LJI-R-83-248

Comparison of Short-Wavelength Internal Wave Transport Theories*

Frank S. Henyey and Neil Pomphrey†
Center For Studies of Nonlinear Dynamics‡
La Jolla Institute
8950 Villa La Jolla Drive Suite 2150
La Jolla, Ca 92037
and
James D. Meiss
Institute for Fusion Studies
University of Texas
Austin, Texas 78712

Abstract

We compare three theoretical approaches to the transport of small scale oceanic internal waves; these are Induced Diffusion, Meiss-Watson transport, and Eikonal Monte-Carlo Simulations. The transport is forced by a background given by the Garrett-Munk spectrum, except that the spectrum is truncated in order to guarantee the correctness of the assumption of scale separation and of the eikonal approach. It is found that the oceanic interactions are too strong for the validity of either Induced Diffusion or the Meiss-Watson theory. This conclusion is also derived in the context of perturbation theory, where the second order term neglected in the transport theories is shown to be larger than the first order term.

* Supported in part by the Office of Naval Research

† Plasma Physics Laboratory, Princeton University, Princeton, NJ

‡ Affiliated with the University of California, San Diego

I. INTRODUCTION

The problem of the transport of small scale internal waves through the ambient internal wavefield of the ocean has generated a considerable interest over the last few years. The importance of this kinematic regime was identified by McComas and Bretherton, (1977) who named the relevant transport process "induced diffusion." Their study and most subsequent studies (Pomphrey, Meiss, and Watson (1980), McComas and Muller (1981)) were based on weakly nonlinear interaction theory (WNIT). Indeed, diffusion in phase space (i.e. wave number and physical space) is a consequence of scale separation and weakly nonlinear transport. More recent approaches have attempted to go beyond WNIT. Meiss and Watson (1982) (MW) derived a transport equation that was apparently different from that of McComas and Bretherton. Henyey and Pomphrey (1983) (HP) carried out a set of Monte-Carlo calculations on the eikonal equations for small scale internal waves. They interpreted their results to mean that a diffusion of small-scale internal wave transport is incorrect. Since HP did not use a transport equation (but instead included the random effects by the Monte Carlo technique), they were able to refrain from any assumption on the strength of the interaction.

The purpose of this paper is to compare the WNIT, the MW theory, and the HP calculations. We restrict ourselves to the question of the adequacy of the assumptions on the strength of nonlinear interactions. We take care to avoid having to resolve another important issue, that of whether a scale separation exists in the ocean between the forced and the forcing waves.

The internal waves responsible for the transport will be referred to as "the background" and the small scale waves whose transport we are following as "the test waves." The background flow is described phenomenologically by the Garrett-Munk model.

The set of approximations made on the test waves in most studies consist of one or more of the following:

1. Scale separation. The background waves have longer wavelengths (both horizontal and vertical) than the test waves.
2. Absence of strong correlations. The properties of the test waves are essentially constant over a period of time long enough to decorrelate the test wave from the background.
3. Weak interactions. Only resonant interactions last long enough to have an important effect on the transport of the test waves.

Approximation 1 has not been questioned in the literature, but its validity for internal-wave transport is not well established. In order to separate the issues of the various approximations this paper will consider only examples in which scale separation is manifestly correct. A future paper will discuss the scale separation issue.

Approximation 3 was made in the early studies. This assumption has been objected to by Holloway (1982) and by Meiss and Watson. Holloway shows that interaction times, appropriately defined, are shorter than wave periods and takes this result to mean that WNIT is wrong. Holloway concentrates on the definition of interaction time, and accepts the common view that the wave period is the proper comparison time. Careful studies of stochastic differential equations (Van Kampen (1981)) have, however, established that the comparison time should be the correlation time, which in general has no particular relationship to the wave period. Van Kampen (1981) refers to the ratio of

correlation time to interaction time as the "Kubo number." We adopt his terminology in this paper.

Thus, Holloway's considerations do not settle the issue of the adequacy of WNIT. One purpose of the present paper is to provide a definitive answer to this issue in the "induced diffusion" kinematic regime.

As a consequence of Van Kampen's considerations, it is seen that the assumption of the absence of strong correlations is a part of the assumption of weak interactions, and does not have to be considered as a separate issue. It might seem reasonable to assume the absence of strong correlations even if the interaction is not weak. Meiss and Watson (MW) re-examined the problem without making the weak interaction approximation. By making a weak correlation approximation they were able to complete the derivation of a transport equation. The essential approximation of MW is that the test wave group velocity (especially in the vertical direction) can be considered constant over a correlation time. Their transport equation has the structure that the density of waves in position wave number phase space, averaged over the ensemble of backgrounds, suffices to close the equation (without a memory kernel). A further purpose of this paper is to understand the relationship of the MW theory to weak interaction theory, and to assess its validity. Meiss and Watson's approximation is suggested by a diffusion - like picture of the transport. In such a picture, the test waves take small steps in phase space, with an uncorrelated forcing in each step. If the diffusion picture is correct, the kinematics of the test wave can be considered constant over each small step.

Henyey and Pomphrey (1983) (HP) have carried out a study in which approximation 2 was not made. As a result, no transport equation was derived, but rather Monte-Carlo studies were made. The numerical results obtained by HP suggested to them a nondiffusive picture of transport, inconsistent with

approximation 2. The HP picture includes a strong correlation between the vertical position of a test wave packet and the phase of the background waves, leading to critical layer phenomena. Since

$$z(t) = z_0 + \int_0^t v_g(t')\, dt' \tag{1}$$

and z_0 is assumed uncorrelated to the background, it follows that v_g cannot be considered constant for a correlation time in this picture.

In this paper we wish to compare the WNIT, the MW, and the HP approaches. We do so in the context of a model which is representative of the features of the oceanic interaction we believe to be important with the possible exception of scale separation which we put into the model but which might not be correct in nature. With the imposed scale separation, the HP approach, up to questions of numerics, is known to give correct results. Therefore the assumptions made by the other approaches can be tested by numerical comparison with the HP results.

The dominant transport of short horizontal wavelength internal waves is in their vertical wave number k_v. Strict diffusion (which is different from the MW transport equation) has the RMS k_v value increasing roughly like $t^{1/2}$ and the mean k_v increasing according to

$$\frac{d}{dt}\langle k_v \rangle = \langle \partial D(k_v)/\partial k_v \rangle \tag{2}$$

where D(k_v) is the k_v-dependent diffusion coefficient. The critical-layer picture has $\langle k_v \rangle$ increase rapidly due to the occurrence of critical layer events, and the RMS k_v increase because of the different times for critical layer events in different members of the ensemble of background flows.

Therefore, we calculate $\langle k_v \rangle$ and $k_{v_{RMS}} = \sqrt{\langle k_v^2 \rangle - \langle k_v \rangle^2}$ by all methods, and compare the results.

We take care that the initial conditions and the specification of the background are identical in the different calculations.

II. THE MODEL

In this section we describe the background ocean model used in our comparison. The model is designed to approximate conditions in the real ocean to the extent that they matter for the transport, with the possible exception of the small scale content of the background. The elimination of these small scales is done in order to enforce scale separation between background and test waves, so as not to confuse the issues of the validity of the various approximations made. The model also makes some simplifications which have been demonstrated in all approaches not to have a significant effect. These simplifications include setting the frequency of all background waves to the inertial frequency f, choosing the Brunt-Väisälä frequency N to be depth independent, and to ignore the horizontal dependence of the background waves.

The value of N is chosen to be the Garrett Munk (1979) (GM) value at one scale depth. GM chose $N = N_0 e^{z/B}$ with $N_0 = 3$ cph and $B = 1.3$ km. So we use $N = 3\ cph\ e^{-1}$. The intensity and vertical wave number spectrum of the background are given by the GM model, with $N/N_0 = e^{-1}$. A wave number cutoff of $K_v = 17.8\ km^{-1}$, corresponding to mode number 20, is imposed. Thus the background is

$$\vec{U} = \sum_j 2\sqrt{<U_j^2>}\ \cos K_v z\ \mathrm{Re}\left[(\hat{x} + i\hat{y})\, e^{-ift}\, a_j \right] , \tag{3a}$$

where a_j is a complex Gaussian random variable with

$$<|a_j|^2> = 1 , \tag{3b}$$

$$K_v = \frac{\pi j}{B} \left(\frac{N}{N_0} \right) , \tag{3c}$$

and

$$<U_j^2> = \frac{3}{2} B^2 E\, N_0 N\, \frac{2j_*}{\pi (j^2 + j_*^2)} \tag{3d}$$

with $j_* = 3$.

The test waves are assumed to obey the Taylor-Goldstein equation[3]

$$\nabla^2 (\partial_t - \vec{U} \cdot \nabla)^2 \psi + N^2 (\partial_x^2 + \partial_y^2) \psi + f^2 \partial_z^2 \psi = 0 \qquad (4)$$

where ψ is some appropriate variable which measures the test wave amplitude. This equation is linear in ψ, and although not quite self-adjoint, becomes so upon multiplication by the inverse Laplacian operator.

We choose initial conditions to be a state with a narrow band spectrum centered on horizontal wave number $k_H = 31\, km^{-1}$ and vertical wave number $k_v = 44\, km^{-1}$. The initial state is statistically independent of the background. As the test wave evolves we discard all components with vertical wave number in excess of $2\pi/10\, m$, roughly at the observed cutoff in the ocean internal wave spectrum (Gregg (1977)). The model assumes that waves that reach this wave number are lost to the internal wave field by breaking or other dissipative processes.

The transport in vertical wave number has been universally recognized as the most important aspect of the transport. Therefore our diagnostics in the comparison consist of the time evolution of the vertical wave number distribution of test wave action intensity, especially the zeroth, first and second moments of the distribution, averaged over the ensemble of random backgrounds (the zeroth is non-trivial because of the assumed dissipation at the high wave number).

III. EIKONAL APPROACH

The eikonal, or ray-tracing, approach is described in detail in HP (Henyey and Pomphrey (1983)). It consists of imagining a decomposition of the small-scale part of the wave field into wave packets, each with a small spread of wavenumbers, centered on a value $\vec{k}$ and occupying a region near some point $\vec{r}$. Wave packets (generally with different $\vec{k}$'s) can overlap, and fill space, and in the approximation that the test wave field is of low intensity, can be regarded as independent of each other. The evolution of the small-scale wave field is constructed from the motion of individual wave packets as they move through the large-scale background.

The ray equations are defined by a frequency function

$$\omega = \sigma + \vec{U} \cdot \vec{k} \quad , \tag{5}$$

and the set of equations

$$\dot{\vec{r}} = \partial_{\vec{k}}\, \omega = \partial_{\vec{k}}\, \sigma + \vec{U} \quad , \tag{6a}$$

$$\dot{\vec{k}} = -\nabla\omega = -\nabla\sigma - \nabla \vec{U} \cdot \vec{k} \quad . \tag{6b}$$

The function ω includes the "intrinsic" frequency (dispersion relation) for internal waves

$$\sigma = \left(\frac{N^2 k_H^2 + f^2 k_v^2}{k_H^2 + k_v^2} \right)^{1/2} \tag{7}$$

(where k_H and k_v are the horizontal and vertical wavenumbers, respectively), and the Doppler shift $\vec{U} \cdot \vec{k}$. The interaction of the wave packet with the background is entirely due to the Doppler shift term.

The ray equations (5) and (6) are of Hamiltonian form, where the Hamiltonian can be chosen as $H = A\,\omega$, and the momentum as $\vec{P} = A\vec{k}$. A is

the wave action of each packet, and is conserved along packet trajectories. The relationship of H to the wave energy and $\vec{P}$ to the Stokes drift is discussed in detail by HP.

Each term in the ray equations has a simple interpretation. The velocity $\dot{\vec{r}}$ is the sum of the wave group velocity, $\vec{v}_g = \partial_{\vec{k}} \sigma$, and advection by the background flow. The evolution of the wavenumber, $\dot{\vec{k}}$, is the sum of the WKB scaling term and the shearing of the waves by the background. The most important part of the transport comes from the vertical shear of horizontal current

$$\dot{k}_v = - \partial_Z \vec{U}_H \cdot \vec{k}_H \quad . \tag{8}$$

The ray equations do not constitute a transport theory; the average over the ensemble of background flows has yet to be done.

If the interaction is assumed sufficiently weak, induced diffusion can be recovered. By any of a number of formalisms, it can be shown that a Hamiltonian system weakly interacting with a random background undergoes diffusion in phase space. Each packet separately undergoes diffusion when averaged over backgrounds and the action of each packet is conserved, so it is action density which diffuses.

In this paper we use Van Kampen's (1974) formalism applied to the Liouville equation, mainly because our discussion of corrections to WNIT is based on it. The Liouville equation for the phase space probability density ρ is

$$0 = \left(\frac{d}{dt} \right)_{phase\ space} \rho \tag{9a}$$

$$= \left(\partial_t + (\vec{v}_g + \vec{U}) \bullet \partial_{\vec{r}} - (\nabla \sigma + \nabla \vec{U} \bullet \vec{k}) \bullet \partial_{\vec{k}} \right) \rho \tag{9b}$$

$$= \partial_t \rho + \mathbf{L}(t) \rho \quad , \tag{9c}$$

where $\mathbf{L}$ is the Liouville operator on phase space. $\vec{v}_g = \partial_{\vec{k}} \sigma$ is the group

velocity of the test wave.

We are interested in $< \rho >$, the ensemble average of the probability density. Since the Liouville equation is linear, the solution ρ can be expressed in terms of an evolution operator G(t):

$$\rho(t) = G(t)\,\rho(0) \quad . \tag{10}$$

Neglecting the time dependence of $\mathbf{L}(t)$ (which arises because the background flow $\vec{U}$ is time dependent), gives

$$G(t) = \exp(-\mathbf{L}t) \quad . \tag{11}$$

But this solution is not particularly useful, since we are interested in <G(t)>, and the ensemble average is hard to evaluate. Therefore, we follow van Kampen and decompose into deterministic and fluctuating parts

$$\mathbf{L} = \mathbf{L}_0 + \mathbf{L}_1 \tag{12a}$$

$$\mathbf{L}_0 = \vec{v}_g \cdot \partial_{\vec{r}} - \nabla\sigma \cdot \partial_{\vec{k}} \tag{12b}$$

$$\mathbf{L}_1 = \vec{U} \cdot \partial_{\vec{r}} - \nabla\vec{U} \cdot \vec{k} \cdot \partial_{\vec{k}} \tag{12c}$$

$$G_0(t) = \exp(-\mathbf{L}_0 t) \tag{13a}$$

$$G(t) = G_0(t)\, g(t) \quad . \tag{13b}$$

It can easily be seen that $g(t)$ obeys the equation

$$\partial_t\, g(t) + \mathbf{L}'(t)\, g(t) = 0 \quad , \tag{14}$$

where

$$\mathbf{L}'(t) = G_0^{-1}(t)\, \mathbf{L}_1(t)\, G_0(t) \quad . \tag{15}$$

The transformation from $\mathbf{L}$, G to $\mathbf{L}'$, g is known as going to the interaction representation. What has been gained is that $<\mathbf{L}'> = 0$. The price paid is that, although it makes sense to ignore the time dependence of $\mathbf{L}_1(t)$, it makes no

sense to ignore the time dependence of $\mathbf{L}'(t)$.

The formal solution of Eq. (14) is

$$g(t) = T\exp\left[- \int_0^t \mathbf{L}'(t')\, dt' \right] \quad , \tag{16}$$

where the time ordering symbol T means that in the Taylor series expansion of the exponential the operators $\mathbf{L}'(t_1')$, $\mathbf{L}'(t_2')$, . . . are ordered so that if $t_i' < t_j'$ then $\mathbf{L}'(t_i')$ is farther to the right than $\mathbf{L}'(t_j')$, i.e. the earliest operators are on the right and the later ones are placed successively to the left. Thus

$$T \frac{1}{n!}\left[\int_0^t dt'\mathbf{L}'(t')\right]^n = \int_0^t dt_1' \int_0^{t_1'} dt_2' \ldots \int_0^{t_{n-1}'} dt_n'\, \mathbf{L}'(t_1')\mathbf{L}'(t_2')\ldots\mathbf{L}'(t_n') \quad . \tag{17}$$

The $\frac{1}{n!}$ has been absorbed in nesting the integrals. From this expression one immediately sees that g(t) obeys the integral equation.

$$g(t) = 1 + \int_0^t \mathbf{L}(t')g(t')dt' \quad , \tag{18}$$

and a differentiation shows that g obeys Eq. (14).

The time ordered exponential of operators can be shown to have the property, as does the ordinary exponential of numbers, that its expectation value over an ensemble is given by the exponential of the sum of cumulants (Van Kampen 1981). In particular, since we are assuming $\vec{U}$ to be a zero-centered Gaussian process,

$$\langle g \rangle = \left\langle T \exp\left[- \int_0^t \mathbf{L}'(t')\, dt' \right] \right\rangle \tag{19a}$$

$$= T \exp \frac{1}{2} \int_0^t dt' \int_0^t dt'' \, \langle \mathbf{L}'(t')\, \mathbf{L}'(t'') \rangle \quad . \tag{19b}$$

By nesting the two integrals,

$$\langle g \rangle = T \exp \int_0^t dt' \int_0^{t'} dt'' \, \langle \mathbf{L}'(t')\, \mathbf{L}'(t'') \rangle \quad . \tag{20}$$

Differentiating, we obtain

$$\partial_t \langle g(t)\rangle = T \langle L'(t) \int_0^t L'(t')\, dt' \rangle \exp \int_0^t dt'' \int_0^{t''} dt''' \langle L'(t'')L'(t''')\rangle \quad . (21)$$

The G_0 can be put back in to obtain

$$\partial_t \langle G(t)\rangle = -L_0 \langle G(t)\rangle + T\langle G_0(t)L'(t)\int_0^t L'(t')dt'\rangle \qquad (22)$$

$$\exp \int_0^t dt'' \int_0^{t''} dt''' \langle L'(t'')L'(t''')\rangle \quad .$$

So far, the expression exactly represents the solution. We are now ready to assume a small Kubo number. The two correlated interactions in each pair have to be within about a correlation time of each other. Thus one of the t'', t''' integrations gives a factor of about τ_c, the correlation time. Each $\mathbf{L}$ is an interaction rate, so $\langle \mathbf{LL} \rangle$ gives a factor $\frac{1}{\tau_i^2}$. The remaining integration gives a factor t. Therefore the exponent is roughly $\frac{t\,\tau_c}{\tau_i^2}$. The dominant terms in the Taylor series for e^x when x is large are $n \approx x$. Therefore if $t >> \tau_i^2/\tau_c$, the dominant number of correlated interaction pairs is $n = \frac{t\,\tau_c}{\tau_i^2}$. The typical spacing in time between pairs is $\frac{t}{n} = \frac{\tau_i^2}{\tau_c}$ = (Kubo number)$^{-2}\,\tau_c$ while the time between interactions of the same pair is τ_c. Thus the probability for a pair to overlap one of its neighbors is on the order of (Kubo number)^2 and this possibility can be neglected if the Kubo number is much smaller than unity. In particular, the leftmost pair in Eq. (27) can be considered to occur later than any of the pairs in the exponent, since one of its interactions occurs at time t. Thus the time ordering symbol can be moved past the first correlation bracket, and acts only on the exponential. But the time ordered exponential is just <g(t)>, or $G_0^{-1}(t)\langle G(t)\rangle$. Therefore, for small Kubo number we obtain

$$\partial_t \langle G(t) \rangle = \left[- L_0 + G_0(t) \langle L'(t) \int_0^t L'(t')\, dt' \rangle G_0^{-1}(t) \right] G(t) \quad . \qquad (23)$$

Expressing $L'(t)$ in terms of the interaction $L_1(t)$ and using $G_0(t_1)\, G_0(t_2) = G_0(t + t_2)$, this can be written as

$$\partial_t \langle G(t) \rangle = \left[- L_0 + \langle L_1(t) \int_0^t dt'\, G_0(t - t')\, L_1(t')\, G_0^{-1}(t - t') \rangle \right] G(t) \quad , \qquad (24)$$

the lowest order van Kampen expression. Since the correlation time is short compared to the interaction time, $G(t)$ does not significantly deviate from the identity operator until $t \gg \tau_c$. Therefore, the lower limit can be replaced by $- \infty$. Moreover, since we assume a stationary process for the random L_1, both L's can be shifted by a time t' so that

$$\partial_t \langle G \rangle = \left[- L_0 + \int_0^\infty d\tau \langle L_1(\tau)\, G_0(\tau)\, L_1(0)\, G_0'(\tau) \rangle \right] \langle G \rangle \quad . \qquad (25)$$

This equation is, in general, a Fokker-Planck equation in phase space. For our purposes it can be written as a diffusion equation. First we remove the WKB scaling term $\nabla\sigma \cdot \partial_{\vec{k}}$ from L_0 either by neglecting the small dependence of the Brunt-Väisälä frequency on the vertical position or by changing variables from vertical wave number to mode number. We use the former way for notational purposes, since it agrees with previous derivations of induced diffusion. Thus $G_0(\tau)$ is

$$\exp(- \tau \vec{v}_g \cdot \partial_{\vec{r}}) \quad .$$

We are only interested in distributions in k_v. We either consider the distribution to be spatially homogeneous or we integrate over space. Thus any terms with $\partial_{\vec{r}}$ on either the left or right can be dropped. (Since we dropped the space dependence of N, $\langle \ldots \rangle$ is always independent of space.) The only $\partial_{\vec{k}}$ terms from commuting $G_0(\tau)$ through $L_1(0)$ is

$$\partial_{\vec{k}} \cdot \int_0^{\infty} d\tau \; \langle \nabla \vec{U}(\vec{r},\tau) \cdot \vec{k} \; \nabla \vec{U}(\vec{r} - \vec{v}_g \tau, 0) \cdot \vec{k} \rangle \cdot \partial_{\vec{k}} \quad .$$

Thus Eq. (30) is the induced diffusion equation

$$\partial_t \langle G \rangle = \partial_{\vec{k}} \, D(\vec{k}) \, \partial_{\vec{k}} \langle G \rangle \quad , \tag{26}$$

where, since $\langle \; \rangle$ is translation invariant

$$D(\vec{k}) = \int_0^{\infty} d\tau \langle \nabla \vec{U}(\vec{v}_g \tau, \tau) \, \nabla \vec{U}(0,0) \rangle \quad . \tag{27}$$

Specializing to the vertical component which is the most important term,[1]

$$\partial_t \langle G \rangle = \partial_{k_v} D_{33} \, \partial_{k_v} (G) \quad , \tag{28}$$

where

$$D_{33} = \int_0^{\infty} d\tau \langle \partial_z \vec{U}(\vec{v}_g \tau, \tau) \, \partial_z \vec{U}(0,0) \rangle \quad . \tag{29}$$

By Fourier transforming, D_{33} can be written as

$$D_{33} = \int dK_v d\Omega \, K_v^2 \langle U^2(\vec{K}, \Omega) \rangle \, \delta(\Omega - \vec{K} \cdot \vec{v}_g) \quad . \tag{30}$$

If the Garrett-Munk spectrum is put in, this expression for the diffusivity agrees with those previously calculated (McComas and Bretherton (1977); McComas and Muller (1981)).

The main result of this section has been to show that the eikonal approach is neutral as to the validity of WNIT. If the WNIT is made, the same induced diffusion equation results as if the WNIT is made in the mode representation with scale separation being applied afterwards. We have also introduced the formalism we will need later on and have shown why the correlation time, rather than the wave period, is the appropriate time scale to compare to the interaction time.

IV. INDUCED DIFFUSION

In this section we obtain the induced diffusion predictions for $< k_v >$ and $k_{v\,rms} = \sqrt{<k_v^2> - <k_v>^2}$ as a function of time by solving the diffusion equation (35) for our model.

Figure 1 shows the diffusivity D_{33} as a function of k_v. An important property of D_{33} in our model is its vanishing for $k_v > 113\ km^{-1}$. The reason is that the delta function in Eq. (30) requires that the test wave vertical group velocity v_g is equal to the background phase velocity f / K_v. K_v cannot exceed our background cutoff. Therefore there is a minimum possible value of $f / K_v = v_g$. Since the test wave horizontal wave number is fixed, this translates into a maximum possible vertical wave number contribution to the integral. As a consequence, since we start the test wave in a region for which D_{33} is nonzero, it never leaves this region. In effect, the cutoff in the background K_v has induced reflecting boundary conditions as $k_v = 113\,km^{-1}$.

We use a spectral method to solve the diffusion equation. We diagonalize the diffusion operator:

$$\partial_{k_v} D_{33}\, \partial_{k_v}\, \varphi_n(k_v) = -\lambda_n\, \varphi_n$$

with the reflecting boundary conditions discussed in the previous paragraph. The four eigenfunctions with the smallest nonvanishing eigenvalues are shown in Figure 2. In addition, there is a constant eigenfunction with $\lambda_0 = 0$. The solution for the distribution in k_v for a unit delta function at $k_v(0)$ at time $t = 0$ is

$$\rho = \sum_{n=0}^{\infty} e^{-\lambda_n t}\, \varphi_n(k_v)\, \varphi_n\left(k_v(0)\right)$$

which relaxes to the uniform distribution $\rho = \varphi_0^2 = \dfrac{1}{2k_{vmax}}$. The quantities

$< k_v >$ and $k_{v\,rms}$ are readily calculated from $\int k_v \varphi_n(k_v) dk_v$ and $\int k_v^2 \varphi_n (k_v)\, dk_v$.

The calculated values of $< k_v >$ and $k_{v\,rms}$ are as a function of time are shown in Figure 3. These will be compared with the predictions of MW and of the "exact" Monte Carlo calculations in later sections.

V. MEISS-WATSON THEORY

Meiss and Watson's (MW) approach is based on an approximation for the time displacement matrix U(t) defined by

$$< b(t) > \; = \; U(t) b(0) \quad . \tag{36}$$

where b is the vector of test wave mode amplitudes, a formal solution for U can be written as the time ordered exponential

$$U(t) \; = \; \sum_n \frac{1}{(2n)!} \int_0^t dt_1 \, dt_2 \cdots dt_{2n} \tag{37}$$

$$T \left[< A(t_1) \, A \, (t_2) \cdots A \, (t_{2n}) > \right]$$

where the A's are the individual interaction matrices consisting of coupling constants, intensities of the background modes and factors relating the time dependence of the interacting modes. The MW approximation consists of commuting the A's in Eq. 37.

The time ordered exponential is dominated by terms with

$$2n \; \approx \; \text{typical eigenvalue of} \int_0^t A(t')dt' \tag{3}$$

which increases linearly with t. There are several ways to see that times equal to the correlation time of the background as observed by the test wave system are the maximum times of interest. The transport equation could be derived for this time, and patched together using the independence of the background for longer periods. Alternatively, the expression for U(t) could be used for longer times, but the reordering necessary to derive the transport equation requires two factors referring to the same background wave to be moved to adjacent positions in the product. In order to accomplish this reordering, no factor need be moved more than a correlation time. Thus, the carefully established result that the Kubo number is the expansion parameter makes intuitive sense.

The A factors contain $e^{i\Delta t}$, where Δ is the frequency mismatch

$$\Delta = \omega_k - \omega_{k\pm l} \pm \omega_l \tag{4}$$

The first two ω's belong to test waves and the last to the background. By scale separation, Δ can be written in terms of the group velocity of the test wave and the phase velocity of the background as

$$\Delta = \pm\left[\vec{v}_g(k) - \vec{V}_p(l)\right]\cdot\vec{l} \tag{5}$$

The two A's to be placed adjacent have test wave wavenumbers k, k' before the reordering, and equal wavenumbers after. Thus, the approximation of MW includes the assumption that the test wave group velocity (especially in the vertical direction) can be considered constant over a correlation time.

The transport theory resulting from the MW approximation differs from induced diffusion (ID) in two ways: MW did not completely make the scale separation approximation and they retained an initial transient in the transport.

The part of the scale separation approximation MW refrained from making was to take the limit that the change in k_v in an interaction (which is the K_v of the background) is infinitesimal. As a result they end up with an integral equation rather than a differential equation which, when specialized to transport in k_v alone, is

$$\partial_t\, \rho(k_v) = 2 \int dK_v\, dk_v'\, G(k_v, k_v', K_v, t)\, \delta\,(k_v + K_v - k_v')$$

$$\left[\rho(k_v') - \rho(k_v)\right] \tag{36}$$

If the remainder of the scale separation approximation is made, the integrand except for the δ function can be expanded in a power series in $k_v' - k_v$. The constant term is not present and the linear term vanishes upon integration leaving

$$\partial_t\, \rho(k_v) = \partial_{k_v}\left[\int dK_v\, K_v^2\, G(k_v, k_v, K_v, t)\right] \partial_{k_v}\rho(k_v) \tag{37}$$

(where symmetry of G in its first two arguments is used). The expression in

brackets is to be identified as D_{33}. Using the MW expression for **G** it is identical to Eq. (35) if t is set to ∞.

Since the model used in this paper has scale separation built into it, we expect this aspect of MW not to be significant. Indeed, we have solved the MW integral equation with t set to ∞ in the expression for **G**, using a spectral technique. Graphs of the eigenfunctions $\varphi_0, \cdots, \varphi_4$ are indistinguishable from those of ID, and the eigenvalues were

$$\lambda_0 = 0,\ \lambda_1 = 4.86 \times 10^{-3},$$

$$\lambda_2 = 2.86 \times 10^{-2},\ \lambda_3 = 8.42 \times 10^{-2},\ \lambda_4 = 1.36 \times 10^{-1};$$

compared to the values

$$0,\ 5.99 \times 10^{-3},$$

$$2.96 \times 10^{-2},\ 9.53 \times 10^{-2},\ 1.38 \times 10^{-1}$$

for ID. The graph of $< k_v >$ and $k_{v\,rms}$ is shown in Figure 4, and is seen to be the same as for ID, as expected.

The other difference between MW and ID is that **G** is time dependent. In fact, the time dependence is exactly the same as retaining the lower limit of integration at $t' = 0$, rather than at $t' = -\infty$ in the expression for the diffusivity (Eq. (34), where $t' = t - \tau$). Physically, the density in vertical position of the test wave is initially uncorrelated with the shear, so the transport cancels out in the ensemble average. As time proceeds, some of this correlation develops, allowing net transport. Thus, as applied to our model, MW theory is essentially a diffusion equation with a time dependent diffusivity given by

$$D_{33}(k_v, t) = \int_0^t dt' \, < \partial_z \vec{U}\,(\vec{v}_g\, t, t)\, \partial_z \vec{U}\,(\vec{v}_g\, t', t') > \quad , \tag{38}$$

which is smaller than $D_{33}(k_v, \infty)$ for small t.

The differences shown by MW (in their Fig. 2) between their theory and ID are due to this initial transient. We present the difference in another way, based

on the spectral decomposition of the diffusion operator. The eigenfunctions and eigenvalues are now time dependent. It turns out that the t dependence of the eigenfunctions is rather small; e.g. the overlap of $\varphi_1(t = 0)$ with $\varphi_1(t = \infty)$ is 90 percent. The eigenvalues are $\lambda_n(t = 0)$ and asymptotically approach their ID level. It is a reasonably good approximation to treat the φ_n as time-independent, making the solution for a unit δ-function at $t = 0$

$$\rho \approx \sum_n e^{-\int_0^t \lambda_n(t')\,dt'} \varphi_n(k_v)\, \varphi_n(k_v(0)) \tag{39}$$

which, after the transient dies down is

$$\rho \approx \sum_n e^{-\lambda_n(\infty)(t - T_n)} \varphi_n(k_v)\, \varphi_n(k_v(0)) \quad . \tag{40}$$

The most significant nonzero eigenvalue is $\lambda_1(t)$, shown in Figure 5. The integral $\int_0^t \lambda_1(t')dt'$ is, for $t \geq 0.22$ inertial periods, $t - T_1$ with $T_1 = 0.032$ inertial periods. T_2, T_3, T_4 have the values 0.047, 0.072, 0.074.

Thus, as applied to the scale-separated regime, MW differs from ID only by having the exponential approach to equilibrium postponed by a small amount of time, about 0.03 inertial period, due to the assumed initial absence of correlation of the test waves with the background.

VI. COMPARISON OF ID WITH EIKONAL RESULTS

Up to this point we have concentrated on the consequences of making the weak interaction assumption. That assumption leads to a diffusion equation, or, if the MW theory is used, to a time dependent diffusion equation in which the ID transport is somewhat postponed. In this section we test the weak interaction assumption by comparing to the eikonal results, which are valid for our model.

The calculation of the eikonal results is described in detail in HP. The method is to numerically integrate the deterministic eikonal equations in a randomly constructed background flow. Ensemble averages are constructed by averaging over a large sample of such backgrounds, always keeping the initial conditions on the test wave the same. The eikonal results do not show any cutoff in k_v ; the weak interaction approximation has not been made, so net transport does not require resonant kinematics. We have imposed an absorbing boundary at a vertical wavelength of 5 m. This boundary is not particularly important for the model we are using, but is more significant for the calculations reported in HP.

The numerical comparison of the eikonal results with ID is shown in Fig. 6. Very significant differences can be seen. The level of the $k_{v\,rms}$ is about a factor of 2 greater than that given by ID, and takes considerably longer to become established. $< k_v >$ has a longer time before beginning a possibly exponential decay, and the decay is much slower than that of ID. The value of $< k_v >$ at the maximum is much greater than given by ID.

Thus it must be concluded that ID is an inadequate representation of the transport properties of our model. Since the MW theory resembles ID so strongly, it too is inadequate. In the next section we examine the adequacy of ID in the context of perturbation theory, of which it is the first term.

VII. SECOND VAN KAMPEN TERM

We have seen in the previous section that there is a severe numerical discrepancy between WNIT and eikonal results, and therefore that WNIT is incorrect for short wavelength internal wave transport. This section is devoted to showing that the same result can be obtained remaining within the framework of perturbation theory. The convergence properties of perturbation theory can be estimated by comparing the second order result to the first order result, obtaining an estimate for the square of the effective Kubo number.

Comparison of first order theory to free propagation is not relevant; the Kubo number is not isolated by such a comparison. We have tried to directly estimate the Kubo number without success, even knowing the answer of this section. The trouble is that the range of kinematic conditions of both test waves and background waves leads to a large range of possible correlation and interaction times. One does not know what sort of average to take.

Thus, an estimate of the second term is needed to be able to assess the convergence of perturbation theory. The second order contains terms of the same form as the lowest order, as well as new terms involving three derivations of the density, and the initial transient effect discussed as the main difference between MW and ID. The only part of this order we look at is the correction to the second derivative term D_{33}. The "size" of third derivative terms cannot be directly compared to second derivative terms, and we wish to make a numerical comparison.

We have been able to calculate the correction to D_{33} in two ways. One way is a detailed rigorous calculation, presented in the appendix, which does not give much insight into the physical processes involved. The other is a physically motivated calculation in which terms are dropped with insufficient justification. This calculation was originally intended to be a rough estimate, but we found the

result to be identical with that of the detailed calculation. The motivated calculation is presented below.

The correction to D_{33} arises because the force $\partial_z \vec{U} \cdot \vec{k}$ acts along the actual trajectory of a wavepacket, whereas ID approximates it by the force along the straight-line trajectory given by the current group velocity. Van Kampen (1974) suggests decomposing the correction into two parts, a part due to the average corrections to the straight-line motion and another part which is whatever is left over, namely details of correlations between the current force and the previous trajectory. It turns out that only the former contributes in our case.

The correction can be calculated from

$$D_{33} = \int_0^\infty d\tau \; \langle \partial_z \vec{U}(z(t)) \cdot \vec{k} \; \partial_z \vec{U}(z(t-\tau), t-\tau) \cdot \vec{k} \rangle \qquad (42)$$

where $z(t-\tau)$ is related to $z(t)$ by both the straight-line motion and by including the diffractive spreading of the ensemble of trajectories between these times.

We rewrite D_{33} in terms of the velocity spectrum $S(K_v, \Omega)$. The ∂_z's give factors of K_v; and the angular integration gives a factor of ½. Therefore, equation 42 can be expressed as

$$D_{33} = \int \frac{k^2}{2} K_v^2 S(K_v, \Omega) \, dK_v \, d\Omega \qquad (43)$$

$$\int_0^\infty d\tau \, \langle \cos \left[\Omega\tau - K_v z(t) + K_v z(t-\tau) \right] \rangle$$

The quantity $z(t) - z(t-\tau)$ is

$$z(t) - z(t-\tau) = v_g(t)\tau + \zeta$$

where the first term on the right alone gives ID and ζ is the correction. For the

next order term we can expand in a power series in ζ, obtaining

$$D_{33} = \int \frac{k^2}{2} K_v^2 S(K_v, \Omega)\, dK_v\, d\Omega \int_0^\infty d\tau \cos\left[(\Omega - K_v v_g(t))\tau\right]\left[1 - \frac{K_v^2 <\zeta^2>}{2}\right] \tag{44}$$

The 1 gives ID and the $\frac{K_v^2 <\zeta^2>}{2}$ the correction.

We evaluate $<\zeta^2>$ as follows:

$$\begin{aligned}\zeta(t-\tau) &= \int_0^\tau d\tau'\, \delta v_g(\tau') \\ &= \int_0^\tau d\tau' \int_0^{\tau'} d\tau''\, \partial_t v_g(t-\tau') \\ &= \int_0^\tau d\tau' \int_0^{\tau'} d\tau''\, \partial_z \vec{U}(z - v_g\tau', t-\tau') \cdot \vec{k} \\ &= \int_0^\tau d\tau' (\tau - \tau'')\, \partial_z \vec{U}(z - v_g\tau', t-\tau') \cdot \vec{k}\end{aligned} \tag{45}$$

where δv_g is the change in group velocity due to the acceleration between $t - \tau'$ and t, and the straight line approximation suffices for $z(t - \tau'')$. Thus

$$<\zeta^2> = \int \frac{k^2}{2} K_v^2 S(K_v, \Omega)\, dK_v\, d\Omega \int_0^\tau d\tau'\, d\tau' \tag{46}$$

$$(\tau - \tau')(\tau - \tau'') \cos\left[(\Omega - K_v v_g)(\tau' - \tau'')\right]$$

Putting this in equation 44, we obtain

$$\begin{aligned}D_{33} - D_{ID} = &- \frac{k^4}{8} \int K_v^4 S(K_v, \Omega)\, dK_v\, d\Omega \int K_v'^2 S(K_v, \Omega')\, dK_v' d\Omega' \\ &\int_0^\infty d\tau \int_0^\tau d\tau' \int_0^\tau d\tau'' \; (\tau - \tau')(\tau - \tau'') \\ &\cos\left[(\Omega - K_v v_g)\tau\right] \cos\left[(\Omega' - K_v' v_g)(\tau' - \tau'')\right]\end{aligned} \tag{47}$$

The integrand is symmetric in τ', τ''. Therefore a factor of two can be absorbed in nesting the τ', τ'' integrals.

The integrals in Eq. 44 are done partly analytically and partly numerically. If we were to use the sharp cutoff in K_v, the extra factors of $\tau - \tau'$, $\tau - \tau''$ would cause the correction to be extremely singular; therefore we smooth out the cutoff in order to draw a graph. We use

$$S(K_v, \Omega) = S_{GM}(K_v, \Omega)\ e^{-K_v^2/2K_c^2} \tag{48}$$

where K_c is the previous cutoff value of K_v, and S_{GM} is the Garrett-Munk spectra.

The numerical evaluation of ID and the correction, with this cutoff spectrum are shown in Fig. 7. It is clear that the correction is not small compared to ID and that the induced cutoff is not respected. Thus perturbation theory shows no signs of convergence. This result is entirely consistent with that of the previous section in which it was shown by comparison to the eikonal that ID is inadequate.

Presumably, higher order corrections are also large and higher derivative terms also make a significant contribution. One must conclude that there is no reason that ID has any relationship to the actual transport of the model.

VIII. CONCLUSIONS

We have demonstrated that induced diffusion, and its refinement to Meiss-Watson theory, do not correctly describe the transport of small scale internal waves in a simplified model of the ocean IW field. Both comparison with the eikonal theory and the direct calculation of corrections give this result.

The reader may be concerned over whether the results for the model extend to the real ocean. There are three issues involved. First, we have made a number of simplifying assumptions. These simplifications change the quantitative features of the graphs we have drawn, so one should not apply these graphs as representing transport in the ocean. The question of the validity of weak interaction theories should not, however, be particularly influenced by the simplifications.

The second issue concerns the adequacy of the description of the ocean by the Garret-Munk spectrum. Again, there might be quantitative features in the transport which depend on IW properties not included in the Garrett-Munk model. However, it does not seem that these features could make ID work, since ID involves only a gross smearing of IW properties, which Garrett-Munk certainly does well with. In contrast, the transport as described by the eikonal does seem to have some sensitivity to more subtle properties of the IW field, and the question of the adequacy of the model for the background as applied for the actual transport is not resolved.

The third issue concerns the background cutoff we made in order to avoid confusion with the question of scale separation. This issue is tricky because the higher wavenumbers which we neglected introduce both interaction times and correlation times which are shorter. The part we have included is still a major part of the transport, and the imposition of the extra interactions actually

lengthen the correlation times for this part, since they cause there to be more high wavenumbers in the test wave ensemble, with correspondingly smaller group velocities.

Since weak interaction theory is not applicable to this part, it is not reasonable that by adding a moderate amount of extra interaction, the interaction should become weak (as measured by the effective Kubo number). Thus, even if the higher order terms were unchanged by the introduction of the higher wavenumber background, the lowest term would have to increase very significantly in order to dominate.

Therefore, it seems reasonable to us to extrapolate to the real ocean our result that weak interaction transport gives wrong results for small wavenumber transport, and some other method, such as Monte-Carlo simulations, is required. Pending the outcome of a study of the validity of the scale separation approximation, the eikonal representation is a good way to implement such calculations.

Our finding that weak interaction theory is not valid in one kinematic regime raises the question of its validity elsewhere. It would be prudent to examine the corrections to weak interaction theory in the remaining regimes. Intuitive feelings about its validity tend to be based on the comparison of interaction times with wave periods, rather than on the correct comparison with correlation times. Perhaps the estimate of the second van Kampen term will be no harder than in our regime.

Wright (private communication) has pointed out that whenever weak interaction theory is valid, the direct interaction theory will agree with it. When it is wrong, the DIA will disagree with it and possibly be wrong itself. It turns out that DIA disagrees with weak interaction theory over a major part of phase space.

The physical phenomena which happen when transport is controlled by strong interactions has yet to be elucidated. Rather than thinking of Markov processes in the phase space density, one should think of large correlations and such phenomena as dominance of transport by approximate critical layers, as suggested by Henyey and Pomphrey. The future of internal wave physics involves new concepts as well as more calculations.

REFERENCES

Garrett, C.J.R. and Munk, W.H., (1979), Internal waves in the ocean. J. Phys. Oceanogr., **12**, 293-296.

Gregg, M.C., (1977), A comparison of finestructure spectra from the main thermocline. J. Phys. Oceanogr., **7**, 33-40.

Henyey, F.S. and Pomphrey, N., (1983), Eikonal description of internal wave interactions: A non-diffusive picture of induced diffusion. Dyn. Atmos. and Oceans, (1983).

Holloway, G., (1982), On interaction time scales of oceanic internal waves. J. Phys. Oceanogr., **12**, 293-296.

McComas, C.H. and Bretherton, F.B., (1977), Resonant interaction of oceanic internal waves. J. Geophys. Res., **82**, 1397-1412.

McComas, C.H. and Müller, P., (1981), The dynamic balance of internal waves, J. Phys. Oceanogr., 11, 970-986.

Meiss, J.D. and Watson, K.M., (1982), Internal wave interactions in the induced diffusion approximation. J. Fluid Mech., **117**, 315-341.

Pomphrey, N., Meiss, J.D. and Watson, K.M., (1980), Description of nonlinear internal waves interactions using Langevin methods. J. Geophys. Res., **85**, 1085-1094.

Van Kampen, N.G., (1974), A cumulant expansion for stochastic linear differential equations. Physica, **74**, 215-239.

Van Kampen, N.G., (1981), *Stochastic Processes in Physics and Chemistry*, North-Holland Publishing Company, New York, N.Y. 10017.

APPENDIX 1

EQUIPARTITION OF ACTION AND THE STATISTICAL MECHANICS OF INDUCED DIFFUSION

Consider the statistical mechanics of induced diffusion. Energy is conserved, and in a linearized treatment of the test waves, so too is the action. Let E_j and A_j denote the energy and action in test wave mode j. Then the probability density for wave action (in a grand canonical ensemble) is $g \exp(- \beta E_j + \mu A_j)$. Here, β is the "inverse temperature" of the system, μ is the "chemical potential" for test wave action, and g is a normalization factor. Thus, the expectation of the test wave action is

$$<A_j>_{TW} = g \int_0^\infty dA_j \cdot A_j \cdot \exp(- \beta E_j + \mu A_j) = \frac{1}{\beta \omega_j + \mu} , \qquad (A.1)$$

while the background waves have

$$<A_j>_{BG} = \frac{1}{\beta \Omega_j} . \qquad (A.2)$$

The development of induced diffusion as a kinetic theory has ignored the effect of the small scale test waves on the background, since the background is much more intense than the test wave field. Thus, built into the theory is the assumption that

$$<A_j>_{TW} \ll <A_j>_{BG}$$

In fact, the test wave action should be infinitesimal compared with the background action. Thus $\beta \omega_{TW} \ll \mu$ and $<A_j> = \frac{1}{\mu}$, independent of j; i.e. action is equipartitioned. The dominance of $\beta \omega_{TW}$ over μ would have given the contrary (familiar) result, that energy is equipartitioned.

APPENDIX 2

DETAILED CALCULATION OF THE SECOND VAN KAMPEN TERM

We show that the expression 47 for the correction to the vertical wavenumber diffusivity that we obtained in an imprecise way in section 7 is identical to that derived more carefully. The careful derivation here does not reveal the physical mechanism responsible in as simple a way.

We use the fact that the interaction operator in equation 26 is the Liouville operator $L'(t)$ in the interaction representation (relative to the current time t). L' is the Liouville operator of the interaction representation Hamiltonian which is the Doppler shift

$$H'(t-\tau) = \vec{U}(z - v_g \tau,\ t-\tau) \bullet \vec{k} \tag{A.3}$$

The Liouville operator is

$$L'(t-\tau) = \partial_z H' \partial_{k_v} - \partial_{k_v} H' \partial_z$$

$$= \partial_z \vec{U} \bullet \vec{k}\ (\partial_{k_v} - \partial_{k_v} v_g \tau \partial_z) \tag{A.4}$$

where horizontal coordinates have been suppressed, since we ignore horizontal dependence of the background. We will have occasion to use the commutator of two L' operators, which is the operator associated with the Poisson brackets of the two H' functions:

$$\{H'(t-\tau_1), H'(t-\tau_2)\} = -\partial_z \vec{U}(z - \tau_1 v_g, t-\tau_1) \bullet \vec{k}$$

$$\partial_z \vec{U}(z - \tau_2 v_g, t-\tau_2) \bullet \vec{k}\ \partial_{k_v} v_g (\tau_1 - \tau_2) \tag{A.5}$$

which we abbreviate as

$$\{H'_1, H'_2\} = -\partial_z \vec{U}_1 \bullet \vec{k}\ \partial_z \vec{U}_2 \bullet \vec{k}\ \frac{\partial v_g}{\partial k_v}(\tau_1 - \tau_2) \tag{A.6}$$

The second order term involves four L' operators, correlated in pairs. We use

a's to denote one pair and b's the other, and put subscripts 1, 2, 3, 4 to denote the time ordering $t - \tau_4 < t - \tau_3 < t - \tau_2 < t - \tau_1 = t$. The induced diffusion operator is, with this notation

$$d_{ID}(t) = \int d\tau_2 \, a_1 \, a_2 \tag{A.7}$$

The solution of ID involves the time ordered product of d's, but the time integrated over in defining D_{ID} cannot be ordered, since D is not a function of it. Therefore the solution of ID contains the τ integrals of:

$$a_1 a_2 b_3 b_4 + a_1 a_3 b_2 b_4 + a_1 a_4 b_2 b_3$$

While the ensemble average of the true solution always has correct time ordering:

$$a_1 a_2 b_3 b_4 + a_1 b_2 a_3 b_4 + a_1 b_2 b_3 a_4$$

The correction d_2 to ID is the difference between these, so that d_2 applied once replaces d_{ID} applied twice with the correct four-L' term. Therefore:

$$\begin{aligned} d_2 &= \int_{0<\tau_2<\tau_3<\tau_4} d\tau_2 d\tau_3 d\tau_4 \left[a_1 [b_2, a_3] b_4 + a_1 [b_2 b_3, a_4] \right] \\ &= \int_{0<\tau_2<\tau_3<\tau_4} d\tau_2 d\tau_3 d\tau_4 \left[a_1 [b_2, a_3] b_4 + a_1 b_2 [b_3, a_4] + a_1 [b_2, a_4] b_3 \right] \\ &= \int_{0<\tau_2<\tau_3<\tau_4} d\tau_2 d\tau_3 d\tau_4 \left[a_1 [b_2, a_3] b_4 + 3 \leftrightarrow 4 - a_1 b_2 [a_4, b_3] \right] \end{aligned} \tag{A.8}$$

The second and third terms are the part discussed in section 7 (as can be seen from the first form of d_2), while the first term is the "what's left" term which we asserted gave no contribution to D_{33}.

Each a_j or b_j consists of a phase space velocity

$$v_{PS,j} = \left(\partial_z \vec{U}_j \cdot \vec{k}, \, \tau_j \, \partial_{k_y} v_g \, \partial_z \vec{U}_j \cdot \vec{k} \right) \tag{A.9}$$

contracted with a phase space derivative, $\partial_{PS} = (\partial_{k_v}, \partial_z)$. By Liouville's theorem, the derivative can appear either to the right or to the left of $v_{PS,j}$, since phase space is incompressible.

We now evaluate the first term. ∂_{PS} is put to the left of $v_{PS,1}$ in α_1. In α_4 only the term $\partial_z \vec{U}_4 \cdot \vec{k}\, \partial_{k_v}$ contributes, since we are not looking at terms which have a ∂_z on the right.

$[\alpha_2, \alpha_3]$ is of the form

$$\partial_{PS} \{H'_2, H'_3\} \partial_{PS} = \partial_{PS}(---)\, \partial_z \vec{U}_3 \cdot \vec{k}\, \partial_{PS} \qquad (A.10)$$

(involving the antisymmetric part of the ∂'s.)

Thus the only part of the first term which can appear is

$$\partial_{k_v}(---) \; < \partial_z \vec{U}_3 \cdot \vec{k} \;\; \partial_{PS}\, \partial_z \vec{U}_4 \cdot \vec{k} > \partial_{k_v}$$

The ∂_{PS} acting on the $\partial_z \vec{U}_4 \cdot \vec{k}$ gives $< \partial_z \vec{U}_3 \cdot \vec{k} \;\; \partial_z^2 \vec{U}_4 \cdot \vec{k} >$ in this term, which vanishes by the assumed up-down symmetry of the GM spectrum. Therefore the term becomes

$$\partial_{k_v}(---)\, \partial_{k_v}^2$$

which is the form in which we call a term a third derivative term (i.e., the flux is proportional to a second derivative of the distribution). As we are interested only in second derivative terms of the form

$$\partial_{k_v}(---)\, \partial_{k_v} \; .$$

The first term in A.8 does not contribute to D_{33}.

Since the second term is the same as the first, except that indicies 3 and 4 are interchanged, it also does not contribute to D_{33}.

The third term is

$$- \partial_{k_v} \; \partial_z \vec{U}_1 \cdot \vec{k} \left[\partial_{k_v} + \tau_2 \frac{\partial v_g}{\partial k_v} \partial_z\right] \partial_z \vec{U}_2 \cdot \vec{k} \left[\partial_z \{H'_3, H'_4\} \partial_{k_v} - \partial_{k_v} \{H'_3, H'_4\} \partial_z\right]$$

with the u_1 correlated to u_4 and u_2 correlated to u_3. The term in the last parenthesis with ∂_z is discarded as not contributing to D_{33}. The ∂_z cannot act on the $\partial_z \vec{U}_3 \cdot \vec{k}$ in $\{H'_3, H'_4\}$ because of the up-down symmetry when correlating u_2 and u_3. We already have a ∂_{k_v} on both left and right; therefore the ∂_z must act on $\partial_z \vec{u}_4 \cdot \vec{k}$. In order that the up-down symmetry not give zero in the $u_1 - u_4$ correlation, the remaining differential operator must also act on $\partial_z \vec{U}_4 \cdot \vec{k}$. Therefore, the relevant part is (removing the ∂_{k_v}'s from left and right)

$$\delta D_{33} = -\int d\tau_2 d\tau_3 d\tau_4 \; \langle \partial_z \vec{U}_2 \cdot \vec{k} \, \partial_z \vec{U}_3 \cdot \vec{k} \rangle$$

$$\langle \partial_z \vec{U}_1 \cdot \vec{k} \left(\partial_{k_v} + \tau_2 \frac{\partial v_g}{\partial k_v} \partial_z \right) \partial_z^2 \vec{U}_4 \cdot \vec{k} \rangle (\tau_3 - \tau_4) \frac{\partial v_g}{\partial k_v}$$

$$= -\int d\tau_2 d\tau_3 d\tau_4 \; \langle \partial_z \vec{U}_2 \cdot \vec{k} \, \partial_z \vec{U}_3 \cdot \vec{k} \rangle$$

$$\langle \partial_z \vec{U}_1 \cdot \vec{k} \, \partial_z^3 \vec{U}_4 \cdot \vec{k} \rangle \; (\tau_2 - \tau_4) \frac{\partial v_g}{\partial k_v} (\tau_3 - \tau_4) \frac{\partial v_g}{\partial k_v} . \qquad \text{(A.11)}$$

This last form is then, by Fourier transforming the correlation functions, the same as equation 47.

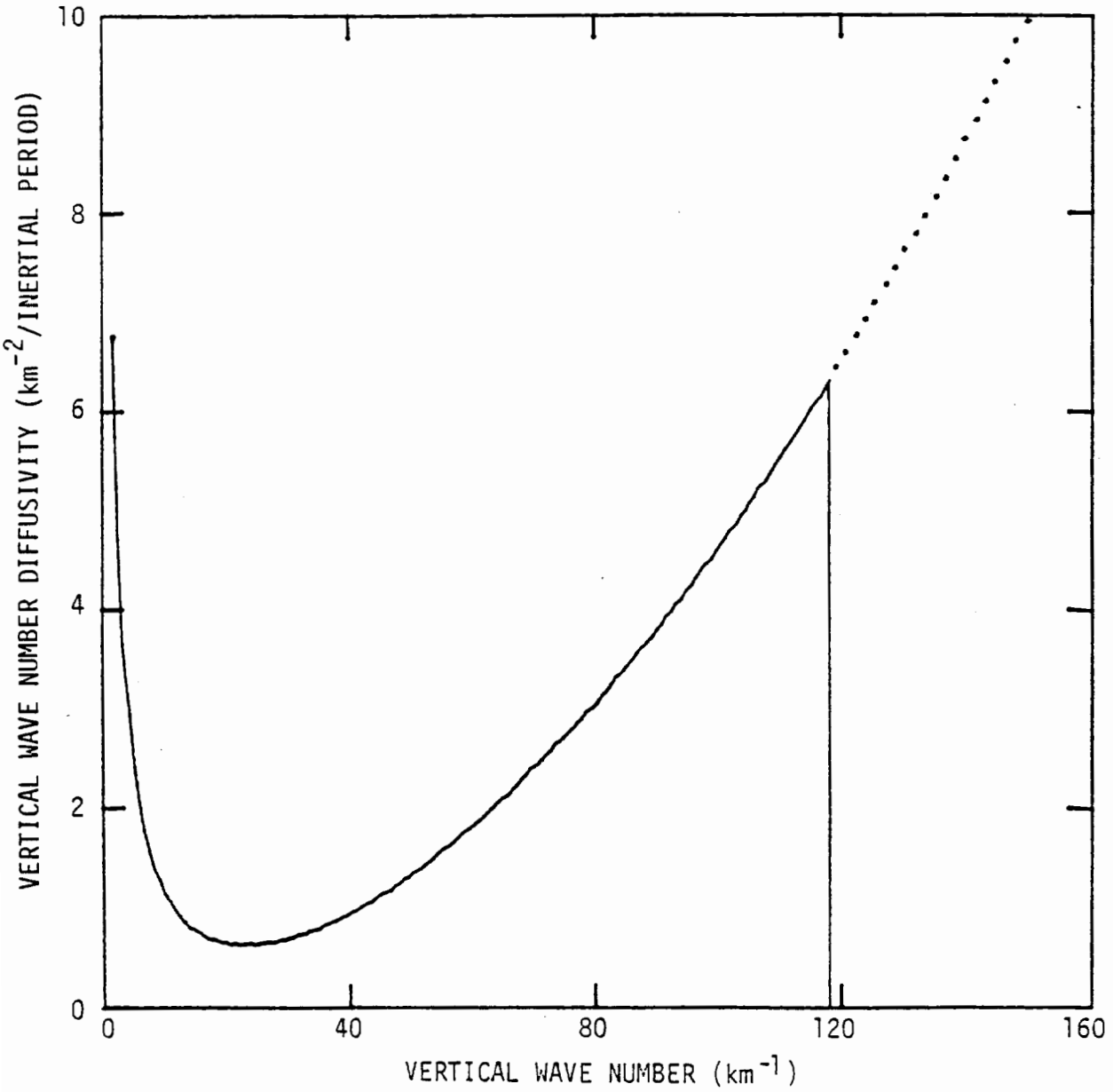

Figure 1. Diffusivity D_{33} (Eq. 35) as a function of vertical wave number k_v. Resonant kinematics and a finite background cut-off at $K_v = 17.8km^{1}$ implies the vanishing of D_{33} beyond $k_v = 113km^{-1}$.

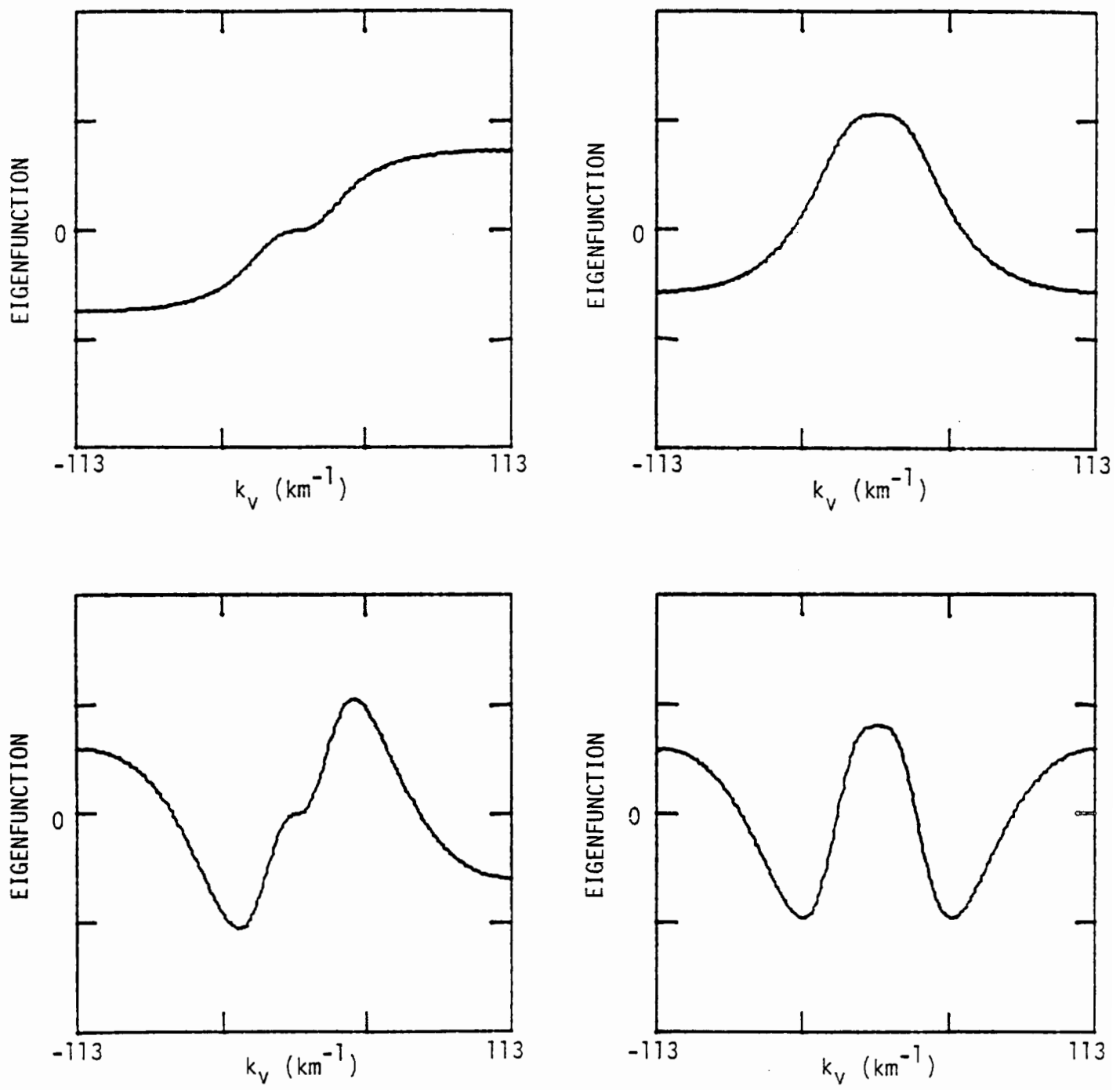

Figure 2. The four lowest eigenfunctions $\varphi_n(k_v)$ of the diffusion operator (Eq. 36). At long times only small eigenvalues and corresponding eigenfunctions are important. These determine the distribution of action in wavenumber space.

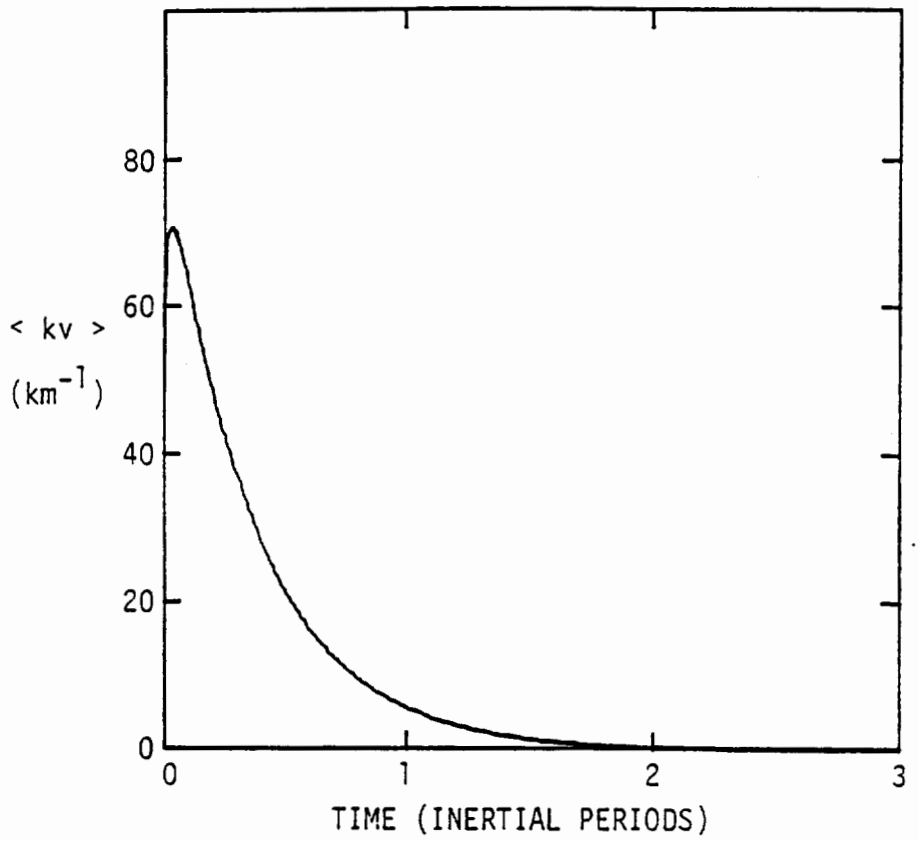

Figure 3a. Calculated value of $\langle k_v \rangle$ as a function of time for induced diffusion.

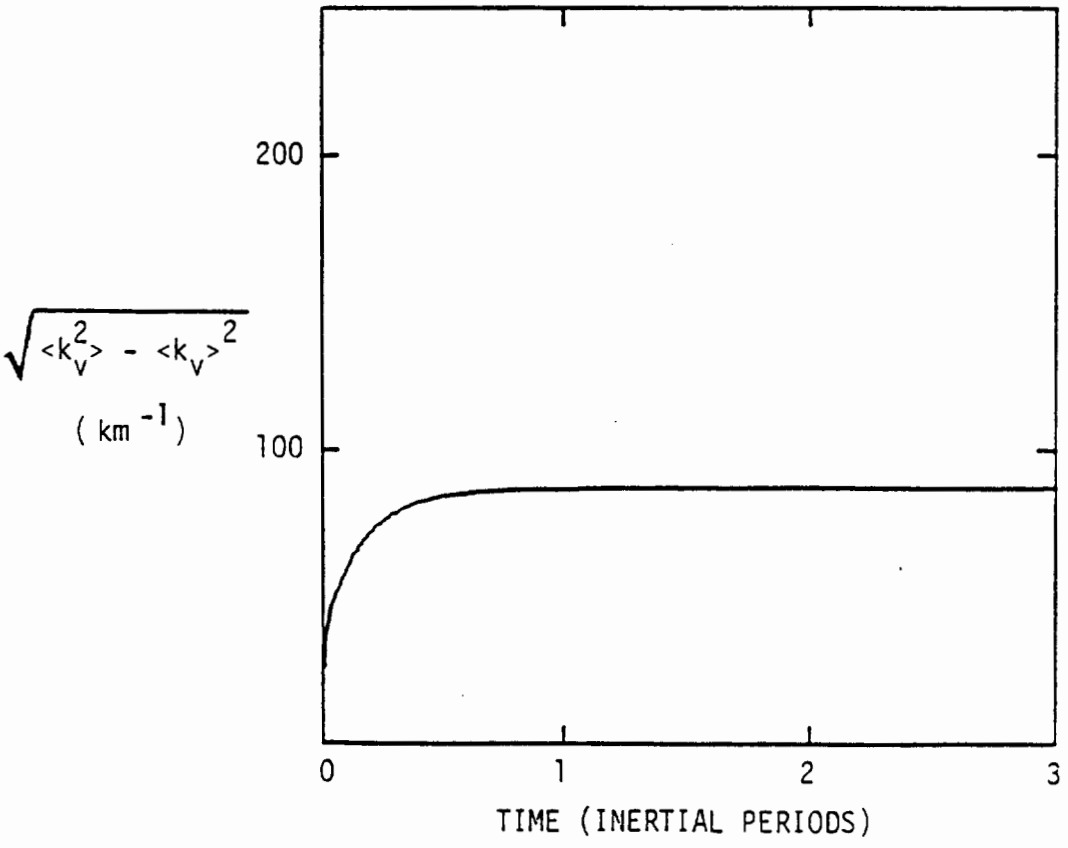

Figure 3b. Calculated value of $k_{v\,rms}(t)$ for induced diffusion.

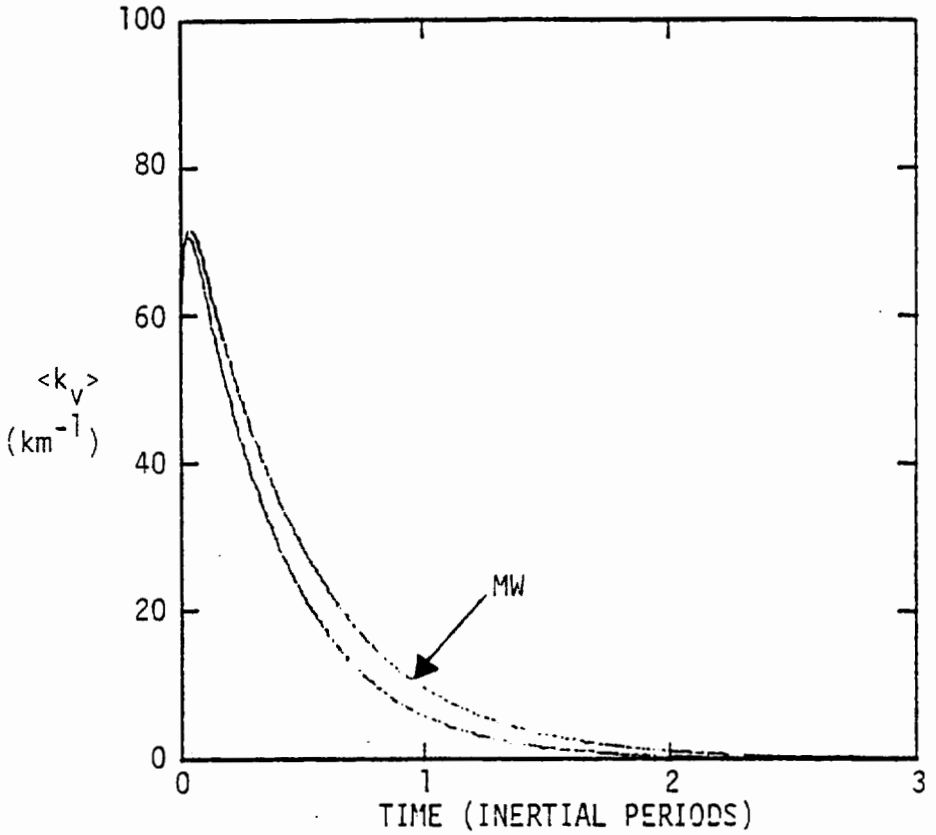

Figure 4a. Comparison of $\langle k_v \rangle$ versus time for induced diffusion and for Meiss-Watson. The slight difference between the curves in due to an initial transient that MW take into account.

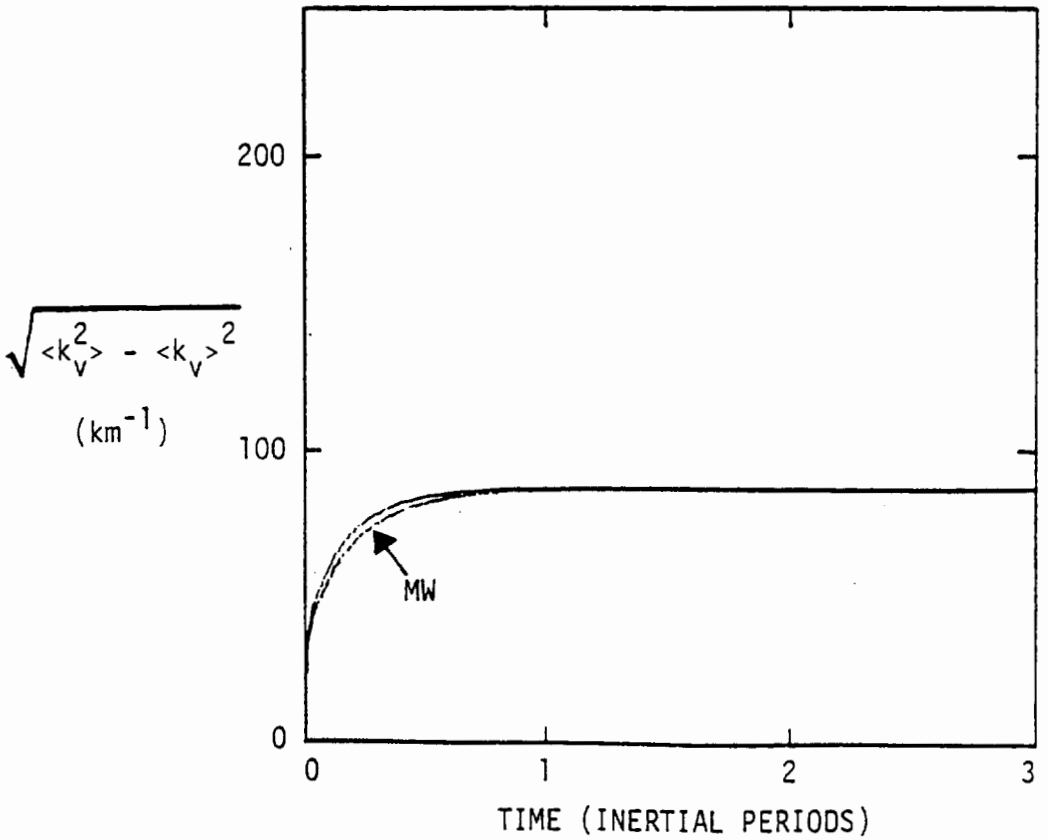

Figure 4b. Comparison of $k_{v\,rms}$ versus time for ID and for MW. The predicted spreading of a wave packet is essentially the same in the two theories.

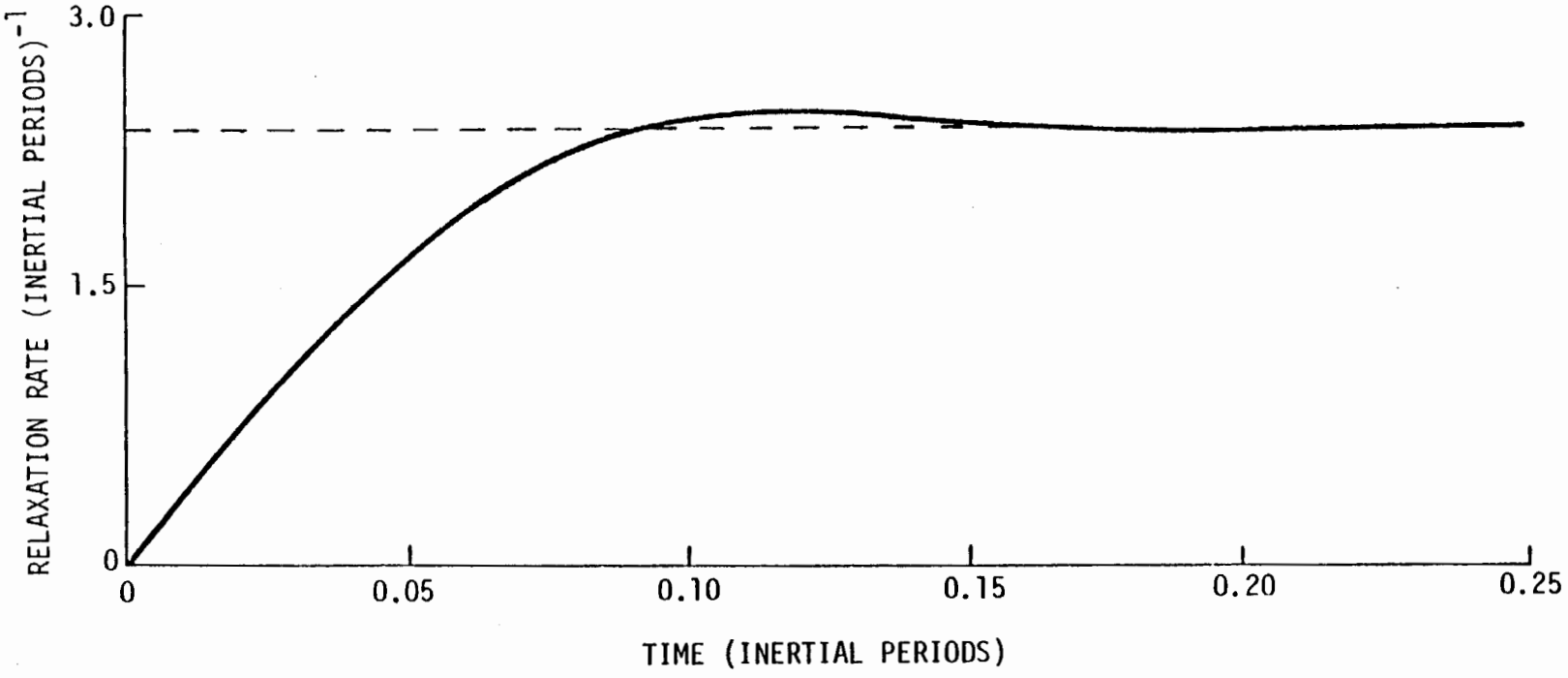

Figure 5. Meiss-Watson's diffusion operator (Eq. 38) is shown as a function of time (solid curve). After $t \geq 0.22$ inertial periods, λ_1 has reached its asymptotic value of 4.86×10^{-3} (dashed line). The area under the solid wave gives a time delay = 0.03 inertial periods for transport. This is the main difference between MW and ID.

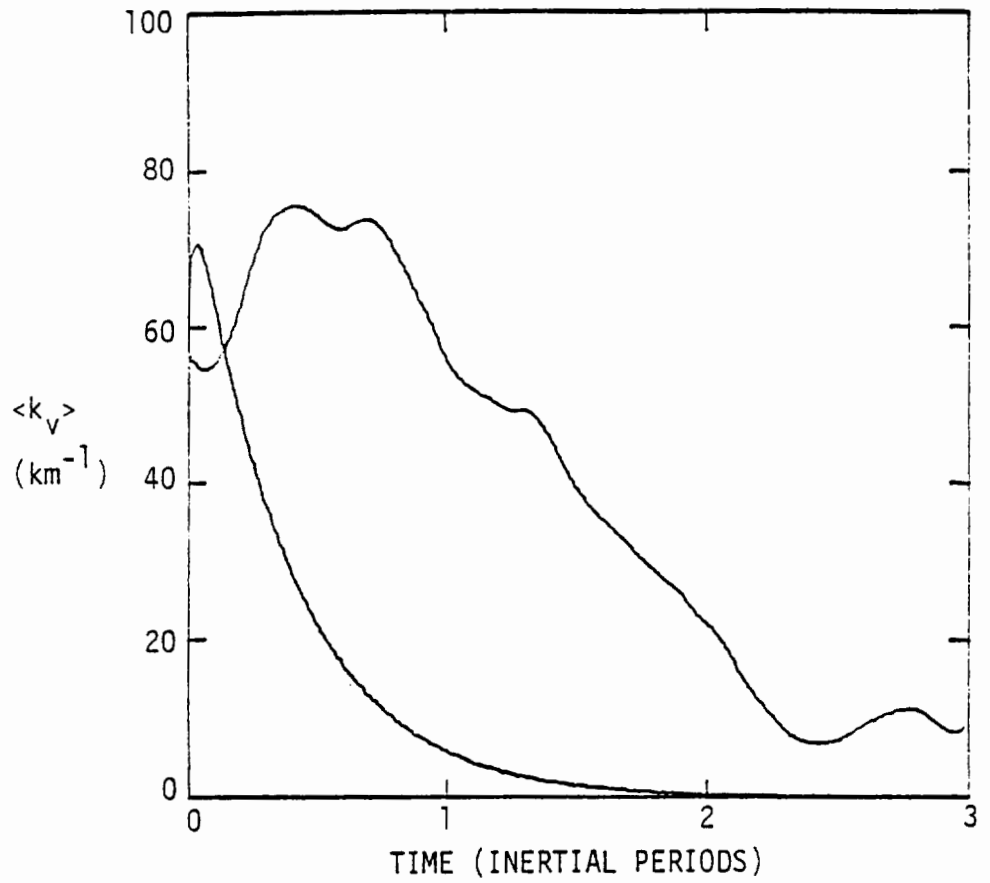

Figure 6a. Comparison of eikonal prediction of $\langle k_v \rangle$ versus time with induced diffusion. The eikonal decay is much slower than that of ID, and it is significantly delayed.

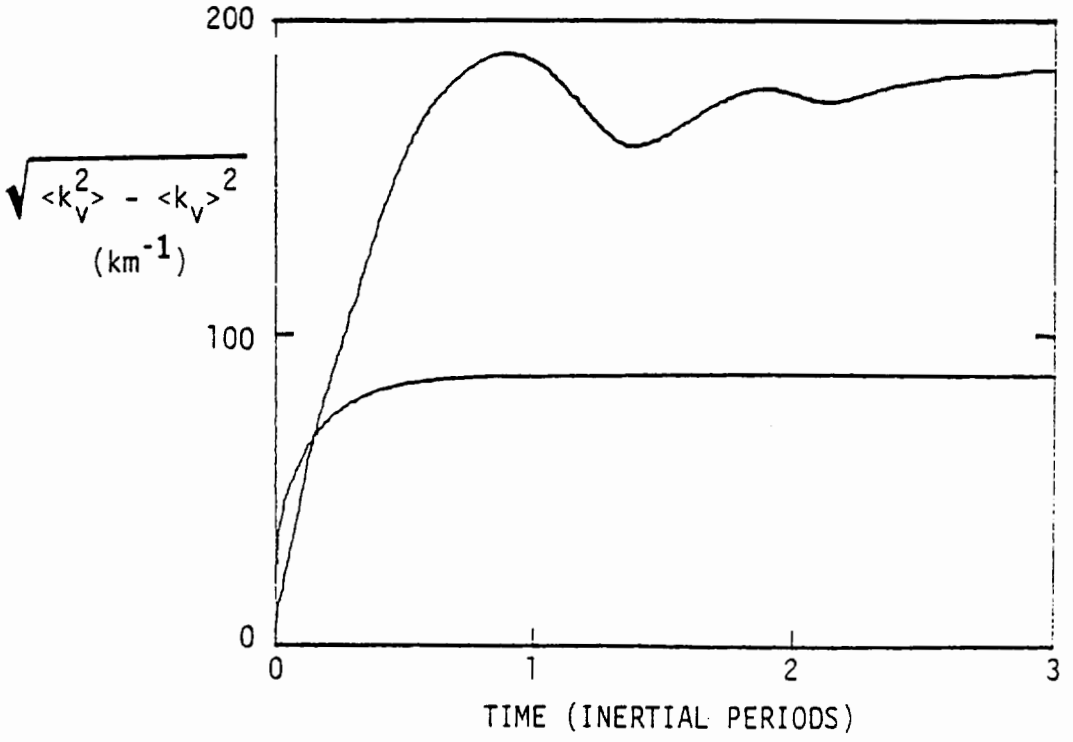

Figure 6b. Comparison of eikonal prediction of $k_{v\,rms}$ versus time with induced diffusion. The eikonal predicts a much higher level of $k_{v\,rms}$ than does ID, and saturation takes considerably longer. Since the eikonal is "exact" for the model calculation, it is concluded that ID is inadequate.

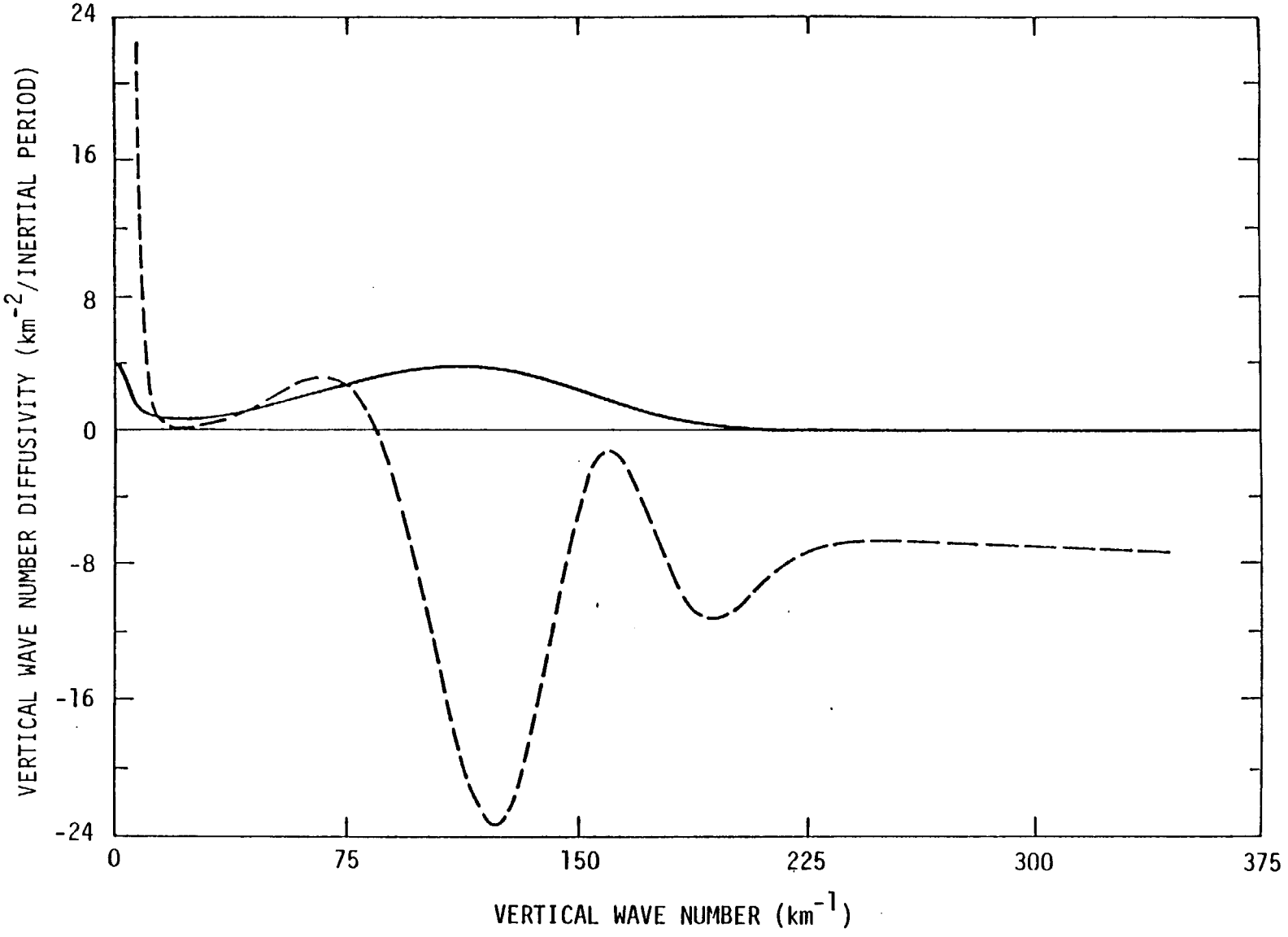

Figure 7. Comparison of ID diffusivity D_{33} (solid curve) with "second Van Kampen" correction (dashed curve). The correction is not small, therefore D_{33} cannot be expected to adequately represent transport. This result is consistent with Figures 6a, 6b.