# Limits on guiding center and gyrokinetic plasma models in 3D magnetic fields


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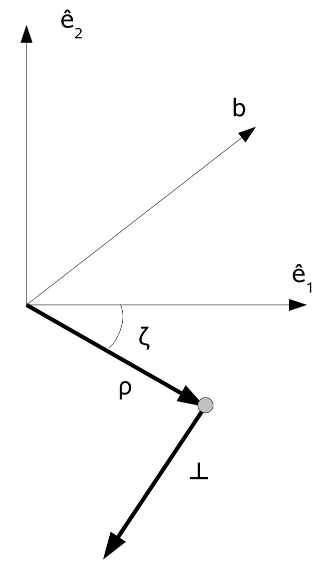
# Topics

- Guiding center model of single particle motion at small gyroradius
- Gyroangle and locally orthogonal coordinates tied to magnetic field
- Second order expansion: Hamiltonian/Lagrangian
- 3D field: magnetic field torsion and coordinate system twisting
- GC higher order validity and global coordinate system existence
- Time-scales, magnetic vector potential, and geometric approximations
- Summary

(for details, see L. Sugiyama, Phys. Plasmas 15, 092112 (2008))

# Guiding Center Model - Single Particle Motion




- **Guiding Center (GC) model for the motion of a single charged particle separates the motion into fast gyration around magnetic field lines and a slowly varying GC motion, with particle position**

$$\mathrm{x} = \mathrm{X} + \frac{\epsilon v_\perp}{\Omega}\hat{\zeta}. \qquad (1)$$

**Particle velocity** $\mathrm{v} = (v_\parallel, \zeta, v_\perp)$ **is written in terms of a gyroangle** $\zeta$ **with direction** $\mathrm{v}_\perp = v_\perp \hat{\zeta} \times \hat{\mathrm{b}}$, **where** $\mathrm{v}_\perp = \hat{\mathrm{b}} \times (\mathrm{v} \times \hat{\mathrm{b}})$, $\hat{\mathrm{b}} = \mathrm{B}/B$, $\mathrm{x}$ **is the particle and** $\mathrm{X}$ **the GC position.**

  – **Expansion in small gyroradius** $\rho/L \sim \epsilon < 1$, **where** $\rho = v_\perp/\Omega$, $\Omega = ZeB/mc$, **and** $L$ **is a system scale length. Fast gyrofrequency** $(\partial/\partial t)/\Omega \sim \epsilon$.

- **Gyroangle defined in local orthogonal coordinates tied to the magnetic field lines at each point in space, axes** $(\hat{\mathrm{e}}_1, \hat{\mathrm{e}}_2, \hat{\mathrm{b}})$. **Originally defined from particle position, transformed to GC position.**

# 3D Magnetic Field

- **Always possible to define locally orthogonal coordinates at each point of the magnetic field. Relation between coordinates at different points not specified.**

- **Gyroaverage defined in terms of cumulative gyroangle, $\langle f\rangle \equiv \oint d\zeta\ f = \int_0^{2\pi} d\zeta\ f$ over non-closed curves. In 3D fields, a globally consistent definition may not exist!**

- **In 2D slab (straight, uniform magnetic field lines), a simple connection exists and the GC expansion in small gyroradius is exact to all orders.**

- **In 3D, the curl of a vector field in a given direction is twice the rate of rotation of the field around that axis, as seen when moving in that direction.**

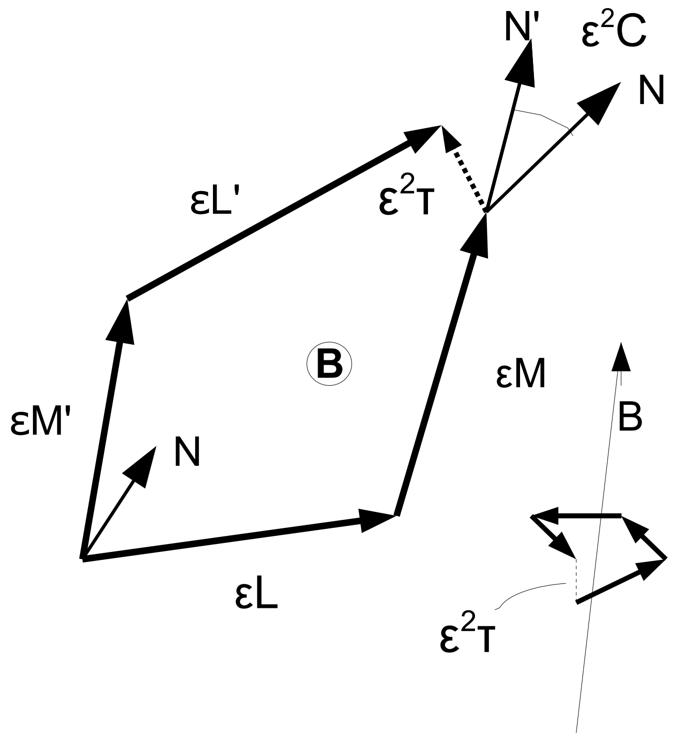
  – **Magnetic field torsion $\tau \equiv \hat{\mathrm{b}}\cdot\nabla\times\hat{\mathrm{b}}$ is the twisting of the field line when moving along itself. (Usually nonzero — plasma parallel current $\mathrm{J}_{\|} = \hat{\mathrm{b}}\cdot\nabla\times\mathrm{B} = B\tau$.)**

  – **Torsion also introduces twisting of the local field-tied orthogonal coordinate system, $\mathrm{R} = (\nabla\hat{\mathrm{e}}_1)\cdot\hat{\mathrm{e}}_2 = -(\nabla\hat{\mathrm{e}}_2)\cdot\hat{\mathrm{e}}_1$. (Def'n: given a vector $\mathrm{x}$, $\mathrm{x}\cdot\mathrm{R} = (\mathrm{x}\cdot\nabla\hat{\mathrm{e}}_1)\cdot\hat{\mathrm{e}}_2$.)**

- **For nonzero $\tau$, gyromotion mixes parallel and perpendicular directions.**
  - **For any small closed curve $C$ surrounding a field line that encloses a surface S that has normal direction $\hat{\mathrm{n}}_S$ along $\hat{\mathrm{b}}$ at one point P on the field line,**

$$\lim_{S\to 0}\frac{1}{S}\oint_C d\mathrm{l}\cdot\hat{\mathrm{b}} = \lim_{S\to 0}\frac{1}{S}\int\int_S dS\,\hat{\mathrm{n}}_S\cdot\nabla\times\hat{\mathrm{b}} = \hat{\mathrm{b}}\cdot\nabla\times\hat{\mathrm{b}}|_P. \qquad (2)$$

- **An infinitesimal path around a field line does not close.**
  **Difference in parallel transport of vector $N$ along two parallel paths around B (B points out of page). Inset: Closed path corresp. to Eq. (2). Torsion contribution is out of plane.**




- **The angle nonuniformity due to torsion is a real physical effect; appears in many areas (Aharonov-Bohm effect, Berry phase, related to Dirac magnetic monopole)**

- **Equations of motion**

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \qquad m\frac{d\mathbf{v}}{dt} = q\mathbf{E}(\mathbf{r},t) + \frac{q}{c}\mathbf{v}\times\mathbf{B}(\mathbf{r},t) \tag{3}$$

- **Transform particle to guiding center phase space coordinates**

  $(\mathbf{r},\mathbf{v},t) \longrightarrow (\mathbf{x}, v_{\parallel}, \zeta, v_{\perp}, t) \longrightarrow (\mathbf{X}, U_{\parallel}, \zeta, w, t)$,
  **where** $w \equiv (\mu B/m)^{1/2}$

- **Time derivative** $d\zeta/dt$ **contains 3D effects**

$$\begin{aligned} d\zeta/dt &= (\partial\zeta/\partial t) + (\mathbf{v}\cdot\nabla)\zeta \\ \partial/\partial\mathbf{r} &= \partial/\partial\mathbf{x} + (\partial v_{\parallel}/\partial\mathbf{x})(\partial/\partial v_{\parallel}) + (\partial v_{\perp}/\partial\mathbf{x})(\partial/\partial v_{\perp}) + (\partial\zeta/\partial\mathbf{x})(\partial/\partial\zeta) \\ \partial\zeta/\partial\mathbf{x} &= (v_{\parallel}/v_{\perp})\,(\zeta\cdot\nabla)\hat{\mathbf{b}} + \mathbf{R}. \end{aligned} \tag{4}$$

- **Poisson brackets used in the Hamiltonian and Lagrangian equations**

$$\{F,G\} = (\partial F/\partial\mathbf{x})(\partial G/\partial\mathbf{v}) - (\partial G/\partial\mathbf{x})(\partial F/\partial\mathbf{v}) + \epsilon^{-1}\mathbf{B}\cdot((\partial F/\partial\mathbf{v})\times(\partial G/\partial\mathbf{v})) \tag{5}$$

  **Bracket** $\{\zeta, w\}$ **contains the torsion.**

# GC expansion in small gyroradius

- $\epsilon^0$: **GC moves along** $\mathrm{B}$.

$$\langle \mathbf{x} \rangle = \mathbf{X}, \qquad \langle \boldsymbol{v} \rangle = \boldsymbol{v}_\parallel \tag{6}$$

- $\epsilon^1$: **GC drifts across** $\mathrm{B}$ **appear**

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= \boldsymbol{\Omega}\left[\mathbf{1} + \frac{\boldsymbol{\epsilon w}}{\boldsymbol{B}}\frac{\boldsymbol{U}_\parallel}{\boldsymbol{w}}\left(\frac{\mathbf{1}}{\mathbf{2}}\hat{\mathrm{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathrm{b}} + \hat{\mathrm{b}} \cdot \mathrm{R}\right)\right] \\ \langle \boldsymbol{v}_\parallel \rangle &= \boldsymbol{U}_\parallel\left[\mathbf{1} - \frac{\boldsymbol{\epsilon w}}{\boldsymbol{B}}\frac{\boldsymbol{w}}{\boldsymbol{U}_\parallel}\left(\frac{\mathbf{1}}{\mathbf{2}}\hat{\mathrm{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathrm{b}} - \hat{\mathrm{b}} \cdot \mathrm{R}\right)\right] \end{aligned} \tag{7}$$

  - **Torsional terms** $\boldsymbol{\tau} = \hat{\mathrm{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathrm{b}}$ **and** $\boldsymbol{\tau}_g = \hat{\mathrm{b}} \cdot \mathrm{R}$ **appear.**
  - **Nonuniform gyroangle due to torsion→ nonuniform gyroperiod, velocity space nonuniformities:**
    **Particle sees longer or shorter gyroperiod depending on whether it moves parallel or anti-parallel to B, and how far it moves along B in one gyroperiod. Due to magnetic torsion, the baseline direction for defining** $\zeta$ **rotates along B by** $(1/2)\boldsymbol{\tau}$.

**(Northrup-Rome 1978:** $\perp$ **motion to O(**$\epsilon^2$**),** $\parallel$ **to O(**$\epsilon$**))**

- $\epsilon^2$: **No direct derivation from equations of motion. Hamiltonian/Lagrangian non-canonical phase-space variable methods were developed to extend the expansion to second and higher orders (Littlejohn 1979-83, Brizard 1989).**

Eliminate the geometrical terms $\hat{\mathrm{b}} \cdot \nabla \times \hat{\mathrm{b}}$ and $\hat{\mathrm{b}} \cdot \mathrm{R}$ from the dynamical equations, keeping them only in the gyroangle time derivative $\dot{\zeta}$ and $\langle v_\parallel \rangle$.

Method: add free functions (gyrogauge) to the Lagrangian and define their gyroaverages appropriately.

Effective magnetic vector potential $\mathrm{A}^*$ simplifies the expression for the guiding center phase space Lagrangian $\Gamma$ (Littlejohn 1983, Brizard-Hahm 2007),

$$\begin{aligned} \mathrm{A}^* &= \mathrm{A} + \epsilon U_\parallel \hat{\mathrm{b}} - \epsilon^2 \mu \mathrm{R} \\ \Gamma &= (1/\epsilon)\mathrm{A}^* \cdot d\mathrm{X} + \epsilon\mu\, d\zeta - ((1/2)U_\parallel^2 + \mu B)dt. \end{aligned} \tag{8}$$

The curl $\nabla \times \mathrm{A}^*$ in GC space coordinates $X$ is needed for the equations of motion.

Problem: The effective magnetic field $\mathrm{B}^* = \nabla \times \mathrm{A}^*$ (Northrup) or the quantity $\nabla \times \mathrm{R}$ (Morozov-Solov'ev) is always defined for GC problem, but the corresponding vector potentials $\mathrm{A}^*$ or $\mathrm{R}$ do not have to exist everywhere.

At $O(\epsilon^2)$, existence of $\mathrm{R}$ requires globally consistent magnetic coordinates (vector fields $\hat{\mathrm{e}}_1$ and $\hat{\mathrm{e}}_2$) such that the gradients $\nabla \hat{\mathrm{e}}_1$ and $\nabla \hat{\mathrm{e}}_2$ are defined.

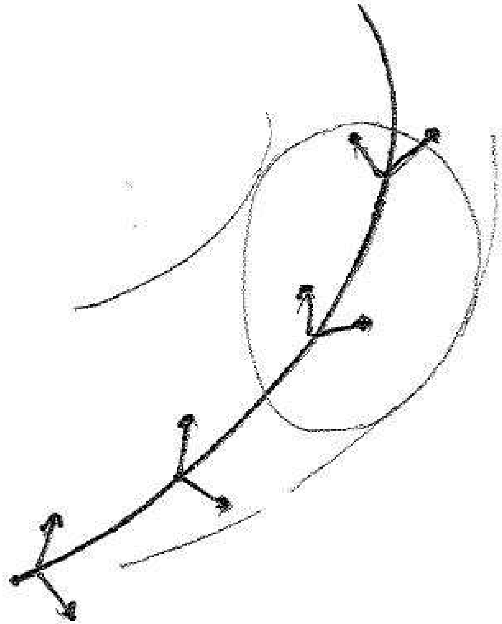
# Torsion and geodesic torsion

- If plasma has good magnetic flux surfaces $\Psi$, $\mathrm{B}\cdot\nabla\Psi = 0$, then $\hat{\mathrm{e}}_1$ can be defined so that $\tau_g = \hat{\mathrm{b}}\cdot\mathrm{R}$ is the (negative) geodesic torsion $\tau_G$ of a field line on a flux surface.

- Vector curve as function of arc-length $s$: Serret-Frenet equations (DoCarmo 1975)

Tangent unit vector $\hat{\mathrm{t}}$, normal $\hat{\mathrm{n}}$ in $\kappa = \hat{\mathrm{b}}\cdot\nabla\hat{\mathrm{b}}$ (curvature) direction, binormal $\hat{\beta} \equiv \hat{\mathrm{t}}\times\hat{\mathrm{n}}$.

$$\begin{aligned} \hat{\mathrm{t}}'(s) &= \kappa\hat{\mathrm{n}} \\ \hat{\mathrm{n}}'(s) &= -\kappa\hat{\mathrm{t}} - \tau\hat{\beta} \\ \hat{\beta}'(s) &= \tau\hat{\mathrm{n}}. \end{aligned} \tag{9}$$



- Curve on an oriented surface: Darboux equations

$\hat{\mathrm{N}}$ is inward normal to surface, $\hat{\mathrm{T}} = \hat{\mathrm{t}}$, binormal $\hat{\mathrm{V}} = \hat{\mathrm{T}}\times\hat{\mathrm{N}}$.

$$\begin{aligned} \hat{\mathrm{T}}'(s) &= \kappa_N\hat{\mathrm{N}} + \kappa_G\hat{\mathrm{V}} \\ \hat{\mathrm{N}}'(s) &= -\kappa_N\hat{\mathrm{T}} - \tau_G\hat{\mathrm{V}} \\ \hat{\mathrm{V}}'(s) &= -\kappa_G\hat{\mathrm{T}} + \tau_G\hat{\mathrm{N}}. \end{aligned} \tag{10}$$

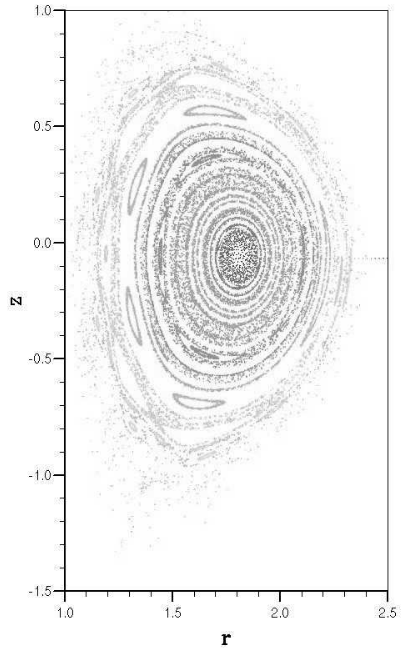
Geodesic torsion $\tau_G$ is rotation of surface-normal and binormal axes around field line.

$\kappa_N$ is the normal curvature and $\kappa_G$ the geodesic curvature. $\tau - \tau_G = d\theta/ds$, where $\cos\theta = \hat{\mathrm{N}}\cdot\hat{\mathrm{n}}$.

Defining $\hat{\mathrm{e}}_1 = -\hat{\mathrm{N}} = \nabla\Psi/|\nabla\Psi|$, then $\tau_g = \hat{\mathrm{b}}\cdot\mathrm{R} \equiv \hat{\mathrm{b}}\cdot(\nabla\hat{\mathrm{e}}_1)\cdot\hat{\mathrm{e}}_2 = -\tau_G$.

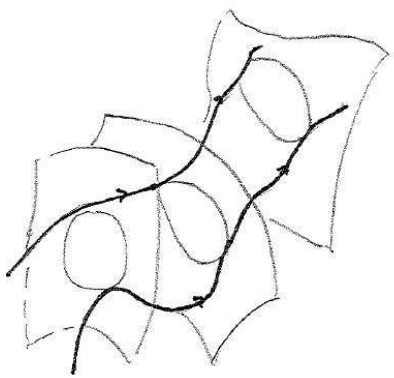
# Locally orthogonal magnetic coordinates for 3D fields

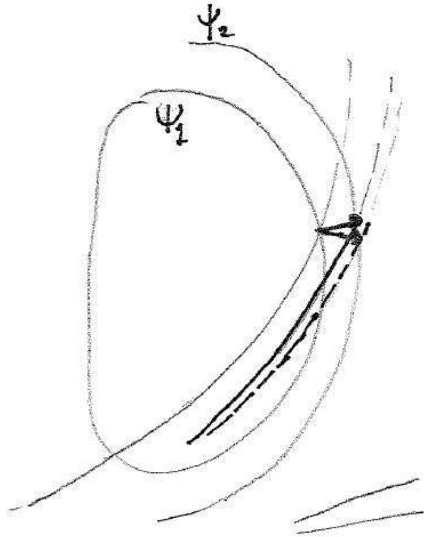
- Existence of locally orthogonal magnetic coordinates throughout a volume is equivalent to existence of a family of triply orthogonal surfaces in the volume, one family aligned with each axis. (Triply orthogonal: a unique surface from each family passes through each point and the three intersecting surfaces are pairwise orthogonal.)




- Fails where a consistent perpendicular directions cannot be defined for field lines across a curve or surface
  - On magnetic axes, O-points, and X-points.
  - On boundaries between magnetic regions of different topologies
  - In truly stochastic fields

- Requires good magnetic flux surfaces, but may still fail.
  - Choose $\hat{\mathrm{e}}_1 = \nabla\Psi/|\nabla\Psi|$ to be the normal to the surface, so $\tau_g = -\tau_G$. The coordinate systems are then consistent on each flux surface. The problem is to match coordinates across flux surfaces.






- For toroidal plasmas, existence is closely related to Newcomb's solvability condition for $\hat{\mathrm{b}} \cdot \nabla\psi = S$. A solution exists in a toroidal plasma if and only if $\oint dl\, S = 0$ on all closed field lines on rational magnetic surfaces.
  - Early Hamiltonian/Lagrangian GC theory defined a gyrophase $\psi$ in the gyroangle, $\zeta' = \zeta + \psi$, where $\hat{\mathrm{b}} \cdot \nabla\psi = (1/2)\hat{\mathrm{b}} \cdot \nabla \times \hat{\mathrm{b}} + \hat{\mathrm{b}} \cdot \mathrm{R}$. It did not satisfy Newcomb solvability (Hagan and Frieman 1985).

- I. If $\hat{\mathrm{b}} \cdot \nabla \times \hat{\mathrm{b}} = 0$ for a vector field with unit direction $\hat{\mathrm{b}}$, then there exists a surface normal to $\hat{\mathrm{b}}$ (Kelvin, 1850's, fluid vorticity). Condition $\mathrm{B} \cdot \nabla \times \mathrm{B} = 0$ or $\nabla \times \mathrm{B} = 0$ is necessary and sufficient for the existence of global planes perpendicular to the magnetic field lines, since the field then has the form $\mathrm{B} = f \nabla g$ for two functions $f$ and $g$.)

- II. Given magnetic flux surfaces, classical differential geometry (DoCarmo, 1975) states that $\tau_g = 0$ is the necessary and sufficient condition for existence of a set of triply orthogonal surfaces, one defined by field lines and one by flux surfaces.

  - On an (oriented) surface, at any point there is a maximum and a minimum value of the surface curvature $\kappa_N$, corresponding to two curves passing through the point. These are the lines of curvature of the surface at the point and have $\tau_G = 0$. Dupin's theorem states that, if three families of surfaces form a triply orthogonal system, then the surfaces must intersect in lines of curvature. Thus the magnetic field lines must have $\tau_G = 0$ and since they densely cover each flux surface, $\tau_g = \hat{\mathrm{b}} \cdot \mathrm{R} = 0$ (almost) everywhere. In general fields, $\tau_g = 0$ is equivalent to $\tau = 0$, zero torsion.
  - $\tau_g = 0$ is equivalent to $\hat{\mathrm{e}}_j \cdot \nabla \times \hat{\mathrm{e}}_j = 0$ for all three axes in Eq. (11), so that three mutually orthogonal surfaces exist, one perpendicular to each axis.

# Magnetic coordinate rotation $\hat{b} \cdot R$

- *Assuming* **that the gradients $\nabla \hat{e}_j$ are defined everywhere, it can be shown that $\hat{b} \cdot R$ depends on all three orthogonal axes,**

$$\hat{b} \cdot R = \frac{1}{2}\left(\hat{b} \cdot \nabla \times \hat{b} - \hat{e}_1 \cdot (\nabla \times \hat{e}_1) - \hat{e}_2 \cdot (\nabla \times \hat{e}_2)\right). \tag{11}$$

**Derivation:**

$$R \equiv (\nabla \hat{e}_1) \cdot \hat{e}_2 = \hat{e}_2 \times (\nabla \times \hat{e}_1) + (\hat{e}_2 \cdot \nabla)\hat{e}_1. \tag{12}$$

**Since $\hat{e}_1 \cdot \hat{e}_2 = 0$, $\hat{b} \cdot (\nabla \hat{e}_1) \cdot \hat{e}_2 = -\hat{b} \cdot (\nabla \hat{e}_2) \cdot \hat{e}_1$, so that**

$$\begin{aligned} \hat{b} \cdot (\nabla \hat{e}_1) \cdot \hat{e}_2 &= -\hat{e}_1 \cdot (\nabla \times \hat{e}_1) + \hat{b} \cdot (\hat{e}_2 \cdot \nabla)\hat{e}_1 \\ -\hat{b} \cdot (\nabla \hat{e}_2) \cdot \hat{e}_1 &= -\hat{e}_2 \cdot (\nabla \times \hat{e}_2) - \hat{b} \cdot (\hat{e}_1 \cdot \nabla)\hat{e}_2. \end{aligned} \tag{13}$$

**Adding and using $\hat{b} \cdot \nabla \times \hat{b} = \hat{b} \cdot \nabla \times (\hat{e}_1 \times \hat{e}_2) = \hat{b} \cdot [(\hat{e}_2 \cdot \nabla)\hat{e}_1 - (\hat{e}_1 \cdot \nabla)\hat{e}_2$ gives the result.**

- **The perpendicular component is**

$$R_\perp = \hat{e}_1(\hat{b} \cdot \nabla \times \hat{e}_1) + \hat{e}_2(\hat{b} \cdot \nabla \times \hat{e}_2) = \hat{b} \times \boldsymbol{\kappa} + \hat{e}_2(\nabla \cdot \hat{e}_1) - \hat{e}_1(\nabla \cdot \hat{e}_2). \tag{14}$$

- **Eqs. (11)–(14) assume that the third coordinate axis $\hat{e}_2$ defines a continuous vector field with well-defined gradient and curl, locally orthogonal to $\hat{e}_1$ and $\hat{b}$.**
  **A nontrivial existence condition involves second derivatives of $\Psi$. Assuming that flux surfaces with surface normals $\hat{e}_1 = \nabla\Psi/|\nabla\Psi|$ exist, so that $\hat{e}_1 \cdot \nabla \times \hat{e}_1 = 0$,**

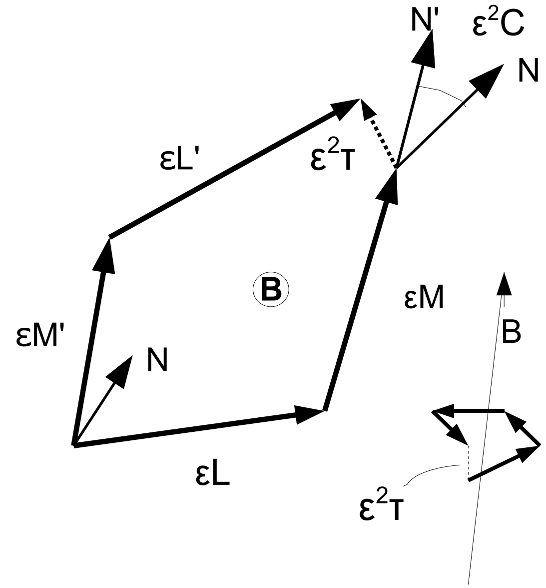
$$\hat{e}_2 \cdot \nabla \times \hat{e}_2 = \hat{b} \cdot \nabla \times \hat{b} - 2\hat{b} \cdot (\hat{e}_2 \cdot \nabla)(\nabla\Psi/|\nabla\Psi|), \tag{15}$$

  **Substituting $\hat{e}_2 = \hat{b} \times \nabla\Psi/|\nabla\Psi|$, Eq. (15) becomes a relation between $\hat{b}$ and $\nabla\Psi$ so that such an $\hat{e}_2$ exists.**

- **Torus: Equilibrium force balance, $J \times B = \nabla p$ implies a natural coordinate $\hat{e}_{2'} = \nabla I/|\nabla I|$ with $\hat{e}_{2'} \cdot \nabla \times \hat{e}_{2'} = 0$, that is not generally orthogonal, $\nabla I \cdot \nabla\Psi_p \neq 0$.**
  **In canonical magnetic coordinates (Boozer), not necessarily orthogonal, $B = \nabla\Psi_p \times \nabla\phi + \nabla\Psi_t \times \nabla\theta$, and good flux surfaces $\Psi_p$ require that $\Psi_t = \Psi_t(\Psi_p)$. Then $B = \nabla \times (I\nabla\Psi_p)$, where $I(\Psi_p, \theta, \phi) = -(d\Psi_t/d\Psi_p)\theta - \phi$.**

# Coordinate existence condition has $n$-dimensional analogue

- **In terms of manifolds and differential forms, the corresponding $n$-dimensional result for the existence of locally orthogonal coordinates tied to a field shows that the problem is one of linking the twisting of the different coordinate systems, ie, the affine connections (Flanders, 1989).**
- **Possible iff the generalized differential curvature form $\Omega \equiv 0$.**
- **In three dimensions, this is equivalent to zero torsion of the vector field.**
- **In four dimensions, non-zero curvature is possible.**
  - **Theories of quantum gravity attempt to attach small scale, locally orthogonal quantum theory to large scale, curved space-time**



# GC/GK Time Dependence

- The time-dependent magnetic vector potential term in the electric field in Ohm's law, $\mathrm{E} + \mathrm{v} \times \mathrm{B} \simeq 0$, also affects geometrical accuracy in 3D.
  - Perpendicular: Ordering $-(1/c)\partial \mathrm{A}_\perp/\partial t \ll \nabla_\perp \Phi$ (electrostatic potential) drops the compressional Alfvén wave and makes the geometrical approximation $\nabla \cdot \left(\hat{\mathrm{b}}\hat{\mathrm{b}} \cdot \nabla\Phi\right) = (\hat{\mathrm{b}} \cdot \nabla)(\hat{\mathrm{b}} \cdot \nabla\Phi) - (1/B)(\hat{\mathrm{b}} \cdot \nabla B)(\hat{\mathrm{b}} \cdot \nabla\Phi) \simeq 0$.
  - Parallel: Ordering $-(1/c)\partial \mathrm{A}_\parallel/\partial t \ll \nabla_\parallel \Phi$ drops the shear Alfvén wave and makes geometrical approximation $\hat{\mathrm{b}} \cdot \nabla\Phi \simeq 0$.

- Analytic GC/GK models drop the compressional wave, keep shear Alfvén. Velocity moments yield reduced MHD.

- Numerical GK particle models usually drop or approximate the parallel $\partial \mathrm{A}_\parallel/\partial t$ for numerical reasons. Part of the shear Alfvén wave appears through the nonlinear polarization drift.

- Both approximations encourage an artificial enhancement of turbulent and zonal poloidal ExB flows with $\nabla_\parallel \Phi \simeq 0$, ie $\Phi \simeq \Phi(r)$.
  - GK simulations see robust zonal flows $v_{E,\theta} \sim E_r B_\phi / B^2$, while experiment is more ambiguous.

# Implications and Connections

- Nonexistence of the GC expansion at higher order implies that the magnetic moment $\mu = (1/2)mv_\perp^2/B$ cannot be shown to be an invariant at that order by the GC analysis; $\mu$ is a first order invariant in general 3D fields.

- Since time-evolving fields will in general break any 2D symmetries, GK models that keep the exact 3D geometry can be at most first order in gyroradius.

- Twisting of field-line-tied coordinate systems in 3D is a real physical effect. Velocity space nonuniformities due to $\tau$ and $\tau_g$ appear in all GC and related models at first order.

- FLR fluid models valid to all orders in $\epsilon$ (Ramos 2005) assume the gyroradius smaller than all other scales, including the fluid element. Unlike GC/GK, yields full, not reduced MHD. Still puzzling.

- Lagrangians are closely connected to vector potentials.
  - Lagrangian formalism describes strictly local relations; existence of (effective) vector potential is a separate, non-local condition.
  - Higher order existence problem involves gradients, not basic variables.

# Summary

- **Guiding Center model: gyroangle around GC introduces field-line-tied coordinates.**
- **In 3D magnetic fields, nonzero field line torsion $\hat{b} \cdot \nabla \times \hat{b}$ imposes strong nonlocal, topological constraints on the validity of the GC expansion.**
  - **First order in $\epsilon$: velocity space nonuniformities**
  - **Second order: existence!**
- **Second order GC equations exist in 3D only when the local orthogonal magnetic coordinate systems defined at each point on a field line can be extended to a global coordinate system in a volume.**
  - **Requires good magnetic flux surfaces and either $\tau = \tau_g = 0$ or a 2D symmetry**
- **Nonexistence of GC expansion at higher order means that it cannot be used to prove that magnetic moment $\mu$ is an invariant at that order; need different proof.**
- **In 3D, geometrically exact GC/GK equations require keeping electromagnetic vector potential terms $\partial \mathrm{A}/\partial t$ in the electric field at the same order as $\nabla \Phi$.**

- Time-dependent, geometrically exact GK model can be at most first order for 3D fields.
- Assuming $-(1/c)\partial \mathrm{A}/\partial t \ll -\nabla\Phi$, as in some numerical GK models, increases poloidal ExB flows with $E \simeq -\nabla\Phi(r)$ and may encourage zonal flows.

• The GC coordinate existence problem has analogies to the problems encountered by unified theories of physics, such as quantum gravity, relative to 4D space-time.