Jacobian-Free Newton-Krylov Method for GKM

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Outline

- Primer on Newton-Krylov Method
- Good preconditioning is the key
  - *Example: Wave-structure based preconditioner for MHD*
- JFNK for GKM

Related Work

- Chacon & co-workers (JCP 2002, 2003, 2006) developed JFNK methods with "physics-based" preconditioners
  - *Parabolization trick for the equations*
  - *Schur complement approach*
- M3d (Strauss, Park et al.) treats fast compressive wave implicitly
  - *Recently Fu & Breslau have extended to treating shear Alfvén wave implicitly*
- Glasser & co-workers use a static-condensation method in their fully implicit SEL code
- Rognlien et al. (J. Nuclear Matter 1992, JCP 2002) for edge plasmas
- Mousseau & Knoll (JCP 2000) - 2d Fokker-Planck for edge plasmas

- Reynolds, Samtaney & Woodward (JCP 2006) developed a fully implicit parallel JFNK method for 3D compressible MHD
  - *Recent work (2008) on development of a wave-structure based preconditioner*
- Excellent review paper by Knoll & Keyes (JCP 2004)

Nonlinearly Implicit : Introduction to Newton-Krylov

- Consider the equations of single-fluid resistive MHD written below in conservation form

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\partial_t (\rho \mathbf{v}) + \nabla \cdot \left(\rho \mathbf{v}\mathbf{v}^T - \mathbf{B}\mathbf{B}^T + \left(p + \tfrac{1}{2}\mathbf{B}\cdot\mathbf{B}\right) \bar{\mathbf{I}} \right) &= \nabla \cdot \overline{\tau}, \\
\partial_t \mathbf{B} + \nabla \cdot \left(\mathbf{v}\mathbf{B}^T - \mathbf{B}\mathbf{v}^T \right) &= \nabla \cdot \left(\eta \nabla \mathbf{B} - \eta \left(\nabla \mathbf{B}\right)^T \right), \\
\partial_t e + \nabla \cdot \left((e + p + \tfrac{1}{2}\mathbf{B}\cdot\mathbf{B})\mathbf{v} - \mathbf{B}(\mathbf{B}\cdot\mathbf{v}) \right) &= \nabla \cdot \left(\overline{\tau}\mathbf{v} + \kappa \nabla T \right) \\
&\quad + \nabla \cdot \left(\eta \left(\tfrac{1}{2}\nabla(\mathbf{B}\cdot\mathbf{B}) - \mathbf{B}(\nabla \mathbf{B})^T \right) \right)
\end{aligned}
$$

- Condensing notation $\mathbf{U} = (\rho,\ \rho\mathbf{v},\ \mathbf{B},\ e)^T$

$$
\partial_t \mathbf{U} = -\nabla \cdot \mathbf{F}_h(\mathbf{U}) + \nabla \cdot \mathbf{F}_v(\mathbf{U}) = \nabla \cdot \mathbf{F}(\mathbf{U}).
$$

Nonlinearly Implicit : Introduction to Newton-Krylov

- The solution at the next time level to the entire system of equations is expressed as the solution to the following nonlinear equation

$$\mathcal{F}(U^{n+1}) = 0$$

$$\mathcal{F}(U^{n+1}) = U^{n+1} - U^{n} + (1-\theta)R(U^{n+1}) + \theta R(U^{n}) = 0$$

R(U) is the entire right hand side (contains divergence of hyperbolic and diffusive fluxes)
The number of unknowns is $8N^2$ for an NxN mesh

- This is solved using Newton's method

$$\delta U^{k} = -\left[\left(\frac{\partial \mathcal{F}}{\partial U}\right)^{n+1,k}\right]^{-1} \mathcal{F}$$

where $J(U^{n+1,k}) \equiv \left(\frac{\partial \mathcal{F}}{\partial U}\right)^{n+1,k}$ is the Jacobian; and $\delta U^{k} \equiv U^{n+1,k+1} - U^{n+1,k}$

The size of the Jacobian matrix is $64N^4$

- The linear system at each Newton iteration is solved with a Krylov method in which an approximation to the linear system $J\,\delta U = -\mathcal{F}$ is obtained by iteratively building a Krylov subspace of dimension m

$$\mathcal{K}(r_0, J) = span\{r_0, Jr_0, J^2 r_0, \cdots, J^{m-1} r_0\}$$

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Nonlinearly Implicit : Introduction to Newton-Krylov

- Commonly used Krylov methods which can handle asymmetric matrices
  - *GMRES (Generalized Minimum Residual)*
    - *Long-recurrence Arnoldi orthogonalization method*
    - *Robust, guaranteed convergence, but heavy on memory requirement*
  - *BiCGStab (Bi-conjugate Gradient Stabilized)*
    - *Short-recurrence Lanzcos biorthogonalization procedure*
    - *Residual not guaranteed to decrease monotonically, but less memory requirement*
- Steps in a Newton-Krylov method
  1. *Guess the solution $U^{n+1,0}$ ($=U^n$)*
  2. *For each Newton iteration k*
     1. *Using a Krylov Method solve for δU^k*
        Solve $J\,\delta U^k = -F(U^{n+1,k})$ until $\|J\,\delta U^k + F(U^{n+1,k})\| < ltol$
  3. *Update the Newton iterate: $U^{n+1,k+1}=U^{n+1,k} + \lambda\,\delta U^k$*
  4. *Check for convergence $\|F(U^{n+1,k+1})\| < ftol$*

- Newton method converges quadratically if the approximate solution $U^{n+1,k+1}$ is close to the actual solution U^* (Constant C is not a fnc($U^{n+1,k+1}$,U^*)) $||U^{n+1,k+1} - U^*|| \leq C||U^{n+1,k} - U^*||^2$

- **Jacobian-Free Newton-Krylov:** Krylov methods require only matrix-vector products to build up the Krylov subspace, i.e., only $J\,\delta U$ is required. This can be approximated as follows. Typically σ is chosen as square-root of machine zero. Thus, the entire method can be built from evaluations of the nonlinear function F(U)

$$J(U^k)\delta U^k \approx \frac{\mathcal{F}(U^{n+1,k} + \sigma\delta U^k) - \mathcal{F}(U^{n+1,k})}{\sigma}$$

Introduction to Newton-Krylov: Preconditoners

- Krylov methods can lead to slow convergence. This is especially true for MHD where the Jacobian is ill-conditioned. Preconditioners help alleviate the problem of slow convergence and are formulated as follows

$$\begin{aligned}
(J(U^k)P^{-1})(P\delta U^k) &= -\mathcal{F}(U^{n+1,k}) \quad (Right), \\
(P^{-1}J(U^k))\delta U^k &= -P^{-1}\mathcal{F}(U^{n+1,k}) \quad (Left), \\
(P_L^{-1}J(U^k)P_R^{-1})(P_R\delta U^k) &= -P_L^{-1}\mathcal{F}(U^{n+1,k}) \quad (Both).
\end{aligned}$$

- The basic idea of preconditioners is that the matrix JP^{-1} or $P^{-1}J$ is close to the identity matrix, i.e., P is a good approximation of J. Furthermore, to make preconditioning effective, P^{-1} should be computationally inexpensive to evaluate

- Two broad classes of preconditioners
  1. ***Algebraic***: *These are of the "black-box" type. Obtained from relatively inexpensive techniques such as incomplete LU, multi-grid etc. These require storage for the preconditioner.*
  2. ***Physics-based***: *These may be derived from semi-implicit methods, and pay close attention to the underlying physics in the problem. Furthermore, these can still operate in the "Jacobian-Free" mode.*
     1. *Chacon, Knoll and co-workers (LANL) championed the "physics-based" preconditioners. Their work involves using "parabolizing" the wave terms and using multi-grid to solve approximate systems.*

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Wave-structure Based Preconditioner - Basic Idea

- The stiffness usually arises from the hyperbolic terms in the MHD equation
- Consider a system of hyperbolic conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

- Linearizing about a background state
  $J(u^0)$ has real eigenvalues $\Lambda=\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$
  with linearly independent left (L) and right (R) eigenvectors

$$\frac{\partial u}{\partial t} + J(u^0)\frac{\partial u}{\partial x} = 0,$$

- Characteristic equations (w=[L]u)

$$\frac{\partial w}{\partial t} + \Lambda\frac{\partial w}{\partial x} = 0$$

- Solve implicitly
  - *Using Crank-Nicholson and 4th order finite differences*

$$\left(\frac{\partial u}{\partial x}\right)_i = a(u_{i+1} - u_{i-1}) + b(u_{i+2} - u_{i-2}),$$

$$a = (1.5\Delta x)^{-1}, \quad b = (-12\Delta x)^{-1}$$

  - *Leads to a linear system of the form* $A\, U^{n+1} = R(U^n)$
    $K = a\Delta t\, J, \quad M = b\Delta t\, J$

$$A = \begin{bmatrix} I & K & M & 0 & 0 & 0 & \cdots & -M & -K \\ -K & I & K & M & 0 & 0 & 0 & \cdots & -M \\ -M & -K & I & K & M & 0 & 0 & \cdots & 0 \\ 0 & -M & -K & I & K & M & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & -M & -K & I & K & M \\ M & 0 & \cdots & 0 & 0 & -M & -K & I & K \\ K & M & 0 & \cdots & 0 & 0 & -M & -K & I \end{bmatrix}$$

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Wave-structure Based Preconditioner - Basic Idea

- Solve for all but the stiffness inducing waves
- Preconditioner Matrix is then

$$\tilde{\Lambda} = diag\{0, 0, \cdots, \lambda_q, \lambda_{q+1}, \cdots, \lambda_n\}$$

$$\tilde{K} = a\Delta t R\tilde{\Lambda}L$$

$$\tilde{M} = b\Delta t R\tilde{\Lambda}L$$

$$P = \begin{bmatrix} I & \tilde{K} & \tilde{M} & 0 & 0 & 0 & \cdots & -\tilde{M} & -\tilde{K} \\ -\tilde{K} & I & \tilde{K} & \tilde{M} & 0 & 0 & 0 & \cdots & -\tilde{M} \\ -\tilde{M} & -\tilde{K} & I & \tilde{K} & \tilde{M} & 0 & 0 & \cdots & 0 \\ 0 & -\tilde{M} & -\tilde{K} & I & \tilde{K} & \tilde{M} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & -\tilde{M} & -\tilde{K} & I & \tilde{K} & \tilde{M} \\ \tilde{M} & 0 & \cdots & 0 & 0 & -\tilde{M} & -\tilde{K} & I & \tilde{K} \\ \tilde{K} & \tilde{M} & 0 & \cdots & 0 & 0 & -\tilde{M} & -\tilde{K} & I \end{bmatrix}$$

- Example: Ideal MHD, linearizing about a background state (low β tokamak parameters) $U^0=\{\rho=1, u=0, B_x=0.1 \cos\alpha, B_y=0.1 \cos\alpha, B_z=1.0, p=0.01\}^T$

- Full matrix A has $\lambda_{max} \approx 342$

  Fast wave preconditioning $\lambda_{max} \approx 22$
  Fast + Alfven wave $\lambda_{max} \approx 2$
  All waves: $\lambda_{max} =1$

- Preconditioner is ***exact*** for a system of linear hyperbolic conservation laws in 1D

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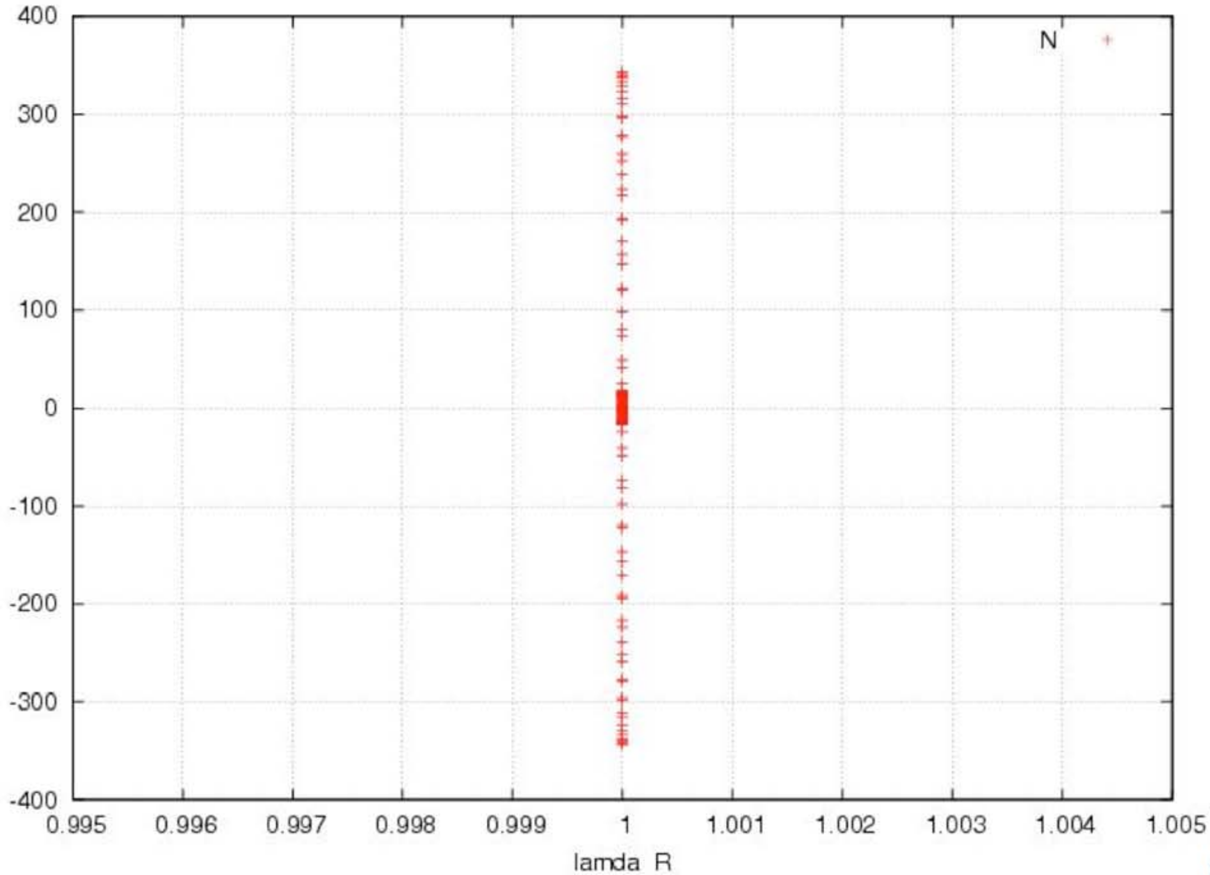

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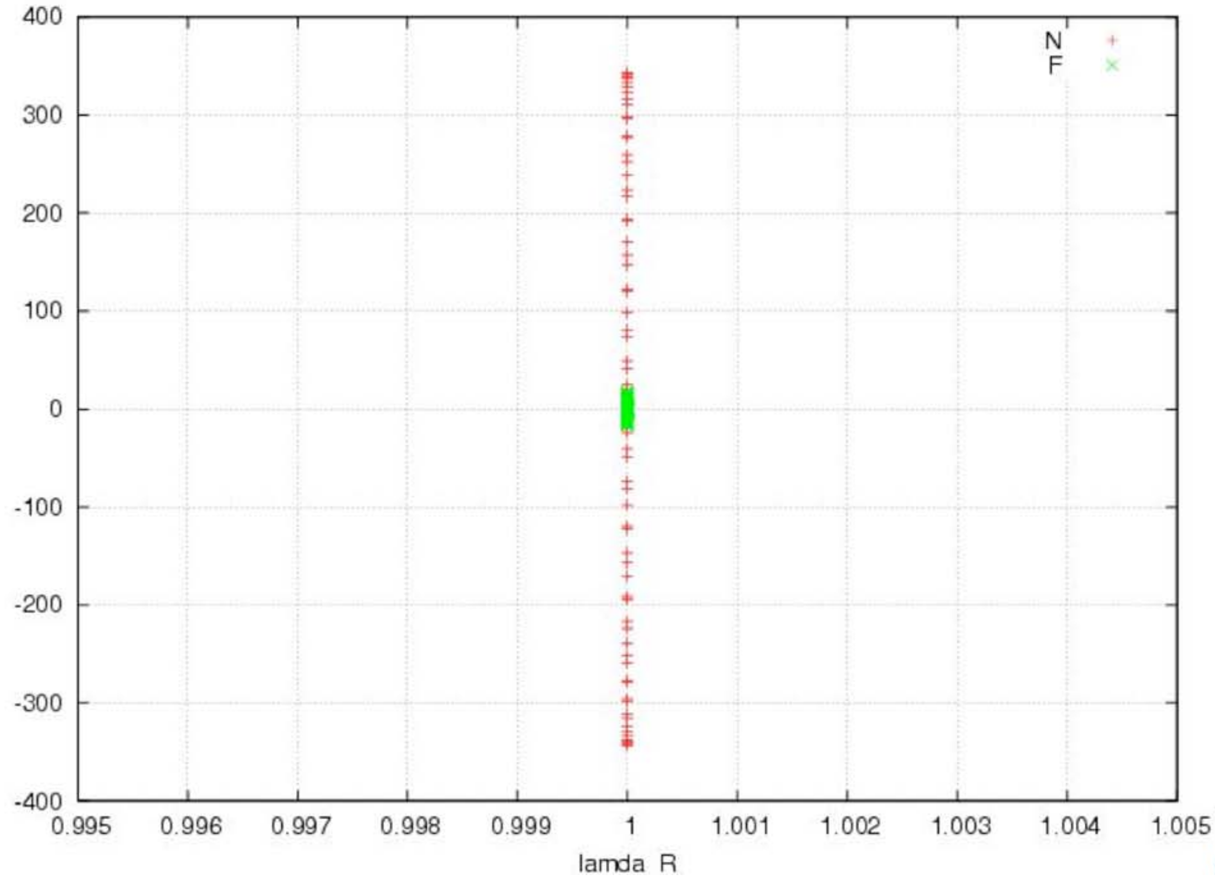

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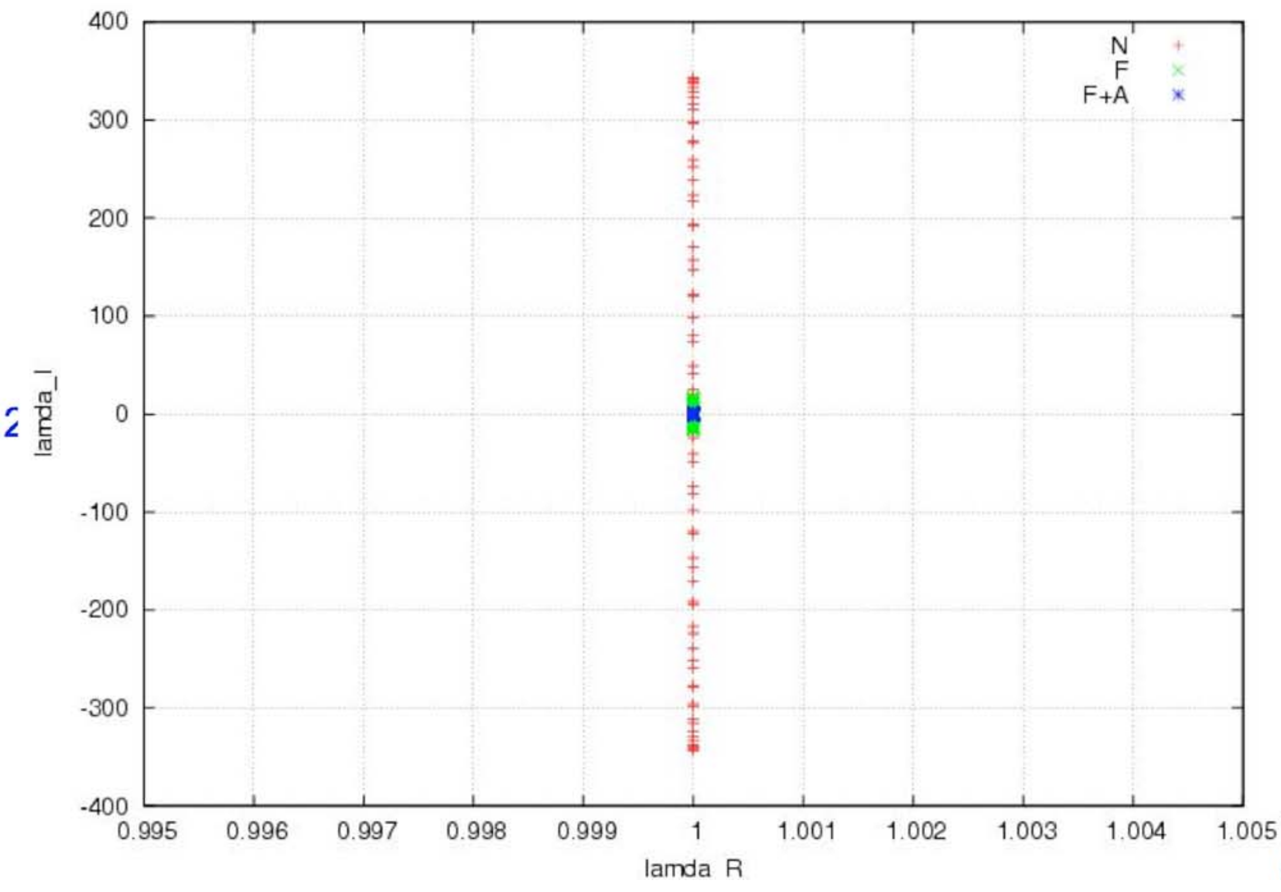

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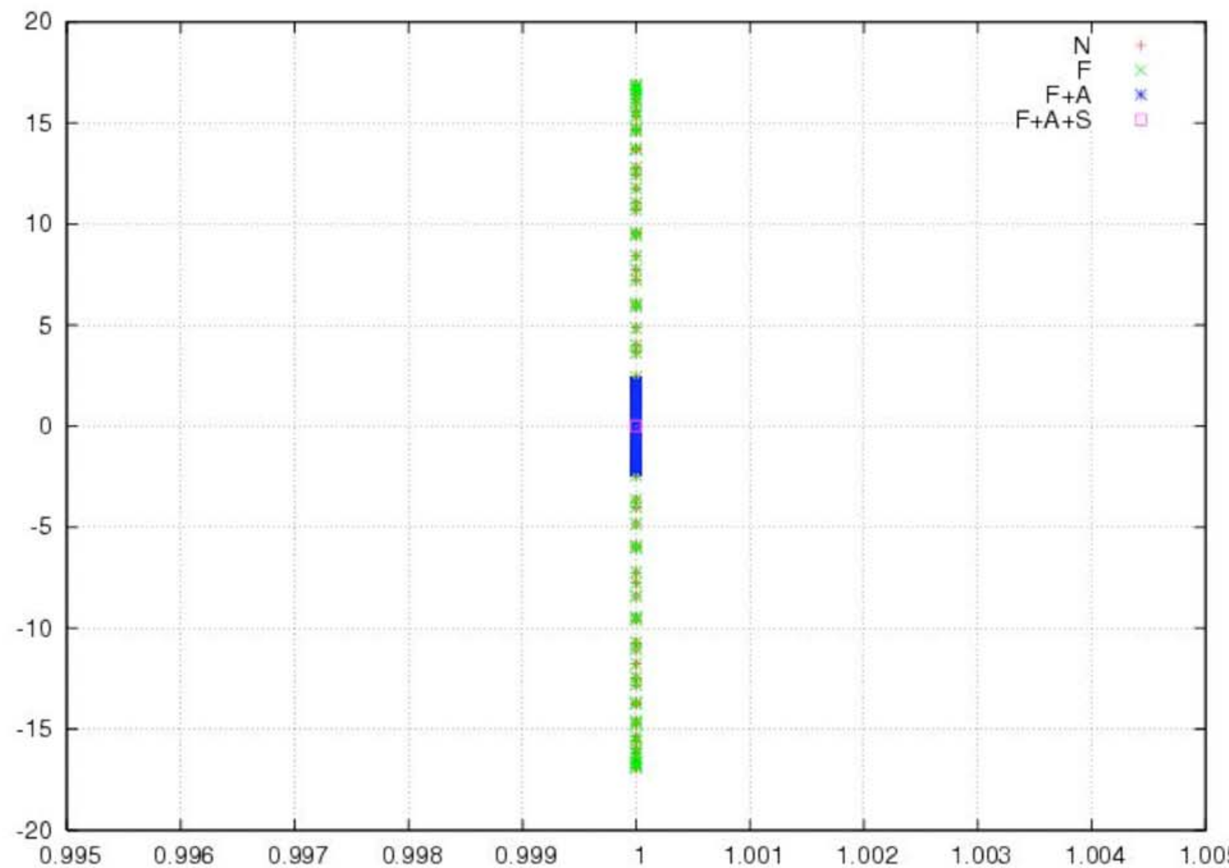

JFNK: Resistive MHD - Preconditioner

- Instead of solving $J \, \delta U = -g$ solve $(J P^{-1})(P \, \delta U) = -g$, i.e., right preconditioning is employed
- The preconditioner is split into a hyperbolic and a diffusive component

$$P^{-1} = P_h^{-1} P_d^{-1} = J(\mathbf{U})^{-1} + \mathcal{O}(\Delta t^2)$$

- Denoting by (.) the location of the linear operator action, the ideal MHD Jacobian is

$$\begin{aligned} J_h(\mathbf{U}) &= I + \overline{\gamma}\,[J_x\partial_x(\cdot) + J_y\partial_y(\cdot) + J_z\partial_z(\cdot)] \\ &= I + \overline{\gamma}\left[J_x L_x^{-1} L_x \partial_x(\cdot) + J_y L_y^{-1} L_y \partial_y(\cdot) + J_z L_z^{-1} L_z \partial_z(\cdot)\right] \\ &= I + \overline{\gamma}\left[J_x L_x^{-1} \partial_x\left(L_x(\cdot)\right) - J_x L_x^{-1}\partial_x(L_x)(\cdot) \right. \\ &\qquad + J_y L_y^{-1} \partial_y\left(L_y(\cdot)\right) - J_y L_y^{-1}\partial_y(L_y)(\cdot) \\ &\qquad \left. + J_z L_z^{-1} \partial_z\left(L_z(\cdot)\right) - J_z L_z^{-1}\partial_z(L_z)(\cdot)\right] \end{aligned}$$

where J_x is the Jacboian of the hyperbolic flux in the x-direction. L_x is the spatially local left eigenvector matrix for J_x. J_y, L_y, J_z, and L_z are similarly defined

JFNK: Resistive MHD - Preconditioner

- Directional splitting is employed to further approximate the preconditioner

$$\begin{aligned} P_h = &\left[I + \overline{\gamma} J_x L_x^{-1} \partial_x (L_x(\cdot))\right] \left[I + \overline{\gamma} J_y L_y^{-1} \partial_y (L_y(\cdot))\right] \left[I + \overline{\gamma} J_z L_z^{-1} \partial_z (L_z(\cdot))\right] \\ &\left[I - \overline{\gamma}\left(J_x L_x^{-1} \partial_x (L_x) + J_y L_y^{-1} \partial_y (L_y) + J_z L_z^{-1} \partial_z (L_z)\right)\right] \\ = &\, P_x \, P_y \, P_z \, P_{\text{corr}}. \end{aligned}$$

- Decoupling into 1D wave equations along characteristics

$$L_i(x) J_i(x) = \Lambda_i(x) L_i(x), \quad \Lambda_i = \text{Diag}(\lambda^1, \ldots, \lambda^8)$$

$$L_i \left[I + \overline{\gamma} J_i L_i^{-1} \partial_i (L_i(\cdot))\right] \xi = L_i \beta \quad \Leftrightarrow \quad \zeta + \overline{\gamma}\, \Lambda_i \, \partial_i \zeta = \chi,$$

where $\zeta = L_i \xi$ and $\chi = L_i \beta$

- Thus along each direction, we get a system of linear wave equations. For each wave family, we now get a sequence of tridiagonal linear systems which can be efficiently solved. In parallel we use the method proposed by Arbenz & Gander (1994)

JFNK: Resistive MHD - Preconditioner

- For spatially varying J(U) a correction solve is involved

$$
\begin{aligned}
P_{\text{corr}} &= I - \overline{\gamma}\left[J_x L_x^{-1}\partial_x\left(L_x\right) + J_y L_y^{-1}\partial_y\left(L_y\right) + J_z L_z^{-1}\partial_z\left(L_z\right)\right] \\
&= I - \overline{\gamma}\left[L_x^{-1}\Lambda_x\partial_x\left(L_x\right) + L_y^{-1}\Lambda_y\partial_y\left(L_y\right) + L_z^{-1}\Lambda_z\partial_z\left(L_z\right)\right]
\end{aligned}
$$

- Since this has no spatial couplings, the resulting local block systems may be solved easily by precomputing the 8x8 block matrices P_{corr} at each location coupled with a LU factorization
- Only the fastest stiffness inducing waves need to be solved. Furthermore, accuracy may be sacrificed because this is done in the context of the preconditioner.
- It can be shown that the error bound (Reynolds, Samtaney, Woodward, 2008) q-fastes waves are preconditioned is

$$
\left\|\chi - \hat{\chi}\right\|_p \le \left\|L_F^{-1}\right\|_p \left[\sum_{l=q+1}^{n}\left(\frac{\left\|\Delta t\lambda^l\partial_x(\cdot)\right\|_p}{1-\left\|\Delta t\lambda^l\partial_x(\cdot)\right\|_p}\right)^p \left\|(L_F b)^l\right\|_p^p\right]^{1/p}
$$

- Error from preconditioning q-fastest waves is dominantly $\frac{|\Delta t\lambda^{q+1}/\Delta x|}{1-|\Delta t\lambda^{q+1}/\Delta x|}$
  - *Omission of waves with small speeds compared to the dynamical time scale will not significantly affect the precondtioner accuracy*

JFNK: Resistive MHD - Preconditioner

- Diffusion Preconditioner P_d: This solves the subsystem $\partial_t \mathbf{U} - \nabla \cdot \mathbf{F}_v = 0$

$$P_d = J_v(\mathbf{U}) = I - \overline{\gamma}\frac{\partial}{\partial \mathbf{U}}(\nabla \cdot \mathbf{F}_v)$$
$$= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I - \overline{\gamma} D_{\rho\mathbf{v}} & 0 & 0 \\ 0 & 0 & I - \overline{\gamma} D_{\mathbf{B}} & 0 \\ -\overline{\gamma} L_\rho & -\overline{\gamma} L_{\rho\mathbf{v}} & -\overline{\gamma} L_{\mathbf{B}} & I - \overline{\gamma} D_e \end{bmatrix}$$

- To solve $P_d\, y = b$ for $y = [\, y_\rho, y_{\rho\mathbf{v}}, y_{\mathbf{B}}, y_e]^T$
  1. Update $y_\rho = b_\rho$
  2. Solve $(I - \overline{\gamma} D_{\rho\mathbf{v}})\, y_{\rho\mathbf{v}} = b_{\rho\mathbf{v}}$ for $y_{\rho\mathbf{v}}$
  3. Solve $(I - \overline{\gamma} D_{\mathbf{B}})\, y_{\mathbf{B}} = b_{\mathbf{B}}$ for $y_{\mathbf{B}}$
  4. Update $\tilde{b}_e = b_e + \overline{\gamma}\,(L_\rho\, y_\rho + L_{\rho\mathbf{v}}\, y_{\rho\mathbf{v}} + L_{\mathbf{B}}\, y_{\mathbf{B}})$
  5. Solve $(I - \overline{\gamma} D_e)\, y_e = \tilde{b}_e$ for y_e.

Steps 2,3 and 5 are solved using a geometric multigrid approach. Step 4 may be approximated with finite differences instead of constructing and multiplying by individual submatrices

$$L_\rho\, y_\rho + L_{\rho\mathbf{v}}\, y_{\rho\mathbf{v}} + L_{\mathbf{B}}\, y_{\mathbf{B}} = \frac{1}{\sigma}\left[\nabla \cdot \mathbf{F}_v(U + \sigma W) - \nabla \cdot \mathbf{F}_v(\mathbf{U})\right]_e + O(\sigma),$$

where $W = [y_\rho, y_{\rho\mathbf{v}}, y_{\mathbf{B}}, 0]^T$

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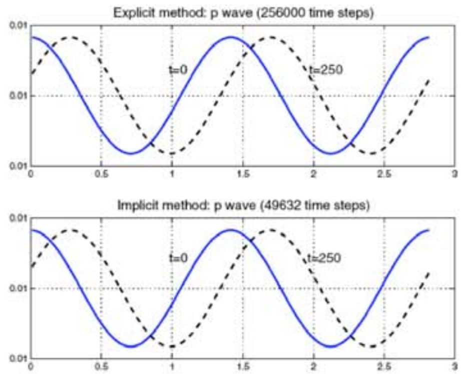
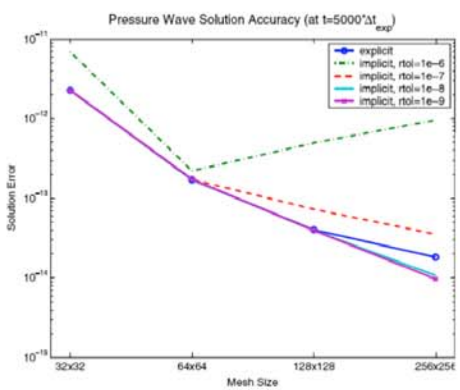
Verification Tests

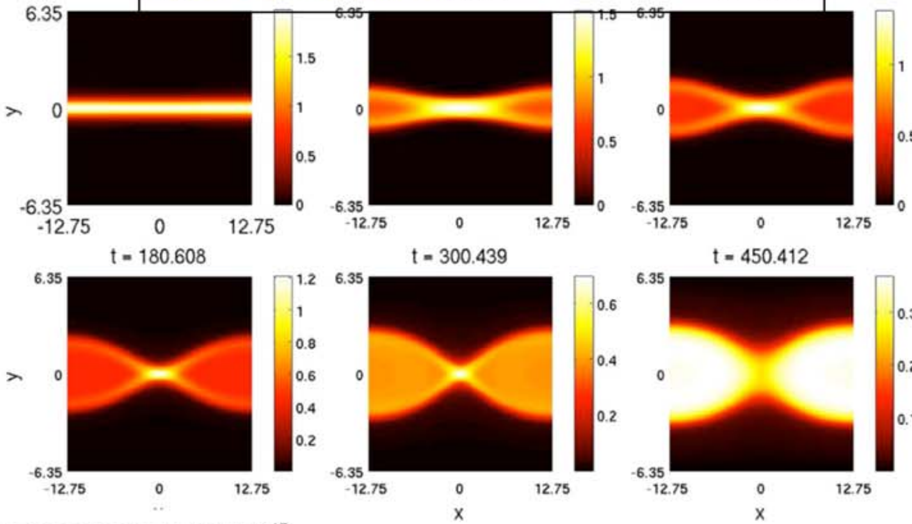
Linear wave propagtion test.



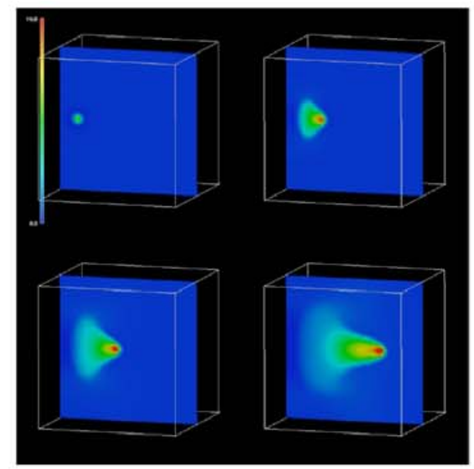


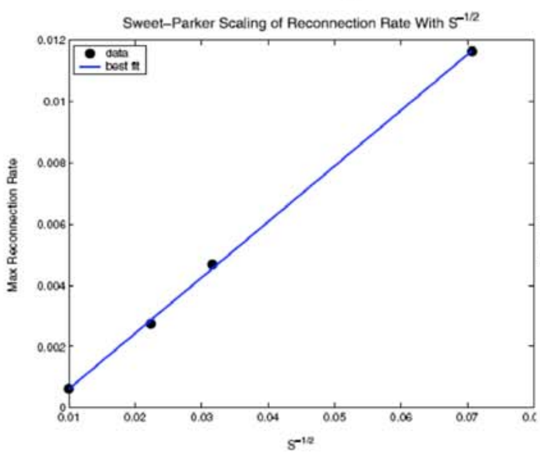
Computational time for an explicit method scales as $S^{3/2}$

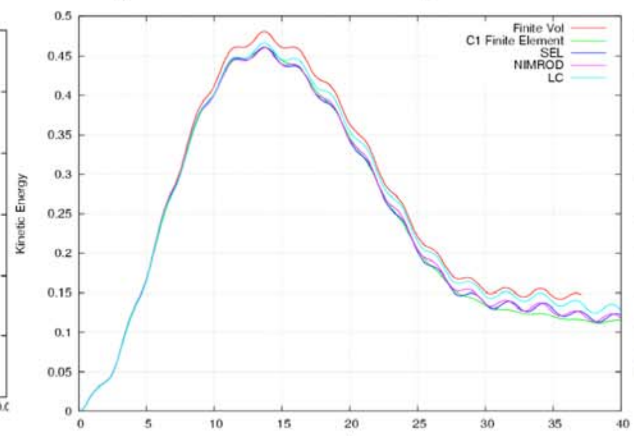
Initial conditions: Perturbed Harris sheet proposed by Birn et al. (J. Geo. Lett. 2001)

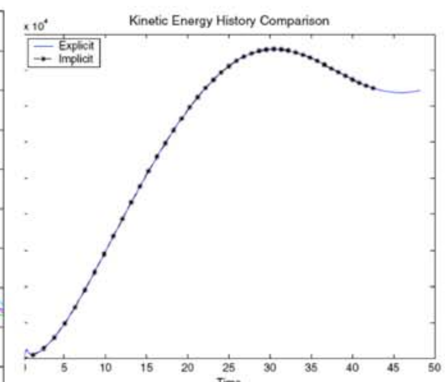


Pellet Model Problem with a similar separation of scale as the tokamak case.









From Reynolds, Samtaney & Woodward, JCP 2006

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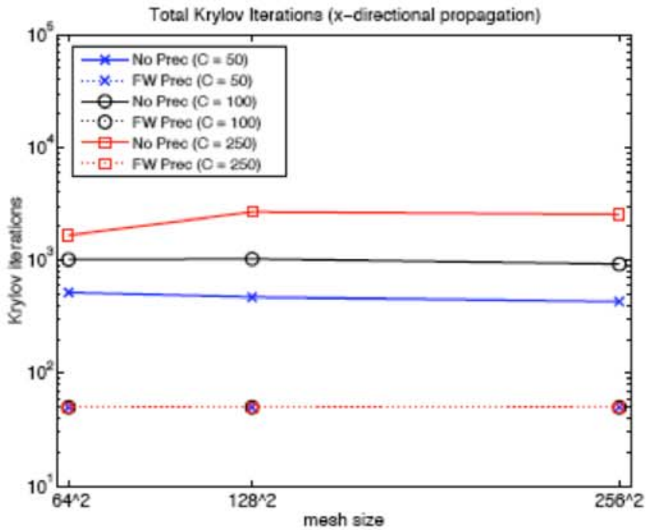
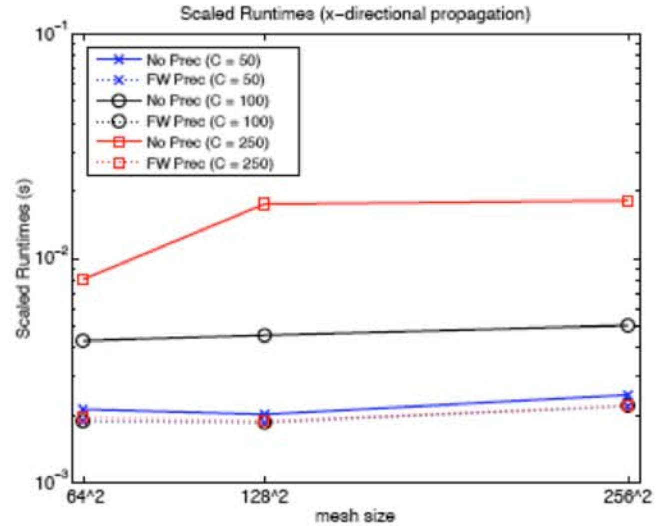
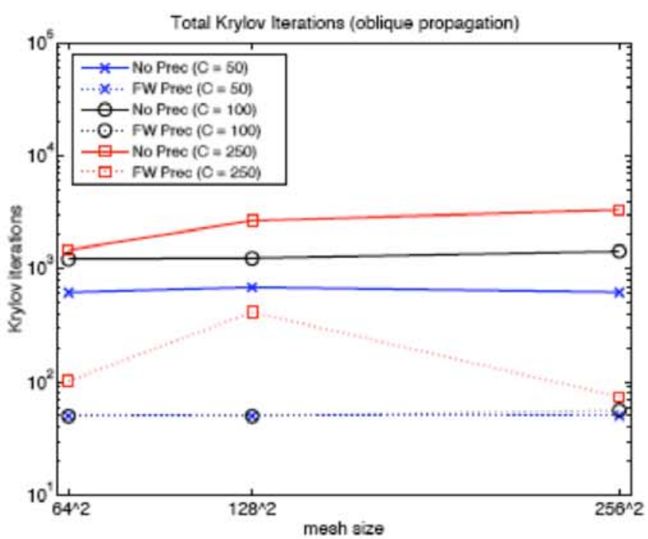
Verification Test: Linear Wave Propagation



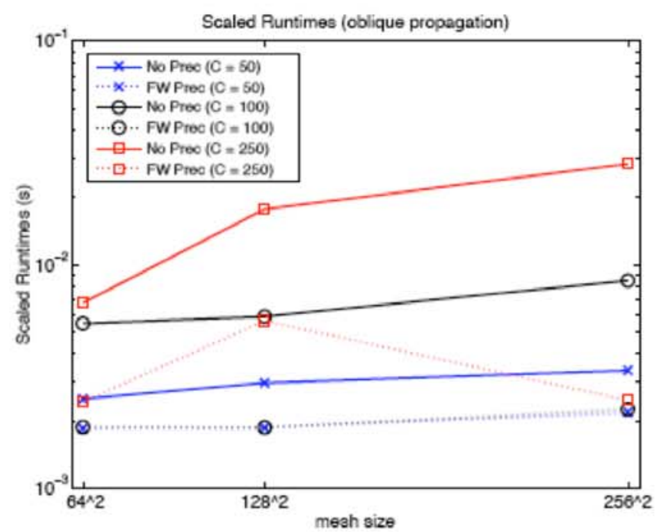
(a) Krylov iterations (x-dir.)



(b) scaled CPU (x-dir.)

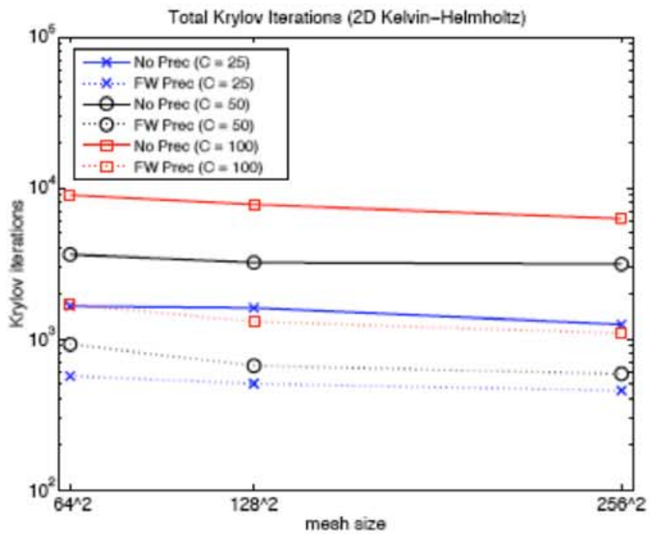


(c) Krylov iterations (oblique)

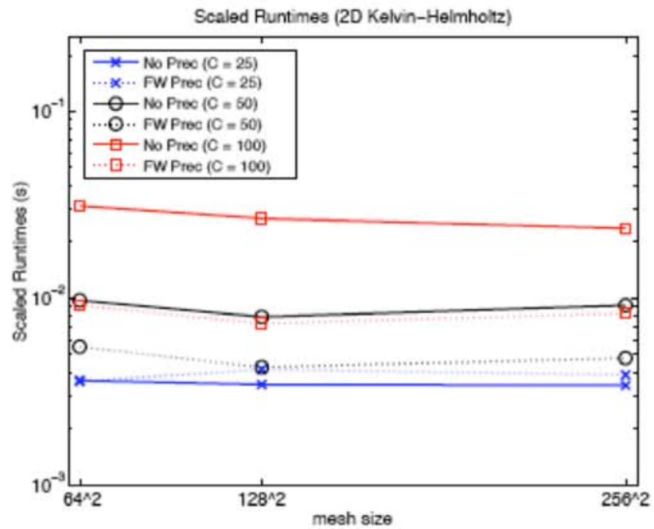


(d) scaled CPU (oblique)

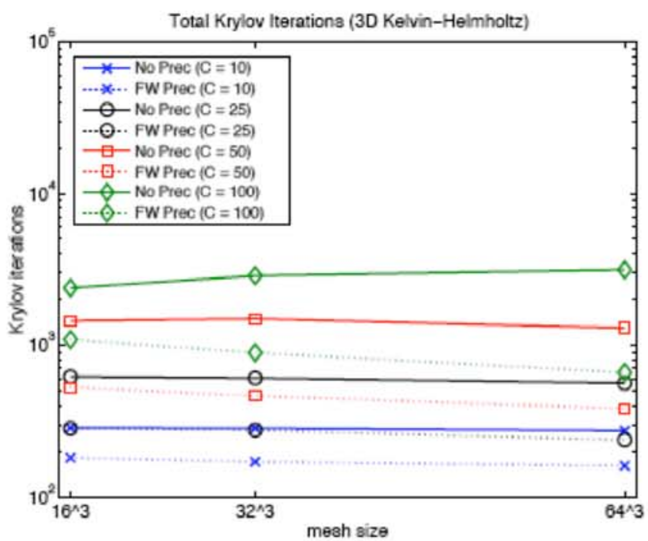
Verification Test: Kelvin-Helmholtz



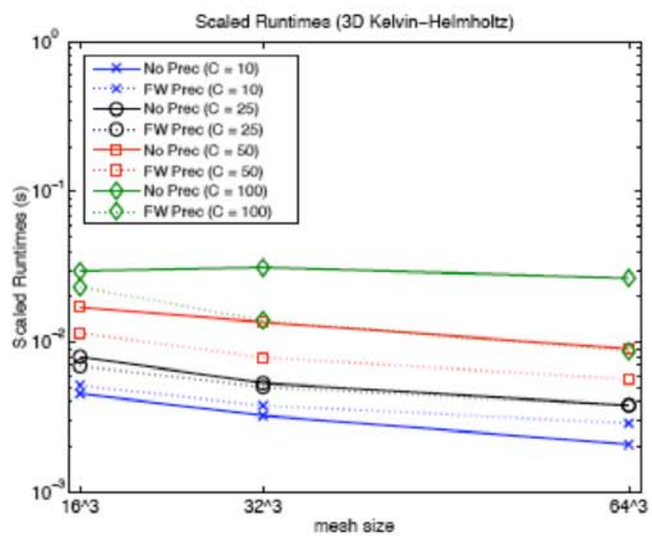
(a) Krylov iterations (2D)



(b) scaled CPU (2D)



(c) Krylov iterations (3D)



(d) scaled CPU (3D)

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CSEPP Equations (from GY-Fu)

- The main equation (equivalent to the perpendicular momentum equation derived from gyrokinetics)

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{V}_{\boldsymbol{E}\times\boldsymbol{B}} \cdot \nabla$$

Shear Alfvén wave term w/o ballooning term

$$-\frac{d}{dt}\nabla \cdot \left(\frac{1}{V_A^2}\nabla_\perp \Phi\right) + \boldsymbol{B} \cdot \nabla \frac{\boldsymbol{B} \cdot \nabla \times \nabla \times \boldsymbol{A}}{B_0^2} + (\nabla \boldsymbol{A}_{||} \times \boldsymbol{b}).\nabla \left(\frac{\boldsymbol{J}_{||0}}{B_0}\right)$$

$$= \frac{1}{V_A^2}\left(\frac{3v_t^2}{4\Omega^2}\right)\nabla_\perp^4 \frac{d\Phi}{dt} + \boldsymbol{b} \times \sum_j \nabla \left(\frac{env_t^2}{2B_0\Omega^2}\right)_j \cdot \nabla\nabla_\perp^2 \Phi - \sum_j \int (e\boldsymbol{v}_d \cdot \nabla f)_j d^3v$$

Ion FLR term

Diamagnetic drift

All other kinetic terms from both thermal and fast ions

Model Equations

- Model assumes isothermal electrons
- If first and third terms in V_e can be neglected, we don't have to solve an equation for $v_{||,i}$
- The equations can then be rewritten as:

$$\frac{\partial L(\Phi)}{\partial t} = R_\Phi(\boldsymbol{A}_{||}, \Phi) + S(f)$$

$$\frac{\partial \boldsymbol{A}_{||}}{\partial t} = R_{\boldsymbol{A}}(\Phi, n_e)$$

$$\frac{\partial \delta n_e}{\partial t} = R_{ne}(\boldsymbol{A}_{||}, n_e)$$

$$L = \frac{1}{V_A^2}\left[\nabla_\perp^2 + \left(\frac{3v_t^2}{4\Omega^2}\right)\nabla_\perp^4\right]$$

$$\frac{\partial \boldsymbol{A}_{||}}{\partial t} = \nabla_{||}\Phi - E_{||}$$

$$\frac{\partial \delta n_e}{\partial t} = \nabla \cdot (\boldsymbol{V}_e n_e)$$

$$E_{||} = -\frac{1}{en_e}\nabla_{||}\delta p_e = -\frac{T_e}{en_e}\nabla_{||}\delta n_e$$

$$V_e = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B_0^2} - \frac{\delta \boldsymbol{J}_{||}}{en_e} + v_{||,i}\boldsymbol{b} \approx -\frac{\delta \boldsymbol{J}_{||}}{en_e}$$

$$\boldsymbol{J} \approx \nabla_\perp^2 \boldsymbol{A}_{||}$$

Implicit Solve for Model

- Define $\Psi = L(\Phi)$
- If we use backward Euler
  - *Unknowns are Ψ, Φ, $\mathbf{A}_{||}$, δn_e*

$$\begin{aligned}
\Psi^{n+1} - \Delta t R_{\Phi}^{n} + 1 - L(\Phi^{n}) - \Delta t S^{n}(f) &= 0 \\
\mathbf{A}_{||}^{n+1} - \Delta t R_{\mathbf{A}}^{n} + 1 - \mathbf{A}^{n} &= 0 \\
\delta n_e^{n+1} - \Delta t R_{ne}^{n} + 1 - \delta n_e^{n} &= 0 \\
\Phi^{n+1} - L^{-1}(\Psi^{n+1}) &= 0
\end{aligned}$$

- This can be cast into the JFNK framework
- It will require an elliptic solve for Ψ
  - *Has been done in the context of reduced MHD where a Poisson operator was inverted during each Newton step (Chacon, Knoll & Finn, JCP 2002)*

Implicit Solve for Model - Issues

- The operator L includes a fourth order operator
  - *Will require an additional auxiliary variable if C0 continuous finite or spectral elements are employed*
  - *A solver to invert L will have to be developed*
- As mentioned earlier Krylov methods can have convergence problems
  - *Especially true if the linear system is ill-conditioned*
  - *Physics-based preconditioners will have to be developed for the linear Krylov phase of the solver*
  - *Such preconditioners are a subject of ongoing research*

Summary & Future Work

- Presented a primer on Jacobian-Free Newton-Krylov methods for nonlinearly implicit solution of PDEs
- Preconditioning is the key to have an effective JFNK method
  - *Presented an example of wave-structure based preconditioner for MHD*
- Discussed a set of model equations relevant to CSEPP

- Future Directions
  - *Recommend developing a code to test the JFNK ideas in a simple geometry*
  - *Develop physics-based preconditioners for the model system*

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- Phillip Colella, the Applied Numerical Algorithms Group, and the Applied Partial Differential Equations Center (APDEC) at Lawrence Berkeley National Laboratory

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