NONCONFORMING TETRAHEDRAL FINITE ELEMENTS FOR FOURTH ORDER ELLIPTIC EQUATIONS

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Abstract. This paper is devoted to the construction of nonconforming finite elements for the discretization of fourth order elliptic partial differential operator in three spatial dimensions. The newly constructed elements include two tetrahedron nonconforming finite elements and one quasi-conforming tetrahedron element. These elements are all proved to be convergent for a model biharmonic equation in three dimensions. In particular, the quasi-conforming tetrahedron element is a modified Zienkiewicz element while the non-modified Zienkiewicz element (a tetrahedral element of Hermite type) is proved to be divergent on a special but regular grid.

1. Introduction

The construction of appropriate finite element spaces for fourth order elliptic partial differential equations is an intriguing subject. This problem has been well-studied in two dimensional spaces. There are a lot of interesting constructions of both conforming and nonconforming finite element spaces for fourth order partial differential equations in two dimensions. In comparison, there has been very little work devoted to three dimensional problems.

A conforming finite element space for fourth order problems consist of piecewise polynomials that are globally continuously differentiable (C^1). This smoothness requirement can only be met with piecewise polynomials of sufficiently high degree. In two dimensions, it is known [20] that at least 5th order polynomial (the well known Argyris element) is needed on a triangular mesh. Such a high order polynomial leads to finite element spaces with a very large degree of freedoms which is not desirable in practical computations. As a result, many lower order nonconforming finite elements have been constructed and used in practice(see [5]).

In three spatial dimensions, even higher degree of polynomials are needed to construct conforming finite element space on, say, a tetrahedral finite element grid. In [19] (see also [9]), a conforming tetrahedral conforming finite element space was first constructed using 9th degree of polynomials. This element requires C^1 globally, C^2 on all element edges and C^4 on all element vertices. The degree of freedoms for this element is huge, 220 on each element!. In order to reduce the degree of polynomials, like in two dimensions, there have been some work on the

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construction of conforming finite element spaces on macro-elements (namely by further partitioning a tetrahedron into sub-tetrahedrons), see [1] and [18] (similar to Clough-Tocher in two dimensions) and [6]. But these elements all still have a very large degree of freedoms and furthermore the macro-elements are often awkward to use in practical applications.

To reduce the order of polynomials and degree of freedoms on each element, one naturally turns to nonconforming elements. Surprisingly, there are very little work on the construction of nonconforming finite elements for fourth order elliptic boundary value problems in three dimensions. The purpose of this work is to fill in this important gap in the literature for this type of elements.

The construction of nonconforming finite elements for fourth order problems in three dimensions is not only important from a mathematical point of view but also potentially important in practical applications. Indeed two dimensional biharmonic equations have been much used in modeling linear plates (see [7]) and such practical applications contributed to the importance and interests of studying efficient numerical methods such as nonconforming finite elements to solve this type of equations. We would like to point out that the three dimensional biharmonic operator also has important application in practice. One notable example is the Cahn-Hilliard diffusion equation that is used in the phase-field method to model and to predict complex microstructure evolution for many important material processes (see [3,11]). In addition to the finite difference method and also the spectral method, the fourth order term in the Cahn-Hilliard equation can also be discretized by the finite element method of mixed type, namely by writing the biharmonic operator as a product of two Laplacian operators. It is conceivable that the biharmonic operator can also be discretized by a direct finite element space as it is often done for biharmonic equations in two dimensions. As discussed above, the existing known conforming finite elements are not very practical and we hope that the nonconforming finite element methods proposed in this paper can be used for such applications.

In this paper, we will propose a family of nonconforming finite elements for 3-dimensional fourth order partial differential equations. We took the natural approach of trying to extend the various nonconforming finite element in two dimensions to three dimensions. In 2 dimensions, there are well-known nonconforming elements, including the elements named after Morley, Zienkiewicz, Adini, Bogner-Fox-Schmit, etc (see [2,5,8,10]). There are some other ways constructing elements, such as quasi-conforming method [16,4]. Extensions of these elements from two dimensions to three dimensions turn out to be not so obvious. In this paper, we will focus on tetrahedron complete or incomplete cubic elements, propose and analyze the following three types of elements:

(1) TNC20 — a tetrahedral element with 20 degrees of freedom and complete cubic polynomial shape function space.
(2) TNC16 — a tetrahedral element with 16 degrees of freedom and incomplete cubic polynomial shape function space.
(3) TQC16 — a tetrahedral element with 16 degrees of freedom similar to 9-parameter quasi-conforming element.

The first two are nonconforming elements, the last one is a quasi-conforming element. For nonconforming elements, the basic mathematical theory has been studied in many papers (see [5,8,13-15,22]). For quasi-conforming elements, detailed discussions can be found in [21,22]. Following these theories, we give the convergence analysis of the elements.

The element of Hermite tetrahedron of type $(3')$ [5], called TE16 in this paper, is also viewed as an element for biharmonic equation just like Zienkiewicz element. In 2-dimensional case, Zienkiewicz element is not convergent for general meshes. We will also show that TE16 element is divergent for a popular grids in three dimensions.

We note that the degree of freedoms of each of these finite elements is substantially smaller than any known conforming elements. We expect that they can be easily used in practice.

The rest of the paper is organized as follows. Section 2 gives the basic descriptions of nonconforming element method. Section 3 gives the detail descriptions of the finite elements. Section 4 shows the convergence of TNC20, TNC16 and TQC16 elements and the divergence of TE16 element. Some concluding remarks will be made in the end of the paper.

2. Preliminaries

In this section, we shall give a brief discussion on a model fourth order elliptic boundary value and how it may be discretized by a nonconforming finite element.

Given a bounded polyhedron domain $\Omega \subset R^3$ with boundary $\partial\Omega$, for nonnegative integer s, let $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ be the usual Sobolev space, norm and semi-norm respectively. Let $H_0^s(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$ and $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$.

For $f \in L^2(\Omega)$, we consider the following fourth order boundary value problem:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

where $\nu = (\nu_1, \nu_2, \nu_3)^\top$ is the unit outer normal to $\partial\Omega$ and Δ is the standard Laplacian operator.

For any function $v \in H^1(T)$, set

$$Dv = \Big(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_3}\Big).$$

When $v \in H^2(\Omega)$, we define

$$E(v) = \Big(\frac{\partial^2 v}{\partial x_1^2}, \frac{\partial^2 v}{\partial x_2^2}, \frac{\partial^2 v}{\partial x_3^2}, \frac{\partial^2 v}{\partial x_1 \partial x_2}, \frac{\partial^2 v}{\partial x_1 \partial x_3}, \frac{\partial^2 v}{\partial x_2 \partial x_3}\Big)^\top. \tag{2.2}$$

Let K be the 6×6 diagonal matrix with the first three elements in diagonal 1 and the last three 2. Define

$$a(v, w) = \int_\Omega E(w)^\top K E(v) dx, \quad \forall v, w \in H^2(\Omega). \tag{2.3}$$

The weak form of problem (2.1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v), \qquad \forall v \in H_0^2(\Omega). \tag{2.4}$$

For a subset $B \subset R^3$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials of degree not greater than r, and let $Q_r(B)$ the space of all polynomials of degree in each coordinate not greater than r.

Let (T, P_T, Φ_T) be a finite element where T is the geometric shape, P_T the shape function space and Φ_T the vector of degrees of freedom, and let Φ_T be P_T-unisolvent (see [5]). Take $\mathcal{T}_h$ a triangulation of Ω with mesh size h. For each element $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T and ρ_T be the diameter of the largest ball contained in T.

Let $\{\mathcal{T}_h\}$ be a family of triangulations with $h \to 0$. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is quasi-uniform, namely it satisfies that $h_T \le h \le \eta\rho_T$, $\forall T \in \mathcal{T}_h$ for a positive constant η independent of h.

For each $\mathcal{T}_h$, let V_{h0} be the corresponding finite element space associated with (T, P_T, Φ_T) for the discretization of $H_0^2(\Omega)$. This defines a family of finite element spaces $\{V_{h0}\}$. In the case of nonconforming element, $V_{h0} \not\subset H_0^2(\Omega)$.

For $v_h \in V_{h0}$ and $T \in \mathcal{T}_h$, denote by v_h^T be the restriction of v_h on T.

For $v, w \in H^2(\Omega) + V_{h0}$, we define

$$a_h(v, w) = \sum_{T \in \mathcal{T}_h} \int_T E(w)^\top K E(v) dx. \tag{2.5}$$

The finite element method for problem (2.4) corresponding to the element (T, P_T, Φ_T) is: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_{h0}. \tag{2.6}$$

We introduce the following mesh dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\left\{ \begin{aligned} \|v\|_{m,h} &= \Big(\sum_{T \in \mathcal{T}_h} \|v\|_{m,T}^2 \Big)^{1/2}, \\ |v|_{m,h} &= \Big(\sum_{T \in \mathcal{T}_h} |v|_{m,T}^2 \Big)^{1/2}, \end{aligned} \right. \qquad \forall v \in V_h + H^m(\Omega).$$

For each element $T \in \mathcal{T}_h$, we denote the interpolation operator of (T, P_T, Φ_T) by Π_T, and define Π_h by $(\Pi_h v)|_T = \Pi_T(v|_T)$, where $T \in \mathcal{T}_h$ and v is piecewisely smooth enough.

3. Tetrahedral Elements

Now let T be a tetrahedron with vertices $a_i = (x_{i1}, x_{i2}, x_{i3})$, $0 \le i \le 3$. Denote by F_i the face opposite a_i and by b_i the centric points of F_i, $0 \le i \le 3$. Let $\lambda_0, \cdots, \lambda_3$ be the barycentric coordinates of T.

Let $\hat{T}$ be the tetrahedron with vertices $\hat{a}_i$ given by

$$\hat{a}_0 = (0,0,0),\ \hat{a}_1 = (1,0,0),\ \hat{a}_2 = (0,1,0),\ \hat{a}_3 = (0,0,1).$$

Define

$$B_T = \begin{pmatrix} x_{11} - x_{01} & x_{21} - x_{01} & x_{31} - x_{01} \\ x_{12} - x_{02} & x_{22} - x_{02} & x_{32} - x_{02} \\ x_{13} - x_{03} & x_{23} - x_{03} & x_{33} - x_{03} \end{pmatrix},$$

and $F_T \hat{x} = B_T \hat{x} + a_0$, $\hat{x} \in R^3$, then

$$T = F_T \hat{T}, \quad a_i = F_T \hat{a}_i, \quad 0 \le i \le 3.$$

Set $B_T^{-1} = (\xi_{ij})_{3\times 3}$. Let B_1, B_2, B_3 be the row vectors of B_T^{-1} and

$$B_0 = -(B_1 + B_2 + B_3),$$

then

$$D\lambda_i = B_i, \quad 0 \le i \le 3. \tag{3.1}$$

3.1. **TNC20 Element.** For TNC20 element, (T, P_T, Φ_T) is defined by

1): T is a tetrahedron,
2): $P_T = P_3(T)$,
3): Φ_T is the vector with its component the following degrees of freedom,

$$v(a_j),\ \frac{\partial v}{\partial \nu}(b_j),\ 0 \le j \le 3,\ Dv(a_i)(a_j - a_i),\ 0 \le i \ne j \le 3,\ \forall v \in C^1(T).$$

For TNC20 element, we define

$$\begin{cases} q_i = \dfrac{9}{4\|B_i\|}\left(\displaystyle\sum_{\substack{0\le j\ne k\ne l\le 3\\ j\ne i, k\ne i, l\ne i}} \lambda_j\lambda_k\lambda_l - \lambda_i \sum_{\substack{0\le j\ne k\le 3\\ j\ne i, k\ne i}} \lambda_j\lambda_k\right), & 0 \le i \le 3 \\ p_i = 3\lambda_i^2 - 2\lambda_i^3 + \displaystyle\sum_{\substack{0\le k\le 3\\ k\ne i}} \frac{4B_iB_k^\top}{3\|B_k\|} q_k, & 0 \le i \le 3 \\ p_{ij} = \lambda_i^2\lambda_j + \dfrac{\|B_j\|}{9} q_j + \displaystyle\sum_{\substack{0\le k\le 3\\ k\ne i, k\ne j}} \frac{(2B_i + B_j)B_k^\top}{9\,\|B_k\|} q_k, & 0 \le i \ne j \le 3. \end{cases} \tag{3.2}$$

It can be verified that

$$\begin{cases} q_i(a_k) = 0, & Dq_i(a_k) = 0, & \dfrac{\partial q_i}{\partial \nu}(b_k) = \delta_{ik}, \\ p_i(a_k) = \delta_{ik}, & Dp_i(a_k) = 0, & \dfrac{\partial p_i}{\partial \nu}(b_k) = 0 \\ p_{ij}(a_k) = 0, & Dp_{ij}(a_k)(a_l - a_k) = \delta_{ik}\delta_{jl}, & \dfrac{\partial p_{ij}}{\partial \nu}(b_k) = 0 \end{cases} \tag{3.3}$$

when $0 \le i \ne j \le 3$ and $0 \le k \ne l \le 3$. Hence q_i, p_i and p_{ij} are basis functions respect to the degrees of freedom. Therefore Φ_T is P_T-unisolvent.

The corresponding interpolation operator Π_T can be written by, $\forall v \in C^1(T)$

$$\Pi_T v = \sum_{0\le i\le 3} p_i v(a_i) + \sum_{0\le i\le 3} q_i \frac{\partial v}{\partial \nu}(b_i) + \sum_{0\le i\ne j\le 3} p_{ij} Dv(a_i)(a_j - a_i). \tag{3.4}$$

For TNC20 element, define the corresponding finite element space V_{h0} as follows. $v \in V_{h0}$ if any only if (1) $\mid v|_T \in P_3(T), \forall T \in \mathcal{T}_h$, (2) v and its first order derivatives are continuous at all vertices of elements in $\mathcal{T}_h$ and vanish at all vertices belonging to $\partial\Omega$, and (3) $\frac{\partial}{\partial \nu} v$ is continuous at the barycentric points of all faces of elements in $\mathcal{T}_h$ and vanishes at barycentric points of all faces on $\partial\Omega$.

Unlike Zienkiewicz element, this complete cubic element space is not contain in $C^0(\bar{\Omega})$. But it has the following property.

Lemma 3.1. *Let V_{h0} be the finite element space of TNC20 element. If $T, T' \in \mathcal{T}_h$ with common face F, then*

$$\int_F Dv_h^T \mathrm{d}s = \int_F Dv_h^{T'} \mathrm{d}s, \quad v_h \in V_{h0}. \tag{3.5}$$

If a face F of $T \in \mathcal{T}_h$ is on $\partial\Omega$ then

$$\int_F Dv_h^T \mathrm{d}s = 0, \quad v_h \in V_{h0}. \tag{3.6}$$

Proof. Let $v_h \in V_{h0}$ and F be a common face of $T, T' \in \mathcal{T}_h$. Denote the unit normal of F relative to T by ν, and chose $\nu, \tau^{(1)}, \tau^{(2)}$ an orthogonal unit basis of R^3. Let $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ be vertices of F and $\tilde{a}_0$ be barycentric point of F.

By the definition of V_{h0}, $\frac{\partial}{\partial\nu} v_h^T$ and $\frac{\partial}{\partial\nu} v_h^{T'}$ are quadratic polynomials on F. Hence

$$\int_F \frac{\partial v_h^T}{\partial\nu}\mathrm{d}s = \frac{|F|}{12}\Big(\sum_{i=1}^3 \frac{\partial v_h}{\partial\nu}(\tilde{a}_i) + 9\frac{\partial v_h}{\partial\nu}(\tilde{a}_0)\Big) = \int_F \frac{\partial v_h^{T'}}{\partial\nu}\mathrm{d}s. \tag{3.7}$$

Denote all sides of F by S_1, S_2, S_3, and the unit out normal of S_i by $n^{(i)}$, viewed as the boundary of a triangle in 2-dimensional space with axes $\tau^{(1)}$ and $\tau^{(2)}$. Then for $i \in \{1,2\}$

$$\int_F \frac{\partial v_h^T}{\partial\tau^{(i)}}\mathrm{d}s = \sum_{j=1}^3 n_i^{(j)} \int_{S_j} v_h^T \mathrm{d}t, \quad \int_F \frac{\partial v_h^{T'}}{\partial\tau^{(i)}}\mathrm{d}s = \sum_{j=1}^3 n_i^{(j)} \int_{S_j} v_h^{T'} \mathrm{d}t.$$

By the definition of V_{h0}, $v_h^T = v_h^{T'}$ on S_j. Therefore

$$\int_F \frac{\partial v_h^T}{\partial\tau^{(i)}}\mathrm{d}s = \int_F \frac{\partial v_h^{T'}}{\partial\tau^{(i)}}\mathrm{d}s, \quad i = 1, 2. \tag{3.8}$$

Equality (3.5) follows from (3.7) and (3.8). By the similar way, we can show (3.6). □

3.2. **TNC16 Element.** For $0 \le i < j < k \le 3$, let $a_{ijk} = (a_i + a_j + a_k)/3$ and ν_{ijk} be the unit out normal of the face with a_i, a_j, a_k as vertices. Set

$$\tilde{\psi}_{ijk}(v) = 3\frac{\partial v}{\partial\nu_{ijk}}(a_{ijk}) - \sum_{l=i,j,k} \frac{\partial v}{\partial\nu_{ijk}}(a_l).$$

Define

$$P_3''(T) = \{\, p \in P_3(T) \,|\, \tilde{\psi}_{ijk}(p) = 0,\ 0 \le i < j < k \le 3 \,\}.$$

It is obvious that $P_2(T) \subset P_3''(T)$. For TNC16 element, (T, P_T, Φ_T) is defined by

1): T is a tetrahedron,
2): $P_T = P_3''(T)$,
3): Φ_T is the vector with its component the following degrees of freedom,

$$v(a_j),\ 0 \le j \le 3, \quad Dv(a_i)(a_j - a_i),\ 0 \le i \ne j \le 3,\ \forall v \in C^1(T)$$

The basis functions of TNC16 element can be derived from ones of TNC20. Set

$$\tilde{p}_{ij} = \lambda_i^2\lambda_j - \frac{2\|B_j\|}{9}q_j + \sum_{\substack{1\le k\le 4\\ k\ne i, k\ne j}} \frac{2(B_i - B_j)B_k^\top}{9\|B_k\|}q_k, \quad 0 \le i \ne j \le 3. \tag{3.9}$$

The corresponding interpolation operator Π_T can be written by,

$$\Pi_T v = \sum_{0\le i\le 3} p_i v(a_i) + \sum_{0\le i\ne j\le 3} \tilde{p}_{ij} Dv(a_i)(a_j - a_i), \quad \forall v \in C^1(T) \tag{3.10}$$

For TNC16 element, define the corresponding finite element space V_{h0} as follows. $V_{h0} = \{v \in L^2(\Omega) \mid v|_T \in P_3''(T), \forall T \in T_h,\ v$ and its first order derivatives are continuous at all vertices of elements in $\mathcal{T}_h$ and vanish at all vertices belonging to $\partial\Omega\}$.

For this element, V_{h0} is still not a subspace of $C^0(\bar{\Omega})$. □

3.3. **TE16 Element.** For $0 \le i < j < k \le 3$, define

$$\psi_{ijk}(v) = 6v(a_{ijk}) - 2\sum_{l=i,j,k} v(a_l) + \sum_{l=i,j,k} Dv(a_l)(a_l - a_{ijk}).$$

Define

$$P_3'(T) = \{\, p \in P_3(T) \,|\, \psi_{ijk}(p) = 0,\ 0 \le i < j < k \le 3 \,\}.$$

Define TE16 element (T, P_T, Φ_T) as follows.

1): The element T is a tetrahedron.

2): The shape function space $P_T = P_3'(T)$.

3): For $v \in C^1(T)$, the vector $\Phi_T(v)$ of its degrees of freedom is

$$\Phi_T(v) = \Big(v(a_0), Dv(a_0), v(a_1), Dv(a_1), v(a_2), Dv(a_2), v(a_3), Dv(a_3)\Big)^\top.$$

The corresponding interpolation operator Π_T is defined by

$$\begin{aligned}\Pi_T v = &\sum_{i=0}^{3}\Big(3\lambda_i^2 - 2\lambda_i^3 + 2\lambda_i \sum_{\substack{0\le j<k\le 3\\ j,k\ne i}} \lambda_j\lambda_k\Big)v(a_i)\\ &+\frac{1}{2}\sum_{0\le i\ne j\le 3} \lambda_i\lambda_j(1+\lambda_i-\lambda_j)Dv(a_i)(a_j-a_i), \quad \forall v \in C^1(T).\end{aligned} \tag{3.11}$$

In Ciarlet [5], TE16 element is the element of Hermite n-simplex of type $(3')$ with $n = 3$. The element with $n = 2$ is the Zienkiewicz element.

For TE16 element, define the corresponding finite element space V_{h0} as follows. $V_{h0} = \{v \in L^2(\Omega) \mid v|_T \in P_3'(T), \forall T \in T_h,\ v$ and its first order derivatives are continuous at all vertices of elements in $\mathcal{T}_h$ and vanish at all vertices belonging to $\partial\Omega\}$. From [5], we know that $V_{h0} \subset H^1(\Omega)$. □

3.4. **TQC16: a modified Zikeniewicz Element.** The Zienkiewicz element is not convergent in general. We will show in next section that TE16 element is also divergent for a special tedrahetral grid. In 2-dimensional case, a convergent element was proposed by the so-called quasi-conforming element technique in [16,4]. Now we use the technique to give a new element by modifying TE16.

Define

$$N^{ii} = \mathrm{span}\,\{1, \lambda_i, \lambda_0\lambda_i\},\ 1 \le i \le 3; \quad N^{ij} = P_0(T), \quad 1 \le i \ne j \le 3.$$

Let Π_T^1 be the linear interpolation operator with the function values at four vertices as degrees of freedom. For $p \in P_T$, define $\partial_T^{ij} p \in N^{ij}$, $1 \le i, j \le 3$, such that

$$\begin{cases} \displaystyle\int_T q\partial_T^{ii} p dx = \int_{\partial T} q\Pi_T^1 \frac{\partial p}{\partial x_i}\nu_i ds - \int_T \frac{\partial q}{\partial x_i}\frac{\partial p}{\partial x_i}dx, \quad \forall q \in N^{ii},\ 1 \le i \le 3,\\ \displaystyle\int_T \partial_T^{ij} p dx = \frac{1}{2}\int_{\partial T}\Big(\Pi_T^1\frac{\partial p}{\partial x_i}\nu_j + \Pi_T^1\frac{\partial p}{\partial x_j}\nu_i\Big)ds,\ 1 \le i \ne j \le 3.\end{cases} \tag{3.12}$$

Set

$$E_T(p) = \Big(\partial_T^{11}p, \partial_T^{22}p, \partial_T^{33}p, \partial_T^{12}p, \partial_T^{13}p, \partial_T^{23}p\Big)^\top. \tag{3.13}$$

For TQC16 element, we use $E_T(p)$ to approximate $E(p)$.

Define

$$N = \begin{pmatrix} \tilde{N}^{11} & & & & & \\ & \tilde{N}^{22} & & & 0 & \\ & & \tilde{N}^{33} & & & \\ & & & 1 & & \\ & 0 & & & 1 & \\ & & & & & 1 \end{pmatrix},$$

where $\tilde{N}^{ii} = (1, \lambda_i, \lambda_0\lambda_i),\ 1 \le i \le 3$,

$$H_T = \begin{pmatrix} \xi_{11}^2 & \xi_{21}^2 & \xi_{31}^2 & 2\xi_{11}\xi_{21} & 2\xi_{11}\xi_{31} & 2\xi_{21}\xi_{31} \\ \xi_{12}^2 & \xi_{22}^2 & \xi_{32}^2 & 2\xi_{12}\xi_{22} & 2\xi_{12}\xi_{32} & 2\xi_{22}\xi_{32} \\ \xi_{13}^2 & \xi_{23}^2 & \xi_{33}^2 & 2\xi_{13}\xi_{23} & 2\xi_{13}\xi_{33} & 2\xi_{23}\xi_{33} \\ \xi_{11}\xi_{12} & \xi_{21}\xi_{22} & \xi_{31}\xi_{32} & \xi_{12}\xi_{21}+\xi_{11}\xi_{22} & \xi_{12}\xi_{31}+\xi_{11}\xi_{32} & \xi_{22}\xi_{31}+\xi_{21}\xi_{32} \\ \xi_{11}\xi_{13} & \xi_{21}\xi_{23} & \xi_{31}\xi_{33} & \xi_{13}\xi_{21}+\xi_{11}\xi_{23} & \xi_{13}\xi_{31}+\xi_{11}\xi_{33} & \xi_{23}\xi_{31}+\xi_{21}\xi_{33} \\ \xi_{12}\xi_{13} & \xi_{22}\xi_{23} & \xi_{32}\xi_{33} & \xi_{13}\xi_{22}+\xi_{12}\xi_{23} & \xi_{13}\xi_{32}+\xi_{12}\xi_{33} & \xi_{23}\xi_{32}+\xi_{22}\xi_{33} \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & -120 & 0 & 0 & 0 & 120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 120 & 0 & 15 & 15 & -120 & 90 & -15 & -15 & 0 & 15 & 0 & 0 & 0 & 15 & 0 & 0 \\ 4 & -6 & 1 & 1 & 4 & 4 & 1 & 1 & -4 & -1 & 2 & 0 & -4 & -1 & 0 & 2 \\ 0 & 0 & -120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 120 & 0 & 0 & 0 & 0 & 0 \\ 120 & 15 & 0 & 15 & 0 & 0 & 15 & 0 & -120 & -15 & 90 & -15 & 0 & 0 & 15 & 0 \\ 4 & 1 & -6 & 1 & -4 & 2 & -1 & 0 & 4 & 1 & 4 & 1 & -4 & 0 & -1 & 2 \\ 0 & 0 & 0 & -120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 120 \\ 120 & 15 & 15 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 & 15 & -120 & -15 & -15 & 90 \\ 4 & 1 & 1 & -6 & -4 & 2 & 0 & -1 & -4 & 0 & 2 & -1 & 4 & 1 & 1 & 4 \\ 0 & -60 & -60 & 0 & 0 & 0 & 60 & 0 & 0 & 60 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -60 & 0 & -60 & 0 & 0 & 0 & 60 & 0 & 0 & 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & -60 & -60 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 60 & 0 & 0 & 60 & 0 \end{pmatrix}$$

$$A^0 = \begin{pmatrix} 50 & -90 & -210 \\ -90 & 570 & -1050 \\ -210 & -1050 & 9450 \end{pmatrix}, \quad A = \begin{pmatrix} A^0 & & & & & \\ & A^0 & & & 0 & \\ & & A^0 & & & \\ & & & 17 & & \\ & 0 & & & 17 & \\ & & & & & 17 \end{pmatrix},$$

and

$$M_T = \begin{pmatrix} 1 & 0 & & & & & & \\ 0 & B_T^\top & & & & 0 & & \\ & & 1 & 0 & & & & \\ & & 0 & B_T^\top & & & & \\ & & & & 1 & 0 & & \\ & & & & 0 & B_T^\top & & \\ & & 0 & & & & 1 & 0 \\ & & & & & & 0 & B_T^\top \end{pmatrix},$$

then

$$E_T(p) = \frac{1}{2040} H_T N A Q M_T \Phi_T(p), \quad \forall p \in P_T. \tag{3.14}$$

Now let V_{h0} be the finite element space corresponding to TE16 element. Define

$$\bar{a}_h(v_h, w_h) = \sum_{T\in\mathcal{T}_h} \int_T E_T(w_h)^\top K E_T(v_h) dx, \quad \forall v_h, w_h \in V_{h0}. \tag{3.15}$$

Instead of solving problem (2.6), TQC16 element finds $\bar{u}_h \in V_{h0}$ such that

$$\bar{a}_h(\bar{u}_h, v_h) = (f, v_h), \qquad \forall v_h \in V_{h0}. \tag{3.16}$$

TQC16 element is a 3-dimensional analogue of one proposed in [16,4] (also see [22]).

For $v_h \in V_{h0}$ and $i, j \in \{1, 2, 3\}$, define $\partial_h^{ij} v_h$ by

$$\partial_h^{ij} v_h|_T = \partial_T^{ij} v_h^T, \quad \forall T \in \mathcal{T}_h.$$

Let Π_T be the interpolation operator of TE16 element.

Lemma 3.2. *TQC16 element has the following properties:*

1. $E_T(p) = E(p)$, $\forall p \in P_2(T)$.
2. *There exist positive constants c_1 and c_2 independent of h such that*

$$c_1|p|_{2,T} \le \sum_{1\le i,j\le 3} |\partial_T^{ij} p|_{0,T} \le c_2|p|_{2,T}, \quad \forall p \in P_T. \tag{3.17}$$

3. *there exists a constant C independent of h such that*

$$\sum_{1\le i,j\le 3} \Big|\frac{\partial^2 v}{\partial x_i \partial x_j} - \partial_T^{ij}\Pi_T v\Big|_{0,T} \le Ch|v|_{3,T}, \quad \forall v \in H^3(T). \tag{3.18}$$

Proof. From Green formula, we have

$$\begin{cases} \displaystyle\int_T q\frac{\partial^2 p}{\partial x_i^2} dx = \int_{\partial T} q\frac{\partial p}{\partial x_i}\nu_i ds - \int_T \frac{\partial q}{\partial x_i}\frac{\partial p}{\partial x_i} dx, \ \forall q \in N^{ii}, \ \forall p \in P_T, \ 1 \le i \le 3, \\ \displaystyle\int_T \frac{\partial^2 p}{\partial x_i \partial x_j} dx = \frac{1}{2}\int_{\partial T} \Big(\frac{\partial p}{\partial x_i}\nu_j + \frac{\partial p}{\partial x_j}\nu_i\Big) ds, \ \forall p \in P_T. \ 1 \le i \ne j \le 3. \end{cases} \tag{3.19}$$

If $p \in P_2(T)$, then $E(p)$ is uniquely determined by (3.19). On the other hand,

$$\Pi_T^1 \frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial x_i}, \quad \forall p \in P_2(T).$$

$E_T(p) = E(p), \forall p \in P_2(T)$, from (3.12).

It can be verified that the rank of matrix Q is 12. Then the rank of AQM_T is 12 too. Let S be the subspace of R^{16} such that $AQM_T d = 0, \forall d \in S$. Then the dimension of S is 4. By the conclusion 1 of the lemma, we have

$$S = span\{\Phi_T(1), \Phi_T(x_1), \Phi_T(x_2), \Phi_T(x_3)\}.$$

If $E_T(p) = 0$ for some $p \in P_T$ then $AQM_T\Phi_T(p) = 0$. It follows that $p \in P_1(T)$. Furthermore, for all $T \in \mathcal{T}_h$,

$$\alpha_{1T}|p|_{2,T} \le \sum_{1\le i,j\le 3} |\partial_T^{ij} p|_{0,T} \le \alpha_{2T}|p|_{2,T}, \quad \forall p \in P_T \tag{3.20}$$

where α_{1T} and α_{2T} are positive constants perhaps dependent on T. By the affine technique, we obtain (3.17).

From the first two conclusions of the lemma, the interpolation theory and the affine technique, we can prove (3.18). □

4. Convergence Analysis

In this section, we discuss the convergence properties of the elements in previous sections. Toward the end of this section, we show that TE16 is not convergent in general.

First, let us derive the error estimates for the interpolation operator.

Theorem 4.1. *Let Π_T be the interpolation operator corresponding to TNC20, TNC16 and TE16 elements. Then there exists a constant C independent of h such that*

$$|v - \Pi_T v|_{m,T} \le Ch^{r-m}|v|_{r,T}, \quad 0 \le m \le r, \ \forall v \in H^r(T), \tag{4.1}$$

where $r = 3$ for TNC16 and TE16 elements, $r = 4$ for THC20 element.

From Lemma 3.1 and the argument [12] for Morley element, we can show the following lemma.

Lemma 4.2. *Let V_{h0} be the finite element space of TNC20 or TNC16 element. Then there exists a constant C independent of h such that for $v \in H^3(\Omega) \cap H_0^2(\Omega)$ with $\Delta^2 v \in L^2(\Omega)$,*

$$|a_h(v, v_h) - (\Delta^2 v, v_h)| \le Ch(|v|_{3,\Omega} + h\|\Delta^2 v\|_{0,\Omega})|v_h|_{2,h}, \quad \forall v_h \in V_{h0}. \tag{4.2}$$

Now let u and u_h be the solutions of problems (2.4) and (2.6) respectively. Combining theorem 4.1 and lemma 4.2, we get the following theorem.

Theorem 4.3. *Let V_{h0} be the finite element space of TNC20 or TNC16 element. Then there exists a constant C independent of h such that*

$$\|u - u_h\|_{2,h} \le Ch(|u|_{3,\Omega} + h\|f\|_{0,\Omega}) \tag{4.3}$$

when $u \in H^3(\Omega)$.

Now let Π_h^1 be the interpolation operator corresponding to linear conforming element for second order partial differential equation and $\mathcal{T}_h$. For TNC20 and TNC16 elements, we can also consider the following finite element method: to find $\tilde{u}_h \in V_{h0}$ such that

$$a_h(\tilde{u}_h, v_h) = (f, \Pi_h^1 v_h), \quad \forall v_h \in V_{h0}. \tag{4.4}$$

For the finite element solution $\tilde{u}_h$ of problem (4.4), we have

Theorem 4.4. *Let V_{h0} be the finite element space of TNC20 or TNC16 element. Then there exists a constant C independent of h such that*

$$\|u - \tilde{u}_h\|_{2,h} \le Ch|u|_{3,\Omega} \tag{4.5}$$

when $u \in H^3(\Omega)$.

For the convergence of TQC16 element, we can follow the way used in [21 or 22]. We give the result without proof.

Theorem 4.5. *For TQC16 element, problem (3.16) has unique solution $\bar{u}_h$, and there exists a constant C independent of h such that*

$$\|u - \bar{u}_h\|_{2,h} + \sum_{1 \le i \ne j \le 3} \Big|\frac{\partial^2 u}{\partial x_i \partial x_j} - \partial_h^{ij}\bar{u}_h\Big|_{0,\Omega} \le Ch|u|_{3,\Omega}. \tag{4.6}$$

when $u \in H^3(\Omega)$.

4.1. **TE16: the Zienkiewicz element and its divergence property.** It is known that Zienkiewicz element is not convergent for all general meshes in 2 dimensions. As a analogue in 3 dimensional case, TE16 element has the same property. In this section, we show that TE16 element is divergent for one special grid.

Now let Ω be the cube $[-1,1]^3$. For $k=1,2,\cdots$, let $\mathcal{T}_k$ be a triangulation of Ω defined as follows. $\mathcal{T}_1$ is shown in Fig. 1, while the subdivision is symmetric with respect to the centric point of Ω. For $k>2$, Ω is first subdivided into equal cubes with side length $h_k=2/k$, then each cube is subdivided into tetrahedrons by the same way used for $\mathcal{T}_1$. Fig. 2 shows the case of $k=2$.

Fig. 1 Fig. 2

Theorem 4.6. *TE16 element is divergent for triangulations $\mathcal{T}_k$.*

Proof. Let V_{10} be the finite element space of TE16 element on $\mathcal{T}_1$. Let $v_h\in V_{10}$ be the function such that v_h is 1 at the centric point of Ω and vanishes at other vertices of $\mathcal{T}_1$, and Dv_h is zero at all vertices of $\mathcal{T}_1$. It is can be computed that

$$\sum_{T\in\mathcal{T}_1}\int_T \frac{\partial^2 v_h}{\partial x_i^2}dx=-\frac{8}{3},\quad i=1,2,3,$$

$$\sum_{T\in\mathcal{T}_1}\int_T \frac{\partial^2 v_h}{\partial x_i\partial x_j}dx=0,\quad 1\le i\le 3,\ i<j\le 3.$$

That is, TE16 element does not pass the patch test. On the other hand, for $\mathcal{T}_k$ the number of the patches reduced from $\mathcal{T}_1$ is k^3 and the number of elements in $\mathcal{T}_k$ is $48k^3$. From the result given in [17], we obtain the conclusion of the lemma. □

5. Concluding remarks

In this paper, we proposed and analyzed several tetrahedron complete or incomplete cubic nonconforming finite elements for fourth order elliptic partial differential operators.

More works need to be done for constructing other types of nonconforming elements. One noticeable element that is missing from our work is a three dimensional extension of Morley triangular element in two dimensions that only makes use of quadratic polynomials. Another type of elements are cuboid nonconforming elements that may be extended from rectangular nonconforming elements in two dimensions. We will discuss them in future papers.

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