Equilibria with Stochastic Regions

Presented by Allan Reiman

References

A. Reiman, M.C. Zarnstorff, D. Monticello, A. Weller, J.Geiger and the W7-AS Team, Nucl. Fusion **47**, 572-578 (2007).

J. A. Krommes and A. H. Reiman, Phys. Plasmas **16**, 072308 (2009).

17th International Stellarator / Heliotron Workshop
10/13/09, Princeton, NJ

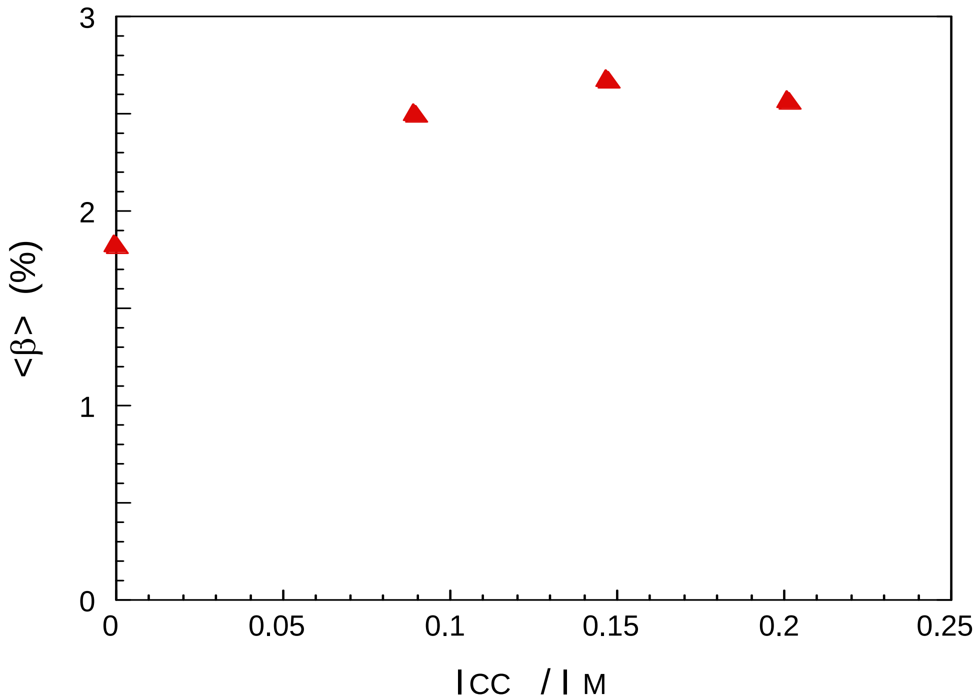
Outline

- Background and motivation:

  Equilibrium reconstruction calculations for W7AS.

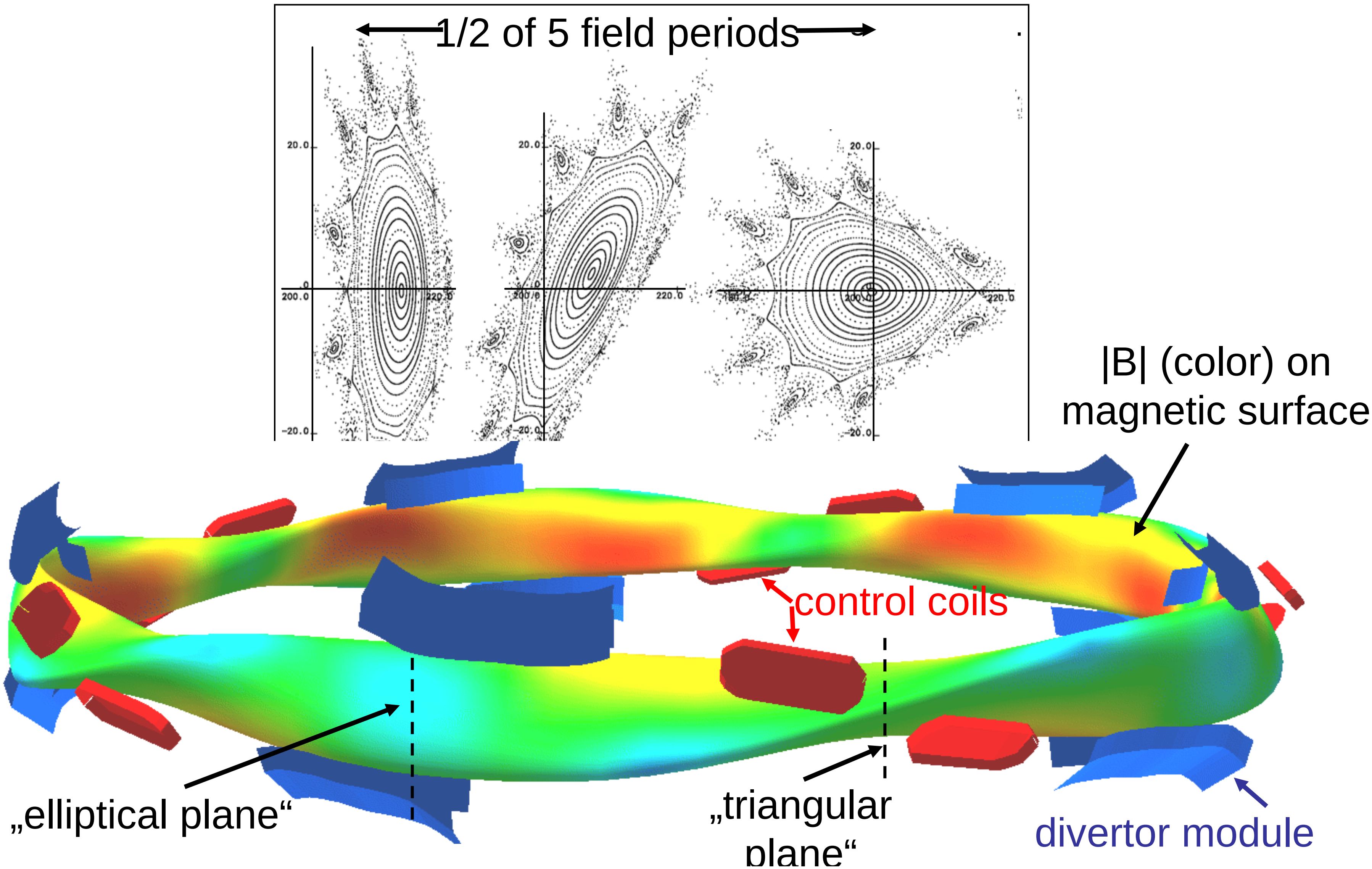
- Equilibria with stochastic regions.

Initially motivated by puzzle on W7AS.
Control coil for island divertor substantially affects achievable β.

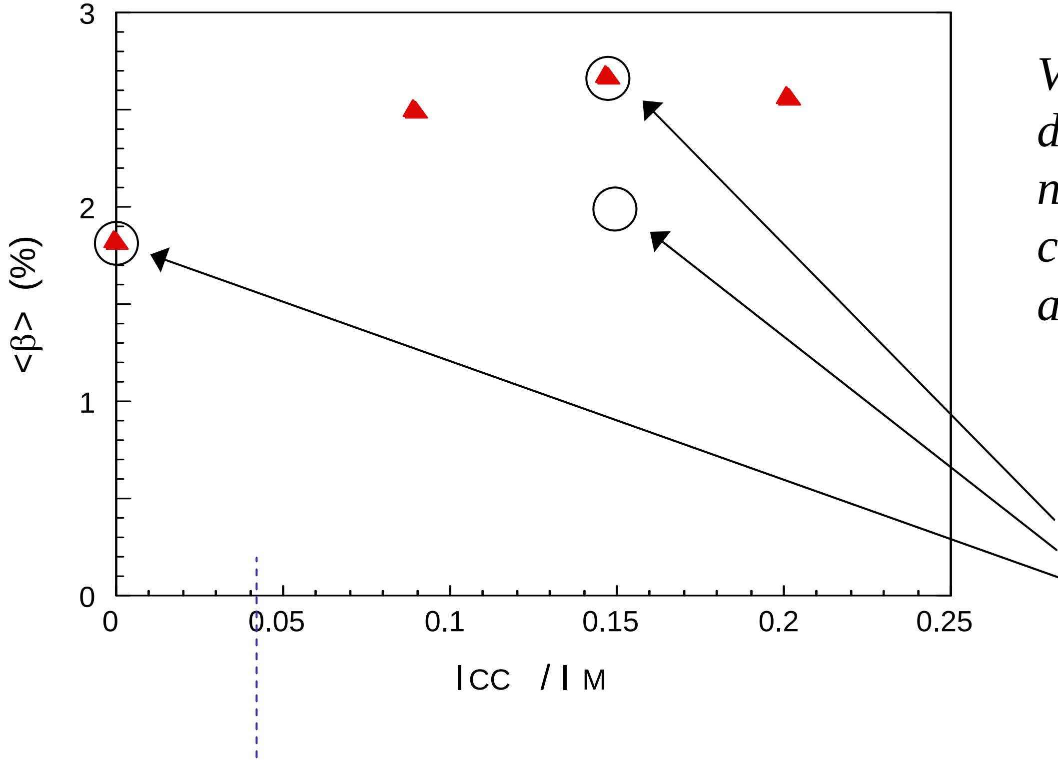


Variation of peak-⟨β⟩ versus the divertor control-coil current I_{CC} normalized by the modular coil current, for B=1.25 T, P_{NB} = 2.8 MW absorbed and ι_{vac}= 0.44.

Divertor control coils affect resonant field at edge. Is the effect produced by islands at edge?



Vacuum island width does not explain optimal Icc. Is this just an issue of finite β effect on ι?

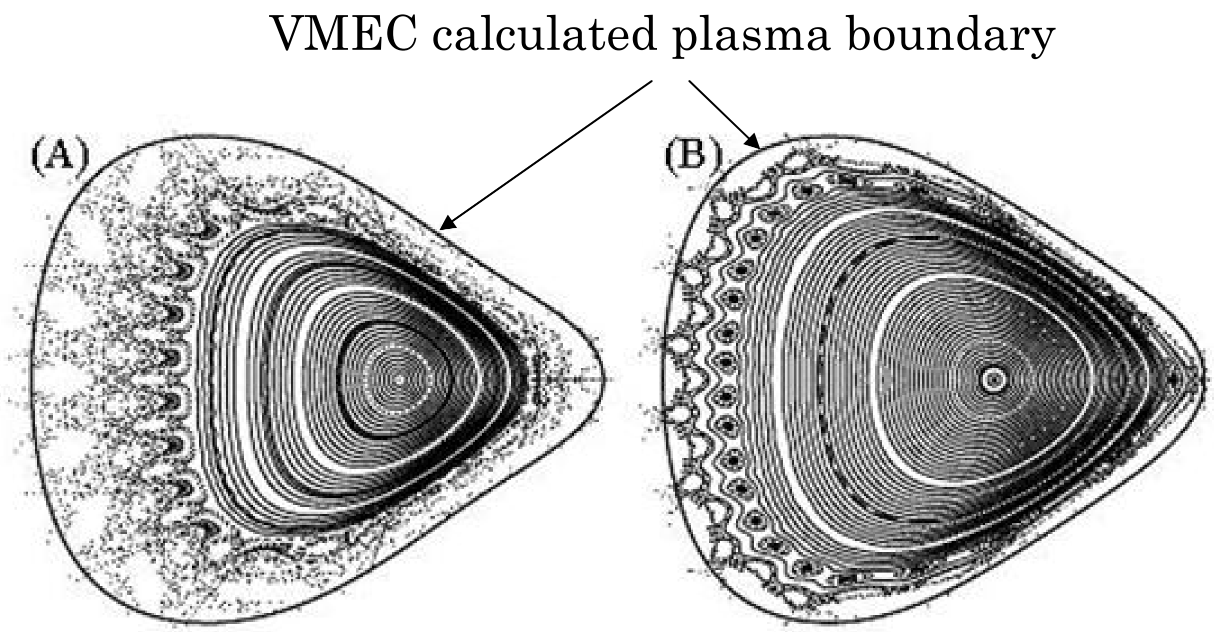


Variation of peak-⟨β⟩ versus the divertor control-coil current I_{CC} normalized by the modular coil current, for B=1.25 T, P_{NB} = 2.8 MW absorbed and ι_{vac}= 0.44.

PIES code run for these parameters.

Island width zero for vacuum field.

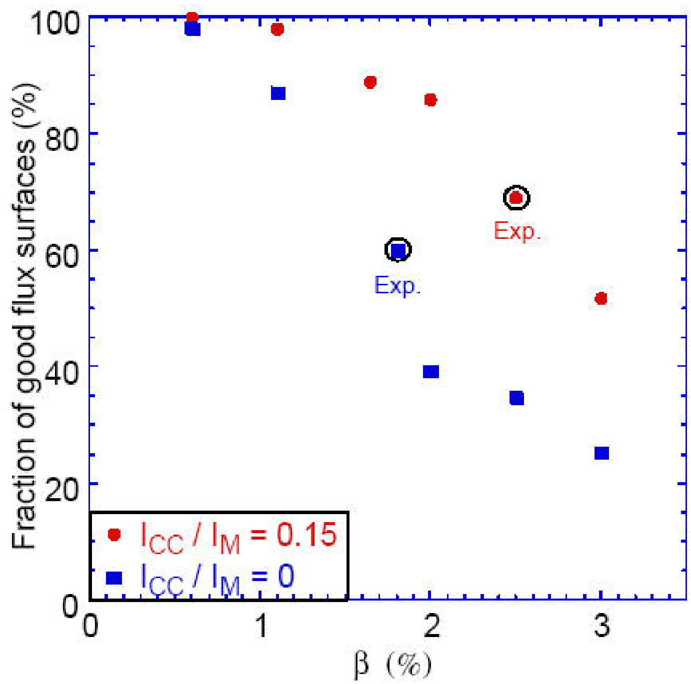
PIES results a surprise: Substantial stochastic region.
Calculations for two different values of control coil current are consistent with observed trend.



$I_{CC} = 0$, $\langle\beta\rangle = 1.8\%$ $\qquad$ $I_{CC} = -2.5\text{kA}$, $\langle\beta\rangle = 2.0\%$

Width of stochastic region larger for $I_{CC} = 0$, even though it has somewhat lower β.

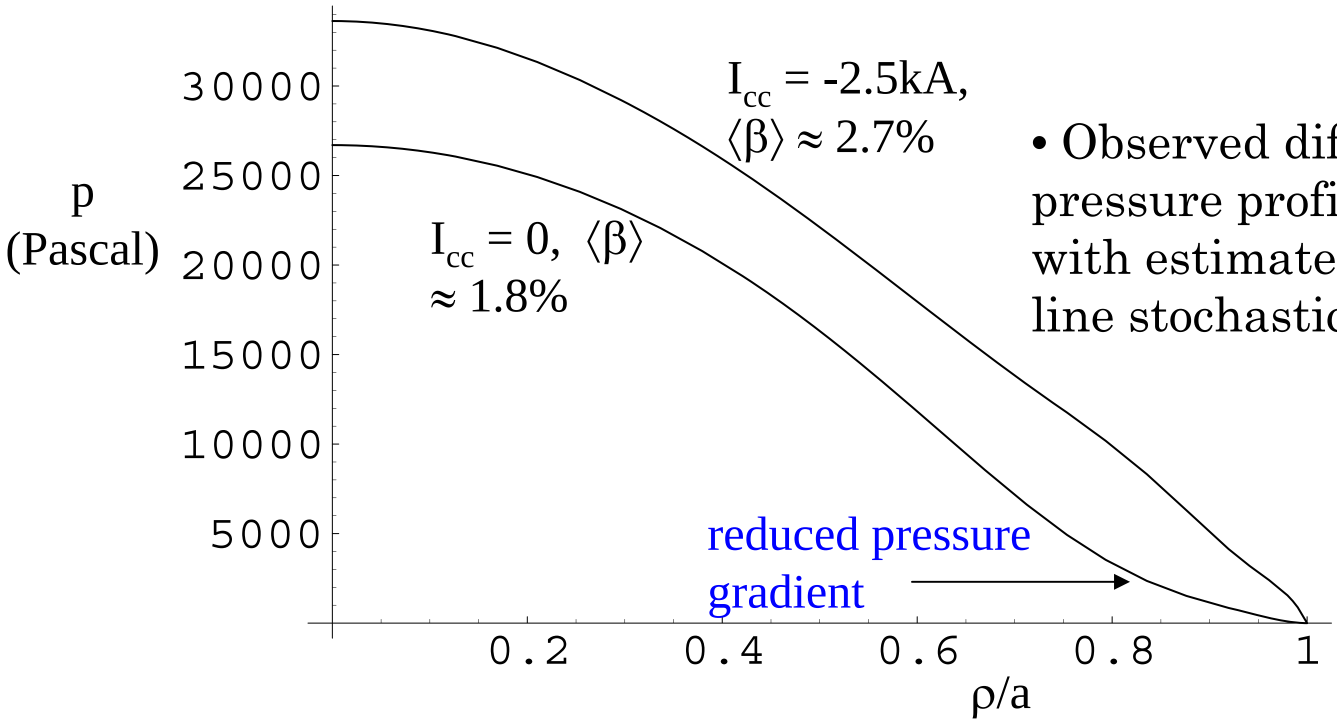
Same trend emerges from PIES β scan for two different values of control coil current.



Fraction of good flux surfaces versus β as predicted by PIES for two different values of the control coil current for the W7AS stellarator. The circles indicate the PIES calculations done for the experimentally achieved value of β.

W7AS experimental observations consistent with picture that field line stochasticity produces enhanced transport in outer region of plasma.

- Rechester-Rosenbluth estimate of enhancement in transport due to field-line stochasticity is consistent. (But $\chi_{stoch} \propto T_e^{5/2}$, so error bars are large.)



- Observed difference in pressure profile consistent with estimated effect of field line stochasticity.

- Require equilibrium calculation with $\nabla p \neq 0$ in stochastic region.

An issue also for tokamaks when field line stochasticity contributes to transport.

PIES code uses equilibrium equations in form that integrates to get B from j rather than differentiating to get j from B.

MHD equilibrium:

$$\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{\nabla} p, \quad \boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{B}.$$

Writing $\boldsymbol{j} = j_{\|} \boldsymbol{B}/B + \boldsymbol{j}_{\perp}$, get

$$\boldsymbol{j}_{\perp} = \boldsymbol{B} \times \boldsymbol{\nabla} p / B^2, \tag{1}$$

$\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$ gives

$$\boldsymbol{B} \cdot \boldsymbol{\nabla}(j_{\|}/B) = -\boldsymbol{\nabla} \cdot \boldsymbol{j}_{\perp}. \tag{2}$$

Alternative form of equilibrium equations:

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j}[\boldsymbol{B}], \tag{3}$$

with $\boldsymbol{j}[\boldsymbol{B}]$ given by Eq. (1) and Eq. (2).

Eq. (3) can be solved numerically by standard methods such as Picard iteration (Spitzer, Grad) or Newton-Krylov.

Plasma with $\nabla p \neq 0$ in stochastic region does not satisfy MHD equilibrium equation.

$\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{\nabla} p \Rightarrow \boldsymbol{B} \cdot \boldsymbol{\nabla} p = 0 \Rightarrow \boldsymbol{\nabla} p = 0$ in stochastic region.

If radial diffusion of field lines is weak, $\boldsymbol{B} \cdot \boldsymbol{\nabla} p$ due to radial pressure gradient is small, can be balanced by small neglected terms.

$$\boldsymbol{j} \times \boldsymbol{B} - \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} - \boldsymbol{\nabla}\, \boldsymbol{\pi} = \boldsymbol{\nabla} p.$$

$$\boldsymbol{B} \cdot \boldsymbol{\nabla} p = -\boldsymbol{B} \cdot \boldsymbol{\nabla} (\rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi})$$

$$\boldsymbol{j}_\perp = \boldsymbol{B} \times \boldsymbol{\nabla} p / B^2 + \boldsymbol{B} \times \boldsymbol{\nabla} (\rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{\nabla} \cdot \boldsymbol{\pi})$$

2nd term on RHS small, can be neglected. (Relative contribution to resonant Fourier components of equation also small.)

$$\boldsymbol{j}_\perp \approx \boldsymbol{B} \times \boldsymbol{\nabla} p / B^2$$

Our equations decouple parallel and perpendicular components of force balance.

$$\boldsymbol{j}_\perp = \boldsymbol{B} \times \boldsymbol{\nabla} p / B^2 \tag{1}$$

$$\boldsymbol{B} \cdot \boldsymbol{\nabla}(j_\| / B) = -\boldsymbol{\nabla} \cdot \boldsymbol{j}_\perp \tag{2}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{j}[\boldsymbol{B}] \tag{3}$$

- Perpendicular force balance determines self-consistent equilibrium magnetic field.
- Parallel component of force balance can be regarded as part of transport problem rather than equilibrium problem.
- Note that formulation also permits calculation of $\boldsymbol{j}_\perp$ by alternative code.

Stochasticity enters through Eq. (2).

Equation for $j_\parallel$ can be cast in same form as plasma turbulence equations.

For a field $\boldsymbol{B}_0$ with good flux surfaces, have

$$(\partial/\partial\phi + \iota\partial/\partial\theta)\left(j_\parallel/B\right) = -\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp/B^\phi.$$

Letting $\mu \equiv j_\parallel/B$ and $g \equiv -\boldsymbol{\nabla}\cdot\boldsymbol{j}_\perp/B^\phi$, have

$$(\partial\mu/\partial\phi + \iota\partial\mu/\partial\theta) = g.$$

Assume $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$, where $\boldsymbol{B}_0$ has good surfaces and $|\delta\boldsymbol{B}| \ll |\boldsymbol{B}_0|$, and assume $B_0^\phi \gg B_0^\theta$. Work in coordinate system (ψ, θ, ϕ) where $\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\psi = 0$ and $\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\theta/\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\phi = \iota(\psi)$ constant on flux surfaces of $\boldsymbol{B}_0$.
If ι monotonic function of ψ in region of interest, can adopt it as radial variable:

$$\frac{\partial\mu(\iota,\theta,\phi)}{\partial\phi} + \iota\frac{\partial\mu}{\partial\theta} + \frac{\delta B^\iota}{B_0^\phi}\frac{\partial\mu}{\partial\iota} + \frac{\delta B^\theta}{B_0^\phi}\frac{\partial\mu}{\partial\theta} = g$$

Compare with drift-kinetic equation with strong B_z, fluctuating $\boldsymbol{E}\times\boldsymbol{B}$ velocity $\delta\boldsymbol{V}_E$, neglecting parallel nonlinearity:

$$\frac{\partial f(\boldsymbol{x}, v_\parallel, t)}{\partial t} + v_\parallel\frac{\partial f}{\partial z} + \delta V_{E,x}\frac{\partial f}{\partial x} + \delta V_{E,y}\frac{\partial f}{\partial y} = 0.$$

Resonance broadening approximation replaces effect of turbulent, fluctuating field with a diffusion operator.

Consider e.g. Vlasov equation with turbulent electrostatic field, with δE random Gaussian white noise:

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}\delta E\frac{\partial f}{\partial v} = 0. \tag{1}$$

Equivalent to the more tractable equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial}{\partial v}D_v(v)\frac{\partial f}{\partial v} = 0, \tag{2}$$

where

$$D_v = \left(\frac{q}{m}\right)^2 \int_0^\infty d\tau \, \langle \delta E\big(\widetilde{x}(\tau),\tau\big)\delta E\big(\widetilde{x}(0),0\big)\rangle$$

and the integral is taken over the turbulent particle trajectories $\widetilde{x}(\tau)$.

Cannot directly apply resonance broadening approximation to our equation because of causality issue.

Our equation:

$$\frac{\partial \mu}{\partial \phi} + \iota(\psi)\frac{\partial \mu}{\partial \theta} + \frac{\delta B^{\psi}}{B_0^{\phi}}\frac{\partial \mu}{\partial \psi} + \frac{\delta B^{\theta}}{B_0^{\phi}}\frac{\partial \mu}{\partial \theta} = g.$$

Vlasov equation with turbulent electrostatic field

$$\frac{\partial f(x,v,t)}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}\delta E\frac{\partial f}{\partial v} = 0, \tag{1}$$

is equivalent to

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial}{\partial v}D_v(v)\frac{\partial f}{\partial v} = 0. \tag{2}$$

- Eq. (1) satisfies causality.
  Eq. (2) not reversible in time. Under $t \rightarrow -t$, get anti-diffusion.
- Physics for $j_{||}$ must look the same whether we integrate backwards or forwards along field lines.
  μ cannot satisfy diffusion equation.

Causality issue can be addressed by working in terms of Green’s functions.

- Causal Green’s function satisfies diffusion equation.
- Anti-causal Green’s function satisfies diffusion equation with sign of the diffusion terms reversed.
- General solution of equation for μ can be written as superposition of solutions constructed from the two types of Green’s functions.
  μ itself does not satisfy a diffusion equation.
- There remains an issue of periodicity in the torus with respect to θ and ϕ.
  - Want Green’s functions to satisfy periodicity, but periodic solutions are not spatially causal.

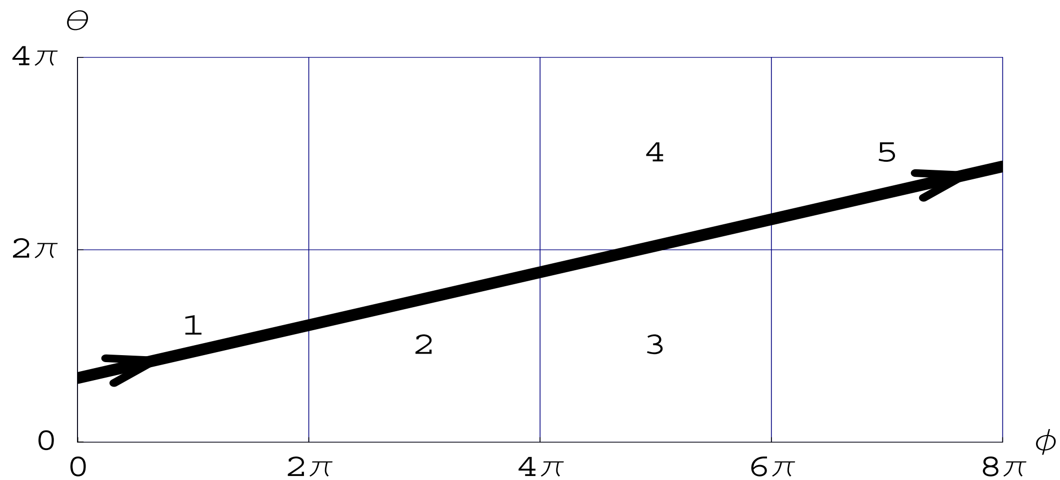


FIG. 1: Portion of covering space for a magnetic flux surface. Numbers indicate consecutive domains that are pierced as field line is traversed in direction of increasing ϕ. Periodicity wraps a line into the fundamental square.

The periodic solution is constructed using methods familiar from ballooning mode theory.

- Work in infinite covering space, $\theta \to \pm\infty$, $\phi \to \pm\infty$, for flux surfaces of $\boldsymbol{B}_0$ (depicted in Fig. 1).
- Construct periodic solutions by using shifted-sum representation (ballooning representation).

We obtain an explicit analytic solution when the field line diffusion is weak.

- When diffusion coefficient small, diffusion term in equation can be neglected, except near rational surfaces, where radial derivative can be large.
- Radial derivatives of metric elements and Jacobian can be neglected relative to radial derivatives of the Greens functions near the resonant surfaces.

Diffusion term for causal (anti-causal) Green's function takes the form:

$$\pm D^{\psi} \frac{\partial^2 G^{\pm}}{\partial \psi^2},$$

where D^{ψ} is radial diffusion coefficient of magnetic field lines.
In boundary layer at resonant surface, solution can be expressed in terms of Airy functions.

- Gives resonance broadening width, in terms of distance along the field line:

$$\zeta_D \approx (m^2 \iota'^2 D^{\psi})^{-1/3}.$$

Summary

Solution for equilibria with stochastic field lines:

- We work with the equilibrium equations in the form $\nabla\times\mathbf{B} = \mathbf{j}[B\}$.
  - Decouples perpendicular and parallel force-balance.
  - Stochasticity enters entirely through solution of magnetic differential equation along stochastic field lines.
- Can cast the magnetic differential equation in the same form as standard equations of turbulence theory.
  - Suggests application of resonance-broadening theory.
  - Work in terms of Greens functions to handle causality issue.
  - Use ballooning representation to handle periodicity issue.
  - Can obtain analytic solution for pressure-driven current in limit of weak diffusion.

Summary (continued)

- Incorporating model in PIES code, have applied the code to calculate reconstructed W7AS equilibria.
  - Find threshold in β above which stochastic region appears at plasma edge.
  - Calculated differences in size of stochastic region and field line diffusion coefficient provide plausible explanation for previously puzzling observations that current in divertor control coils has large effect on achievable β.
  - Rechester-Rosenbluth estimate for contribution of field-line stochasticity to energy transport is consistent with observations. (But sensitive to local temperature.)
  - Difference in pressure profile in presence of divertor control coil current also consistent with this picture.