Intrinsic ambipolarity and bootstrap currrents in stellarators

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1 Introduction

A basic and simple question that one may ask about a stellarator plasma is whether it should rotate – and, if so, in what direction and at what speed. One feels that a simple, macroscopic property like rotation, which is easily measured and potentially important for MHD stabilisation and turbulence suppression, should be predicted by theory. In this paper we investigate the basic question of how the magnetic configuration affects plasma rotation.

2 Rapid rotation

Nearly all theory of plasma confinement rests upon an expansion in the smallness of the ion gyroradius $\rho_i = v_{Ti}/\Omega_i$, compared with the macroscopic length scale, $\delta = \rho_i/L \ll 1$. It is therefore natural to distinguish between rapid plasma rotation, $V \sim v_{Ti}$, and slow rotation, $V \sim \delta v_i$. Rapid rotation is relatively easily dealt with, as it turns out that it is governed by a simple theorem [1]. If the magnetic field is written (locally) as

$$\mathbf{B} = \nabla\psi \times \nabla\alpha, \tag{1}$$

then rapid plasma rotation can only occur if the field strength in lowest order only depends on ψ and the arc length l along the field, i.e., if

$$B \simeq F(\psi, l) \tag{2}$$

for some function F. Moreover, the plasma flow velocity is then related to the radial electic field $\mathbf{E} = -\nabla\Phi(\psi)$ by the relation

$$\mathbf{V} = -\frac{d\Phi}{d\psi}\frac{\nabla\psi \times \nabla B}{\mathbf{B}\cdot\nabla B}, \tag{3}$$

so that the flow occurs along lines of constant magnetic field strength, $B = |\mathbf{B}|$. The condition (2) is satisfied by quasi-helically and quasi-axisymmetric fields. Conversely, one can show that this condition implies quasi-symmetry if the rotational transform is irrational [2].

The conditions (2) and (3) are approximate in the sense that they only need to be satisfied to lowest order in δ, but they follow directly *in zeroth order* from an expansion of the Vlasov equation. They are therefore valid in all collisionality regimes and are independent of the cross-field transport, whether this is neoclassial or turbulent.

3 Slow rotation

In most stellarator configurations, then, only slow rotation can occur, $V \sim \delta v_{Ti}$, and we now proceed to discuss what determines the rate of such rotation. Consider the momentum equation, summed over all plasma species,

$$\mathbf{J} \times \mathbf{B} - \nabla p = \nabla \cdot (\rho \mathbf{V}\mathbf{V} + \boldsymbol{\pi}) + \frac{\partial(\rho \mathbf{V})}{\partial t}, \tag{4}$$

where $\mathbf{J}$ is the plasma current, p the total (electron + ion) pressure, ρ the density, and $\boldsymbol{\pi}$ the viscosity tensor. We write $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$, $\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_1$ and $p = p_0(\psi) + p_1$, where

$$\mathbf{J}_0 \times \mathbf{B}_0 = p_0'(\psi)\nabla\psi, \tag{5}$$

and ψ is the toroidal flux. The quantities p_1, $\mathbf{B}_1$ and $\mathbf{J}_1$ thus represent turbulent fluctuations present in the plasma as well as any small deviation of the equilibrium from Eq. (5). Equation (4) now becomes

$$\frac{\partial(\rho \mathbf{V})}{\partial t} - \mathbf{J}_1 \times \mathbf{B}_0 = \mathbf{J}_0 \times \mathbf{B}_1 + \mathbf{J}_1 \times \mathbf{B}_1 - \nabla p_1 - \nabla \cdot (\rho \mathbf{V}\mathbf{V} + \boldsymbol{\pi}). \tag{6}$$

We multiply this equation by $\mathbf{B}_0$ and $\mathbf{J}_0$, respectively, and take the flux-surface average, $\langle \cdots \rangle$, giving

$$\frac{\partial \langle \rho \mathbf{V} \cdot \mathbf{J_0} \rangle}{\partial t} + \langle \mathbf{J}_1 \cdot \nabla p_0 \rangle = - \langle (\nabla \cdot \mathsf{S}) \cdot \mathbf{J}_0 \rangle, \tag{7}$$

$$\frac{\partial \langle \rho \mathbf{V} \cdot \mathbf{B_0} \rangle}{\partial t} = - \langle (\nabla \cdot \mathsf{S}) \cdot \mathbf{B}_0 \rangle, \tag{8}$$

where $\mathsf{S} = \rho \mathbf{V}\mathbf{V} + \boldsymbol{\pi} + \mathsf{M}$ is the total stress tensor, with

$$\mathsf{M} = \frac{1}{\mu_0}\left(\frac{\mathsf{B}_1^2}{2}\mathsf{I} - \mathbf{B_1}\mathbf{B_1} \right),$$

the Maxwell stress associated with $\mathbf{B}_1$. Equations (7) and (8) are exact: no approximations have been made, but in Eq. (7) the term

$$\langle \mathbf{J}_1 \cdot \nabla p_0 \rangle = - \frac{p_0'(\psi)}{\mu_0 c^2} \frac{\partial \langle \mathbf{E} \cdot \nabla\psi \rangle}{\partial t}$$

can be neglected, since it is of order $(v_A/c)^2$ times the left-hand side. Equations (7) and (8) govern the flow in the two "natural" directions within the flux surface, tangentially to $\mathbf{J}_0$ and $\mathbf{B}_0$ respectively, and show that this flow is damped or driven by Reynolds stress, $\rho \mathbf{V}\mathbf{V}$, viscous stress, $\boldsymbol{\pi}$, and Maxwell stress, M. In a turbulent plasma, these stresses can be decomposed into average and fluctuating parts, e.g.,

$$\boldsymbol{\pi} = \boldsymbol{\pi}_{\rm nc} + \boldsymbol{\pi}_{\rm turb},$$

where the average part is determined by neoclassical theory. The fluctuating part depends on the nature of the turbulence, which we shall assume obeys the conventional assumptions made in gyrokinetics. We thus take $k_\perp \rho_i \sim 1$ and for each ion species a we assume

$$\frac{V}{v_{Ta}} \sim \frac{\tilde{f}_a}{f_a} \sim \frac{B_1}{\beta B_0} \sim \frac{e_a \tilde{\phi}}{T_a} \sim \delta, \tag{9}$$

where e_a is the charge, $v_{Ta} = (2T_a/m_a)^{1/2}$ the thermal velocity and $\tilde{\phi}$ is the fluctuating electrostatic potential. The distribution function is denoted by f_a and its turbulent fluctuation by $\tilde{f}_a$. It follows from the assumptions (9) that the various turbulent stresses in Eqs. (7) and (8) are of order

$$\begin{aligned}\nabla \cdot (\rho \mathbf{V}\mathbf{V})_{\text{turb}} &\sim k_\perp \delta^2 p, \\ \nabla \cdot \boldsymbol{\pi}_{\text{turb}} &\sim k_\perp \delta p, \\ \nabla \cdot \mathsf{M}_{\text{turb}} &\sim k_\perp \delta^2 \beta p.\end{aligned} \tag{10}$$

The Reynolds and Maxwell stresses are therefore small in comparison with the viscous stress. However, they are comparable to the viscous stress produced by neoclassical effects in the absence of turbulence,

$$\nabla \cdot \boldsymbol{\pi}_{\text{nc}} \sim \delta p/L,$$

where we have recognised that the neoclassical stress tensor,

$$\boldsymbol{\pi}_{\text{nc}} = (p_\parallel - p_\perp)(\mathbf{bb} - \mathsf{I}/3) + O(\delta^2 p), \tag{11}$$

varies on the macroscopic length scale L rather than the turbulent length scale $k_\perp^{-1} \sim \rho_i$. Here $p_\parallel - p_\perp \sim \delta p$ is the pressure anisotropy and $\mathbf{b} = \mathbf{B}/B$ is the unit vector along the magnetic field. Locally, the largest force thus comes from the turbulent fluctuating viscous force $\nabla \cdot \boldsymbol{\pi}_{\text{turb}}$. Nevertheless, it is still the neoclassical viscosity (11) that determines the plasma rotation on large scales.

To establish this result, we recall that the viscosity tensor for each species is equal to [3]

$$\boldsymbol{\pi}_a = \boldsymbol{\pi}_{a\parallel} + \boldsymbol{\pi}_{ag},$$

where $\boldsymbol{\pi}_{a\parallel}$ is of the form (11), we have neglected the collisional cross-field viscosity, and the gyroviscosity is given by

$$\boldsymbol{\pi}_{ag} = \frac{1}{\Omega}\left[\mathbf{b} \times \mathsf{K} \cdot (\mathsf{I} + 3\mathbf{bb}) - (\mathsf{I} + 3\mathbf{bb}) \cdot \mathsf{K} \times \mathbf{b}\right],$$

$$\mathsf{K} = \nabla \cdot \left(m_a \int \mathbf{vvv} f_a d^3 v \right),$$

in leading order. The average (neoclassical) part of the gyroviscosity is thus smaller than $\boldsymbol{\pi}_{a\parallel}$ whilst the turbulent part is locally of the order indicated in Eq. (10). However, since the turbulent gyroviscosity involves the gradient of $\tilde{f}$, its local value is much larger than its average over any length scale exceeding the gyroradius.

Keeping this result in mind, we now integrate Eqs. (7) and (8) over the volume ΔV between two flux surfaces, ψ_1 and ψ_2, several gyroradii apart but still close to each other on the macroscopic length scale, so that the distance between them Δr satisfies

$$\rho_i \ll \Delta r \ll L.$$

This gives

$$\frac{\partial}{\partial t} \int_{\Delta V} \rho \mathbf{V} \cdot \mathbf{G} \, dV = - \left[V'(\psi) \left\langle \mathbf{G} \cdot \mathsf{S} \cdot \nabla \psi \right\rangle \right]_{\psi_1}^{\psi_2} + \int_{\Delta V} \mathsf{S} : \nabla \mathbf{G} \, dV, \tag{12}$$

where $V(\psi)$ is the volume enclosed by the flux surface ψ and $\mathbf{G}$ denotes either $\mathbf{J}_0$ or $\mathbf{B}_0$. Because the gyroviscosity has a small spatial average, as discussed above, it does not contribute much to Eq. (12). If $\Delta V \ll V$, the contribution from the turbulent Reynolds and Maxwell stresses is dominated by the first term on the right, which is of order

$$\left[V'(\psi)\left\langle \mathbf{G} \cdot (\rho \mathbf{V}\mathbf{V} + \mathsf{M}) \cdot \nabla\psi \right\rangle\right]_{\psi_1}^{\psi_2} \sim \delta^2 p |G| L^2,$$

and exceeds the corresponding second term on the right by a factor of about $L/\Delta r$. It is therefore much smaller than the contribution from parallel viscosity

$$\int_{\Delta V} \boldsymbol{\pi}_\parallel : \nabla \mathbf{G}\, dV \sim \delta p |G| L \Delta r.$$

We can thus conclude that, in leading order, the macroscopic rotation is determined by parallel viscosity alone,

$$\frac{\partial}{\partial t}\int_{\Delta V} \rho \mathbf{V} \cdot \mathbf{G}\, dV \simeq \int_{\Delta V} \boldsymbol{\pi}_\parallel : \nabla \mathbf{G}\, dV, \tag{13}$$

On shorter length scales, turbulent Reynolds and Maxwell stresses may affect the rotation and give rise to zonal flows, but the large-scale rotation is governed by parallel viscosity. The only exception occurs if its contribution to Eq. (13) for some reason vanishes in leading order, so that it becomes $\boldsymbol{\pi}_\parallel \sim \delta^2 p$ instead of $\boldsymbol{\pi}_\parallel \sim \delta p$. This can only happen in magnetic configurations that are intrinsically ambipolar, since when $\mathbf{G} = \mathbf{J}$, the right-hand side of Eq. (13) represents the neoclassical radial current,

$$\langle \mathbf{J} \cdot \nabla\psi \rangle = \left\langle \boldsymbol{\pi}_\parallel : \nabla \mathbf{G} \right\rangle / p_0'(\psi),$$

as follows from Eq. (7) in steady state. But intrinsic ambipolarity holds if [4] and only if [5] the magnetic field is quasisymmetric, and we can thus conclude that only then is stellarator rotation tokamak-like in the sense that gyrokinetic turbulence can affect the plasma rotation. Otherwise the rotation is determined by neoclassical theory on radial length scales exceeding ρ_i.

4 Quasi-isodynamic configurations

Having established that plasma rotation is governed by neoclassical theory in most stellarators, we finally investigate the consequences for quasi-isodynamic configurations [6]. In a quasi-isodynamic magnetic field the cross-field drift vanishes on a time average, i.e., the field is omnigenous, and the trapped particles precess poloidally around the torus rather than toroidally or helically. In such a field, it can be shown [7] that at low collisionality the solution of the first-order drift kinetic equation consists of two terms: a tokamak-like term and a term specific to stellarators, i.e.,

$$f_a = \left(1 + \frac{e_a \phi_1}{T_a}\right) f_{a0} + f_{at} + f_{as},$$

with

$$f_{at} = g_a + \frac{\mu_0 J(\psi) v_\parallel}{2\pi \Omega_a}\frac{\partial f_{a0}}{\partial \psi},$$

$$f_{as} = -\frac{\partial f_{a0}}{\partial \psi}\frac{\partial}{\partial \alpha}\int_B^{\min(B_{\max},\lambda^{-1})} h \frac{\partial}{\partial B'}\left(\frac{v_\parallel'}{\Omega_a'}\right) dB'$$

Here $J(\psi)$ denotes the toroidal current enclosed by the flux surface labelled by ψ, $B_{\max}(\psi)$ denotes the maximum magnetic field strength on that surface, $\lambda = v_\perp^2/v^2B$, $\Omega_a' = e_aB'/m_a$, $v_\parallel' = \sigma v(1-\lambda B')^{1/2}$, the function h is defined by

$$-\frac{(\mathbf{B}\times\nabla\psi)\cdot\nabla B}{\mathbf{B}\cdot\nabla B} = \frac{\mu_0 J(\psi)}{2\pi} + \frac{\partial h}{\partial\alpha},$$

and the function g_a is determined by a kinetic equation that is identical to that solved in the theory for banana-regime transport in tokamaks. More importantly, the entire tokamak-like term f_{at} is proportional to the enclosed toroidal current $J(\psi)$, whilst the stellarator term, f_{as}, is independent of the collision operator and does not carry any net toroidal current. It follows that if the total toroidal current enclosed by a certain flux surface vanishes, then the bootstrap current on that surface also vanishes. Quasi-isodynamic stellarators are thus inherently current-free: if one does not specifically drive a current (Ohmically or non-inductively), then there is no net bootstrap or Pfirsch-Schlüter current either. In fact, the current carried by each species then vanishes separately on a flux-surface average. There is therefore *no net toroidal rotation*. Locally, the flow velocity is order δv_{Ti}, but the flow passing through any poloidal section of the torus is much smaller.

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