Confinement of Particles and the Non-Conservation of the Magnetic Moment in Mirror Machines

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November 2020

1 Abstract

Adiabatic invariants have been used in Physics for a very long time and one such invariant is the magnetic moment for a charged particle gyrating around a magnetic filed whose constancy has been central in the development of a lot of concepts in Plasma Physics, such as the loss cone. Here we build upon basic electrodynamics principles this concept of the magnetic moment and derive how we come to the conclusion that it's constant while using the magnetic mirror machine as an example of a confining device. Since it's been established that the magnetic moment is actually not conserved in the real world, we use the RMF Single Particle Hamiltonian Simulation Code to simulate particles in a mirror machine while looking at their magnetic moments and seeing how much they differ from the theoretical values. We then discuss the consequence of this discrepancy between theory and the real world and what that means for confinement devices

2 Introduction

Magnetic mirror confinement machines have been around since the 1950s and they have been essential to the development of our current understanding of Plasma Physics. In the operation of this machine, numerous parameters have to be considered so as to make sure the mirror works as prescribed. In this paper we're going to be discussing one parameter that's widely considered as conserved in most considerations of mirror machines, the magnetic moment μ. For a charged particle moving in a magnetic field this μ is defined as the its perpendicular Kinetic Energy divided by the magnetic field strength at that point,$\frac{\frac{1}{2}mv_{\perp}^2}{|B|}$. Here when we speak of the perpendicular kinetic energy we me mean the kinetic energy that's due to the component of velocity perpendicular to the magnetic field line at that point that the particle is spiraling around as we will see later in greater detail. Kulsrud discusses the idea of an adiabatic invariant saying "There are many problems in physics in which there exist quantities which change so slowly that they may be taken as constants of the motion to a high degree of accuracy. Any such quantity whose change approaches zero asymptotically as some physical parameter approaches zero or infinity is an adiabatic Invariant."[3] The magnetic moment is widely regarded as one such adiabatic invariant many machines built in Plasma Physics today and Kruskal explains this by saying " the theory of virtually every prospective device for the production of useful energy from controlled thermonuclear fusion3 has leaned very heavily on the constancy of this magnetic moment"[4]. This concept has also been crucial in explaining a lot of astrophysical processes with a very good example being the Fermi Acceleration [1] to explain the production of high energy cosmic rays. Given how crucial this μ parameter a lot of work went into proving to how much of an extent it's an adiabatic invariant and Kruskul [2] in 1957 proved that it's constant to all orders of magnitude for a particle gyrating around a magnetic field. This constancy of the magnetic moment is central in the development of the idea of a loss cone which dictates which particles are lost and which ones are confined in the velocity space in magnetic confinement machines. It is important to note that the magnetic moment is defined for a single particle and that turbulence, scattering, waves, collisions and many such collective effects in plasmas will need to be considered when building real world confinement devices but that is beyond the scope of this paper. In this paper we will look at this μ conservation, derive how it arises, look at the consequences of this conservation and test the accuracy of this approximation using a single particle Hamiltonian Simulation Code called the RMF code authored by Alan Glasser. It turns out that in the real world μ is not totally conserved and we will mean for the loss cone, as well as the general confinement of particles. It is important to note that even if the losses are small for the single particles we will look at,these small losses easily turn into huge losses once we have a large collection of particles in the plasma, as well as over large periods of time which is what's required for thermonuclear fusion devices.

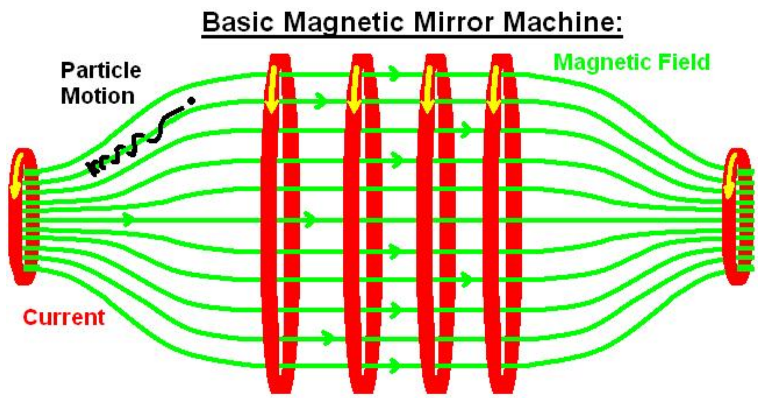


Figure 1: Mirror Machine Layout

3 Mirror Machine

The magnetic mirror machine has the layout from fig 1. The central green line runs along the z-axis of the machine while the other lines run from the right to the left ends of the figure in curvy lines. These green lines are the magnetic fields that will confine our particles. When charged particles are introduced in the machine with horizontal and perpendicular components of their velocity to the magnetic field, $v_\perp$ and $v_\|$, they will move along the magnetic field lines while circling these magnetic field lines thus resulting in a spiraling motion and we will, of course discuss the mathematics of this confinement in greater detail in the following section. The red circles in fig 1 are coils which create the magnetic field, and we can adjust the current through them to weaken or strengthen the field. For most of the paper we will use cylindrical coordinates where the z-axis known as the axial length runs horizontally from left to right and the radial axis is centered at the middle of the machine and radiates outward. If we slice the machine into two equal parts one to the left and one to the right the cross section the left and the right halves is known as the mid-plane. The field is therefore strongest at the two ends on the z-axis and weakest in the middle of the machine.

4 Confinement Mechanics

Before we go any further it is important that we derive the principles that govern how particles move in this machine which will conveniently lead to the definition of our parameter of interest, the magnetic moment, μ. For now we will define these equations for a homogeneous straight magnetic field. For a particle with charge q, we start with the Lorentz force it experiences $\vec{F}$.

$$\vec{F} = q\vec{v} \times \vec{B} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{1}$$

Here the $\frac{d\vec{v}}{dt}$ term is perpendicular to the magnetic field $\vec{B}$. If we dot product multiply the rightmost side equation by $\vec{v}$, we get

$$mv\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}(\frac{1}{2}mv^2)}{\mathrm{d}t} = 0 \tag{2}$$

This shows that Kinetic Energy, KE, is conserved and this makes sense since $\vec{B}$ does no work because the Lorentz force is perpendicular to $v_\perp$. The horizontal velocity is also not changing because there is no force in that direction, that is fine for now but we will see later why that is not exactly true when we finally make our magnetic field vary in space. We break down our $\vec{v}$ to $v_\perp$ and $v_\parallel$ and this reduces the Lorentz equation to two independent equations

$$m\frac{\mathrm{d}v_\perp}{\mathrm{d}t} = qv_\perp \times B \tag{3}$$

$$m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = 0 \tag{4}$$

It is best that we move to cylindrical coordinates where $\vec{r}$ is the radius made by the circle from our particle going around the magnetic field. For this we will only work with the perpendicular component of the Lorentz force. The Lorentz force reduces to

$$\frac{mv_\perp^2}{|r|}(\frac{-\vec{r}}{|r|}) = qv_\perp \times \vec{B} \tag{5}$$

Here we have the negative sign because $\vec{r}$ points out of the circle but our centripetal is pointing inwards. We get the radius as

$$|r| \equiv R = |\frac{mv_\perp}{qB}| \tag{6}$$

This then is the radius that our particles will be spiraling around and from this we can calculate the frequency the particles are going around the magnetic field at and this is known as the cyclotron frequency, w_c.

$$w_c = \frac{2\pi}{T} = \frac{2\pi}{2\pi R/v_\perp} = \frac{v_\perp}{R} = \frac{v_\perp}{mv_\perp/qB} = \frac{qB}{m} \tag{7}$$

The magnetic moment, μ, determines how much force the magnetic field exerts on an electric current and its defined $\mu = IA$ where I is the current going around the magnetic field and A is the area made by the current carrying loop. From our $/w_c$ we can calculate our current of the charge going around the field.

$$I = \frac{q}{T} = \frac{q}{2\pi/w_c} = \frac{q^2B}{2\pi m} \tag{8}$$

From this we can then calculate μ,

$$\mu = IA = \frac{q^2B}{2\pi m}\pi R^2 \tag{9}$$

From this equation we can clearly see that $B\pi R^2$ is equal to the magnetic flux, ϕ, going through our loop and it's important to note this as it will become important later.

$$\mu = \frac{q^2}{2\pi m}\phi \tag{10}$$

Going back to equation 9 we can go in another direction with μ, by replacing R with R from equation 6,

$$\mu = IA = \frac{q^2 B}{2m}(\frac{mv_\perp}{qB})^2 = \frac{\frac{1}{2}mv_\perp^2}{B} \tag{11}$$

Since KE is always constant, it follows that μ will be constant if B is constant. This means that the magnetic flux enclosed in our loop is constant and this is the reason why μ is always constant.

5 Argument for conserved μ

It is important that we examine the argument for why μ is taken as conserved in many approximations of the mirror machines. The key here is that B is taken as changing slowly in time, which would lead to a conserved μ which is why the magnetic moment is known as an adiabatic invariant – conserved over a period of motion – not an exact invariant, conserved at all times, in a mirror machine. For us, our B is not changing in time but space, however since our particle is moving in the machine, in this particle's inertial frame where it's at rest, the magnetic field can be seen as changing in time. What does a slow change in time even mean? It means that per cyclotron period, the change in B is very very small which we can state mathematically as

$$\frac{2\pi}{w_c}|\frac{\partial B}{\partial t}/B| << 1 \approx \frac{1}{w_c}|\frac{\partial B}{\partial t}/B| << 1 \tag{12}$$

For a slowly changing B in time the induced emf, ε, difference across the two ends of the loop can be derived from Maxwell's equations as

$$\varepsilon = -\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial t}\int\int_S \vec{B}\cdot d\vec{A} = \oint_c \vec{E}\cdot dl \tag{13}$$

Therefore the change in perpendicular Kinetic Energy, $K_\perp$, is

$$\Delta K_\perp = q\varepsilon = -q\frac{\partial \phi}{\partial t} = -q\int\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A} = q\oint_c E\cdot dl \tag{14}$$

If we have our loop of radius, r, barely changing per cyclotron period according to our adiabatic approximation then

$$\Delta K_\perp = q\pi r^2\frac{\partial B}{\partial t} \tag{15}$$

Over one cyclotron period, T, the change in the magnetic field ΔB is given by,

$$\Delta B = T\frac{\partial B}{\partial t} = \frac{2\pi}{w_c}\frac{\partial B}{\partial t} \tag{16}$$

Combining these two equations and our adiabatic approximation in equation 12 we get,

$$\Delta\mu = \Delta(K_\perp/B) \approx 0 \tag{17}$$

Since we said $\mu = \frac{q^2}{2\pi m}\phi$ in equation 10 then in this adiabatic approximation, where μ is constant, then as B gets stronger(at the two z-axis ends of the mirror machine), the flux through the orbit is always constant thus the area of the orbit has to become smaller. This is the reason why the radius of orbit is smaller at the ends and the largest when the particle is in the middle of the machine. Now to be able to appreciate how the particle manages to bounce back from both ends of the mirror and stays inside the machine we will need to derive the forces acting on the particle

6 How the Mirror Machine Works

One of the most important questions here is to ask how the particle's parallel velocity changes to be able to stop and be reflected at the two ends which keeps it confined and bouncing between the two ends. This means that there is a parallel force which in addition to the centripetal one we found earlier and to get this Force will allow B to vary in space as we change r which is true for the magnetic mirror machine, and to do so we Taylor Expand our B in r.

$$B(r) = B_o + (\vec{r}\cdot\nabla)B + H.O \tag{18}$$

H.O represents higher order terms and we drop it. We then calculate the force on our particle as

$$\vec{F} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m}(\vec{v}\times(B_o + (\vec{r}\cdot\nabla)B)) = \frac{q}{m}(\vec{v}\times B_o) + \frac{q}{m}(\vec{v}\times(\vec{r}\cdot\nabla)B_o) \tag{19}$$

Here we recover the F we had initially plus an additional term from the Taylor Expansion which we will treat as a perturbation since its an order of magnitude smaller than the first term. we take v as $v = v_o + v_1$ with where $v_o$is the solution to $\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m}(\vec{v}\times B_o)$ and v_1 is the perturbation.

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m}(\vec{v}\times B_o) + \frac{q}{m}(\vec{v_o}\times(\vec{r}\cdot\nabla)B_o) \tag{20}$$

We can talk about the second term as coming from an external additional force, F, to the original equation 1 we had and we want to investigate how this affects the mechanics of the particle.

$$F = q(\vec{v_o}\times(\vec{r}\cdot\nabla)B_o) \tag{21}$$

Since this force is momentary in time, we will need to work with the time averaged force per cyclotron period.

$$F = \langle q\vec{v_o} \times (\vec{r} \cdot \nabla)B_o \rangle \tag{22}$$

We break this F into the parallel and perpendicular coordinates

$$F_{\|} = \langle q\vec{v_o} \times \vec{r}\frac{\partial B_r}{\partial r} \rangle \tag{23}$$

$$F_{\perp} = \langle q\vec{v_o} \times |\vec{r}|\frac{\partial B_z}{\partial r}\hat{z} \rangle \tag{24}$$

We will then simplify the parallel component, using the cross product form of mu we derived earlier.

$$F_{\|} = q\vec{v_o} \times \vec{r}\langle\frac{\partial B_r}{\partial r}\rangle = 2\mu\langle\frac{\partial B_r}{\partial r}\rangle \tag{25}$$

We're going to have to replace the time averaged term and we can do so by invoking Maxwell's second equation,

$$\nabla \cdot \vec{B} = 0$$

(26)

It follows that,

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{\partial B_z}{\partial z} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} = 0 \tag{27}$$

The last term evaluates to zero when averaged over a cyclotron period because B is single valued over the orbit, and in the limit when r $\rightarrow 0$, $\frac{B_r}{r} \rightarrow \frac{\partial B_r}{\partial r}$

$$2\langle\frac{\partial B_r}{\partial r}\rangle = -\langle\frac{\partial B_z}{\partial z}\rangle = -(\frac{\partial B_z}{\partial z}) \tag{28}$$

$$F_{\|} = -2\mu(\frac{\partial B_z}{\partial z})\hat{z} = -\frac{\mu}{B}((\vec{B} \cdot \nabla)\vec{B}) = -(\nabla\frac{B^2}{2})_{\|} \tag{29}$$

For the perpendicular part Force

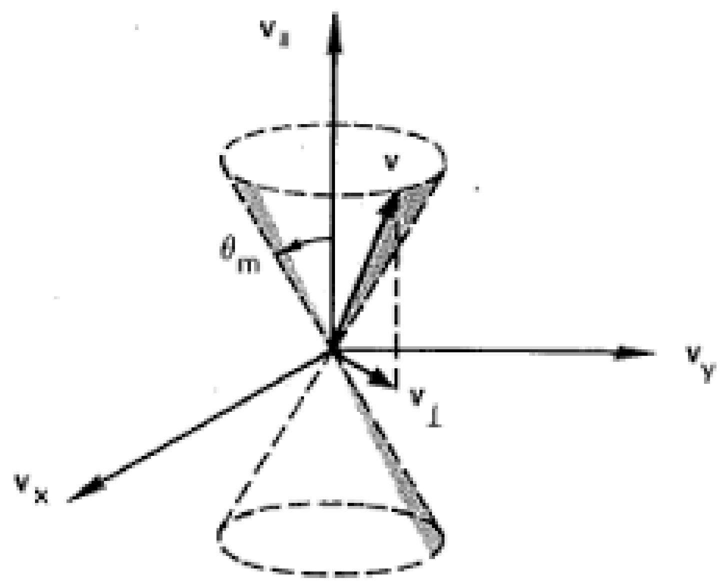
$$F_{\perp} = -qv\langle\vec{r}\frac{\partial B_z}{\partial r}\rangle \tag{30}$$

We can convert to 2d cartesian coordinates to make our calculations simpler,

$$F_{\perp} = -qvr\hat{x}\langle cos^2(\phi)\frac{\partial B_z}{\partial x} = \frac{-qvr}{2}\frac{\partial B_z}{\partial x}\hat{x} = -\mu\frac{\partial B_z}{\partial x}\hat{x} \tag{31}$$

Since $(B_x)_o = (B_y)_o = 0$,

$$\frac{\partial B_z}{\partial x} = \frac{1}{2B_z}\frac{\partial B_z^2}{\partial x} \tag{32}$$



The loss cone.

Figure 2: Mirror Machine Layout

We can replace that in the equation for $F_\perp$

$$F_\perp = -\frac{\mu m}{2B} \nabla_\perp B^2 \tag{33}$$

Combining the perpendicular and parallel components of F we get,

$$F = -\frac{\mu}{B} \nabla \frac{\vec{B}^2}{2} = -\mu m \nabla \vec{B} \tag{34}$$

Here we see that the force is in the direction of the decreasing magnetic field strength, so as B, gets stronger towards the end of the machine, the particle is pushed back toward the center which provides the retardation force. The particle is then accelerated towards the center and when it gets there is decelerates towards the other end of the machine. It's then accelerated back to the middle of the machine after it reaches the other end and that cycle continues, thus the particle is confined, bouncing back and forth between the two ends, (mirror).

7 Loss Cone

The mechanism by which the particle is confined and the magnetic moment is conserved in this approximation gives rise to an interesting parameter of the loss cone. Here the loss cone describes a cone in velocity space which dictates which particles are lost and which ones are confined. If μ is conserved, Mathematically we can show this as follows,

$$\mu = (\frac{\frac{1}{2}mv_{\perp}^{2}}{B})_o = (\frac{\frac{1}{2}mv_{\perp}^{2}}{B})_t \tag{35}$$

Where the t subscript represents μ at the turning point(mirror), and the o subscript represents the mid-plane of the machine.

$$(\frac{1}{2}mv_{\perp}^{2})_o = (\frac{1}{2}mv_{\perp}^{2})_t \tag{36}$$

$$(Sin(\theta))_o = (\frac{v_{\perp}}{v})_o \tag{37}$$

$$(Sin(\theta))_t = 1 \tag{38}$$

Combining the last 4 equations we get,

$$(Sin^2(\theta))_o = \frac{B_o}{B_t} \tag{39}$$

This shows that all particles with the same θ in velocity space get reflected from the same B_t no mater how fast they're moving. The ratio $\frac{B_o}{B_m}$ where the subscript m represents the maximum B reached by a particle is given by mirror ratio, Rm. Of course particles with $B_t \leq B_m$ will be confined and the critical angle,θ_c for confinement is given by,

$$(Sin^2\theta_c) = \frac{B_o}{B_m} \tag{40}$$

The loss cone will contain any angle $\theta < \theta_c$. All lost particles are going to be inside the cone, not dependent on mass. Collisions will scatter particles into the loss cone but electrons, being very light, will scatter more than ions. This all comes from the fact that μ is conserved and if we questioned that then the loss cone would have to be revisited as well. For this, I will use the RMF code to simulate and see how often μ is conserved and how many times it's violated in order to test the accuracy of the adiabatic approximation in practical applications.

8 Why Is μ Not Conserved

As we have stated before, mu is not conserved exactly in mirror machines and in this section we will delve into exactly why that is. Several explanations were proposed over the years up until 1994 [5] when the following model, that makes use of the centrifugal impulse which change the parallel velocity of the particle, was proposed. The picture is based on the fact that the magnetic fields in mirror machines are actually curved rather than straight and this wasn't very important when we derived the concept of μ in earlier sections but here it

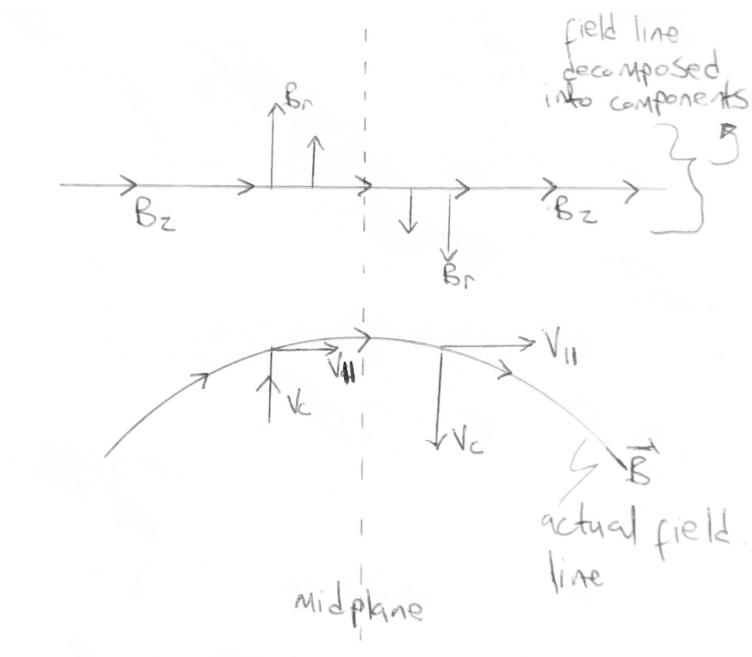


Figure 3: Field Line Decomposed

becomes crucial. As can be seen in figure 3, we can break down our magnetic field line into two components, the axial component which runs along the z-axis of our machine and the radial component which runs along the radial axis of the machine, which we'll label (B_z and B_r) respectively. As we go from left to right on fig 3, the axial field doesn't change direction but the radial field abruptly changes from outward pointing to pointing inward as soon as we cross the mid-plane, and is 0 at the mid-plane. This is a direct consequence of the curvature of this field.

Since the field line is curved, the particle experiences a centrifugal force as it negotiates this curved path and this is accompanied by the velocity v_c which switches directions as the particle crosses the mid-plane. As we approach the mid-plane, $(v_c \times B_z)$ leads to a motion in the azimuthal direction (out of or into the page in figure 3) which can increase or decrease the perpendicular energy of the particle. For this example we're going to assume it increases but the opposite can also be true and that would lead to results that are exactly opposite to the ones we'll get here. If the perpendicular energy increases, the parallel energy has to decrease for energy to be conserved. Here, $(v_\parallel \times B_r)$ also produces motion in the azimuthal direction but however opposite to the motion produced by $(v_c \times B_z)$ so the effects of this cross product are directly opposite as well, which means it reduces perpendicular energy and increases parallel energy. It's important to note here that these changes in motion produced by $(v_c \times B_z)$ and $(v_\parallel \times B_r)$ do not necessarily cancel out. After the particle crosses the mid-plane B_r and v_c switch directions which means the effects on the particle are directly opposite to those we described when they were on the other side of the mid-plane, but since this change is so abrupt, and since the parallel speed of the particle changes, any extra time spent on either side of the mid-plane by this particle will lead to imbalances in the perpendicular energy gained or lost leading to jumps in the magnetic moment μ.

9 Simulations

For the Simulations I used the RMF code which is a single particle Hamiltonian simulation code authored by Alan Glasser. The code simulates a mirror machine whose parameters can be changed and the input variables can also be changed.

9.1 Example of μ evolution for arbitrary conditions

Initially I will just show that the magnetic moment is not constant by providing results from a simulation showing the evolution of μ given certain arbitrary input variables. I will then go on to show how $(\Delta\mu)_m/\mu$, where the m subscript stands for the maximum change in μ during the simulation, evolves with changing pitch angle (initial angle of particle with the z axis) and particle energy. Here I chose $(\Delta\mu)_m/\mu$ as our parameter of interest because it allows us to compare changes on particles even if they have different initial μ and total energy. The simulation stops when the particle escapes from the machine or when the Taumax, which

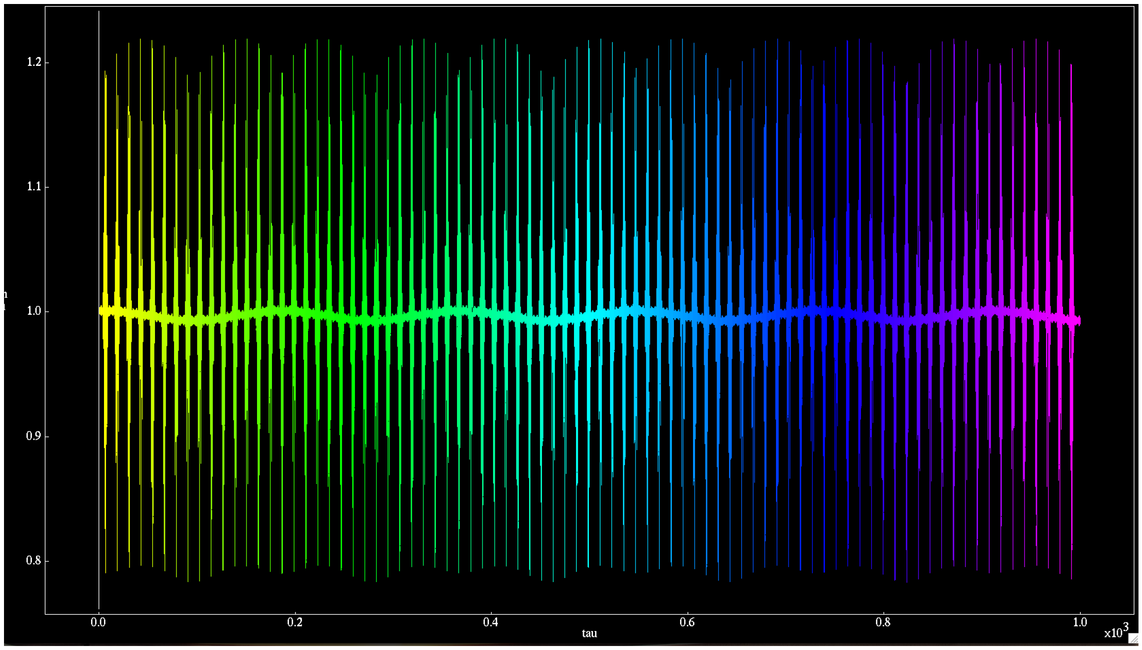


Figure 4: μ evolution in time

is just the maximum time allowed for the simulation, is reached. For all the simulations in this paper I will cap Taumax at 1000.

Figure 4 shows how the magnetic moment evolves in a mirror machine with the following conditions.
Particle - Electron
Pitch angle - 70 degrees
Initial Phase - 30 degrees
Starting radius of particle - 7 cm
Starting z position - 24.5 cm
FRC separatrix radius - 7 cm
Axial Length of machine - 35 cm
Magnetic field strength at coils - 3000 gauss
Axial position of mirror field coil - +38cm and -38cm
Initial energy of particle - 3000 eV
Max Tau - 1000

Figure 4 shows the evolution of μ, which is normalized to it's initial value, in tau, which is time. The spikes in the figure are from when the particle crosses the mid-plane and the steady line in the middle shows a more stable μ from when the particle is away from the mid-plane. In the following simulations when we are recording the maximum change in μ versus several parameters we will ignore the spikes and look at the more steady μ. From the figure we can

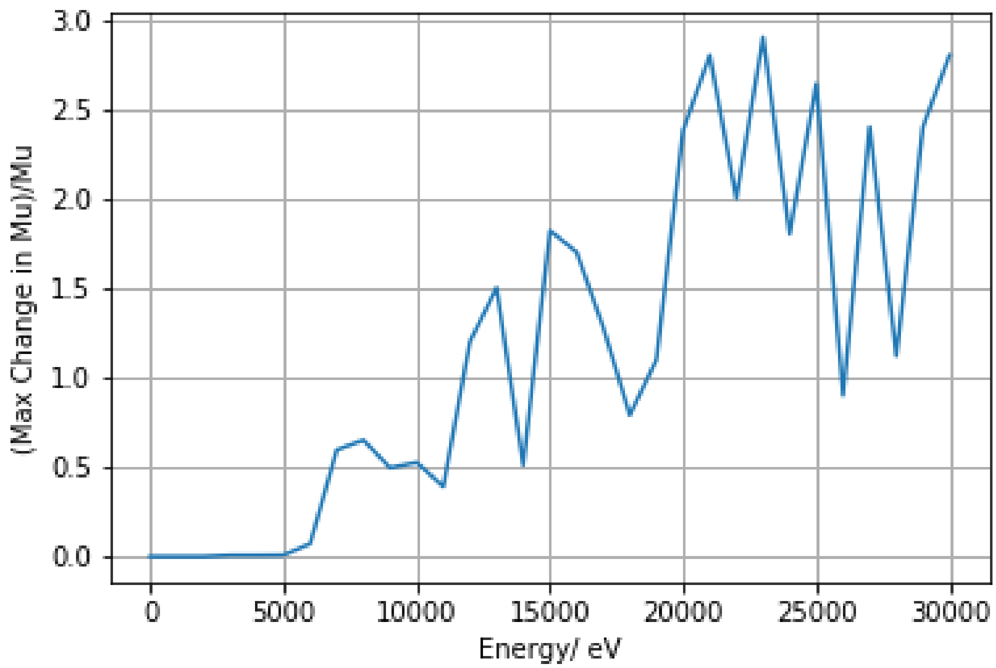


Figure 5: $(\Delta\mu)_m/\mu$ vs Energy

see that μ is not a constant, but rather that it changes based on a variety of conditions in the mirror machine. We will now look at how $(\Delta\mu)_m/\mu$ changes with the changing Initial Energy of the Electron.

9.2 Varying Initial Energy of Electron

Since the μ term has energy in it, the first natural place to go is to see how varying the energy of the injected electron affects the evolution μ. I used similar conditions to the ones that provided figure 4 except that now I was varying the Initial Electron Energy instead of having it constant. I recorded the $(\Delta\mu)_m/\mu$ for energies in the range 0 eV to 30 000 eV, in 1000 eV increments which means I had 31 data points for the entire simulation.

Figure 5 shows the plot showing the data I just described above. The data points are all over the place but a general trend can be seen showing that the $(\Delta\mu)_m/\mu$ term is increasing as we increase the initial electron energy. From this we see that at low energies μ is relatively constant and as the energies increase, the changes become bigger and bigger.

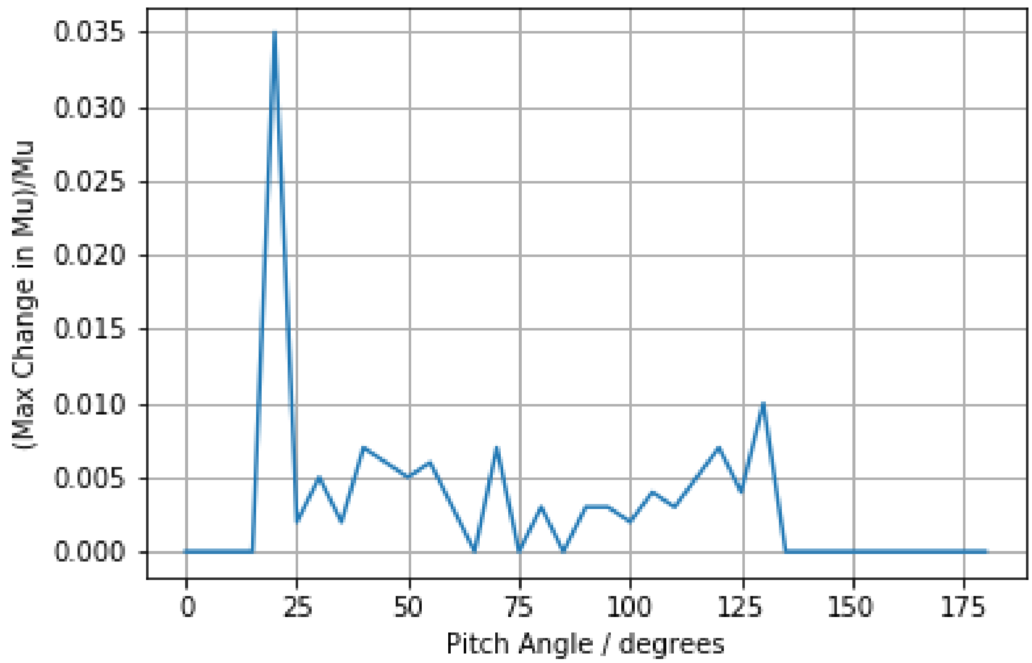


Figure 6: $(\Delta\mu)_m/\mu$ vs Pitch Angle

9.3 Varying Pitch Angle

Another curious parameter is the Pitch Angle which is directly linked with how much proportion of our initial electron energy is apportioned to μ since μ only incorporates the perpendicular component of energy of the particle, which in turn depends on the Pitch Angle. I also had the same set up as the Simulation we just previously described except that now I was varying the Pitch Angle with the Energy fixed at 3000 eV. By the end, I had 37 Data points which I then plotted.

Figure 6 shows the resulting graph and we can see that the data points seem to be random and don't follow a well defined trend. We have to note that the changes in μ are very small and thus μ stays relatively constant. As we've seen in the Energy simulations, at low energies like 3000 eV where we did this Pitch Angle simulation at, changes in μ are very very small so this might be an explanation for these very small changes. It is also important to note that even if these changes are small, they may lead to very large losses when we are operating a real world plasma confinement device since we'll have a lot more particles and operating over very long periods of time.

10 Conclusion

There are several other parameters I could have chosen to vary and see how they affected $(\Delta\mu)_m/\mu$ but since the purpose of the simulations was to illustrate that μ is actually not conserved, the two I chose more than sufficed. Since μ actually varies, it follows that the loss cone is not constant and that a particle can easily be ejected out of confinement at unexpected moments. Future papers may try to model a theoretical framework of how μ changes so that one can predict it at any point in time but that might be very difficult to do, and if so, then computer code and simulations are a very important tool we can use to accomplish these goals. One might ask what's the use of having an adiabatic invariant if it's not constant, but this invariant has been very useful thus far because in some situations it is very close being constant. However, if we need to improve the accuracy of our machines we can, instead of having a single defined loss cone for a machine, have one that evolves in time with the changing μ. Since this can become very completed very quickly, computers are probably the only way we can make that work. It's not as concise or beautiful as the adiabatic invariants method which can be worked out by hand is, but sometimes science is messy and complicated and this is one of those times.

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