CONSTRUCTING CUBATURE FORMULAE FOR THE SPHERE AND FOR THE TRIANGLE

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July 1, 1997; revised, January 25, 1998

Abstract. It has been shown recently that $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetric cubature formulae for the unit sphere S^d are characterized by cubature formulae on the standard simplex Σ^d. In particular, cubature formulae for the surface measure on S^2 correspond to the symmetric cubature formulae for the weight function $(u_1u_2u_3)^{-1/2}$, where $u_3 = 1 - u_1 - u_2$, on the triangle Σ^2. In this paper we construct cubature formulae for the weight function $(u_1u_2u_3)^{-1/2}$ on the triangle and, using the correspondence, cubature formulae for the surface measure on the unit sphere.

1. Introduction

Finding effective cubature formula for integrals over a region in $\mathbb{R}^d$ is a problem of vast dimensions. It is often necessary to limit the scope to constructing cubature formulae for a particular setting. Indeed, most of the results in the literature deal with formulae with respect to the unit weight function over one of the standard regions. The regions that attract most of the attention are cubes, balls, simplices and spheres. We refer to [5, 22, 24] for some of the references. Recently we have shown that cubature formulae for spheres, for balls and for simplices are very closely related and even equivalent in many cases. In [27], a correspondence between cubature formulae on the unit sphere S^d of $\mathbb{R}^{d+1}$ and on the unit ball B^d of $\mathbb{R}^d$ is discovered, which is used to construct new formulae for S^2 in [7], and in [28] a correspondence between $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetric cubature formulae on S^d and cubature formulae on the simplex Σ^d of $\mathbb{R}^d$ is revealed, which also implies an equivalence between $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetric formulae on B^d and cubature formulae on Σ^d. In both cases the results are established for a large class of weight functions; in particular, the cubature formulae with respect to the surface measure on S^2 correspond to the formulae with respect to the weight functions $(1 - x_1^2 - x_2^2)^{-1/2}$ on B^2 and $(u_1u_2u_3)^{-1/2}$ on Σ^2, where $u_3 = 1 - u_1 - u_2$. The correspondence allows us to obtain new cubature formulae on one region from formulae over another region.

1991 *Mathematics Subject Classification.* 65D32, 41A55, 41A63.

Key words and phrases. Cubature formulae, on the unit sphere, on the triangle, symmetric formula on a triangle, octahedral symmetry.

Supported by the National Science Foundation under Grant DMS-9500532.

In this paper we construct cubature formulae for $(u_1u_2u_3)^{-1/2}$ on Σ^2 and use the result to generate formulae for the surface integral on S^2. First we state the correspondence. Throughout this paper we denote by Π_n^d the space of polynomials of degree at most n in d variables ($d = 2$ or 3), and we denote by Σ^2 the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. Let W be a weight function defined on $\mathbb{R}^3$, normalized so that $\int_{S^2} W(y_1^2, y_2^2, y_3^2)d\omega = 1$. Associate to W we define a weight function W_Σ on the triangle Σ^2 by

$$W_\Sigma(u_1,u_2) = 2W(u_1,u_2,1-u_1-u_2)/\sqrt{u_1u_2(1-u_1-u_2)}, \quad (u_1,u_2) \in \Sigma^2.$$

Then the correspondence in [28] states that

Theorem 1.1. *Let W and W_Σ be defined as above. Suppose that there is a cubature formula of degree M on Σ^2 given by*

$$\int_{\Sigma^2} f(u_1,u_2)W_\Sigma(u_1,u_2)du_1du_2 = \sum_{k=1}^N \lambda_k f(u_{k,1},u_{k,2}), \qquad f \in \Pi_M^2, \tag{1.2}$$

whose N nodes lie on the simplex Σ^2. Then there is a cubature formula of degree $2M+1$ on the unit sphere S^2,

$$\begin{aligned}\int_{S^2} & g(y_1,y_2,y_3)W(y_1^2,y_2^2,y_3^2)d\omega \\ &= \sum_{k=1}^N \lambda_k \sum_{\varepsilon_i=\pm1} g(\varepsilon_1 v_{k,1}, \varepsilon_2 v_{k,2}, \varepsilon_3 v_{k,3})/2^{a_k}, \quad g \in \Pi_{2M+1}^3,\end{aligned} \tag{1.3}$$

where a_k is the number of nonzero elements among $v_{k,1}$,$v_{k,2}$ and $v_{k,3}$, and the nodes $(v_{k,1},v_{k,2},v_{k,3}) \in S^2$ are defined in terms of $(u_{k,1},u_{k,2})$ by

$$(v_{k,1},v_{k,2},v_{k,3}) = (\sqrt{u_{k,1}}, \sqrt{u_{k,2}}, \sqrt{1-u_{k,1}-u_{k,2}}). \tag{1.4}$$

On the other hand, if there exists a cubature formula of degree $2M+1$ on S^2 in the form of (1.4), then there is a cubature formula of degree M on the simplex Σ^2 in the form of (1.3) whose nodes $(u_{k,1},u_{k,2}) \in \Sigma^2$ are defined by $(u_{k,1},u_{k,2}) = (v_{k,1}^2, v_{k,2}^2)$.

The formula (1.3) is invariant under the change of signs, or invariant under the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The theorem establishes the equivalence between (1.2) and (1.3). In [28] this theorem is proved more generally for formulae on the sphere S^d and the simplex Σ^d for all d. When $W(\mathbf{x}) = 1/\omega_2 = 1/4\pi$ is the surface area of S^2, the corresponding weight function on Σ^2 is the multiple of the weight function $(u_1u_2u_3)^{-1/2}$, which we will denote by W_0; that is,

$$W_0(u_1,u_2) = (u_1u_2(1-u_1-u_2))^{-1/2}/2\pi, \qquad (u_1,u_2) \in \Sigma^2. \tag{1.5}$$

The construction of cubature formulae in this paper will be carried out only for W_0. Most of the cubature formulae for Σ^2 in the literature are constructed for the unit weight function; the correspond formulae (1.3) on S^2 are with respect to $|y_1y_2y_3|d\omega$.

This theorem allows us to construct cubature formulae for the surface measure on S^2 by working with cubature formulae for W_0 on the triangle. In the literature almost all cubature formulae for the simplex are constructed for the unit weight function, see the recent survey [16]. In Section 2, we will construct a few minimal cubature formulae of lower degrees for W_0 on Σ^2 and discuss the corresponding formulae on S^2. The main part of this paper is in Section 3, in which we adopt the method by Lyness and Jespersen in [17] to construct symmetric cubature formulae on Σ^2, which are formulae that are invariant under the symmetric group of the triangle. When the formula (1.2) on Σ^2 is symmetric, the corresponding formula (1.3) in Theorem 1.1 is invariant under the octahedral group, which is the symmetric group of the unit cube $\{\pm1,\pm1,\pm1\}$ in $\mathbb{R}^3$. In this case, the formula (1.3) is of the form

$$\begin{aligned} \int_{S^2} & g(y_1,y_2,y_3)W(y_1^2,y_2^2,y_3^2)d\omega \\ &= \sum_{k=1}^{N}\mu_k \sum_{\sigma}\sum_{\varepsilon_i=\pm1} g(\varepsilon_1 v_{k,\sigma_1},\varepsilon_2 v_{k,\sigma_2},\varepsilon_3 v_{k,\sigma_3}), \quad g\in\Pi^3_{2M+1}, \end{aligned} \tag{1.6}$$

where the second sum is taken over all permutations of $\sigma=(\sigma_1,\sigma_2,\sigma_3)$. Formulae of this type have been constructed by Lebedev [12 – 15]. It is called *fully symmetric* in [24, 10] and has been studied for S^d in [8], which contains another correspondence between fully symmetric formulae on S^d and cubature formulae on Σ^d, namely, a correspondence between the consistent rule structure on these two regions.

Numerical integration on the sphere has attracted a lot of attentions, we refer to [1, 2, 5, 8, 12–15, 19, 22, 24] and the references there. Most formulae have been constructed by making use of symmetry to reduce the number of moment equations that have to be solved (see, for example, Sobolev [23] and MacLaren [18]). The fundamental result of Sobolev states that a cubature formula invariant under a finite group is exact for all polynomials in a subspace $\mathcal{P}$ if, and only if, it is exact for all polynomials in $\mathcal{P}$ that are invariant under the same group. The group that has been under intense study is the octahedral group; Lebedev constructed in [12 –15] cubature formulae of degree up to 59, many of them have the smallest number of nodes among all formulae that are known. Working with symmetric cubature formulae on Σ^2, we are able to find many formulae on S^2 that Lebedev did not consider (see Section 3). There are also formulae that are invariant under the icosahedral group, which have, however, no correspondence on the triangle, since they are not symmetric under $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ in the first place. We refer to [5, 22, 24] and the references there for other papers that deal with cubature formulae on the sphere, see also [1] in which formulae are constructed making use of symmetry and a Taylor expansion formula.

2. Cubature formulae of lower degree on triangle and on sphere

In this section we present a list of minimal cubature formulae of lower degrees for the weight function W_0 on Σ^2. A cubature formula of degree M is minimal if its number of nodes is minimal among all cubature formulae of degree M. For W_0 on Σ^2 it is known that the lower bound for the number of nodes is given by

$$N_{2n}(\Sigma^2) \ge \binom{n+2}{2} \quad \text{and} \quad N_{2n-1}(\Sigma^2) \ge \binom{n+1}{2} + \left[\frac{n}{2}\right].$$

(cf. [20, 22]). We show here that for n up to 7 the above lower bound is attained for W_0.

There is a close relation between common zeros of orthogonal polynomials and cubature formulae. For example, if a cubature formula attains Möller's lower bound of odd degree formula ([20]), then its nodes are common zeros of a family of linearly independent polynomials $Q_1, \ldots, Q_r$ of degree n, where $r = n+1-[n/2]$, and each Q_i is orthogonal to polynomials of lower degrees (see [20]). There are far more general results in this direction, we refer to [20, 22, 25, 26]. For the weight function W_0, one basis of orthonormal polynomials can be given explicitly in terms of Jacobi polynomials $P_k^{(\alpha,\beta)}$,

$$P_k^n(x_1,x_2) = h_k^n P_{n-k}^{(2k,-1/2)}(2x_1-1)(1-x_1)^k P_k^{(-1/2,-1/2)}(2x_2(1-x_1)^{-1}-1), \quad 0 \le k \le n,$$

where h_k^n are constants chosen so that P_k^n are normalized. The ordinary spherical harmonics invariant under the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are characterized by orthogonal polynomials with respect to W_0. This fact is a corollary of a general result proved in [28] (see also [4]) that, together with Sobolev's theorem, leads to Theorem 1.1.

One interesting fact is that some of the minimal formulae for W_0 can be obtained from known cubature formula on S^2 in [22, 24] via the correspondence in Theorem 1.1. We found minimal cubature formulae of degree 3 and 5 for W_0 on Σ^2 this way from [24, U_n:7-2] (specify $n=3$) and [24, U_3: 11-2], respectively. We also constructed minimal formulae of degree 4 and 7, as well as a new minimal formula of degree 3 by following the method that has worked for the unit weight function on Σ^2. In the following we present the new minimal formulae for W_0. Each formula is given by its nodes and weights; we list them in lines with each line containing one weight on the right and node(s) associated with the weight on the left. We also give polynomials whose common zeros are the nodes. It should be mentioned that we have normalized the weight function W_0 so that it has integral 1. If one uses just $1/\sqrt{u_1u_2u_3}$, then there will be a multiplication factor 2π in the weights λ_i of the cubature formula.

Degree 3: $N = 4$,

$$\begin{array}{llll} (1/3,1/3), & & & -9/40 \\ (1/7,1/7), & (1/7,5/7), & (5/7,1/7), & 49/120 \end{array}$$

This formula is minimal and also symmetric on Σ^2. It is obtained from [24, p. 301, U_n: 7-2], specifying $n=3$, via Theorem 1.1. It has one negative weight.

Degree 3: $N = 4$,

$$\begin{array}{ll} ((15+2\sqrt{30})/35, (10-\sqrt{30} \pm \sqrt{65-10\sqrt{30}})/35), & (18-\sqrt{30})/72 \\ ((15-2\sqrt{30})/35, (10+\sqrt{30} \pm \sqrt{65+10\sqrt{30}})/35), & (18+\sqrt{30})/72 \end{array}$$

This is a minimal formula. It is found by computing the common zeros of orthogonal polynomials P_0^2 and P_2^2. It is not symmetric on Σ^2 but all weights are positive.

Degree 4: $N = 6$, the nodes $(x_1, y_1), \dots, (x_6, y_6)$ are given by

$$\begin{array}{llll} (0,0), & & 1/15 & \\ (0, 2(7+\sqrt{7})/21), & & \lambda_1 & \\ (a_2, b_2), & (b_2, a_2), & \lambda_2 & \\ (a_i, b_i), & & \lambda_i, & i = 3, 4 \end{array}$$

with

$$a_2 = \frac{1}{21}\left(7 - \sqrt{7} - \sqrt{28 - 7\sqrt{7}}\,\right), \quad b_2 = \frac{1}{21}\left(7 - \sqrt{7} + \sqrt{28 - 7\sqrt{7}}\,\right),$$
$$a_3 = \frac{1}{21}\left(28 - 6\sqrt{7} - \sqrt{714 - 259\sqrt{7}}\,\right), \quad b_3 = \frac{1}{21}\left(-14 + 8\sqrt{7} + \sqrt{714 - 259\sqrt{7}}\,\right),$$
$$a_4 = \frac{1}{21}\left(28 - 6\sqrt{7} + \sqrt{714 - 259\sqrt{7}}\,\right), \quad b_4 = \frac{1}{21}\left(-14 + 8\sqrt{7} - \sqrt{714 - 259\sqrt{7}}\,\right),$$

λ_1	0.1122079561300387
λ_2	0.1892374781489235
λ_3	0.2227545696261541
λ_4	0.2198958512792939

This is a minimal cubature formula. It is obtained by the method of reproducing kernel (cf. [21]). We choose the first point as $(0,1)$ and the second point as $(0, 2(7+\sqrt{7})/21))$, which is one zero of $K_2((x_1, x_2), (0,0)) = 855(1 - 14t + 21t^2)$ where $t = 1 - x_1 - x_2$.

Degree 5: $N = 7$,

$$\begin{array}{llll} (1/3, 1/3), & & & 9/70 \\ (a,b), & (b,a), & (b,b), & \lambda_1 \\ (c,d), & (d,c), & (d,d), & \lambda_2 \end{array}$$

with

$$a = (9 - 4\sqrt{3})/33, \quad b = (15 + 8\sqrt{3})/33,$$
$$c = (15 + 8\sqrt{3})/33, \quad d = (9 + 4\sqrt{3})/33,$$
$$\lambda_1 = (122 + 9\sqrt{3})/840, \quad \lambda_2 = (122 - 9\sqrt{3})/840.$$

This formula is minimal and also symmetric on Σ^2. It is obtained from [24, p. 301, U_3: 11-2] via Theorem 1.1. Its nodes are common zeros of orthogonal polynomials P_1^3, P_3^3 and $P_0^3 + P_2^3/\sqrt{7}$.

Degree 7: $N = 12$,

$$(x_i, y_i), \quad (1 - x_i - y_i, x_i), \quad (y_i, 1 - x_i - y_i), \qquad \lambda_i$$

i	x_i	y_i	λ_i
1	0.03580765622	0.03752772328	0.09458016209
2	0.66398055628	0.03149046699	0.08858749646
3	0.26858535175	0.19042737706	0.08088547446
4	0.68672614839	0.29513413158	0.06928020033

This formula is minimal. When it is transformed to the equilateral triangle $\triangle$ (see the next section), it leads to a formula that is invariant under the rotation by $2\pi/3$. Formulae with this symmetry are discussed in [6], which we followed to find this formula. A minimal formula of degree 7 is constructed in [6] for the unit weight function.

In the Table 2.1 below, we list these formulae of degree 3 to 7 on Σ^2 together with others that can be obtained from the formulae on S^2 via Theorem 1.1. Whenever a formula is obtained via Theorem 1.1, we give reference from [24, p. 295-301]. The minimal formulae are marked by an asterisk.

Table 2.1. Formulae on Σ^2

Degrees	N	Reference
3	4^*	
	4^*	$U_n : 7-2$, $n=3$
	7	$U_3 : 7-2$
4	6^*	
	7	$U_3 : 9-1$
	9	$U_4 : 9-2$
	10	$U_3 : 9-3$
5	7^*	$U_3 : 11-2$
	10	$U_3 : 11-1$
	13	$U_3 : 11-3$
7	12^*	

The three new minimal formulae of degree 3, 4 and 7 yield new cubature formulae on S^2 of degree 7, 9 and 15. The result is presented in the following table, in which we also include the minimal number of nodes, N^*, among all $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric formulae of the same degree that are known.

Table 2.2. Formulae on S^2

Degrees	N	N^*
7	32	26, $[24, U_3 : 7-2]$
9	42	32, $[24, U_3 : 9-1]$
15	96	86, [12]

The table shows that these formulae on S^2 use far more nodes than the minimal ones, although they are obtained from the minimal cubature formulae on Σ^2. In fact, the number of nodes of the formula (1.3) in the Theorem 1.1 depends on how many nodes of (1.2) are on the boundary of Σ^2. Let N_0, N_1 and N_2 be the number of nodes in (1.2) that are on the vertices, on the edges, and in the interior of the triangle. Then the number of nodes of (1.2) is equal to $N_0 + N_1 + N_2$. Let $N_M(\Sigma^2)$ and $N_{2M+1}(S^2)$ denote the number of nodes of the formula (1.2) and the formula (1.3), respectively. Then the following relation holds,

$$(2.1)\qquad N_M(\Sigma^2) = N_0 + N_1 + N_2 \quad \Longleftrightarrow \quad N_{2M+1}(S^2) = 8N_2 + 4N_1 + 2N_0.$$

This formula also shows that minimal cubature formulae on S^2 may not lead to minimal formulae on Σ^2.

3. Symmetric formulae on Σ^2 and fully symmetric formulae on S^2

In this section, we consider symmetric formulae with respect to the weight function W_0 on Σ^2, which correspond to cubature formulae with octahedral symmetry on S^2. In the first part of the section we present a method of constructing symmetric formulae given by Lyness and Jespersen in [17]. Our findings of cubature formulae are discussed in the Subsection 3.2, and we discuss the numerical computation in Subsection 3.3.

3.1. Symmetric formulae on Σ^2. Instead of Σ^2, Lyness and Jespersen used the equilateral triangle

$$\triangle = \{(x,y) : x \le 1/2, \quad \sqrt{3}y - x \le 1, \quad -\sqrt{3}y - x \le 1\},$$

whose symmetric group $\mathcal{S}_3(\triangle)$ is generated by a rotation through an angle $2\pi/3$ and a reflection about the x-axis. The triangle Σ^2 can be transformed into $\triangle$ by the affine transformation

$$(3.1)\quad \varphi : (x_1, y_1) \in \Sigma^2 \mapsto (x,y) \in \triangle, \qquad x = 3(x_1 + x_2)/2 - 1, \quad y = \sqrt{3}(x_2 - x_1)/2.$$

It is easy to see that invariance is preserved under φ; in particular, if a function f defined on $\triangle$ is invariant under $\mathcal{S}_3(\triangle)$, then the function $f \circ \varphi$ defined on Σ^2 is invariant under $\mathcal{S}_3(\Sigma^2)$. The weight function W_0 on Σ^2 becomes

$$W^0_\triangle(x,y) = 3^{-3/2}((1+x)^2 - 3y^2)^{-1/2}(1-2x)^{-1/2}/(2\pi)$$

A basis for the class of $\mathcal{S}_3(\triangle)$-invariant polynomials of degree at most n, denoted by Π_n^G, can be written down in terms of the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ as follows,

$$(3.2)\qquad r^{2i}(r^3\cos 3\theta)^j, \qquad 0 \le 2i + 3j \le n.$$

Moreover, working with functions $g(r,\theta) = f(r\cos\theta, r\sin\theta)$ in polar coordinates, a basic invariant cubature formula takes the form

$$Q(r,\theta)g = \frac{1}{6}\sum_{j=1}^{3}\Big\{g\big(r, \theta + \frac{2\pi j}{3}\big) + g\big(r, -\theta + \frac{2\pi j}{3}\big)\Big\},$$

which is just a sum over the $\mathcal{S}_3(\triangle)$-orbit of the point (r, θ). Because of the invariance of $Q(r, \theta)$, we assume that r can take negative value and $0 \le \theta < \pi/3$. Three distinct types of orbits occur according to $r = 0$ (center of triangle); $r \ne 0$, $\cos 3\theta = 1$ (median of triangle); and $r \ne 0$, $\cos 3\theta \ne 1$. These three types are denoted as **type 0, type 1** and **type 2**, whose corresponding $Q(r, \theta)g$ requires 1, 3, or 6 function evaluations, respectively. Let n_i denote the number of orbits of type i in a symmetric cubature formula. The standard (*holistic*) type cubature formula takes the form

$$Q(g) = n_0 \lambda_0 g(0,0) + \sum_{i=1}^{n_1} \lambda_i Q(r_i, 0) g + \sum_{i=n_1+1}^{n_1+n_2} \lambda_i Q(r_i, \alpha_i) g. \tag{3.3}$$

The number of nodes of this formula, denoted by $\mu(Q)$, is $\mu(Q) = n_0 + 3n_1 + 6n_2$. It is shown in [17] that the cubature formula $Q(g)$ is of degree M if its nodes and weights satisfy the following system of equations:

$$\begin{aligned}
\lambda_0 + \sum_{i=1}^{n_1} \lambda_i + \sum_{i=n_1+1}^{n_1+n_2} \lambda_i &= v_{0,0}, \\
\sum_{i=1}^{n_1} \lambda_i r_i^j + \sum_{i=n_1+1}^{n_1+n_2} \lambda_i r_i^j \cos 3k\theta_i &= v_{j,3k}, \quad 2 \le j \le M, \ k = k_0, \\
\sum_{i=n_1+1}^{n_1+n_2} \lambda_i r_i^j (\cos 3k_0\theta_i - \cos 3k\theta_i) &= v_{j,3k}, \quad j = 6, \text{ or } 8 \le j \le M, \\
& \qquad 6 \le 3k \le j, \ j + k \,\text{even},
\end{aligned} \tag{3.4}$$

where $k_0 = 0$ if j is even and $k_0 = 1$ if j is odd, the numbers $v_{j,3k}$ are defined by

$$v_{j,3k} = \int_{\triangle} r^{j+1} \cos 3k\theta \, W_{\triangle}^0 (r\cos\theta, r\sin\theta) dr d\theta, \tag{3.5}$$

the integral is over the region defined by $(r\cos\theta, r\sin\theta) \in \triangle$. For each M, the system contains

$$E(M) = [(M^2 + 6M + 12)/12]$$

equations, where $[x]$ denote the smallest integer less than or equal to x.

It is often useful to construct cubature formulae that have some nodes on the edges or at the vertices of the triangle. To describe such a formula, we use a sub-classification of the types of basic formula $Q(r, \alpha)$. The type 1 ($r \ne 0$, $\cos 3\theta = 1$) is split into three sub-types according to $r = -1$ (vertex), $r = 1/2$ (median of edge), and $-1 < r < 1/2$; the type 2 ($r \ne 0$, $\cos 3\theta \ne 0$) is split into two sub-types according to $r\cos\theta = 1/2$ (on an edge but not at median of the edge nor at a vertex), and $r\cos\theta \ne 1/2$. Accordingly, n_1, the number of orbits of type 1, is split as $n_1 = m_1 + m_2 + m_3$, and n_2, the number of orbits of type 2, is split as $n_2 = m_4 + m_5$. Such a formula is called the *cytolic* type. A cytolic formula is identified by $[n_0; m_1, m_2, m_3; m_4, m_5]$. We note that n_0, m_1 and m_2 can only be either 0 or 1.

According to the discussion in the end of section 2, see (2.1), a cytolic cubature formula is preferable for obtaining, via Theorem 1.1, a cubature formula on the sphere with fewer nodes. It is not hard to see that a cytolic formula $[n_0; m_1, m_2, m_3; m_4, m_5]$ leads to a cubature formula on S^2 whose number of nodes is equal to

$$N(S^2) = 8n_0 + 6m_1 + 12m_2 + 24m_3 + 24m_4 + 48m_5. \tag{3.6}$$

A formula that has nodes on the vertices and edges of Σ^2 uses more nodes than a formula that has all nodes in the interior, but it leads to a formula on S^2 with fewer nodes.

The nonlinear system of equations (3.4) remains in the same form for the cytolic type formulae, we only need to assign proper values of certain r_i and θ_i according to the given type. To form the nonlinear system equations (3.4), we choose n_0 and m_i so that the number of parameters matches with the number of equations. For the type $[n_0; m_1, m_2, m_3; m_4, m_5]$, this means

$$n_0 + m_1 + m_2 + 2m_3 + 2m_4 + 3m_5 = [(M^2 + 6M + 12)/12], \tag{3.7}$$

where M, as before, is the degree of the cubature formula. For each fixed M there may be a number of integer solutions to the above equation, leading to different types of cubature formulae. In this regard, the *consistency conditions* are very useful. Following the argument in [17] for the holistic type, the conditions for the cytolic type are

$$\begin{aligned} 2m_4 + 3m_5 &\geq E(M-6), \\ m_1 + m_2 + 2(m_3 + m_4) + 3m_5 &\geq E(M) - 1, \\ n_0 + m_1 + m_2 + 2(m_3 + m_4) + 3m_5 &\geq E(M). \end{aligned} \tag{3.8}$$

They are also included in the conditions found in [8] for d-dimensional simplex. These conditions ensure that there are enough unknown parameters to match part (properly defined) or all of nonlinear equations, although they are neither necessary nor sufficient for solving the equations. Another useful restriction is as follows.

Theorem 3.1. *A formulae of degree M is of type $[n_0; m_1, m_2, m_3; m_4, m_5]$ only if*

$$m_5 > \begin{cases} (M-9)/4, & \text{if } m_4 \neq 0 \text{ and } M \geq 9, \\ (M-6)/4, & \text{if } m_4 = 0 \text{ and } M \geq 6, \\ (M-3)/4, & \text{if } m_3 = 0 \text{ and } M \geq 3. \end{cases} \tag{3.9}$$

Proof. Let ℓ_i, $i = 1, 2, 3$, be the linear polynomials such that $\ell_i = 0$ give the equations of the sides, and we choose the sign so that ℓ_i are nonnegative on $\triangle$. Let h_i, $i = 1, 2, 3$, be the linear polynomials such that $h_i = 0$ give the equations of the medians of $\triangle$. Furthermore, let g_i, $i = 1, 2, \ldots, n_5$, be the quadratic polynomials so that $g_i = 0$ gives the equation of the circle that has center at origin and radius r_i. If $m_4 \neq 0$, then the polynomial

$$\ell_1 \ell_2 \ell_3 h_1^2 h_2^2 h_3^2 g_1^2 \ldots g_{m_5}^2$$

will vanish on all nodes of the formula. Since the polynomial is positive on $\triangle$, its degree has to be bigger than the degree of the cubature formula, which leads to the desired inequality. If $m_4 = 0$, then the factors $\ell_1\ell_2\ell_3$ can be dropped from the polynomial, leading to the desired inequality in this case. If $m_3 = 0$, then $h_1^2h_2^2h_3^2$ can be dropped from the polynomial. $\square$

This theorem and its proof are extensions of the result in [17, p. 26], which deals with the cases of $M = 5, 6, 9$. There are other conditions that can be derived this way; for example, if both m_3 and m_4 are zero, then $m_5 > M/4$. For fixed M, it is possible to identify all possible integer solutions of (3.7) which also satisfy the restriction (3.8) and (3.9); the number of the solutions, however, is still large even for moderate M.

Some particular choices of the types lead to a system (3.4) that is split into subsystems with independent variables; the smaller size of the subsystem makes them easier to solve. Such a split is possible since the third group of the equations in (3.4) does not contain r_i and λ_i for $i = 1, 2, \ldots, n_1$; and it occurs whenever m_4 and m_5 satisfy the equation

$$2m_4 + 3m_5 = E(M-6) = [(M^2 - 6M + 12)/12], \tag{3.10}$$

because the third group of equations contain $E(M-6)$ independent parameters. It is not hard to check that the integer solutions of the above equation exist for every $M \geq 7$, except $M = 10$; hence, the splitting occurs for each $M \neq 10$. One important class of formulae that admits the splitting corresponds to the cubature formulae constructed by Lebedev in [12–15] on S^2 with octahedral symmetry. Apart from a few lower degree cases, Lebedev consider the formulae on S^2 that correspond to the types

$$[1; 1, 0, 3m; m, m(m-1)] \quad \text{and} \quad [1; 1, 1, 3m+1; m, m^2], \tag{3.11}$$

which are of degree $6m + 2$ and $6m + 5$, respectively; and he has constructed formulae for $m = 1, 2, 3, 4$.

Our strategy for choosing the type $[n_0; m_1, m_2, m_3; m_4, m_5]$ is as follows. We search for types whose corresponding formulae on S^2 have fewer nodes. This means finding n_0 and m_i, which satisfy (3.7), (3.8) and (3.9), so that N in (3.6) is minimal or close to minimal. To this end, we choose n_0, m_1 and m_2 with value one whenever possible, and then m_5 as small as possible. As a starting point, we choose m_4 and m_5 satisfy (3.10) so that the system (3.4) is split into subsystems.

3.2. Fully symmetric cubature formulae on S^2. We have attempted to find symmetric cubature formulae of degree up to 20 on the triangle. There are some nonlinear systems that we found no solution. For each $M \leq 20$, however, we found at least one type of cubature formula that has all nodes inside $\triangle$ and have all positive weights, they correspond to cubature formulae on S^2 of degree up to 41. Some of the formulae, however, have nodes outside $\triangle$, they will not lead to cubature formulae on S^2 via Theorem 1.1.

We report our findings as fully symmetric cubature formulae on S^2 and list the results in Table 3.1 below. Each formula is identified by its $[n_0; m_1, m_2, m_3; m_4, m_5]$ type and we give its number of nodes N. If a formula has all positive weight, we write P in the last column, otherwise, we write N.

Table 3.1. Fully symmetric cubature formula on S^2

Degrees	Type	# of Nodes	Quality
3	0;1,0,0;0,0	6	P [S]
	1;0,0,0;0,0	8	P [S]
5	1;1,0,0;0,0	14	P [S]
7	1;1,1,0;0,0	26	P [S]
	1;0,0,1;0,0	32	N [S]
9	1;1,0,0;1,0	38	P [L]
11	1;1,1,1;0,0	50	P [S]
13	1;1,1,1;1,0	74	N [L]
	0;1,0,2;1,0	78	P
15	1;1,0,2;1,0	86	P [L]
	0;1,1,2;1,0	90	P
17	1;1,0,3;1,0	110	P [L]
	1;1,0,2;2,0	110	N
19	1;1,1,3;0,1	146	P [L]
	1;1,1,2;1,1	146	P
	1;0,0,4;0,1	152	P
21	1;1,1,3;1,1	170	N
	1;1,1,2;2,1	170	N
	1;0,0,3;2,1	176	N
	0;0,0,3;1,2	192	P (2)
	1;0,0,2;0,3	200	P
23	1;1,1,4;1,1	194	P [L]
	0;1,0,4;2,1	198	P
	1;0,0,5;1,1	200	P
	1;0,0,4;2,1	200	N
25	1;1,0,5;2,1	230	N [L]
	1;0,0,5;1,2	248	P
	0;0,0,5;0,3	264	P
27	1;1,1,5;1,2	266	N [L]
	1;0,0,6;1,2	272	N
	1;1,0,5;0,3	278	N
	0;0,0,5;1,3	288	P
29	1;1,0,6;2,2	302	P [L]
	0;0,0,6;0,4	336	P
31	1;0,0,4;3,4	368	P
33	1;0,0,6;1,5	416	P
35	1;1,1,7;2,4	434	P [L]
	1;0,0,8;2,4	440	P
37	1;0,0,5;1,8	536	P
39	0;0,0,4;1,10	600	P (2)
41	1;1,0,9;3,6	590	P [L]

The types marked by [S] correspond to formulae in Stroud's book [24], types marked by [L] correspond to formulae on S^2 found by Lebedev. The types [0;1,0,2;1,0] of degree 13 and [1;1,0,2;2,0] of degree 17 have been constructed by Keast in [8], but the numerical values of the nodes and weights are not given there. All other formulae in the table are new; in particular, these include formulae of degrees 21, 31, 33, 37 and 39, where no formulae of the same degree are known previously, and formulae of degrees 25 and 27 with all positive weights, where only formulae with negative nodes are known. We note that all formulae found by Lebedev have smaller number of nodes on S^2, although their corresponding formulae on Σ^2 are not. Lebedev also found one formula for each of degree 47, 53 and 59 (the degree 41, 47, 53 are found in [15] joint with Skorokhodov). Because the system of equations (3.3) is nonlinear, its solution may not be unique. Indeed, in the cases of $[0; 0, 0, 3; 1, 2]$ of degree 21 and $[0; 0, 0, 4; 1, 10]$ of degree 39, we found two solutions in each case and we mark these cases by (2) in the table.

For each formula that is not marked by [S] or [L], we give the numerical values of the weights and nodes in the Appendix.

In the Table 3.1, we only include the types that we found solution and the solution leads to a fully symmetric formula on the sphere. The types that we attempted and found solution with some nodes outside of the triangle are reported in the Table 3.2 below, those types that we attempted but could not find a solution are not recorded. Moreover, we stopped when one formula with all positive weights and all nodes inside $\triangle$ was found following our strategy, we did not attempt all possible types.

Table 3.2. Cubature formula on triangle (PO or NO)

Degrees	Type	# of Nodes	Quality
6	1;0,0,2;1,0	13	NO
7	1;0,0,2;0,1	13	PO
8	1;0,0,3;0,1	16	PO
10	0;0,0,4;0,2	24	PO
	1;0,0,4;1,1	25	NO
11	0;0,0,5;0,2	27	PO
13	1;0,0,5;2,2	40	PO
15	0;0,0,6;0,5	48	PO
	1;0,0,5;2,4	52	PO
	0;0,0,4;2,5	54	PO
16	1;0,0,7;0,5	52	PO
	1;1,0,7;1,4	55	NO
19	1;0,0;5;1,9	76	PO

In the table we use the notation of [1], the symbol PO (or NO) means that the nodes are outside of the region and the weights are all positive (or some negative).

Numerical computation that leads to symmetric cubature formulae for the unit weight function is carried out in [17] for $M \leq 11$ and in [3] for $M \leq 20$. The equations (3.4) in the cases of the unit weight function and the weight function W_0 are of the same form, except that the moments are different, which only change the right hand side of the equations. Since the equations are nonlinear, formulae of the same type may possess different quality for different weight functions. For example, for the type $[1; 0, 0, 6; 1, 2]$ of degree 13, we found a formula for W_0 with some negative weights, while the formula for the unit weight function has all positive weights. The most interesting case, however, is perhaps the type $[1; 0, 0, 8; 1, 7]$ of degree 19, which we found no solution for the weight function W_0, but a solution is found for the unit weight function in [3]. This shows that the nonlinear system (3.4) is sensitive to the change of weight functions.

3.3. Remarks on numerical computation. The numerical computation was carried out on a DEC Alphastation 500 in double precision, using the DUNLSF Fortran subroutine in the IMSL Math/Library (Visual Numerics, Inc., 1994); however, moments $v_{j,3k}$ in (3.5) were computed exactly using Maple. The subroutine DUNLSF employs iterative techniques which require an initial estimate of the solution. For solving the nonlinear system (3.4), this means that we need to provide initial values for the weights λ_i and for the parameters r_i and θ_i that determine nodes. To determine the initial values, we have followed the strategy in [3] for solving the systems for the unit weight function. The node locations of the formulae for the weight function W_0 appear to be similar to those for the unit weight function: nodes are located closer to the edge of the triangle than the centroid, and are located closer to the median $\theta = \pi/3$ than $\theta = 0$. Our computation shows that whenever a formula of a given type has a solution, then even a rough initial estimate leads to the solution in reasonable computing time. For example, finding a formula of degree 19 needs less than 30 minutes.

For each formula of degree M, we compute the relative error and the absolute error of $\mathcal{I}(f) - \mathcal{I}_n(f)$ for all invariant polynomials f of degree $\le M$, where $\mathcal{I}(f)$ stands for the integral of f with respect to W_0 on the triangle and $\mathcal{I}_n(f)$ stands for the cubature formula. The result shows that

$$\sup\{|\mathcal{I}(f) - \mathcal{I}_M(f)|/\mathcal{I}(f) : f \in \Pi_M^G\} \le 0.5 \times 10^{-13}$$

for formulae of degree up to 19 and $\le 0.5 \times 10^{-12}$ for degree 20. The numerical values of the parameters are given to 12 digits. The DUNLSF subroutine solves the nonlinear equations in the least square sense; that is, it finds the minimal solution of $\sum f_i^2(\mathbf{x})$, where $f_i = 0$ are nonlinear equations. In our computation, equations in (3.4) involve high powers of polynomials which are sensible to perturbations; for example, for $M = 20$, a perturbation in the 5-th decimal place of our solution did not change the order of the relative error 10^{-12}. For M large, the accuracy of the solution found by DUNLSF subroutine is limited by the machine accuracy. Because the computer we used has limited precision of 15 digits, we stopped at $M = 21$.

4. Final comments

We comment on some perspectives that are not covered in the present paper.

Remark 4.1. Theorem 1.1 establishes the connection between cubature formulae on Σ^d and $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetric cubature formulae on S^d. In [27] we also establish a connection between cubature formulae on the ball B^d and on S^d, and the connection has been used to construct cubature formulae on S^2 in [7]. Together, these results yield a correspondence between cubature formulae on Σ^d and $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetric formulae on B^d. In particular, a cubature formula for the weight function W_0 on Σ^2 corresponds to a formula for the weight function $1/\sqrt{1-x_1^2-x_2^2}$ on B^2. Thus, the results in Sections 2 and 3 also lead to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric formulae on B^2. On the other hand, those formulae constructed in [7] that are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric lead to formulae on Σ^2. However, not all formulae on B^2 in [7] are fully symmetric. In fact, the correspondence between formulae on B^d and on S^d is not restricted to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric formulae.

Remark 4.2. The connection between cubature formulae on S^d, Σ^d and B^d works for a large class of weight functions. In particular, cubature formulae for the unit weight function on Σ^2 corresponds to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric formula for $|x_1x_2x_3| d\omega$ on S^2 and for weight function $|x_1x_2|$ on B^2; and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric cubature formulae for the unit weight function on B^2 correspond to formula for $(1/\sqrt{x_1x_2})$ on Σ^2. For examples of formulae for the unit weight function on these domains, see the references in [22,24].

Remark 4.3. The connection between formulae on the three domains works also in higher dimension. Although a number of formulae of lower degrees have been constructed for the unit weight function in the literature (see [2, 5, 22, 24]), it may be of interests to construct formulae for the weight function $(u_1 \cdots u_d(1-u_1-\ldots-u_d))^{-1/2}$ on Σ^d and use them to generate cubature formulae on S^d. To our knowledge, the calculation of symmetric cubature formulae for this weight function on Σ^d for $d>2$ has not been taken previously, although the consistency conditions have been studied in [8] and [17]. For the unit weight function, some symmetric formulae of lower degrees on Σ^d have been constructed, see [2, 8, 22, 24] and ther references there.

Ackowledgement: The authors thank referees for their useful comments.

Appendix

We give the weights and nodes for the cubature formulae described in Section 3. The cubature formulae on S^2 are of the form (1.6) with $W(\mathbf{x}) = 1/4\pi$. Because of the symmetry, for each weight μ_k we need to give only one node $(v_{k,1}, v_{k,2}, v_{k,3})$. For a formula of type $[n_0; m_1, m_2, m_3; m_4, m_5]$, the nodes corresponding to n_0, m_1 and m_2 are

$$(\sqrt{1/3}, \sqrt{1/3}, \sqrt{1/3}), \quad (1,0,0), \quad (\sqrt{1/2}, \sqrt{1/2}, 0),$$

respectively; the weights corresponding to them are μ_0, μ_1 and μ_2. Note that some or all of μ_0, μ_1, μ_2 could be zero, which means that the corresponding node does not show up in the formula.

For each formula we give the value of nonzero μ_i, $i = 0, 1, 2$, first, those that are not given are understood as zero. We then given the table for the other nodes $(v_{i,1}, v_{i,2}, v_{i,3})$ and the corresponding weights μ_i start with $i = 3$ and follow the order of m_3, m_4 and m_5; that is, the type m_3 nodes are listed first, then the m_4 type and followed by the m_5 type.

Degree 13: [0;1,0,2;1,0]; $N = 78$

$\mu_1 = 0.013866592105$

i	x_i	y_i	z_i	μ_i
3	0.286640146767	0.914152532416	0.286640146767	0.013050931863
4	0.659905001656	0.659905001656	0.359236381200	0.013206423223
5	0.539490098706	0.841991943785	0.0	0.011942663555

Degree 15: [0;1,1,2;1,0]; $N = 90$

$\mu_1 = 0.013191522874, \qquad \mu_2 = 0.011024070845$

i	x_i	y_i	z_i	μ_i
3	0.337785899794	0.878522265967	0.337785899794	0.010538971114
4	0.658511676782	0.658511676782	0.364314072036	0.011656960715
5	0.399194381765	0.916866318264	0.0	0.010660818696

Degree 17: [1;1,0,2;2,0]; $N = 110$

$\mu_0 = 0.009103396603, \qquad \mu_1 = -0.002664002664$

i	x_i	y_i	z_i	μ_i
3	0.357406744337	0.862856209461	0.357406744337	0.010777836655
4	0.678598344546	0.678598344546	0.281084637715	0.009161945784
5	0.542521185161	0.840042120165	0.0	0.009798544912
6	0.222866509741	0.974848972321	0.0	0.009559874447

Degree 19-1: [1;1,1,2;1,1]; $N = 146$

$\mu_0 = 0.008559575701, \qquad \mu_1 = 0.006231186664, \qquad \mu_2 = 0.007913582691$

i	x_i	y_i	z_i	μ_i
3	0.201742306653	0.958436269875	0.201742306653	0.007736373931
4	0.675586904541	0.675586904541	0.295236631918	0.004644831902
5	0.443668207806	0.896191118781	0.0	0.007625284540
6	0.496188289109	0.814892033188	0.299579965948	0.006646198191

Degree 19-2: [1;0,0,4;0,1]; $N = 152$

$\mu_0 = 0.006159164865$

i	x_i	y_i	z_i	μ_i
3	0.154480689145	0.975843959536	0.154480689145	0.007661426126
4	0.414167295917	0.810512740174	0.414167295917	0.006632044977
5	0.667293171280	0.667293171280	0.330816636714	0.006075982031
6	0.703446477338	0.703446477338	0.101617454410	0.005261983872
7	0.449332832327	0.882270011260	0.140355381171	0.006991087353

Degree 21-1: [1;1,1,3;1,1]; $N = 170$

$\mu_0 = -0.056995598467, \qquad \mu_1 = 0.005570590570 \qquad , \mu_2 = 0.004620905358$

i	x_i	y_i	z_i	μ_i
3	0.186798108665	0.964475470501	0.186798108665	0.006173897540
4	0.366886721514	0.854861548529	0.366886721514	0.006304034638
5	0.607095656232	0.607095656232	0.512708229276	0.025447255860
6	0.399651971962	0.916666952228	0.0	0.006599388582
7	0.573253885705	0.795085657737	0.198037318162	0.006218761274

Degree 21-2: [1;1,1,2;2,1]; $N = 170$

$\mu_0 = -0.007545260195, \qquad \mu_1 = -0.004709932317, \qquad \mu_3 = 0.006599231780$

i	x_i	y_i	z_i	μ_i
3	0.295937832153	0.908207905163	0.295937832153	0.007200919394
4	0.519472253003	0.678452029786	0.519472253003	0.008304183973
5	0.446007176001	0.895029384409	0.0	0.006872624447
6	0.165319162227	0.986240120154	0.0	0.006895630527
7	0.566806527713	0.782784716286	0.256862702801	0.006393131123

Degree 21-3: [1;0,0,3;2,1]; $N = 176$

$\mu_0 = -0.059097949898$

i	x_i	y_i	z_i	μ_i
3	0.136045412794	0.981317120668	0.136045412794	0.005922907575
4	0.321321668532	0.890788847406	0.321321668532	0.006504946198
5	0.547239633521	0.633291060261	0.547239633521	0.025578972384
6	0.645751079582	0.763547996670	0.0	0.003940466271
7	0.407685091182	0.913122591128	0.0	0.006943311404
8	0.568997367119	0.784986512893	0.245026877683	0.006237689734

Degree 21-4: [0;0,0,3;1,2]; $N = 192$

i	x_i	y_i	z_i	μ_i
3	0.121942991996	0.985017671621	0.121942991996	0.004843132969
4	0.405172013544	0.819555537399	0.405172013544	0.005906722557
5	0.635692088835	0.635692088835	0.437939649249	0.005570538352
6	0.601743299291	0.798689552804	0.0	0.004679374357
7	0.595006226182	0.774595852605	0.214403488618	0.004559352813
8	0.368090580737	0.919422462557	0.138461762661	0.005774096403

Degree 21-5: [0;0,0,3;1,2]; $N = 192$

i	x_i	y_i	z_i	μ_i
3	0.114731000078	0.986749003163	0.114731000078	0.004256407290
4	0.505750398065	0.698879867870	0.505750398065	0.005470376179
5	0.682741864646	0.682741864646	0.260244293924	0.006041707753
6	0.590785303447	0.806828807884	0.0	0.006478529177
7	0.467118005205	0.838405331367	0.280850973913	0.004792379399
8	0.344252417343	0.930159244514	0.127648160971	0.004917443735

Degree 21-6: [1;0,0,2;0,3]; $N = 200$

$\mu_0 = 0.005200472756$

i	x_i	y_i	z_i	μ_i
3	0.124787616061	0.984304882522	0.124787616061	0.005028347403
4	0.382642965364	0.840933244743	0.382642965364	0.004910500721
5	0.580333662379	0.729548523912	0.361900250852	0.004214767940
6	0.600802371831	0.789769929774	0.123693039525	0.005272844391
7	0.371677251651	0.918768165963	0.133120538678	0.005509551481

Degree 23-1: [0;1,0,4;2,1]; $N = 198$

$\mu_1 = 0.005026500922$

i	x_i	y_i	z_i	μ_i
3	0.176588660459	0.968314458218	0.176588660459	0.005279416073
4	0.339207318490	0.877426230612	0.339207318490	0.003732271633
5	0.498904016243	0.708653346251	0.498904016243	0.006051284349
6	0.679838773734	0.679838773734	0.275024514281	0.005561610887
7	0.615520670749	0.788120741943	0.0	0.005177363547
8	0.364554848325	0.931181917008	0.0	0.005381929440
9	0.491903042583	0.842732170863	0.218709590304	0.004613082753

Degree 23-2: [1;0,0,5;1,1]; $N = 200$

$\mu_0 = 0.005651017861$

i	x_i	y_i	z_i	μ_i
3	0.115535209070	0.986561316356	0.115535209070	0.004259841569
4	0.282433358777	0.916767579979	0.282433358777	0.005294395887
5	0.441560530469	0.781056077285	0.441560530469	0.005588219406
6	0.670525624125	0.670525624125	0.317475628644	0.005591297404
7	0.706832372661	0.706832372661	0.027856667351	0.002936895883
8	0.345770219761	0.938319218138	0.0	0.005051846065
9	0.525118572444	0.836036015482	0.159041710538	0.005530248916

Degree 23-3 [1;0,0,4;2,1]; $N = 200$

$\mu_0 = -0.013079151392$

i	x_i	y_i	z_i	μ_i
3	0.083820743273	0.992949226292	0.083820743273	0.002613177651
4	0.208890425565	0.955368818946	0.208890425565	0.004696071564
5	0.527146296056	0.666508488400	0.527146296056	0.010074474289
6	0.684194032927	0.684194032927	0.252501585372	0.005747985286
7	0.599358474983	0.800480742096	0.0	0.005781262714
8	0.364297979633	0.931282439454	0.0	0.005394066441
9	0.461647695180	0.847134675079	0.263143017794	0.005859672926

Degree 25-1: [1;0,0,5;1,2]; $N = 248$

$\mu_0 = 0.004313243133$

i	x_i	y_i	z_i	μ_i
3	0.111691690919	0.987446166816	0.111691690919	0.003986365505
4	0.315067166823	0.895245977808	0.315067166823	0.003663031548
5	0.459462014542	0.760124538735	0.459462014542	0.004204049922
6	0.660753497156	0.660753497156	0.356103400703	0.004269004376
7	0.702154945166	0.702154945166	0.118139180447	0.004203472415
8	0.532020255731	0.846731626604	0.0	0.004142483118
9	0.519695051509	0.822359911686	0.231605762210	0.004090305599
10	0.329337385202	0.938200966027	0.106375909186	0.003789950437

Degree 25-2: [0;0,0,5;0,3]; $N = 264$

i	x_i	y_i	z_i	μ_i
3	0.107086858755	0.988465886798	0.107086858755	0.003694297843
4	0.333879222938	0.881504015295	0.333879222938	0.003835709610
5	0.515654412063	0.684252186435	0.515654412063	0.004019086734
6	0.668811941305	0.668811941305	0.324624666862	0.003295936329
7	0.702834079289	0.702834079289	0.109765723158	0.004023268501
8	0.520513375926	0.792782385892	0.317114985615	0.003428775431
9	0.523649799991	0.845078822417	0.107854860213	0.003917073182
10	0.320409052387	0.940317942977	0.114630734376	0.004053335212

Degree 27-1: [1;0,0,6;1,2]; $N = 272$

$\mu_0 = 0.004205508418$

i	x_i	y_i	z_i	μ_i
3	0.110768319347	0.987654169666	0.110768319347	0.003927799571
4	0.222696255452	0.949111561206	0.222696255452	-0.000407112852
5	0.322320222672	0.890067046976	0.322320222672	0.003694205329
6	0.462107300704	0.756910619077	0.462107300704	0.004136341725
7	0.660712667463	0.660712667463	0.356254883628	0.004202512176
8	0.702450665599	0.702450665599	0.114569301299	0.004176738239
9	0.525731112119	0.850650808352	0.0	0.004229582701
10	0.524493924092	0.819343388819	0.231479015871	0.004071467594
11	0.323348454269	0.939227929750	0.115311201101	0.004080914226

Degree 27-2: [1;1,0,5;0,3]; $N = 278$

$$\mu_0 = 0.004145413998, \qquad \mu_1 = -0.001001399850$$

i	x_i	y_i	z_i	μ_i
3	0.103271889407	0.989277430106	0.103271889407	0.004007770760
4	0.326015048812	0.887371610937	0.326015048812	0.003609101265
5	0.464476869175	0.754004294419	0.464476869175	0.004065377803
6	0.661042838405	0.661042838405	0.355027789879	0.004167019227
7	0.702427075066	0.702427075066	0.114858210103	0.004176827652
8	0.523633133581	0.818640861296	0.235871748274	0.003963497360
9	0.526385749330	0.849564329470	0.034036641931	0.002254418391
10	0.324014265315	0.939146832366	0.114096376493	0.004036641877

Degree 27-3: [0;0,0,5;1,3]; $N = 288$

i	x_i	y_i	z_i	μ_i
3	0.110332978624	0.987751622452	0.110332978624	0.003893829077
4	0.319075401552	0.892402250249	0.319075401552	0.003606286203
5	0.453117779552	0.767703429527	0.453117779552	0.003808504359
6	0.614431551407	0.614431551407	0.494921950685	0.002421634085
7	0.702545464357	0.702545464357	0.113400798163	0.004077606558
8	0.530118512908	0.847923559216	0.0	0.004062279727
9	0.622283431805	0.708196634681	0.333497911729	0.002263516691
10	0.517277385525	0.826495160269	0.222103256339	0.003782934834
11	0.326096877477	0.939054487386	0.108800258362	0.003851811803

Degree 29: [0;0,0,6;0,4]; $N = 336$

i	x_i	y_i	z_i	μ_i
3	0.072505573442	0.994729050365	0.072505573442	0.001894697146
4	0.310481161354	0.898444709979	0.310481161354	0.003783811492
5	0.440992773670	0.781697350093	0.440992773670	0.003012182159
6	0.527582710733	0.665817517546	0.527582710733	0.002724403361
7	0.659924119492	0.659924119492	0.359166135688	0.003559107906
8	0.699729990451	0.699729990451	0.144069014458	0.003227454716
9	0.594169250176	0.802748431448	0.050575270178	0.002024884516
10	0.531107498266	0.806604859340	0.259448311183	0.003467968868
11	0.426665999347	0.898209890946	0.105712425044	0.003229411196
12	0.247454899976	0.963715742772	0.100090157417	0.003010240364

Degree 31: [1;0,0,4;3,4]; $N = 368$

$\mu_0 = 0.000578329494$

i	x_i	y_i	z_i	μ_i
3	0.097855721318	0.990377966037	0.097855721318	0.003061522104
4	0.336734048041	0.879329495570	0.336734048041	0.002631322890
5	0.521197545604	0.675800441634	0.521197545604	0.002821112765
6	0.658338802723	0.658338802723	0.364938408033	0.002963046287
7	0.635957331845	0.771724220219	0.0	0.002854170188
8	0.475363041502	0.879789735547	0.0	0.002771653333
9	0.291089031268	0.956695968360	0.0	0.002626546226
10	0.622288265573	0.760613900822	0.184996779450	0.002893976172
11	0.505117362658	0.786669092004	0.354976322629	0.002837749321
12	0.461391815178	0.869489953690	0.176365567271	0.002715804588
13	0.289362045905	0.943297248354	0.162665016636	0.002424728107

Degree 33: [1;0,0,6;1,5]; $N = 416$

$\mu_0 = 0.002848140682$

i	x_i	y_i	z_i	μ_i
3	0.087642362514	0.992289087205	0.087642362514	0.002449327062
4	0.244826259453	0.938147219451	0.244826259453	0.002179377451
5	0.373613468997	0.849014694553	0.373613468997	0.002653179507
6	0.483960471286	0.729084716933	0.483960471286	0.002817792197
7	0.648358940509	0.648358940509	0.399075642609	0.002832643891
8	0.692365152833	0.692365152833	0.203128014525	0.002776115721
9	0.424680986592	0.905343061843	0.0	0.002634851612
10	0.663530717023	0.747173970345	0.038184363366	0.001044959201
11	0.560342562719	0.821357965819	0.106711313324	0.002552267148
12	0.544103582859	0.784667458480	0.297066104970	0.002771951327
13	0.408251323723	0.892327016644	0.192570382056	0.002540900745
14	0.259902436546	0.962160309302	0.081842914665	0.002276921078

Degree 35: [1;0,0,8;2,4]; $N = 440$

$\mu_0 = 0.002515482567$

i	x_i	y_i	z_i	μ_i
3	0.069156813118	0.995205843230	0.069156813118	0.001527515529
4	0.175148458557	0.968837465693	0.175148458557	0.002054028840
5	0.285287793163	0.914998224121	0.285287793163	0.002318417781
6	0.392405552644	0.831886870018	0.392405552644	0.002451618442
7	0.491306203394	0.719191510665	0.491306203394	0.002504293398
8	0.645641884498	0.645641884498	0.407790527065	0.002513606412
9	0.690921150829	0.690921150829	0.212734404066	0.002529886683
10	0.707101476221	0.707101476221	0.003873584007	0.001275574306
11	0.471598691154	0.881813287778	0.0	0.002417442376
12	0.210272522872	0.977642811115	0.0	0.001910951282
13	0.590515704894	0.799927854385	0.106801826076	0.002512236855
14	0.555015236112	0.771746262687	0.310428403520	0.002496644054
15	0.450233038264	0.868946032283	0.205482369646	0.002416930044
16	0.334436314543	0.937180985852	0.099217696370	0.002236607760

Degree 37: [1;0,0,5;1,8]; $N = 536$

$\mu_0 = 0.001436589472$

i	x_i	y_i	z_i	μ_i
3	0.181665204347	0.966434429777	0.181665204347	0.002233871811
4	0.303427242195	0.903251801763	0.303427242195	0.002119180525
5	0.483529149430	0.729656853119	0.483529149430	0.002281458727
6	0.625463680619	0.625463680619	0.466465827743	0.001864035223
7	0.705074619796	0.705074619796	0.075759890693	0.001858409063
8	0.444572576129	0.895742833940	0.0	0.002336486555
9	0.630713490341	0.746274388124	0.212779300527	0.001818751796
10	0.596705492981	0.724560936181	0.344897092487	0.001961713367
11	0.587046661123	0.807169311670	0.062079948160	0.001611967438
12	0.505464222397	0.844656624412	0.176241614590	0.001942087580
13	0.458746299673	0.824994538226	0.330054305281	0.002245940979
14	0.369539839929	0.915094022626	0.161379169844	0.001967307858
15	0.277639225331	0.958974697835	0.057306103258	0.001561575961
16	0.116956662074	0.992865475735	0.023222538401	0.001137835823

Degree 39-1: [0;0,0,4;1,10]; $N = 600$

i	x_i	y_i	z_i	μ_i
3	0.067102856429	0.995487023179	0.067102856429	0.001461069347
4	0.338906615896	0.877658596154	0.338906615896	0.002081064425
5	0.448234256905	0.773415866060	0.448234256905	0.002003883131
6	0.701808440747	0.701808440747	0.122187663015	0.001868396116
7	0.633404727273	0.773820684311	0.0	0.001974704892
8	0.563818465348	0.651676642607	0.507371946024	0.001120631568
9	0.628884552131	0.731765201351	0.262724019044	0.001626513803
10	0.568134999733	0.808896715135	0.151356289337	0.001822889111
11	0.490729623093	0.818248053726	0.299423712476	0.001943309150
12	0.589988659798	0.705342105582	0.392945155719	0.001766607242
13	0.493911282058	0.868009534637	0.051098857469	0.001361185877
14	0.415275416293	0.891987868825	0.178616825894	0.001852624903
15	0.350457251849	0.935213276436	0.050555338040	0.001383169283
16	0.278497898816	0.940831443705	0.193067643306	0.001483291414
17	0.205495716318	0.975518961862	0.078321552738	0.001778552026

Degree 39-2: [0;0,0,4;1,10]; $N = 600$

i	x_i	y_i	z_i	μ_i
3	0.076392926087	0.994146991992	0.076392926087	0.001861255447
4	0.338772022760	0.877762515257	0.338772022760	0.001793884349
5	0.454420421720	0.766161967633	0.454420421720	0.001974756953
6	0.688478852474	0.688478852474	0.228021357313	0.001692017041
7	0.519110405221	0.854707193834	0.0	0.001863462235
8	0.570664193557	0.653131915106	0.497756044325	0.001291965571
9	0.595919741511	0.712690180359	0.370070761473	0.001931700943
10	0.646378012450	0.759163172837	0.076594660580	0.001947222607
11	0.588862981291	0.775828653959	0.226561887708	0.001433019874
12	0.385384000208	0.903175728662	0.189084043588	0.001705425390
13	0.476387123224	0.823112602544	0.309096995068	0.001832278975
14	0.512379521668	0.846931416658	0.142036619409	0.001646505349
15	0.255068480104	0.944108152303	0.208805812209	0.001064763221
16	0.227252272934	0.970698830744	0.078103677500	0.001894382989
17	0.375230766664	0.925197335485	0.056672410922	0.001493380402

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