Derivation of the Paschen curve law
ALPhA Laboratory Immersion

Arturo Dominguez

July 3, 2014

1 Objective

If a voltage differential is supplied to a gas as shown in the setup (Figure 1), an electric field is formed. If the electric field applied is strong enough, an avalanche process (the Townsend avalanche) is started which leads to the breakdown of the gas and the formation of plasma. This document describes the physics of this process and derives the law (Paschen's law) that predicts the voltage differential that needs to be supplied in order to create the plasma.

2 Experimental Setup

In Figure 1 the experimental setup for the Paschen curve experiment is shown. A pair of parallel plate electrodes are placed inside a vessel that contains a gas which can be air but can also be a more pure gas, like He, Ar, Ne, etc. While the geometry of the setup is irrelevant to the qualitative behavior of the breakdown voltage, the simple 1D geometry results in a simpler comparison with theory. The variables that can be controlled are: the pressure of the gas, p, the distance between the electrodes, d, and the voltage between the electrodes, V, as well as the gas contained in the vessel.

3 Qualitative description

As the voltage difference is applied, the electric field will accelerate any free charges that exist. Free electrons exist in the system due to random events from a variety of mechanisms including the triboelectric effect or through astronomical particles traversing the vessel and ionizing neutral particles. If an electron can gain more than the ionizing energy of the gas, U_I (approximately $14eV$ for Nitrogen), the electron can ionize the neutral particle and create a new free electron and a free ion. The new free electron can repeat the process and create a chain reaction called the Townsend Avalanche. The ion is accelerated towards the cathode and as it hits it there is a chance that it will

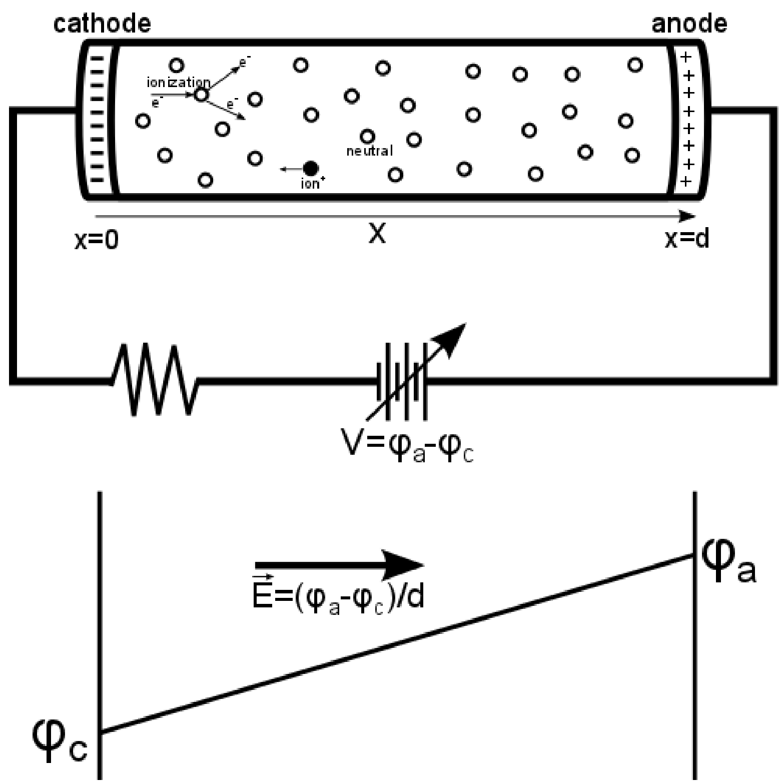

Figure 1: The standard DC discharge setup shows an electron ionizing a neutral particle. When the process becomes self sustaining, the Townsend Avalanche is initiated and a plasma is formed.

free an electron from the cathode. This process is called *secondary emission* and is responsible for the production of electrons that can sustain the plasma. Once a threshold condition is met, the secondary electrons suffice to begin the Townsend Avalanche.

4 Derivation of Paschen's Law

Quantitatively, the electron avalanche can be described using the *rate of ionization per unit length*, α. For example, if $\alpha = 2cm^{-1}$ then within a cm of electron travel along the x-axis, there will, on average, be 2 collisions with neutral particles that result in ionizing events. This results in the following equation describing the increase of electron current density, $\Gamma_e(x)$:

$$d\Gamma_e(x) = \Gamma_e(x)\alpha dx, \tag{1}$$

which integrates to:

$$\Gamma_e(x) = \Gamma_e(0)e^{\alpha x}. \tag{2}$$

Since there is no current leaving or entering the vessel except through the electrodes, and there is no charge accumulation, the continuity equation results in:

$$\Gamma(0) = \Gamma(d), \tag{3}$$

that is, the current density at the cathode ($x = 0$) is the same as that at the anode ($x = d$). Assuming a single ion species, Equation 3 can be rewritten as:

$$\Gamma_e(0) + \Gamma_i(0) = \Gamma_e(d) + \Gamma_i(d) \tag{4}$$

$$\Gamma_i(0) = \Gamma_e(d) - \Gamma_e(0) + \Gamma_i(d). \tag{5}$$

Using the fact that there is no input of ions from the anode, $\Gamma_i(d) = 0$. Using this constraint, and substituting Equation 2 into Equation 5 results in:

$$\Gamma_i(0) = \Gamma_e(0)(e^{\alpha d} - 1). \tag{6}$$

As an ion reaches the cathode, the probability of it releasing a secondary electron is given by the coefficient γ, therefore:

$$\Gamma_e(0) = \gamma\Gamma_i(0). \tag{7}$$

At the threshold point where the secondary emission sustains the plasma and the Townsend Avalanche is formed, Equations 6 and 7 are balanced and $\Gamma_i(0)$ can be substituted, leading to:

$$\frac{\Gamma_e(0)}{\gamma} = \Gamma_e(0)(e^{\alpha d} - 1) \tag{8}$$

$$\frac{1}{\gamma} = e^{\alpha d} - 1 \tag{9}$$

$$\alpha d = \ln(1 + 1/\gamma). \tag{10}$$

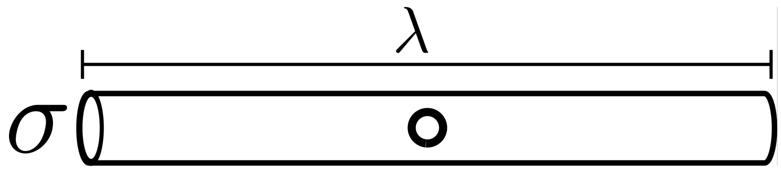

Figure 2: The volume associated with an individual neutral gas particle is given by $vol = 1/n = \sigma\lambda$.

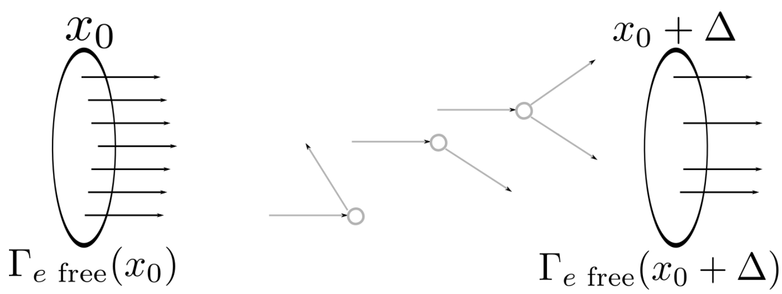

Figure 3: The volume associated with an individual neutral gas particle is given by $vol = 1/n$.

Equation 10 gives one constraint on the ionization coefficient.
While we've defined the role of α, we haven's discussed a physical derivation of what its value should be. In the following section we will tackle that.

As an electron is accelerated through the gas, the distance it travels, on average, before it collides with a neutral particle is given by the *mean free path* (mfp), λ. The volume occupied by a single neutral particle is given by $vol = 1/n$, where n is the neutral density, as shown in Figure 2. The face of the cylinder is the cross section of the collision between the neutral particle and the electron, or $\sigma \approx \pi(r_e + r_n)^2 \approx \pi r_n^2$ where r_e and r_n are the electron and neutral particle's radii respectively. Therefore, the volume occupied per neutral is: $vol = 1/n = \sigma\lambda$. Since the neutral gas follows the ideal gas law, $p = nk_BT$, where p is the neutral pressure, T is the temperature of the neutrals and k_B is the *Boltzmann constant*, the mean free path can be rewritten as:

$$\lambda = \frac{1}{\sigma n} = \frac{k_BT}{\sigma p} \tag{11}$$

λ is the rate of collisions per length in the x-axis of electrons hitting neutral particles. Note that if the electron is energetic enough, the collision will ionize the particle, but this is not generally true.
If you follow the flow of electrons at a position x_0, one can define the current density of *free* electrons, $\Gamma_{e\ \text{free}}(x_0 + \Delta)$ which is the current density of free electrons from x_0 that have reached $x_0 + \Delta$ without having collided with a neutral.

In Figure 3 it is observed that as the electrons travel across the length, Δ, some will collide with neutrals and are lost from the $\Gamma_{e\text{ free}}$. The rate of collisions per unit length is, as was discussed earlier, λ. Therefore, the differential equation that determines $\Gamma_{e\text{ free}}(x)$ is given as:

$$d\Gamma_{e\text{ free}}(x) = -\Gamma_{e\text{ free}}(x)\frac{dx}{\lambda} \tag{12}$$

$$\Gamma_{e\text{ free}}(x_0+\Delta) = \Gamma_{e\text{ free}}(x_0)e^{-\Delta/\lambda} \tag{13}$$

$$\frac{\Gamma_{e\text{ free}}(x_0+\Delta)}{\Gamma_{e\text{ free}}(x_0)} = e^{-\Delta/\lambda} \tag{14}$$

Note that the RHS of Equation 14 is independent of x_0, hence, at any point in x, the probability that a free electron has traversed a distance of *at least* Δ is given by:

$$P(\text{distance traveled} \geq \Delta) = e^{-\Delta/\lambda} \tag{15}$$

Now, let's get to α. If every collision between an electron and a neutral resulted in an ionization, then $\alpha = 1/\lambda$, but only the proportion of those electrons that have enough energy to ionize the neutral particle, U_I, will cause the ionization, therefore:

$$\alpha = \frac{P(\text{electrons with energy } \geq U_I)}{\lambda} \tag{16}$$

Since the electrons gain energy by being accelerated down the electric field, E, then we can define λ_I using:

$$U_I = eE\lambda_I \tag{17}$$

$$\lambda_I = \frac{U_I}{eE} \tag{18}$$

$$\lambda_I = \frac{U_I d}{eV} \tag{19}$$

where $E = V/d$ has been used. λ_I is, therefore, the distance that an electron must travel in the electric field in order to gain the necessary energy, U_I, to ionize the neutral upon collision. Therefore, Equation 20 can be rewritten as:

$$\alpha = \frac{P(\text{distance traveled} \geq \lambda_I)}{\lambda} = \frac{e^{-\lambda_I/\lambda}}{\lambda}, \tag{20}$$

where Equation 15 has been used. Multiplying d on both sides of Equation 20, we obtain:

$$\alpha d = d\frac{e^{-\lambda_I/\lambda}}{\lambda} \tag{21}$$

$$\ln(1+1/\gamma) = \frac{d}{k_B T/\sigma p}e^{(-U_I d/eV)/(k_B T/\sigma p)} \tag{22}$$

$$\ln(1+1/\gamma) = \left(\frac{\sigma}{k_B T}\right)(pd)e^{(-U_I\sigma/ek_B T)(pd/V))} \tag{23}$$

$$\ln(\ln(1+1/\gamma)) - \ln(Apd) = \frac{-Bpd}{V} \tag{24}$$

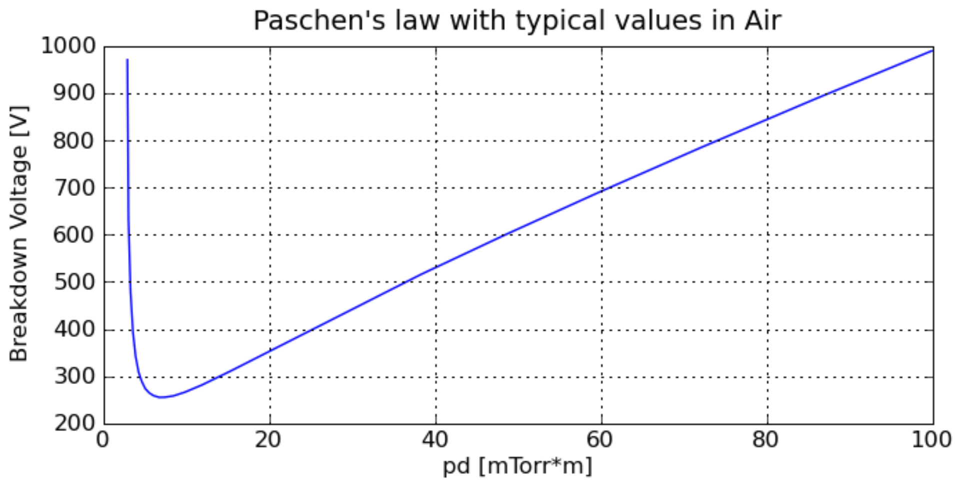

Figure 4: Equation 25 using $A = 1.5mTorr^{-1}m^{-1}$, $B = 36V/(mTorr \times m)$ and $\gamma = 0.02$

where $A \equiv \sigma/k_B T$ and $B \equiv U_I \sigma / e k_B T$ and we have used Equations 10, 11 and 19. Finally, since V is really the voltage right at the point of breakdown, we can change its notation to V_{BD} (breakdown voltage) and put it on the LHS:

$$V_{BD} = \frac{Bpd}{\ln(Apd) - \ln(\ln(1 + 1/\gamma))} \tag{25}$$

This is the final form of Paschen's Law. If we take pd as the abscissa $(x - axis)$ and V_{BD} as the ordinate $(y - axis)$ then the range (as we will see) is $pd = (\ln(1 + 1/\gamma)/A, \infty)$.

For $A = 1.5mTorr^{-1}m^{-1}$, $B = 36V/(mTorr \times m)$ and $\gamma = 0.02$ which are typical values in air at room temperature using stainless steel electrodes, Equation 25 leads to the plot shown in Figure 4.

Note the existence of a minimum in the curve. The pd and V_{BD} at the minimum can be found using the fact that at the critical value, $\frac{dV_{BD}}{d(pd)} = 0$:

$$\frac{dV_{BD}}{d(pd|_{\min})} = \frac{BD - \frac{1}{pd|_{\min}}(Bpd|_{\min})}{D^2} = 0 \tag{26}$$

$$B(D-1) = 0 \tag{27}$$

$$D = \ln(\ln(1 + 1/\gamma)) - \ln(Apd|_{\min}) = 1 \tag{28}$$

$$pd|_{\min} = \frac{\ln(1 + 1/\gamma)}{A} e \tag{29}$$

$$V_{BD\ \min} = \frac{B \ln(1 + 1/\gamma)}{A} e \tag{30}$$

where $D \equiv \ln(\ln(1 + 1/\gamma)) - \ln(Apd|_{\min})$ is the denominator of Equation 25 and e in Equations 29 and 30 is *Euler's number*.

Using Equations 10 and 11, Equation 29 can be rewritten as:

$$\alpha|_{\min} = \frac{e^{-1}}{\lambda} \tag{31}$$

Or, comparing it to Equation 20, at the minimum: $\lambda_I = \lambda$. This makes sense intuitively, it's taking the electrons the mfp to accelerate just enough to ionize the neutrals. If you were to increase λ_I, a lot of the collisions would not lead to ionization, if you increase λ then the rate of collisions would drop. (from the denominator in Equation 20).
We also see that when $D = 0$, $V_{BD} \to \infty$, this occurs when:

$$\ln\left(\ln\left(1 + 1/\gamma\right)\right) - \ln\left(Apd|_{\text{inf}}\right) = 0 \tag{32}$$

$$\ln\left(1 + 1/\gamma\right) = Apd|_{\text{inf}} \tag{33}$$

$$\alpha|_{\text{inf}} = \frac{1}{\lambda} \tag{34}$$

This result also makes sense, since the rate of ionizing collisions can't be greater than the rate of collisions ($1/\lambda$). This sets the lower limit on pd as $pd > \ln(1 + 1/\gamma)/A$ as discussed before.

5 Discussion of the minimum

The conditions for the minimum are shown in Equations 29 and 30. Its existence can be understood by looking at the extremes: if pd is too big, since $\lambda \sim 1/p$, λ is very small, therefore the electrons will collide too much and won't acquire the ionizing energy U_I necessary. On the other hand, if pd is too small, that is λ is big compared to the distance between the electrodes d, the electrons will collide with the anode before they have a chance of ionizing the gas.
There are many implications of Paschen's law, e.g., when constructing a fluorescent light bulb, where the distance between the electrodes d, is set by the application, the pressure is set so as to be close to the minimum in order to minimize the necessary voltage needed, leading to lower power consumption. The typical pressure inside a fluorescent bulb is $\approx 2Torr.$.
Another consequence of Paschen's law is that if we're dealing with atmospheric plasmas ($p \approx 760Torr$) the distance between electrodes for the minimum condition is calculated (using the same conditions as in Figure 4) to be $\sim 10\mu m$. This leads to the label of *microplasmas* for these type of discharges and their consequential difficulty in attaining experimentally.