR equations can be obtained, however, by use of a Poisson backet isomorphism (Hazeltine et al. 1987). This consists in taking the Poisson bracket for the Hamiltonian formulation of the $T_i = 0$ equation and shifting the ion velocity by the magnetization velocity.

- The bracket isomorphism picks out the important terms among all those obtained by expansion procedures.

- The situation is analogous to that for particle motion, where the guiding center equations lose the Hamiltonian property of the original equation. The Hamiltonian property can be regained by keeping selected higher order terms.

Summary

- Conservation laws play an important role in the phenomenological properties of fluid closures. They can both guide the formulation of improved closures and the solution of the resulting equations.
- Violation of the conservation laws, and thus of the integrability conditions for the equilibrium equations, could be responsible for the difficulties encountered in the numerical implementation of NeoMHD.
- Hamiltonian closures of the reduced equations can be used in the implicit equations, but the Hamiltonian property will most likely be lost.