Max-Planck-Institut für Plasmaphysik

Intrinsic ambipolarity and bootstrap currents in stellarators

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Questions

- Can a stellarator plasma rotate?
- Effect of turbulence?
- If so,how quickly?
  - Fast rotation

$$V \sim v_{Ti} = \text{ion thermal speed}$$

  - Slow rotation

$$V \sim \delta v_{Ti}, \qquad \delta = \frac{\rho_i}{L}$$

Tokamak rotation

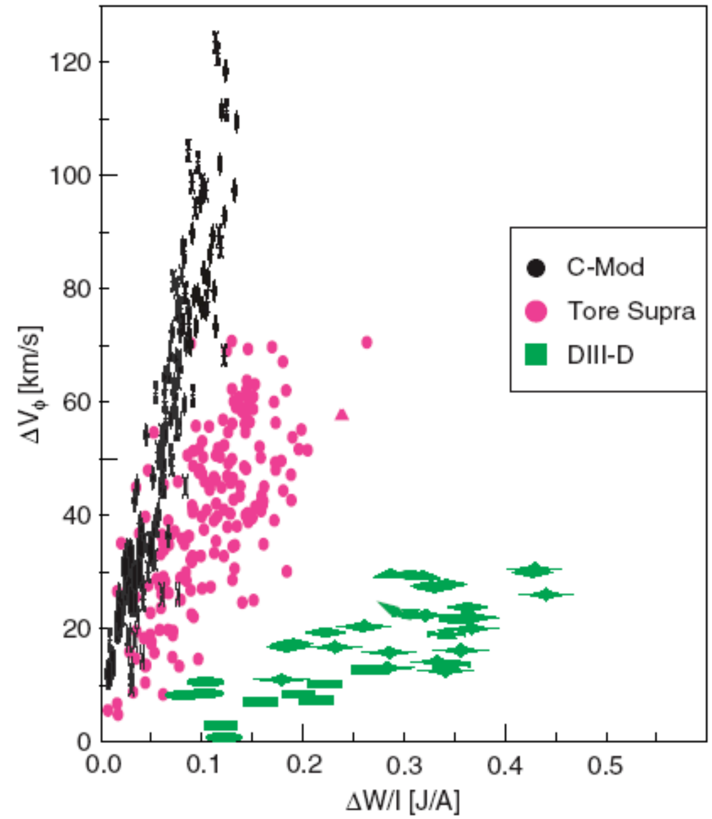
- Tokamak plasmas rotate freely, even spontaneously, in the toroidal direction.
  - Mach numbers ~1/3, up to ~1 in spherical tokamaks with unbalanced NBI

- Poloidal rotation much slower
  - Damped by collisions
  - Friction between trapped and passing ions

- Toroidal rotation determined by E_r

$$\mathbf{V} = u(\psi)\mathbf{B} - \left(\frac{d\Phi}{d\psi} + \frac{1}{ne}\frac{dp_i}{d\psi}\right) R\hat{\varphi}$$

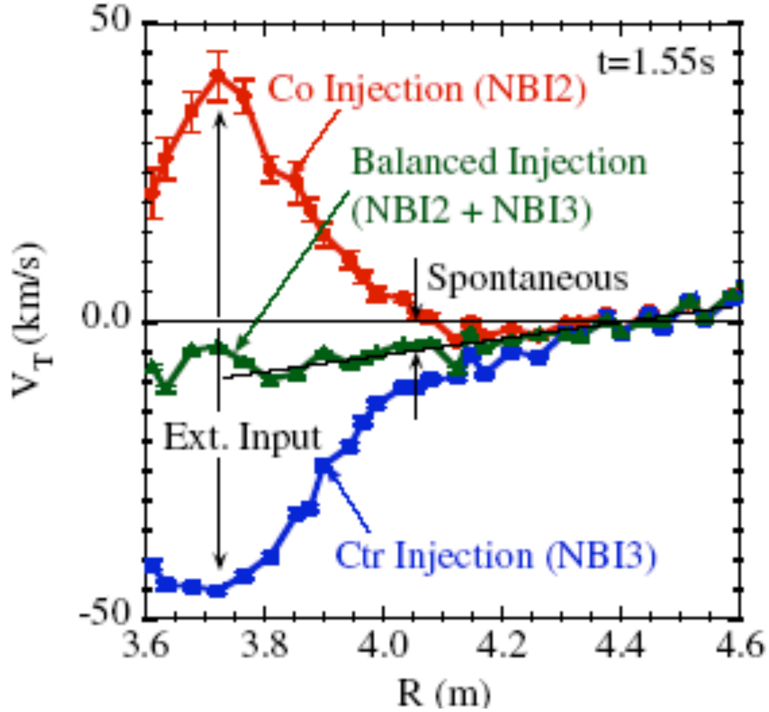
small



Rice et al, Nucl. Fusion 2007

Rotation in LHD

- Much smaller rotation in LHD



Yoshinumo et al, Nucl. Fusion 2009

Fast rotation in stellarators

- **Theorem:** fast rotation is only possible in quasi-symmetric configurations, and then only in the direction quasisymmetry.

- Follows from the 0th-order (!) drift kinetic equation
  - requires only small gyroradius and fluctuations
  - independent of collisionality and turbulence

- Mathematically, the Vlasov equation

$$\frac{\partial f}{\partial t} + (\mathbf{V}+\mathbf{v})\cdot\nabla f + \frac{e}{m}\left(\mathbf{E} + (\mathbf{V}+\mathbf{v})\times\mathbf{B} - \frac{\partial \mathbf{V}}{\partial t} - (\mathbf{V}+\mathbf{v})\cdot\nabla\mathbf{V}\right)\cdot\frac{\partial f}{\partial \mathbf{v}} = C(f)$$

and the ordering

$$\Omega^{-1}\frac{\partial}{\partial t} \ll \rho_i \nabla \ll 1$$

Fast rotation in stellarators

- **Theorem:** fast rotation is only possible in quasi-symmetric configurations, and then only in the direction quasisymmetry.

- Follows from the 0th-order (!) drift kinetic equation
  - requires only small gyroradius and fluctuations
  - independent of collisionality and turbulence

- Mathematically, the drift kinetic equation

$$\frac{\partial f_0}{\partial t} + (v_{\|}\mathbf{b} + \mathbf{V}) \cdot \nabla f_0 + \dot{w}\frac{\partial f_0}{\partial w} = C(f_0)$$

$$\mathbf{V}(\mathbf{r}, t) = V_{\|}\mathbf{b} + \frac{\mathbf{B} \times \nabla \Phi_0(\psi)}{B^2}$$

$$\dot{w} = eE_{\|}v_{\|} - mv_{\|}\mathbf{V} \cdot \nabla \mathbf{V} \cdot \mathbf{b} - mv_{\|}^2\mathbf{b} \cdot \nabla \mathbf{V} \cdot \mathbf{b} + \mu B\mathbf{V} \cdot \nabla \ln B$$

- implies that
  - either V << ion thermal speed
  - or B is quasi-symmetric + O(δ) corrections

Slow rotation

Projections of the momentum equation

$$\frac{\partial \langle \rho \mathbf{V} \cdot \mathbf{J} \rangle}{\partial t} = - \langle \nabla \cdot (\rho \mathbf{V}\mathbf{V} + \pi + \mathsf{M}) \cdot \mathbf{J} \rangle$$
$$\frac{\partial \langle \rho \mathbf{V} \cdot \mathbf{B} \rangle}{\partial t} = - \langle \nabla \cdot (\rho \mathbf{V}\mathbf{V} + \pi + \mathsf{M}) \cdot \mathbf{B} \rangle$$

where

$$\rho \mathbf{V}\mathbf{V} = \text{Reynolds stress}$$

$$\pi = \text{viscous stress}$$

$$\mathsf{M} = \frac{1}{\mu_0} \left(\frac{\tilde{\mathsf{B}}^2}{2} \mathsf{I} - \tilde{\mathbf{B}}\tilde{\mathbf{B}} \right) = \text{Maxwell stress}$$

Gyrokinetics

The gyrokinetic ordering

$$\frac{V}{v_{Ta}} \sim \frac{\tilde{f}_a}{f_a} \sim \frac{\tilde{B}}{\beta B} \sim \frac{e_a \tilde{\phi}}{T_a} \sim \delta \ll 1, \qquad k_\perp \rho_i = O(1)$$

gives

$$\nabla \cdot \pi_{\rm neocl} \sim \nabla \cdot (\rho \mathbf{V} \mathbf{V})_{\rm turb} \sim \nabla \cdot \mathsf{M}_{\rm turb} \ll \nabla \cdot \pi_{\rm turb}$$

$$\pi_{\rm neocl} = (p_{\|} - p_\perp)(\mathbf{bb} - \mathsf{I}/3) \sim O(\delta p)$$

The turbulent gyroviscous force dominates locally.

Global rotation

- However, on a volume average over a volume ΔV
  - between two flux surface several gyroradii apart

$$\rho \ll \Delta r \ll r$$

  - neoclassical viscosity dominates

$$\frac{\partial}{\partial t} \int_{\Delta V} \rho \mathbf{V} \cdot \begin{pmatrix} \mathbf{J} \\ \mathbf{B} \end{pmatrix} dV \simeq \int_{\Delta V} \pi_{\mathrm{neocl}} : \nabla \begin{pmatrix} \mathbf{J} \\ \mathbf{B} \end{pmatrix} dV$$

- Hence, gyrokinetic turbulence cannot affect macroscopic rotation!
  - one exception…

Intrinsic ambipolarity

- The radial neoclassical current

$$\langle \mathbf{J} \cdot \nabla \psi \rangle = \langle \pi_{\|} : \nabla \mathbf{J} \rangle / p'(\psi)$$

vanishes in case of intrinsic ambipolarity (for any E_r).

- In what configurations does intrinsic ambipolarity hold?

- **Theorem:** B is intrinsically ambipolar if, and only if, it is quasisymmetric.

Boozer, PoP 1983

Intrinsic ambipolarity = quasisymmetry

- **Proof:** From the drift kinetic equation

$$v_\| \nabla_\| \bar{f}_{a1} + \mathbf{v}_d \cdot \nabla f_{a0} = C_a(\bar{f}_{a1}) \qquad (1)$$

In an isothermal plasma follows an entropy production law

$$\phi_0'(\psi) \langle \mathbf{J} \cdot \nabla\psi \rangle = \sum_a T_{a0} \left\langle \int d^3v \bar{f}_{a1} C_a(\bar{f}_{a1})/f_{a0} \right\rangle \leq 0.$$

But then the H theorem implies

$$\langle \mathbf{J} \cdot \nabla\psi \rangle = 0 \quad \Rightarrow \quad \bar{f}_{a1} = (\alpha_a + \beta_a v_\| + \gamma_a v^2) f_{a0}$$

Re-insert into (1):

$$\nabla_\| \left(\frac{(\mathbf{B} \times \nabla\psi) \cdot \nabla \ln B}{\nabla_\| B} \right) = 0 \qquad \Rightarrow$$

B is quasisymmetric

Skovoroda and Shafranov, 1995

Summary so far

- Large-scale rotation determined by neoclassical theory
  - E_r set by ambipolarity
  - is much easier to calculate in stellarators than in tokamaks!

- Gyrokinetic turbulence unimportant except on short radial length scales
  - zonal flows

- Exception: quasisymmetric B
  - free (even sonic) rotation in the symmetry direction
  - E_r set by momentum transport (neoclassical + turbulent)

Proximity to quasisymmetry

- How close to perfect quasisymmetry must a stellarator come, in order to have tokamak-like rotation?
  - depends on collisionality and on the radial length scale considered

- In the 1/ν regime, the diffusion coefficient is

$$D \sim \epsilon_h^{3/2} \delta^2 \frac{T_i}{m_i \nu_i}$$

- The current becomes

$$\langle \mathbf{j} \cdot \nabla \psi \rangle \sim \epsilon_h^{3/2} \delta_i^2 \frac{p_i \Omega_i}{\nu_i}$$

and its torque exceeds the Reynolds stress, averaged over the volume between two flux surfaces N ion gyroradii apart if

$$\epsilon_h > \left(\frac{\nu_*}{N} \right)^{2/3}$$

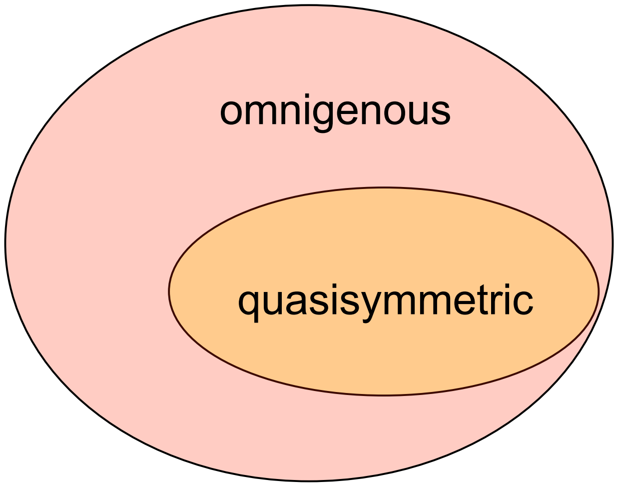
Terminology

Special magnetic fields:

- Quasisymmetric
  - |B| symmetic in Boozer coordinates:

$$B = B(\psi, m\theta - n\varphi)$$

  - Neoclassical properties identical to those in a tokamak
  - Isomorphism between stellarator and tokamak drift kinetic equations (Boozer 1984)

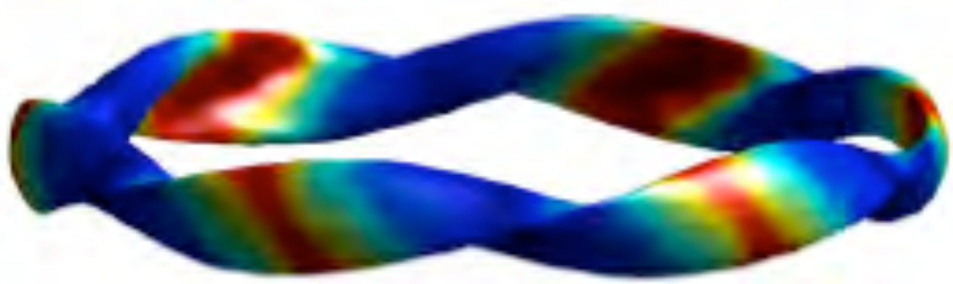


- Omnigenous (Hall and McNamara 1972)
  - No radial magnetic drift on a bounce average

$$\int_0^{\tau_b} \mathbf{v}_d \cdot \nabla\psi \; dt = 0$$

Terminology, cont‘d

- Quasi-isodynamic fields (Nührenberg) are, by definition, those that
  - are omnigenous and
  - have poloidal precession of trapped particles
- Examples
  - W7-X at high beta
  - More recently found configurations (Subbotin et al 2006)

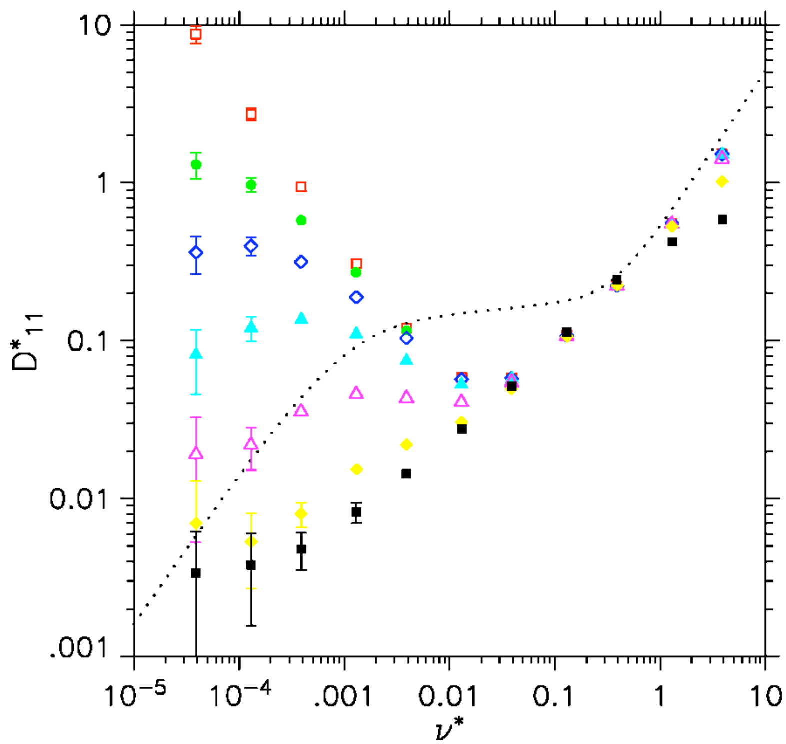


- What can we say about rotation in such configurations?

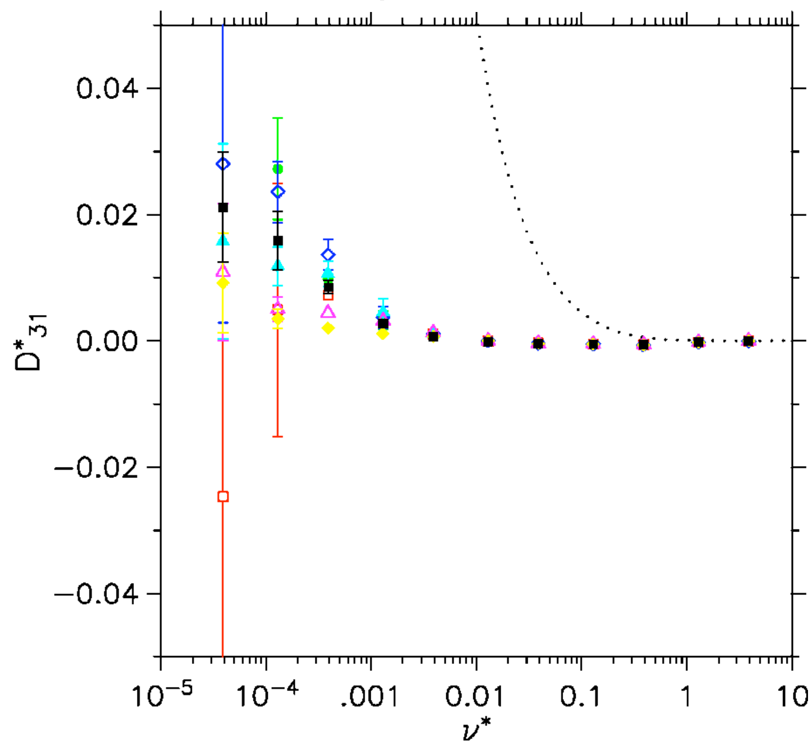
Numerical calculation

DKES calculation with the usual approximations:

Particle diffusivity



Bootstrap current coefficient



dashed lines = equivalent (elongated) tokamak

Orbits

It can be shown that the radial departure from a flux surface is

$$\Delta\psi = -\frac{\mu_0 I(\psi)}{2\pi}\frac{v_\parallel}{\Omega_a} + \frac{\partial}{\partial\alpha}\int_B^{B_0} h(\psi,\alpha,B)\frac{\partial}{\partial B'}\left(\frac{v_\parallel}{\Omega_a'}\right)dB'$$

$$= \Delta\psi_{\rm tok} + \Delta\psi_{\rm stell}$$

with B_o a reference field, and *h* a function encapsulating geometric information

$$\mathbf{B} = \nabla\psi \times \nabla\alpha$$

$$\frac{\partial h}{\partial\alpha} = -\frac{(\mathbf{B}\times\nabla\psi)\cdot\nabla B}{\mathbf{B}\cdot\nabla B} - \frac{\mu_0 I(\psi)}{2\pi}$$

$$I(\psi) = \text{toroidal current enclosed by } \psi$$

in Boozer coordinates.

Drift kinetic equation

First-order drift kinetic equation for species a

$$v_\parallel \nabla_\parallel f_{a1} + \mathbf{v}_d \cdot \nabla f_{a0} + \frac{e_a v_\parallel \nabla_\parallel \phi_1}{T_a} f_{a0} = C_a(f_{a1})$$

Solution at low collisionality:

$$f_a = \left(1 - \frac{e_a \phi_1}{T_a}\right) f_{a0} + f_a^{\rm stell} + f_a^{\rm tok}$$

$$f_a^{\rm stell} = -\Delta\psi_{\rm stell} \frac{\partial f_{a0}}{\partial \psi}$$ ← carries no net toroidal flow

$$f_a^{\rm tok} = g_a - \Delta\psi_{\rm tok} \frac{\partial f_{a0}}{\partial \psi}$$ ← proportional to the enclosed toroidal current I(ψ)

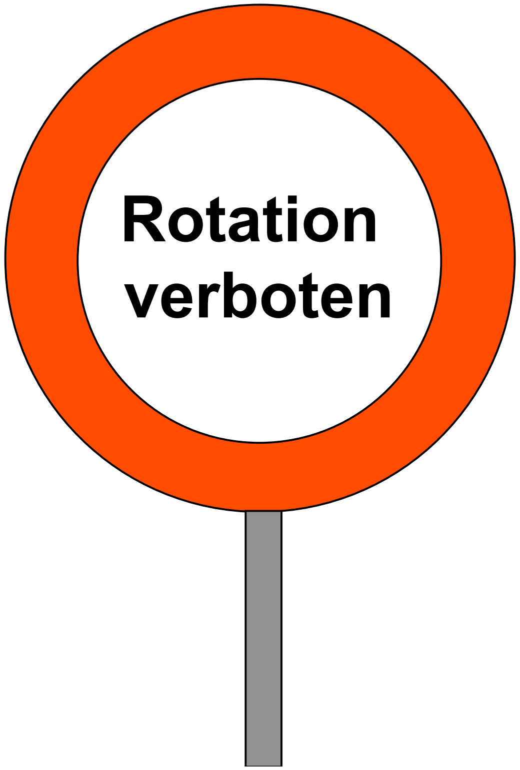
where g_a satisfies an equation identical to that solved in tokamak theory:

$$\int_{l_-(B_{\max})}^{l_+(B_{\max})} C_a\left(\frac{\mu_0 I v_\parallel}{\Omega_a}\frac{\partial f_{a0}}{\partial \psi} + g_a\right)\frac{dl}{v_\parallel} = 0$$

Currents and toroidal rotation

- Therefore, in the absence of external current drive
  - the net bootstrap current = 0
  - the net Pfirsch-Schlüter current = 0
- In fact, the net toroidal toroidal rotation of each species = 0
  - toroidal rotation is very slow

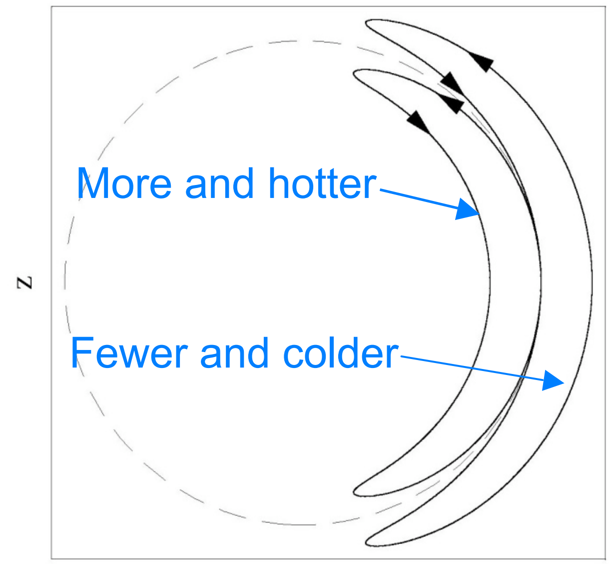
$$V_{\text{net}} \sim \delta^2 v_{Ti}$$



Physical picture

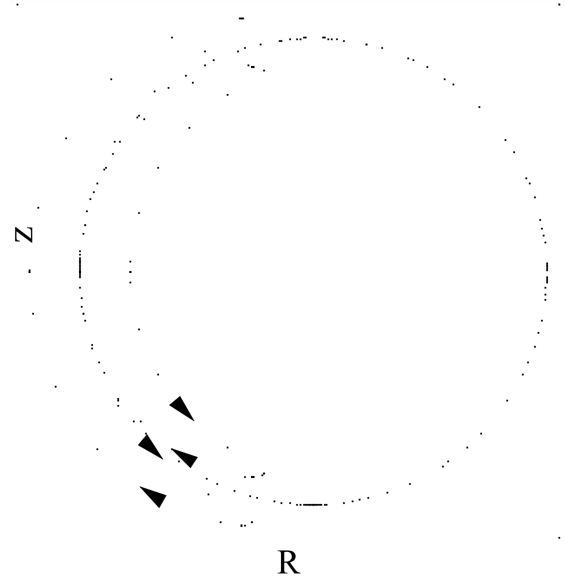
- Diamagnetic effect of banana orbits: on each flux-surface there are
  - more counter-moving trapped electrons than co-moving ones,
  - more co- than counter-moving trapped ions
  - density discrepancy

$$\Delta n_{\text{trapped}} \sim -\epsilon^{1/2} \frac{dn}{dr} \Delta r_{\text{banana}}$$



- These particles produce a drag on the ciculating ones. Collisional equilibrium occurs when

$$\Delta n_{\text{circ}} \sim -\frac{dn}{dr} \Delta r_{\text{banana}} \qquad \Rightarrow \qquad j \sim -\frac{\epsilon^{1/2}}{B_p} \frac{dp}{dr}$$



- In stellarators, trapping occurs on the inboard side, too.
  - exact cancellation in quasi-isodynamic B

Poloidal projection of orbits

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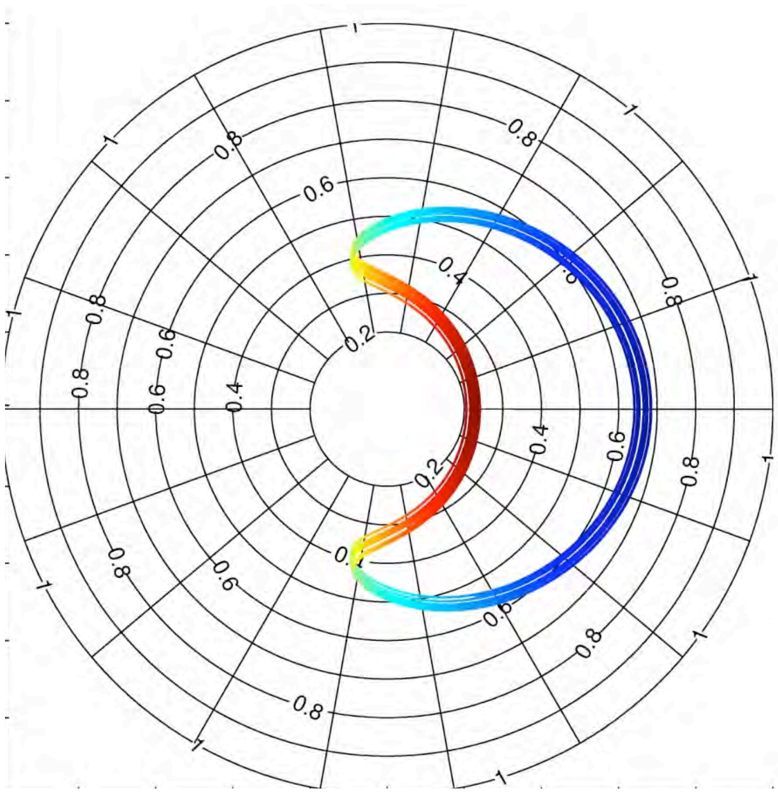
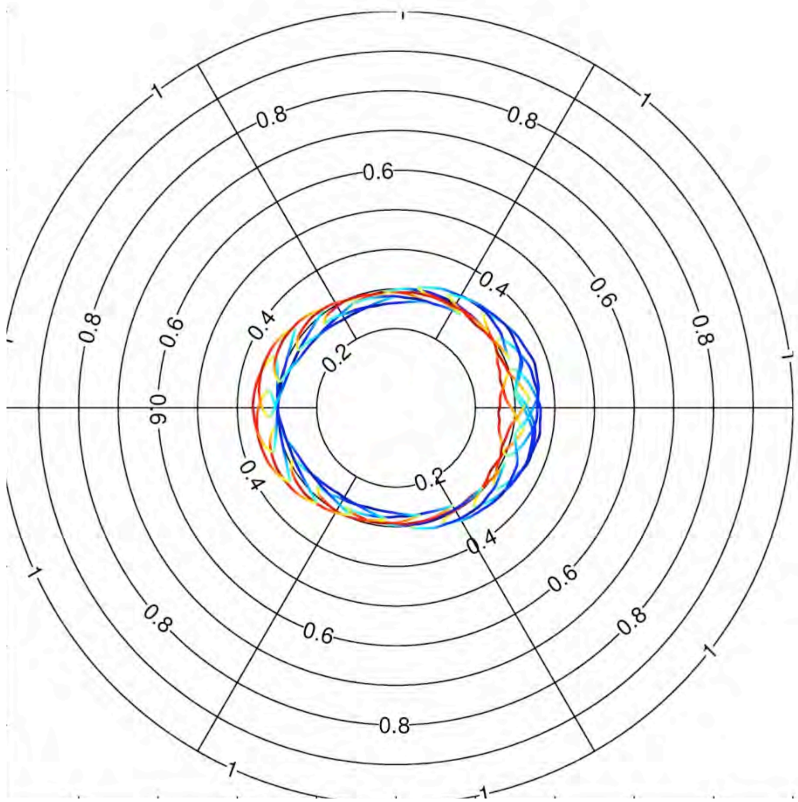
- Cooper et al (EPS 2003)



Tokamak



Quasi-isodynamic stellarator

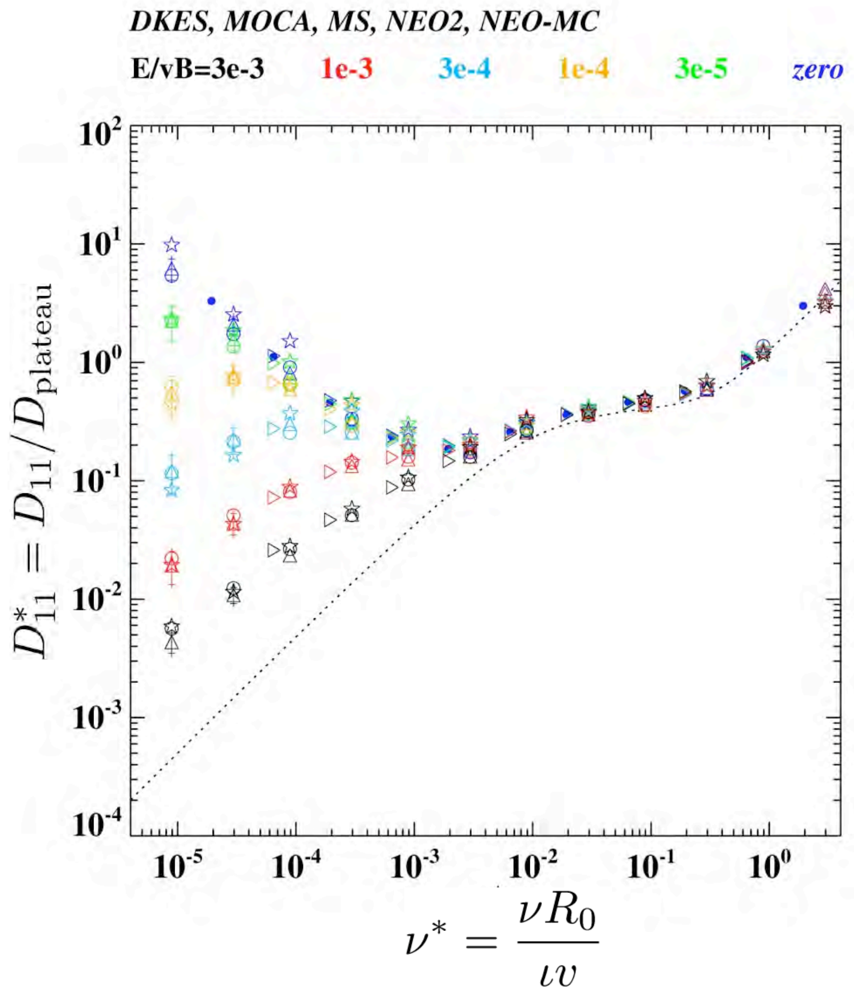
Summary

- Only in quasi-symmetric configurations can the plasma rotate rapidly or freely
- In all other stellarators
  - E_r and rotation are clamped at the value required for neoclassical ambipolarity
  - gyrokinetic turbulence only matters on small scales
- In a quasi-isodynamic stellarator, the distribution function consists of two parts:
  - a term identical to that in tokamak and proportional to the enclosed toroidal current
  - a term specific stellarators that can be calculated exactly. This term carries no current.
- As a result, such a configuration is intrinsically current-free (unless one specifically drives current)
  - no net bootstrap current or Pfirsch-Schlüter current
  - no net toroidal rotation.

Extra material

NCSX

- Monoenergetic particle diffusivity



Omnigeneity

- In an exactly omnigenous magnetid field
  - The second adiabatic invariant

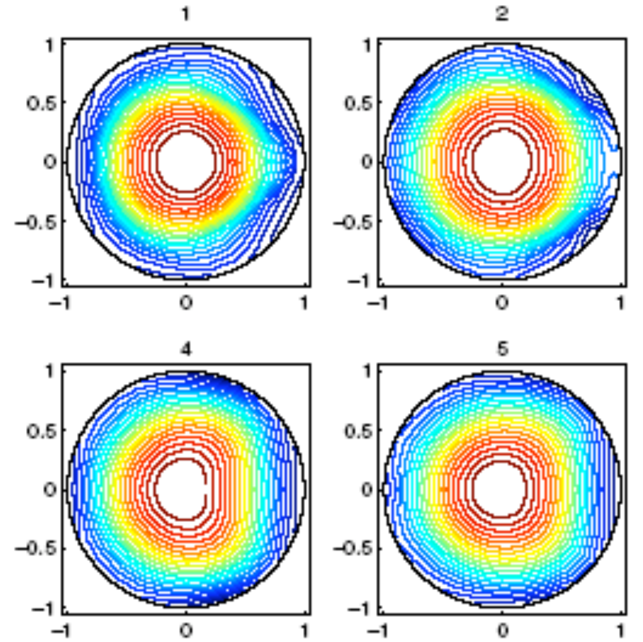
$$J(\psi,\alpha,\lambda) = \int_{l_1}^{l_2} m v_{\|} dl$$

with

$$B(l_1) = B(l_2), \qquad \alpha = \theta - \iota\varphi$$

must be constant on flux surfaces,

$$\frac{\partial J}{\partial \alpha} = 0$$



  - Specifically, for deeply trapped particles *J=0*. Since

$$E = \frac{mv^2}{2} \qquad \text{and} \qquad \mu = \frac{mv_\perp^2}{2B} = \frac{E}{B}$$

are constant, it follows that <u>the minimum value of B must be same for all field lines on each flux surface</u> (Mynick, Chu and Boozer 1983).

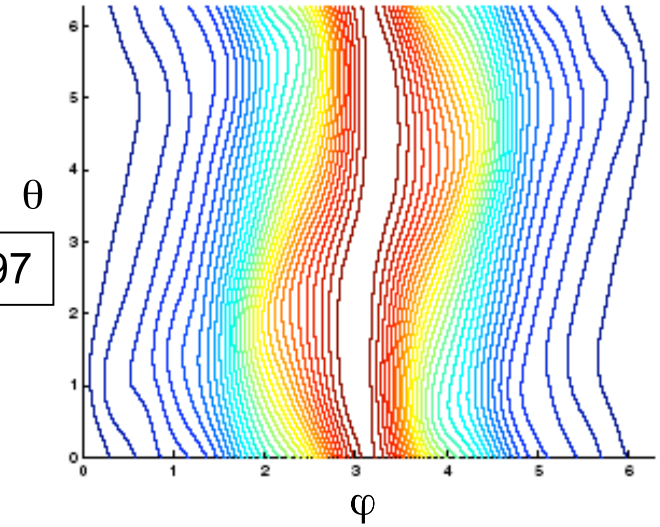
Further properties of omnigenous fields

- The minimum and maximum |B| is the same for all field lines on the same flux surface

$$\frac{\partial B_{\min}}{\partial \alpha} = \frac{\partial B_{\max}}{\partial \alpha} = 0$$

Mynick, Chu and Boozer 1982

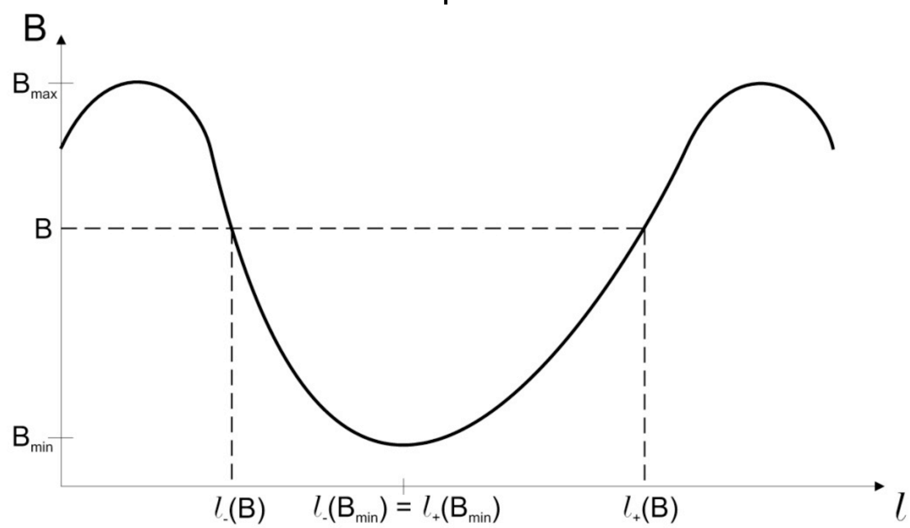
Cary and Shasharina 1997



- Also, for any function $f(\psi, B)$

$$\int_{l_-(\psi,\alpha,B)}^{l_+(\psi,\alpha,B)} f(\psi, B) dl$$

is independent of α.



Negative results

- Theorems that seem to be of little practical importance:

- Exactly quasisymmetric equilibria do not exist. (Garren and Boozer, 1990)
  - It is possible to make B exactly quasisymmetric on <u>one</u> flux surface, but not on others in general.

$$\text{OK up to order } O(\epsilon^2), \text{ but not in order } O(\epsilon^3)$$

- Exactly omnigenous fields are quasisymmetric. (Cary and Shasharina, 1997)
  - B can be approximately omnigenous whilst being far from quasisymmetric.
  - Possible to make B arbitrarily close to exactly omnigenous?

Quasisymmetry

When expressed in so-called Boozer coordinates,

$$\mathbf{B} = \nabla\varphi \times \nabla\psi_p + \nabla\psi_t \times \nabla\theta = \beta(\psi_t, \theta, \varphi)\nabla\psi_t + I(\psi_t)\nabla\theta + J(\psi_t)\nabla\varphi$$

the guiding centre Lagrangian

$$L = \frac{mv_\parallel^2}{2} + e\mathbf{A}\cdot\mathbf{v} - \mu B$$

depends only on the modulus B

$$L = \frac{m}{2B^2}\left(I\dot{\theta} + J\dot{\varphi}\right)^2 + Ze\left(\psi_t\dot{\theta} - \psi_p\dot{\varphi}\right) - \mu B$$

For example, if B is „axisymmetric“ in Boozer coordinates

$$B = B(\psi_t, \theta)$$

there is a correponding constant of motion

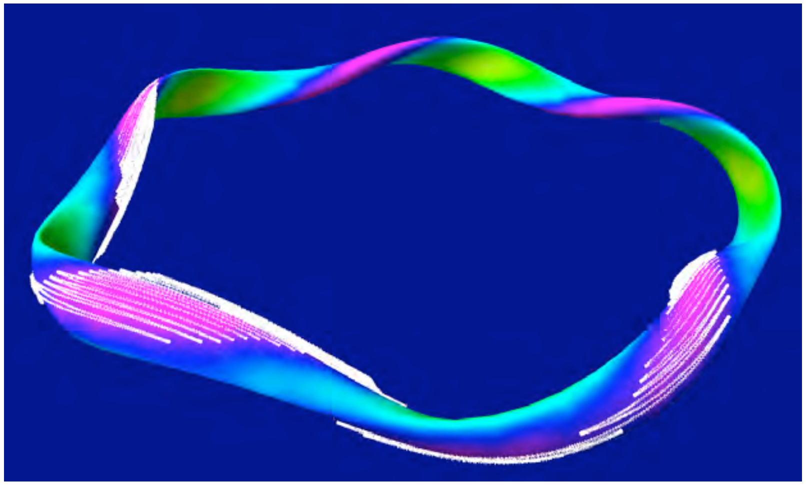
$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \frac{mIv_\parallel}{B} - e\psi_p = \text{constant}$$

Orbit confinement and symmetry

- In the tokamak, trapped particles precess toroidally around the torus
- Fundamental reason for confinement: |B| is toroidally symmetric.
  - gyro-averaged Lagrangian

$$L = \frac{mv_{\|}^2}{2} + e\mathbf{A} \cdot \mathbf{v} - \mu B$$

- Topologically, precession can only occur in three ways:
  - Toroidally
  - Poloidally
  - Helically
- In quasisymmetric fields B is constant in one of these directions.



Current

The current consists of three parts:

$$J(\psi) = \sum_a \int \left(j_{a\|}\mathbf{b} + \mathbf{j}_{a\perp} \right) \cdot d\mathbf{S} + J_{\mathrm{aux}}(\psi)$$

where the diamagnetic current

$$\int \mathbf{j}_{a\perp} \cdot d\mathbf{S} = \int \frac{\mathbf{b} \times (\nabla p_a + n_a e_a \nabla \phi_0)}{B} \cdot d\mathbf{S}$$

can be calculated as a surface integral over a surface of constant B

$$\int \mathbf{j}_{a\perp} \cdot d\mathbf{S} = -\frac{\mu_0}{B^2} \int_0^{\psi} \left(\frac{dp_a}{d\psi'} + n_a e_a \frac{d\phi_0}{d\psi'} \right) J(\psi') d\psi'$$

Like the bootstrap current, the diamagnetic current on a flux surface ψ is proportional to the total toroidal current enclosed by ψ.