PDF tails and self-organization of shear flows

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Outline

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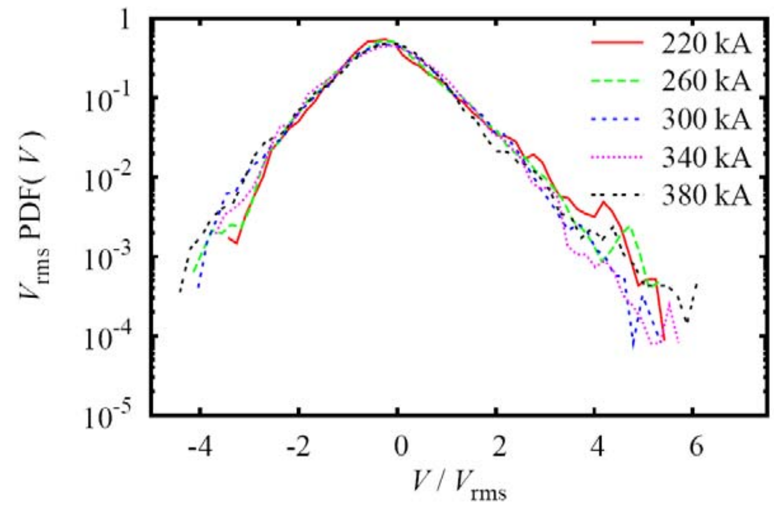
- Probability distribution functions
- Previous results
- Instanton method
- Coherent structures
- Model equations + instanton solutions
- Model of shear flows
- Results and discussion

Goals and motivation

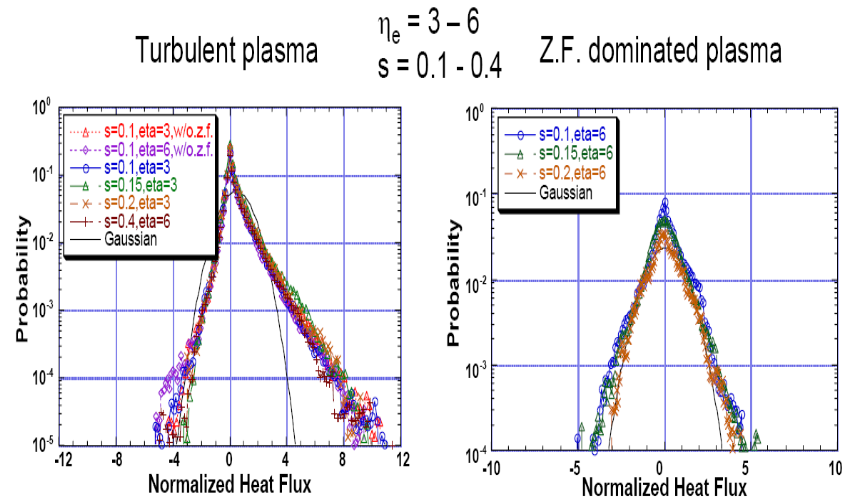
- Goal: To find the generic analytical expression for the PDF tails.
- The tails are often qualitatively different from a Gaussian distribution. The method used is the so-called instanton method.
- Motivation: There are theoretical and experimental evidence that for understanding transport (involving many scales and amplitudes) a probabilistic description is needed.
- Intermittenct systems are badly described by mean field theory and the turbulent transport coefficients are invalid.
- Note that the term ”intermittent” will be used for all phenomena that exhibit strong non-linear features.

Probability distribution function

- Near the center, the PDF is often close to Gaussian but reveals a significant deviation from Gaussianity at the tails (intermittency - the events contributing to the tails are strongly non-linear.).
- Rather than a transport coefficient, a flux PDF is required in order to substantively characterize the transport process.
- PDF tail – rare events, but large amplitude (e.g. large heat load on the wall.)

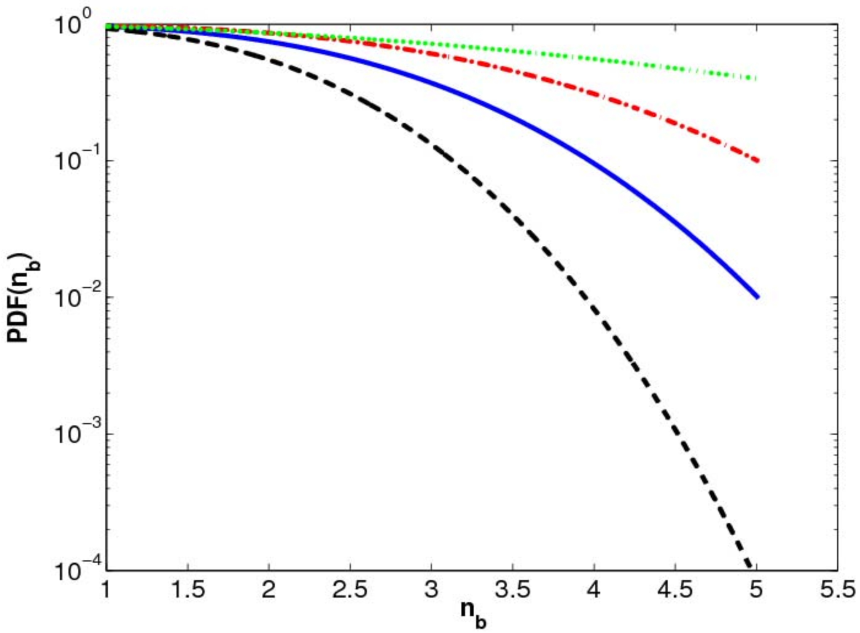


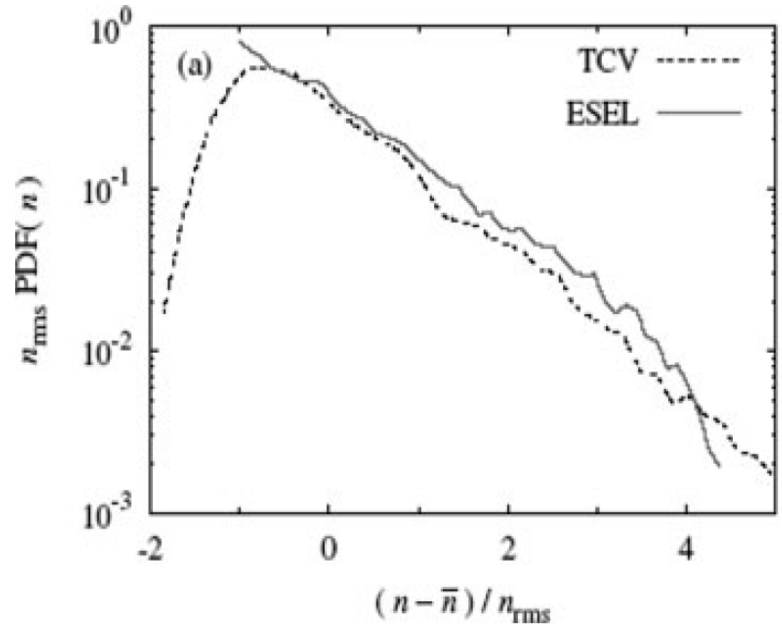
Radial velocity PDF measured at TCV, Garcia EPS2006



Matsumoto, USJP Workshop 2007

Previous results – Blob density PDF





$$P(n) \propto e^{-cn^3}$$

$$c = \frac{4}{27\alpha^2}$$

$$\propto e^{-cn^\Gamma}$$

$$\frac{1}{\alpha^2} = 1.5 \pm 0.5 \quad \Gamma = 2.5 - 4.0$$

Anderson et al PoP **15** 122303 (2008)

A coupled drift-wave zonal flow system

Anderson et al submitted to NF 2008

A coupled system of drift waves ϕ_1 and zonal flows ϕ_0 (ξ is the X-coordinate and ζ is the time-coordinate.

$$C_1 \frac{\partial \tilde{\phi}_1}{\partial \zeta} + i C_2 \frac{\partial^2 \tilde{\phi}_1}{\partial \xi^2} + C_3 \tilde{\phi}_0 \tilde{\phi}_1 = -i\nu C_4 \tilde{\phi}_1 + f$$

$$D_1 \frac{\partial \bar{\phi}_0}{\partial \zeta} + D_2 \frac{\partial \tilde{\phi}_0}{\partial \xi} = D_3 \frac{\partial |\tilde{\phi}_1|^2}{\partial \xi}$$

PDF tails of ZF formation

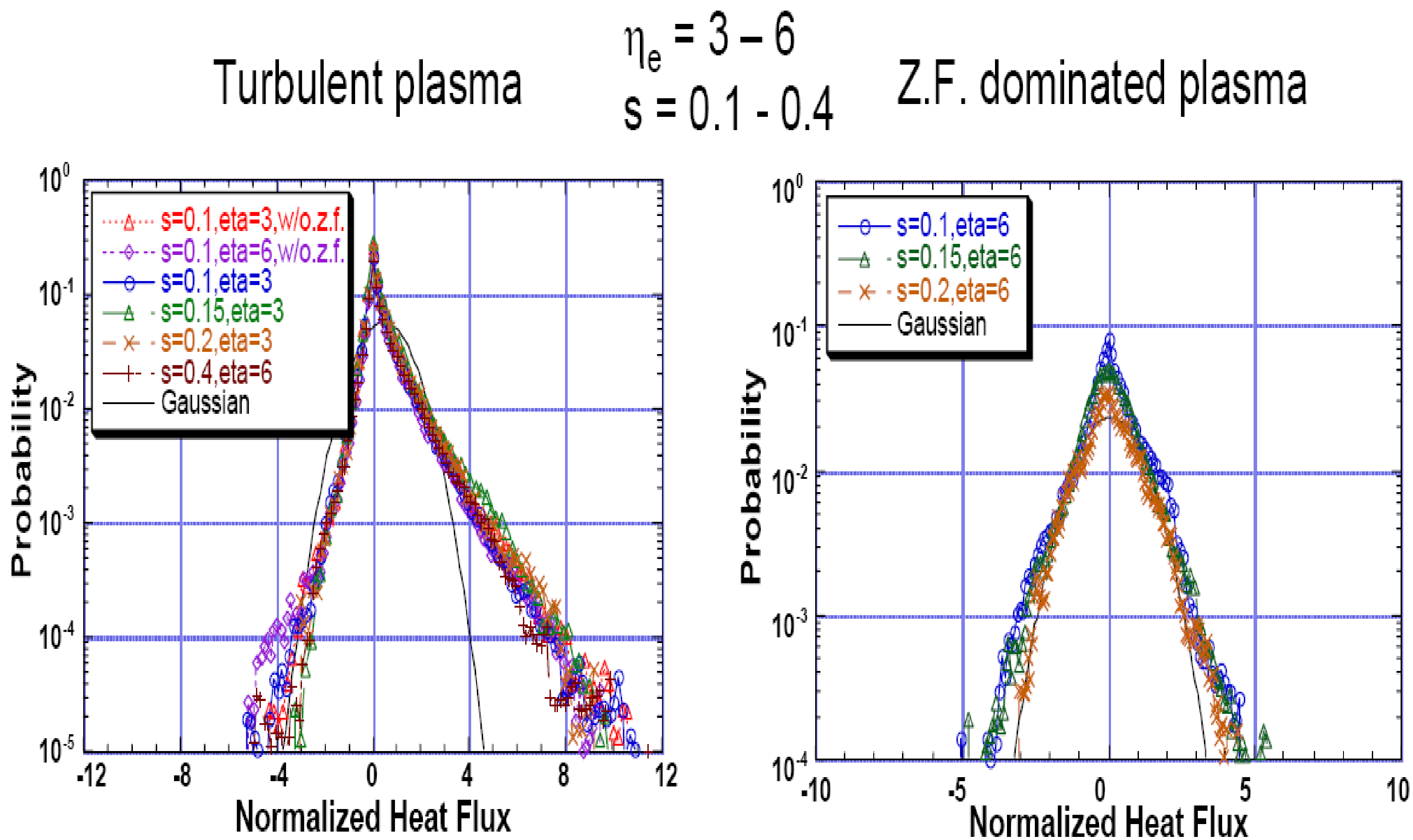
$$P(\phi_0) \propto e^{-c\phi_0^3}$$

PDF tails of Momentum flux

$$P(R) \propto e^{-cR^2}$$

Without the influence of ZF the PDF tails of momentum flux is $\propto e^{-cR^{3/2}}$
Also found experimentally in CSDX by Z. Yan APS2007

PDFs of fluctuations vs ZF dominated plasma

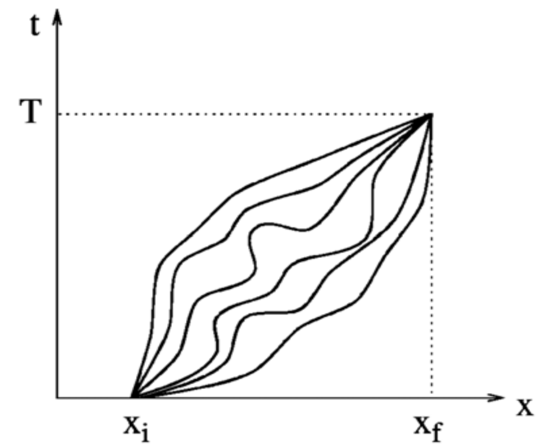


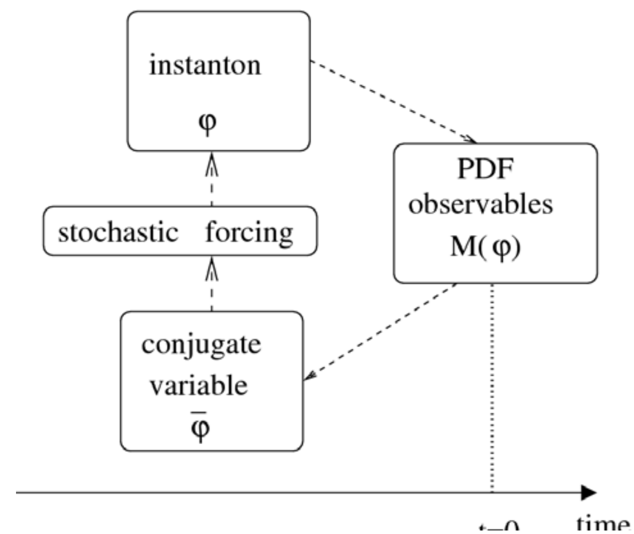
Matsumoto, USJP Workshop 2007

Instanton method

Kim and Anderson PoP in press (2008)

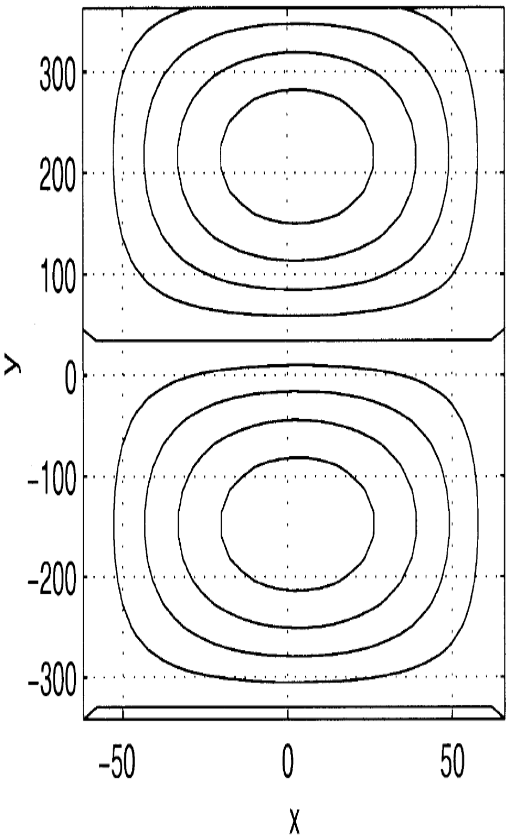
- The instanton method is a non-perturbative way of calculating the Probability Distribution Function tails.
- The PDF tail is viewed as the transition amplitude from a state with no fluid motion to a final state governed by the coherent structure.
- The creation of the coherent structure is associated with the bursty event.
- The optimum path is found by using the saddle-point method.

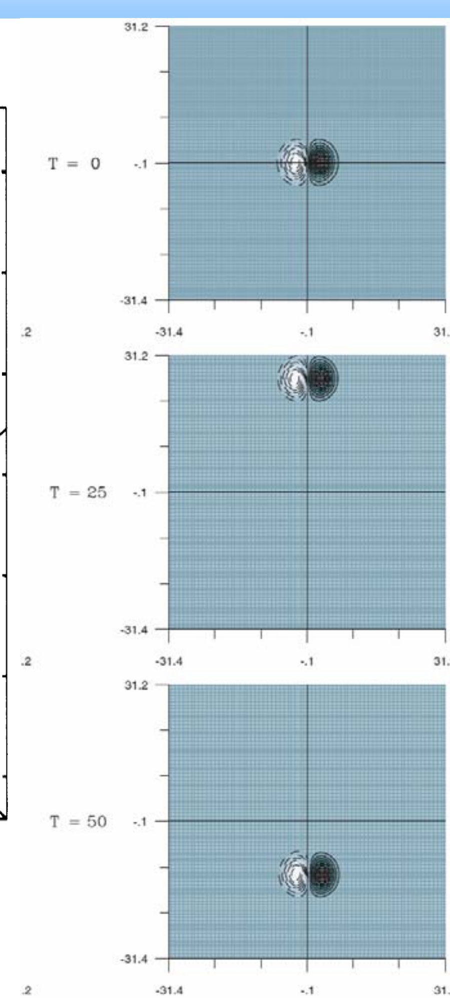




Coherent structures

- Coherent structures are major players in transport dynamics through the formation of avalanche-like events with large amplitude.
- There are several examples of coherent structures (c.f. modon or bipolar vortex soliton) to the non-linear governing equations.
- Strong theoretical evidence that a probabilistic formulation is needed to characterize the problem.





Left:Dastgeer IEEE TPS 2003, Right: Waelbroeck et al PPCF 46 1331 (2004)

Model Equations

Time evolution of the potential is governed by an N order Interaction term.

$$\frac{\partial \phi}{\partial t} + N(\phi) = f \qquad N(\phi) \propto \phi^n$$

The forcing is assumed to be Gaussian with a short correlation time:

$$\left\langle f(x,t) f(x',t') \right\rangle = \delta(t-t')\kappa(x-x')$$

$$\left\langle f \right\rangle = 0$$

The model for calculating the PDF tails

The PDF of the mth moment can be defined as:

$$P(R) = \langle \delta(\langle v_x v_y \rangle - R) \rangle = \int d\lambda e^{i\lambda R} \langle -i\lambda v_x v_y \rangle = \int d\lambda e^{i\lambda R} I_\lambda$$

The integrand can be re-written as a path-integral:

$$I_\lambda = \int D\phi D\bar{\phi} e^{-S_\lambda}$$

Here, the effective action can be written:

$$S_\lambda = -i \int dx dt \bar{\phi} (\frac{\partial \phi}{\partial t} + N(\phi))$$

$$+ \frac{1}{2} \int dx dx' dt \bar{\phi}(x) \kappa(x - x') \bar{\phi}(x')$$

$$+ i\lambda \int dx dt M(\phi) \delta(t) \delta(x - x_0)$$

$$M(\phi) \propto \phi^m$$

The forcing κ is a Gaussian with a delta-correlation in time.

Instanton (saddle-point) solutions

- The path-integral will be solved using a saddle point method.
- Assuming that the coherent structure has a spatial profile $\phi_0(x)$ and a temporal evolution F(t) and similarly for the conjugate variable.

$$\phi(x,t) = \phi_0(x)F(t)$$

$$\bar{\phi}(x,t) = \bar{\phi}_0(x)\mu(t)$$

The action can be recast as:

$$S_\lambda = -i\int dt\mu c_1(\dot{F} + c_2 F^2) + \frac{1}{2}\int dt\mu^2 c_1 c_3 + i\lambda\int dtF^m\delta(t)c_1c_4$$

Constants c are projections on the conjugate structures

$$c_1 = \int dx\,\bar{\phi}_0(x)\phi_0(x)$$

$$c_1c_2 = \int dx\,\bar{\phi}_0(x)\phi_0{}^n(x)$$

$$c_1c_3 = \int dxdy\,\bar{\phi}_0(x)\kappa(x-y)\phi_0(y)$$

$$c_1c_4 = \int dx\,\phi_0^m(x)\delta(x-x_0)$$

Steepest descent method or saddle point integral

Assume the function f has a unique global maximum at x_0

$$\int dx e^{Mf} \approx e^{Mf(x_0)} \int e^{-M|f''(x_0)|(x-x_0)^2/2} dx = \sqrt{\frac{2\pi}{M|f''(x_0)|}}\, e^{Mf(x_0)}$$

As $M \to \infty$

In the case of the path-integral we have a similar situation, however instead Stationary points we must look for functions that optimize the action. Consider the functional derivatives:

$$\frac{\delta S_\lambda}{\delta F} = 0 \qquad \frac{\delta S_\lambda}{\delta \mu} = 0$$

This gives us two equations in F and μ, respectively

$$\dot{F} + c_2 F^n = -ic_3 \mu$$

$$\dot{\mu} - nc_2 F^{n-1} \mu = -\lambda c_4 m F^{m-1} \delta(t)$$

$F(-\infty) = 0$ and $\mu(t>0) = 0$

Instanton solutions

We find the solution:

$$F^{-n+1} = F_0^{-n+1} + c_2(-n+1)t$$

$$F_0^{n-m+1} = q\lambda$$

To calculate the path-integral we now have to evaluate the saddle point action (input the solution above into the action).

$$S_\lambda = -i\int dt\mu c_1(\dot{F} + c_2 F^n) + \frac{1}{2}\int dt\mu^2 c_1 c_3$$

$$+ i\lambda\int dxdtF^m \delta(t) c_1 c_4$$

$$= \int dtc_1(\dot{F} + c_2 F^n)^2 + i\lambda F_0^m c_1 c_4$$

$$= Q\lambda^{(n+1)/(n-m+1)}$$

The PDF tails + Corrections

To determine the PDF tails we have to evaluate the λ integral:

$$P(\xi) = \int d\lambda e^{-\lambda\xi + Q\lambda^{(n+1)/(n-m+1)}}$$

put

$$f(\lambda) = -\lambda\xi + Q\lambda^{(n+1)/(n-m+1)}$$

And find the saddle point:

$$f'(\lambda_0) = 0 \qquad \lambda_0 = \left(\frac{\xi(n-m+1)}{Q(n+1)} \right)^{(n-m+1)/(n+1)}$$

This gives the PDF tails as:

$$P(\xi) \propto \xi^{(n+1-2m)/(2m)} e^{-c\xi^{(n+1)/m}}$$

The first factor comes from the Gaussian integral correction in the steepest descent method.

PDF tail of a general moment

The PDF tails of moment (m) and with the order of the highest non-linear interaction term (n)

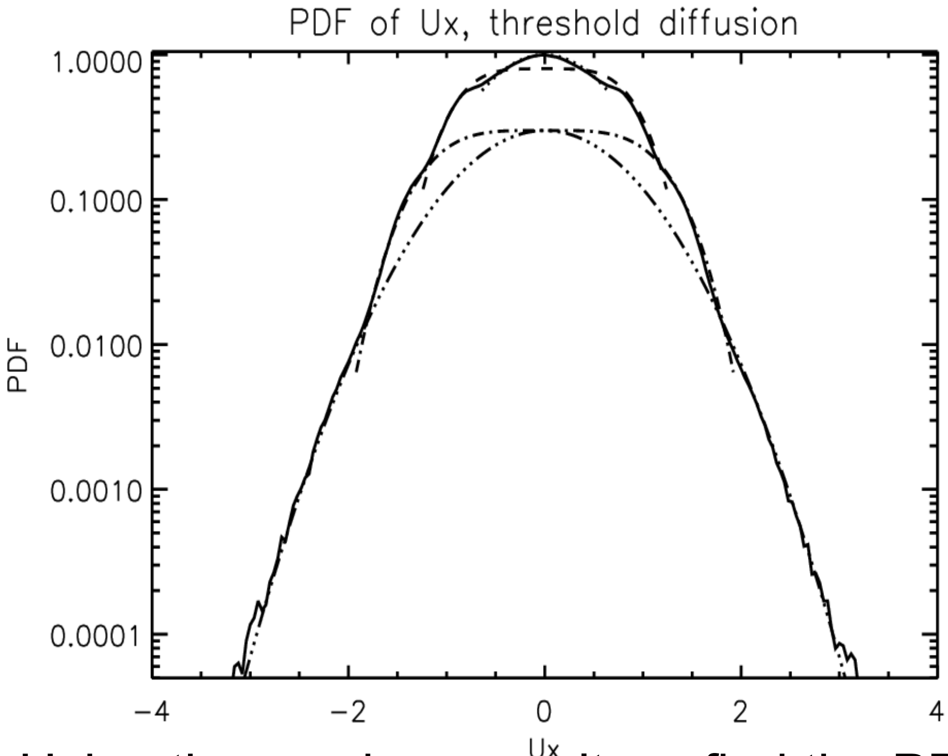
$$P(Z) \propto \exp(-\xi Z^s) \qquad s = \frac{n+1}{m}$$

Examples:

1. Linear system with PDF tails of first moment – Gaussian s=2.
2. Linear system with PDF tails of flux (n*v) – s=1
3. Hasegawa–Mima system with PDF tails of momentum flux – s=3/2

The PDFs tails can be calculated provided that the integral mean value over the considered coherent structure is non-zero. A coherent structure For the HM system is the modon. The mean value of Reynolds stress over this structure is zero. This is solved by having several coupled modons.

Threshold diffusion



A forced model of shear flow that Been used for a wide range of phenomena (solar/atmospheric)

$$\frac{\partial u_x}{\partial t} = \frac{\partial^2}{\partial x^2}(D(u)u_x) + f$$

$$D(u) = \nu + \beta u_x^2$$

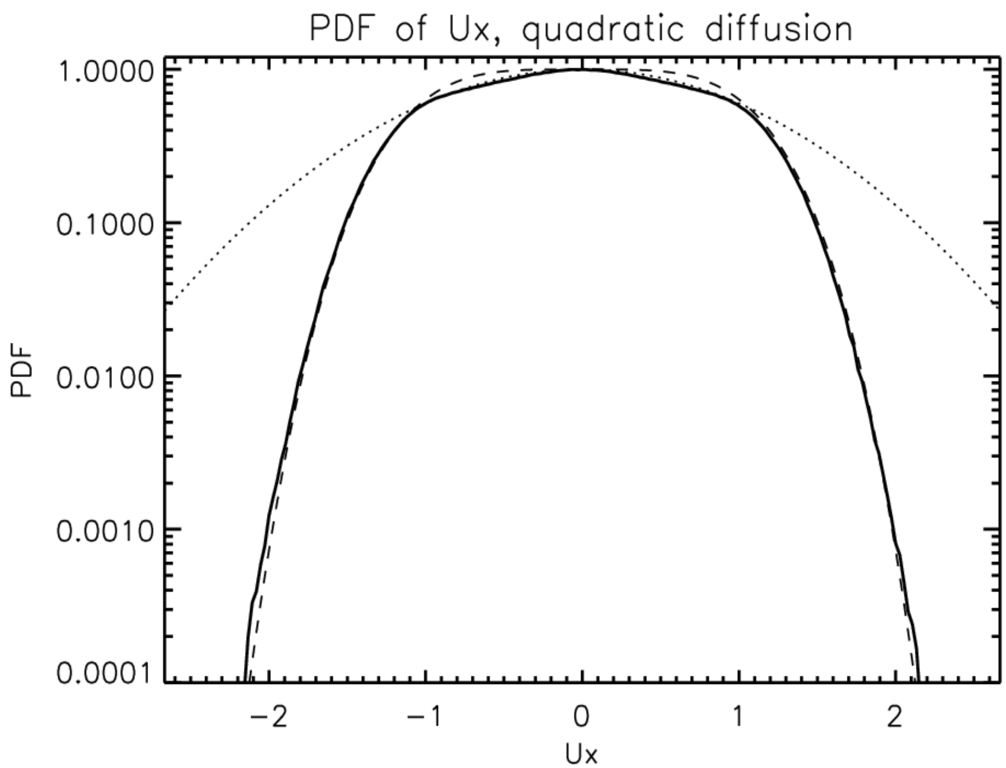
HL Liu J. Atmos. Sci., 64, 579-593, 2007.

Using the previous result we find the PDF: $P(\xi) \propto e^{-cU_x^4}$

A comparison of numerical calculations and the estimated PDFs. In the numerical calculations a Gaussian forcing is used. First an extreme case: Apply forcing only when $|u_x| < u_{xc}$ to allow for relaxation is faster than the disturbance. Result PDF mainly Gaussian!

Quadratic diffusion

Submitted to PRL 2008



Solid line: Non-linear numerical calculation.
Dotted line: Gaussian fit
Dashed line: A fit to the PDF.

$$P(\xi) \propto e^{-cU_x^4}$$

The same PDF may be found
Using the Fokker-Planck
PDF equation.

Close to 0 the PDF is close to Gaussian whereas the tails are strongly intermittent.
There is a cross-over between occurs roughly at the expected critical gradient $u_{xc} \cong \mathrm{sqrt}(\nu/\beta) = 0.98$.

Future work

- Work out corrections to some specific physical equations (HM, Burgers etc)
- PDF tails in an electromagnetic model.
- Self consistent forcing. No external forcing instead use the linear instability as the driving force.
- PDF tails of multi-structures and multi-instantons.
- PDF tail for L-H transition.

Summary

- We have found the PDF tails for general n (degree of the highest non-linear term) and m (moment) as well as subleading corrections coming from the saddle point method.
- By calculating PDFs we may easily discriminate between models and experiments. We need only to know the slope in the log-log plot.
- We have presented a statistical theory of self-organisation of shear flows by a simplified non-linear diffusion model for the shear flow.
- We have compared the PDF tails from numerical calculations and two analytical methods and found very good agreement.

References

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