BEYOND LINEAR POLARIZATION IN GYROKINETIC THEORY

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Polarization effects play a crucial role in the nonlinear gyrokinetic description of turbulent magnetized plasmas

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OUTLINE

- Physics of Polarization and Magnetization
  - High-frequency Oscillation-center Polarization
- Top-down Approach: Variational Formulation
  - Reduced Hamiltonian
- Bottom-up Approach: Push-forward Formulation
  - Near-identity Transformation
- Guiding-center Polarization and Magnetization
- Linear and Nonlinear Gyrocenter Polarization

PHYSICS OF POLARIZATION AND MAGNETIZATION

- Macroscopic Description of Electromagnetic Fields

  - Microscopic $(\mathbf{E}, \mathbf{B}) \rightarrow$ Macroscopic $(\mathbf{D}, \mathbf{H})$

$$\mathbf{D} \equiv \mathbf{E} + 4\pi\, \mathbf{P} \quad \text{and} \quad \mathbf{H} \equiv \mathbf{B} - 4\pi\, \mathbf{M}$$

  - Microscopic & Macroscopic Charge and Current

$$\varrho \equiv \varrho_R - \nabla \cdot \mathbf{P} \quad \text{and} \quad \mathbf{J} \equiv \mathbf{J}_R + \frac{\partial \mathbf{P}}{\partial t} + c\, \nabla \times \mathbf{M}$$

  - Macroscopic Maxwell Equations

$$\nabla \cdot \mathbf{D} = 4\pi\, \varrho_R \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = 4\pi\, \mathbf{J}_R \qquad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$$

- **Oscillation-center Polarization & Magnetization** (Brizard, 2009) $\rightarrow$ Laser-plasma interactions

  - **Eikonal Representation** $(\omega \equiv -\partial_t \Theta,\; \mathbf{k} \equiv \nabla\Theta)$

$$(\mathbf{E}_1, \mathbf{B}_1) \;=\; \left(\overline{\mathbf{E}}_1,\, \overline{\mathbf{B}}_1\right)\, \exp\left(\epsilon^{-1}\, i\Theta(\epsilon\mathbf{r}, \epsilon t)\right) \;+\; \text{c.c.}$$

  - **Oscillation-center Displacement** $(\omega' \equiv \omega - \mathbf{k}\cdot\mathbf{v})$

$$\overline{\boldsymbol{\xi}}_1 \;\equiv\; \frac{-\,e}{m\,\omega'^2}\left(\overline{\mathbf{E}}_1 \;+\; \frac{\mathbf{v}}{c}\times\overline{\mathbf{B}}_1\right)$$

  - **Oscillation-center Polarization & Magnetization**

$$\begin{pmatrix} \overline{\boldsymbol{\pi}}_{2\mathrm{oc}} \\ \\ \overline{\boldsymbol{\mu}}_{2\mathrm{oc}} \end{pmatrix} = \begin{pmatrix} e\,\mathbf{k}\times\left(i\,\overline{\boldsymbol{\xi}}_1\times\overline{\boldsymbol{\xi}}_1^*\right) \\ \\ \frac{e}{c}\,\omega'\,\left(i\,\overline{\boldsymbol{\xi}}_1\times\overline{\boldsymbol{\xi}}_1^*\right) \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \overline{\mathbf{P}}_{2\mathrm{oc}} \\ \\ \overline{\mathbf{M}}_{2\mathrm{oc}} \end{pmatrix} = \sum\int d^3\overline{p}\;\overline{F}\begin{pmatrix} \overline{\boldsymbol{\pi}}_{2\mathrm{oc}} \\ \\ \overline{\boldsymbol{\mu}}_{2\mathrm{oc}} \;+\; \overline{\boldsymbol{\pi}}_{2\mathrm{oc}}\times\overline{\mathbf{v}}/c \end{pmatrix}}$$

TOP-DOWN APPROACH: VARIATIONAL FORMULATION

- **Reduced Lagrangian Density**

$$\mathcal{L} \;=\; \frac{1}{8\pi}\left(|\mathbf{E}|^2 \;-\; |\mathbf{B}|^2\right) \;+\; \underbrace{\mathcal{L}_R(\cdots;\Phi,\mathbf{A};\mathbf{E},\mathbf{B},\;\cdots)}_{\text{Reduced plasma dynamics}}$$

- **Euler-Poincaré Equations**

$$\nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right)+\frac{\partial\mathcal{L}_R}{\partial\Phi} \;=\; 0 \;=\; \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{E}}\right)+\nabla\times\left(\frac{\partial\mathcal{L}}{\partial\mathbf{B}}\right)+\frac{\partial\mathcal{L}_R}{\partial\mathbf{A}}$$

- **Reduced (Dipole) Polarization and Magnetization**

$$\left(\mathbf{P},\,\mathbf{M}\right) \;\equiv\; \left(\frac{\partial\mathcal{L}_R}{\partial\mathbf{E}},\;\frac{\partial\mathcal{L}_R}{\partial\mathbf{B}}\right)$$

- **Reduced Charge and Current Densities**

$$\left(\rho_R,\;\frac{1}{c}\mathbf{J}_R\right) \;\equiv\; \left(-\,\frac{\partial\mathcal{L}_R}{\partial\Phi},\;\frac{\partial\mathcal{L}_R}{\partial\mathbf{A}}\right)$$

- **Oscillation-center (Quadratic) Hamiltonian**

  - **Canonical Transformation** (weak background $\epsilon_0 \ll 1$)

$$-i\,\omega'\,\overline{S}_1 + \epsilon_0\,\frac{e}{m}\left(\mathbf{E}_0 + \frac{\mathbf{v}}{c}\times\mathbf{B}_0\right)\cdot\frac{\partial\overline{S}_1}{\partial\mathbf{v}} = e\left(\overline{\Phi}_1 - \frac{\mathbf{v}}{c}\cdot\overline{\mathbf{A}}_1\right)$$

  - **Oscillation-center Hamiltonian**

$$\begin{aligned}\overline{H}_{2\mathrm{oc}} &= -\frac{e}{2m}\left\langle\left(\mathbf{E}_1 + \frac{\overline{\mathbf{v}}}{c}\times\mathbf{B}_1\right)\cdot\frac{\partial S_1}{\partial\overline{\mathbf{v}}}\right\rangle \\ &= m\,\omega'^2\,|\overline{\boldsymbol{\xi}}_1|^2 - \epsilon_0\left[\mathbf{E}_0\cdot\overline{\boldsymbol{\pi}}_{2\mathrm{oc}} + \mathbf{B}_0\cdot\left(\overline{\boldsymbol{\mu}}_{2\mathrm{oc}} + \overline{\boldsymbol{\pi}}_{2\mathrm{oc}}\times\frac{\overline{\mathbf{v}}}{c}\right)\right]\end{aligned}$$

  - **Oscillation-center Polarization & Magnetization**

$$\boxed{\begin{pmatrix}\overline{\mathbf{P}}_{2\mathrm{oc}} \\ \overline{\mathbf{M}}_{2\mathrm{oc}}\end{pmatrix} = -\,\epsilon_0^{-1}\sum\int d^3\overline{p}\,\overline{F}\begin{pmatrix}\partial\overline{H}_{2\mathrm{oc}}/\partial\mathbf{E}_0 \\ \partial\overline{H}_{2\mathrm{oc}}/\partial\mathbf{B}_0\end{pmatrix}}$$

BOTTOM-UP APPROACH:
PUSH-FORWARD FORMULATION

- **Dynamical Reduction by Near-identity Phase-space Transformation**

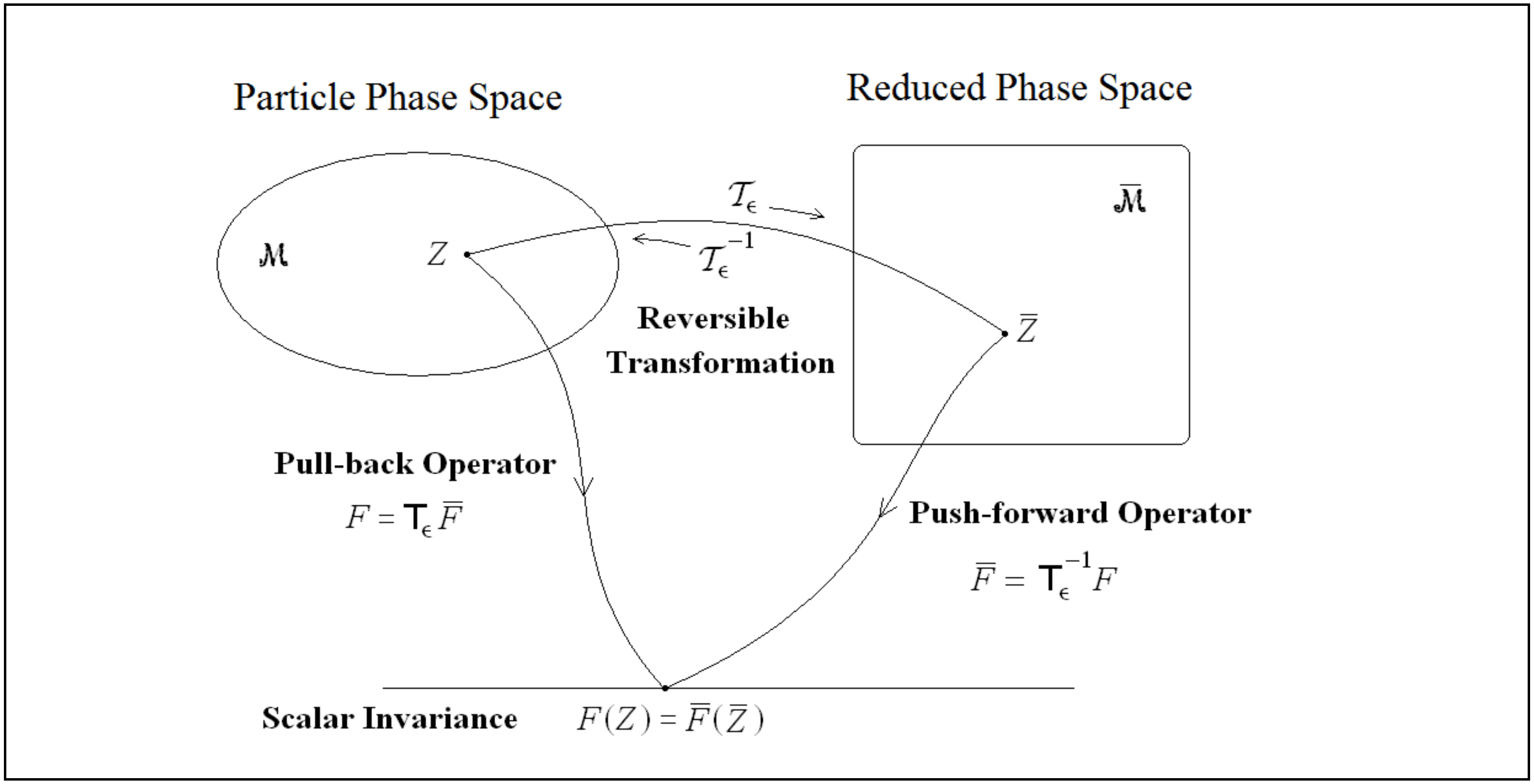
∘ **Transformation** $\mathcal{T}_\epsilon : \mathcal{Z} \rightarrow \overline{\mathcal{Z}}(\mathcal{Z};\epsilon) \equiv \mathcal{T}_\epsilon \mathcal{Z}$

$\longrightarrow$ Pull-back operator $\mathsf{T}_\epsilon : \overline{\mathcal{F}} \rightarrow \mathcal{F} \equiv \mathsf{T}_\epsilon \overline{\mathcal{F}}$

∘ **Inverse Transformation** $\mathcal{T}_\epsilon^{-1} : \overline{\mathcal{Z}} \rightarrow \mathcal{Z}(\overline{\mathcal{Z}};\epsilon) \equiv \mathcal{T}_\epsilon^{-1}\overline{\mathcal{Z}}$

$\longrightarrow$ Push-forward operator $\mathsf{T}_\epsilon^{-1} : \mathcal{F} \rightarrow \overline{\mathcal{F}} \equiv \mathsf{T}_\epsilon^{-1}\mathcal{F}$

- **Pull-back and Push-forward Operators**



- Pull-back Operation: Iterative Vlasov Solver

- Push-forward Operation: Reduced Displacement

- **Reduced Polarization and Magnetization**

  - **Displacement** $\boldsymbol{\rho}_\epsilon \equiv \mathsf{T}_\epsilon^{-1}\mathbf{x} - \overline{\mathbf{x}}$ (e.g., gyroradius)

$$\boxed{\boldsymbol{\rho}_\epsilon \;=\; -\,\epsilon\, G_1^{\mathbf{x}} \,-\, \epsilon^2 \left(G_2^{\mathbf{x}} - \frac{1}{2}\, \mathsf{G}_1 \cdot \mathsf{d}G_1^{\mathbf{x}} \right) + \cdots}$$

  - **Push-forward representation for four-current**

$$\begin{aligned} J^\mu(\mathbf{r}) \;&=\; \sum e \int_{x,p} v^\mu\, \delta^3(\mathbf{x} - \mathbf{r})\, \mathcal{F} \;\equiv\; \sum e \int_{x,p} (c, \mathbf{v})\, \delta^3(\mathbf{x} - \mathbf{r})\, \mathcal{F} \\ &=\; \sum e \int_{\overline{x},\overline{p}} \left(\mathsf{T}_\epsilon^{-1} v^\mu \right) \delta^3(\overline{\mathbf{x}} + \boldsymbol{\rho}_\epsilon - \mathbf{r})\, \overline{\mathcal{F}} \\ &=\; \sum e \int_{\overline{p}} \left[\left(\mathsf{T}_\epsilon^{-1} v^\mu \right) \overline{\mathcal{F}} - \nabla \boldsymbol{\cdot} \left(\boldsymbol{\rho}_\epsilon\, \mathsf{T}_\epsilon^{-1} v^\mu\, \overline{\mathcal{F}} \right) + \cdots \right] \end{aligned}$$

  - **Push-forward of Particle Velocity**

$$\mathsf{T}_\epsilon^{-1}\mathbf{v} \;=\; \mathsf{T}_\epsilon^{-1}\left(\frac{d\mathbf{x}}{dt} \right) \;=\; \left[\mathsf{T}_\epsilon^{-1} \frac{d}{dt} \mathsf{T}_\epsilon \right] \left(\mathsf{T}_\epsilon^{-1}\mathbf{x} \right) \;\equiv\; \frac{d_\epsilon \overline{\mathbf{x}}}{dt} \;+\; \frac{d_\epsilon \boldsymbol{\rho}_\epsilon}{dt}$$

∘ Charge-current Push-forward

$$\rho \equiv \overline{\rho} - \nabla\cdot\mathbf{P}_\epsilon \;\text{ and }\; \mathbf{J} = \overline{\mathbf{J}} + \frac{\partial\mathbf{P}_\epsilon}{\partial t} + c\,\nabla\times\mathbf{M}_\epsilon$$

⋆ Reduced Polarization

$$\mathbf{P}_\epsilon = \sum e \int_{\overline{p}} \left[\boldsymbol{\rho}_\epsilon\,\overline{\mathcal{F}} - \frac{1}{2}\nabla\cdot\left(\boldsymbol{\rho}_\epsilon\,\boldsymbol{\rho}_\epsilon\,\overline{\mathcal{F}}\right) + \cdots \right]$$

$$\equiv \sum \int_{\overline{p}} \boldsymbol{\pi}_\epsilon\,\overline{\mathcal{F}} + \cdots$$

⋆ Reduced Magnetization

$$\mathbf{M}_\epsilon = \sum \frac{e}{c} \int_{\overline{p}} \left[\boldsymbol{\rho}_\epsilon \times \left(\frac{1}{2}\frac{d_\epsilon\boldsymbol{\rho}_\epsilon}{dt} + \frac{d_\epsilon\overline{\mathbf{x}}}{dt}\right) + \cdots \right] \overline{\mathcal{F}}$$

$$\equiv \sum \int_{\overline{p}} \left(\boldsymbol{\mu}_\epsilon + \boldsymbol{\pi}_\epsilon \times \frac{1}{c}\frac{d_\epsilon\overline{\mathbf{x}}}{dt}\right) \overline{\mathcal{F}}$$

- **Oscillation-center Displacement**

$$\boldsymbol{\rho}_\epsilon \equiv \epsilon\,\boldsymbol{\xi} + \frac{\epsilon^2}{2}\left(\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi} + m\,\frac{d\boldsymbol{\xi}}{dt}\cdot\frac{\partial\boldsymbol{\xi}}{\partial\mathbf{p}}\right) - \epsilon^2\,G_2^{\mathbf{x}} + \cdots$$

◦ **Eikonal-averaged Electric-dipole Moment**

$$e\,\langle\boldsymbol{\rho}_\epsilon\rangle = \epsilon^2\,\overline{\boldsymbol{\pi}}_{2\mathrm{oc}} = \epsilon^2\,e\,\mathbf{k}\times\left(i\,\overline{\boldsymbol{\xi}}_1\times\overline{\boldsymbol{\xi}}_1^*\right)$$

◦ **Eikonal-averaged Magnetic-dipole Moment**

$$\frac{e}{2c}\left\langle\boldsymbol{\rho}_\epsilon\times\frac{d_\epsilon\boldsymbol{\rho}_\epsilon}{dt}\right\rangle = \epsilon^2\,\overline{\boldsymbol{\mu}}_{2\mathrm{oc}} = \epsilon^2\,\frac{e}{c}\,\omega'\left(i\,\overline{\boldsymbol{\xi}}_1\times\overline{\boldsymbol{\xi}}_1^*\right)$$

GUIDING-CENTER POLARIZATION AND MAGNETIZATION

- **Variational Formulation**

○ **Perturbed Particle Phase-space Lagrangian**

$$\Lambda_1 \;\equiv\; (e/c)\,\mathbf{A}_1\cdot\mathsf{d}\mathbf{x} \;-\; e\,\Phi_1\,\mathsf{d}t$$

○ **Perturbed Guiding-center Phase-space Lagrangian**

$$\begin{aligned}
\Lambda_{1\mathsf{gc}} \;&\equiv\; \mathsf{T}_{\mathsf{gc}}^{-1}\Lambda_1 \;+\; \mathsf{d}\sigma_1 \\
&=\; \frac{e}{c}\,\mathbf{A}_1(\mathbf{X}+\boldsymbol{\rho}_{\mathsf{gc}},t)\cdot\left(\mathsf{d}\mathbf{X} \;+\; \mathsf{d}\boldsymbol{\rho}_{\mathsf{gc}}\right) \\
&\quad -\,e\,\Phi_1(\mathbf{X}+\boldsymbol{\rho}_{\mathsf{gc}},t)\,\mathsf{d}t \;+\; \mathsf{d}\sigma_1
\end{aligned}$$

○ **Guiding-center Displacement**

$$\boldsymbol{\rho}_{\mathsf{gc}} \;=\; \boldsymbol{\rho}_0 \;+\; \epsilon_B\,\boldsymbol{\rho}_1 \;+\; \cdots$$

⋆ Dipole expansion

$$\Lambda_{1\mathrm{gc}} = \frac{e}{c}\left(\mathbf{A}_1 + \boldsymbol{\rho}_{\mathrm{gc}}\cdot\nabla\mathbf{A}_1 + \cdots\right)\cdot\left(\mathsf{d}\mathbf{X} + \mathsf{d}\boldsymbol{\rho}_{\mathrm{gc}}\right) - e\left(\Phi_1 + \boldsymbol{\rho}_{\mathrm{gc}}\cdot\nabla\Phi_1 + \cdots\right)\mathsf{d}t + \mathsf{d}\sigma_1$$

$$= \frac{e}{c}\mathbf{A}_1\cdot\mathsf{d}\mathbf{X} - e\,\Phi_1\,\mathsf{d}t + e\,\boldsymbol{\rho}_{\mathrm{gc}}\cdot\left[\mathbf{E}_1\,\mathsf{d}t + \frac{1}{c}\left(\mathsf{d}\mathbf{X} + \frac{1}{2}\mathsf{d}\boldsymbol{\rho}_{\mathrm{gc}}\right)\times\mathbf{B}_1\right]$$

where

$$\sigma_1 = -\frac{e}{c}\mathbf{A}_1\cdot\boldsymbol{\rho}_{\mathrm{gc}} - \frac{e}{2c}\boldsymbol{\rho}_{\mathrm{gc}}\cdot\nabla\mathbf{A}_1\cdot\boldsymbol{\rho}_{\mathrm{gc}}$$

⋆ Hamiltonian representation ($\Lambda_{1\mathrm{gc}} \equiv -\,H_{1\mathrm{gc}}\,\mathsf{d}t$)

$$H_{1\mathrm{gc}} = e\left(\phi_1 - \frac{1}{c}\mathbf{A}_1\cdot\frac{d_{\mathrm{gc}}\mathbf{X}}{dt}\right) - e\,\boldsymbol{\rho}_{\mathrm{gc}}\cdot\left[\mathbf{E}_1 + \frac{1}{c}\left(\frac{d_{\mathrm{gc}}\mathbf{X}}{dt} + \frac{1}{2}\frac{d_{\mathrm{gc}}\boldsymbol{\rho}_{\mathrm{gc}}}{dt}\right)\times\mathbf{B}_1\right]$$

⋆ First-order gyrocenter Hamiltonian $H_{1\mathrm{gy}} \equiv \langle H_{1\mathrm{gc}} \rangle$

$$
\begin{aligned}
H_{1\mathrm{gy}} \;&=\; e\left(\phi_1 \;-\; \frac{1}{c}\,\mathbf{A}_1 \cdot \frac{d_{\mathrm{gc}}\mathbf{X}}{dt}\right) \\
&\quad -\, e\,\langle \boldsymbol{\rho}_{\mathrm{gc}} \rangle \cdot \left(\mathbf{E}_1 \;+\; \frac{1}{c}\,\frac{d_{\mathrm{gc}}\mathbf{X}}{dt} \times \mathbf{B}_1\right) \\
&\quad -\, \frac{e}{2c}\,\left\langle \boldsymbol{\rho}_{\mathrm{gc}} \times \frac{d_{\mathrm{gc}}\boldsymbol{\rho}_{\mathrm{gc}}}{dt} \right\rangle \cdot \mathbf{B}_1
\end{aligned}
$$

∘ **Guiding-center Electric-dipole Moment**

$$
\boldsymbol{\pi}_{\mathrm{gc}} \;\equiv\; -\,\frac{\partial H_{1\mathrm{gy}}}{\partial \mathbf{E}_1} \;=\; e\,\langle \boldsymbol{\rho}_{\mathrm{gc}} \rangle
$$

∘ **Guiding-center Magnetic-dipole Moment**

$$
\boldsymbol{\mu}_{\mathrm{gc}} \;\equiv\; -\,\frac{\partial H_{1\mathrm{gy}}}{\partial \mathbf{B}_1} \;=\; \frac{e}{c}\,\langle \boldsymbol{\rho}_{\mathrm{gc}} \rangle \times \frac{d_{\mathrm{gc}}\mathbf{X}}{dt} \;+\; \frac{e}{2c}\left\langle \boldsymbol{\rho}_{\mathrm{gc}} \times \frac{d_{\mathrm{gc}}\boldsymbol{\rho}_{\mathrm{gc}}}{dt} \right\rangle
$$

- **Push-Forward Formulation**

◦ **Guiding-center Polarization**

$$\mathbf{P}_{\rm gc} \equiv \sum e \int \overline{F}\,\langle \boldsymbol{\rho}_{\rm gc}\rangle\, d^3P - \nabla\cdot\left[\sum \frac{e}{2}\int \overline{F}\,\left\langle \boldsymbol{\rho}_{\rm gc}\boldsymbol{\rho}_{\rm gc}\right\rangle\, d^3P\right] + \cdots$$

⋆ Lowest-order contribution

$$\mathbf{P}_{\rm gc}^{(1)} = \epsilon_B \sum e \int \overline{F}\left[\langle\boldsymbol{\rho}_1\rangle - \nabla\cdot\left(\frac{1}{2}\langle\boldsymbol{\rho}_0\boldsymbol{\rho}_0\rangle\right)\right] d^3P$$

$$- \epsilon_F \sum \frac{e}{2}\int \left\langle \boldsymbol{\rho}_0\,\boldsymbol{\rho}_0\right\rangle\cdot\nabla\overline{F}\, d^3P$$

⋆ Lie-transform perturbation analysis

$$\boxed{\langle\boldsymbol{\rho}_1\rangle - \nabla\cdot\left(\frac{1}{2}\left\langle\boldsymbol{\rho}_0\,\boldsymbol{\rho}_0\right\rangle\right) = \frac{\widehat{\mathsf{b}}}{\Omega}\times\left[\frac{d_1\mathbf{X}}{dt} - \frac{c}{e}\nabla\times\left(\frac{1}{2}\mu\,\widehat{\mathsf{b}}\right)\right]}$$

⋆ Guiding-center Polarization

$$\mathbf{P}_{\mathrm{gc}}^{(1)} \;\equiv\; \sum \frac{\widehat{\mathsf{b}}}{\Omega}\times\left[\int \overline{F}\left(e\,\frac{d_1\mathbf{X}}{dt}\right) d^3P \;-\; c\,\nabla\times\left(\frac{1}{2}\int \mu\,\widehat{\mathsf{b}}\,\overline{F}\,d^3P\right)\right]$$

Guiding-center polarization is often ignored in standard gyrokinetic theory.

∘ **Guiding-center Magnetization**

$$\mathbf{M}_{\mathrm{gc}} \;\equiv\; \sum \frac{e}{c}\int \overline{F}\left(\frac{1}{2}\left\langle \boldsymbol{\rho}_{\mathrm{gc}}\times\frac{d_{\mathrm{gc}}\boldsymbol{\rho}_{\mathrm{gc}}}{dt}\right\rangle \;+\; \langle\boldsymbol{\rho}_{\mathrm{gc}}\rangle\times\frac{d_{\mathrm{gc}}\mathbf{X}}{dt}\right) d^3P$$

⋆ Standard parallel guiding-center magnetization

$$\mathbf{M}_{\mathrm{gc}}^{(0)} \;\equiv\; -\left(\sum\int \overline{F}\,\mu\,d^3P\right)\widehat{\mathsf{b}}$$

LINEAR AND NONLINEAR GYROCENTER POLARIZATION

- **Gyrocenter Hamiltonian** (Mishchenko & Brizard, 2011)

$$
\begin{aligned}
H_{\rm gy} \;=\; & H_{\rm gc} + \epsilon\, e\,\langle\Phi_{1\rm gc}\rangle - \frac{\epsilon^2}{2}\, e\,\left\langle \left\{ S_1,\, \Phi_{1\rm gc} \right\}_{\rm gc} \right\rangle \\
& + \epsilon^3 \left[\frac{e}{2} \left\langle \left\{ S_1,\, \{S_1,\, \langle\Phi_{1\rm gc}\rangle\}_{\rm gc} \right\}_{\rm gc} \right\rangle \right. \\
& \qquad \left. + \frac{e}{3} \left\langle \left\{ S_1,\, \{S_1,\, \widetilde{\Phi}_{1\rm gc}\}_{\rm gc} \right\}_{\rm gc} \right\rangle \right] + \cdots
\end{aligned}
$$

- **Gyrocenter Generating Functions**

$$
\frac{d_{\rm gc}S_1}{dt} \;=\; e\,\widetilde{\Phi}_{1\rm gc} \quad\rightarrow\quad S_1 \;=\; \left(\frac{d_0}{dt}\right)^{-1} e\,\widetilde{\Phi}_{1\rm gc} \;\simeq\; \frac{e}{\Omega}\int \widetilde{\Phi}_{1\rm gc}\,d\theta
$$

$$
\frac{d_{\rm gc}S_2}{dt} \;=\; -\frac{e}{2}\left(\left\{ S_1,\, \Phi_{1\rm gc} \right\}_{\rm gc} - \left\langle \left\{ S_1,\, \Phi_{1\rm gc} \right\}_{\rm gc} \right\rangle \right) - e\,\{S_1,\, \langle\Phi_{1\rm gc}\rangle\}_{\rm gc}
$$

- **Variational Derivation**

$$\epsilon^{-1}\,\frac{\delta H_{\rm gy}}{\delta \Phi_1(\mathbf{r})} \;\equiv\; e\,\left\langle {\sf T}_{\rm gy}^{-1}\,\delta^3_{\rm gc}\right\rangle$$

$$= \; e\,\left\langle \delta^3_{\rm gc}\right\rangle \;-\; \epsilon\,e\,\left\langle \left\{ S_1,\,\delta^3_{\rm gc}\right\}_{\rm gc}\right\rangle$$

$$-\;\epsilon^2\left[e\,\left\langle \left\{ S_2,\,\delta^3_{\rm gc}\right\}_{\rm gc}\right\rangle \;-\; \frac{e}{2}\,\left\langle \left\{ S_1,\,\{S_1,\;\delta^3_{\rm gc}\}_{\rm gc}\right\}_{\rm gc}\right\rangle\right]$$

= **Guiding-center** (1st Hamiltonian)

+ **Linear gyrocenter** (2nd Hamiltonian)

+ **Nonlinear gyrocenter** (3rd Hamiltonian)

- **Linear (First-order) Displacement** (gc $\leftrightarrow$ gy)

$$\boldsymbol{\rho}_{1{\rm gy}} \;\equiv\; -\underbrace{\left\langle \left\{ S_1,\,\boldsymbol{\rho}_{\rm gc}\right\}_{\rm gc}\right\rangle}_{\epsilon,\;\epsilon_B\,\epsilon,\;\cdots} \;\rightarrow\; -\,\frac{\partial}{\partial J}\left\langle \boldsymbol{\rho}_{\rm gc}\,\frac{\partial S_1}{\partial \theta}\right\rangle \;+\;\cdots \simeq \frac{e\,\mathbf{E}_{1\perp}}{m\,\Omega^2}$$

- **Cubic Gyrocenter Hamiltonian**

$$H_{3\mathrm{gy}} \;\rightarrow\; \frac{e^3}{2\,\Omega^2}\,\frac{\partial}{\partial J}\left\langle \left(\widetilde{\Phi}_{1\mathrm{gc}}\right)^2\,\frac{\partial \Phi_{1\mathrm{gc}}}{\partial J}\right\rangle \simeq -\,\frac{e}{2}\,\mathbf{E}_{1\perp}\cdot\nabla_\perp\left(\frac{1}{2}\,|\boldsymbol{\rho}_{1\mathrm{gy}}|^2\right)$$

- **Nonlinear (Second-order) Polarization**

$$\boldsymbol{\pi}_{2\mathrm{gy}} \;=\; -\,\frac{e}{2}\left[\boldsymbol{\rho}_{1\mathrm{gy}}\left(\nabla_\perp\cdot\boldsymbol{\rho}_{1\mathrm{gy}}\right) \;-\; \boldsymbol{\rho}_{1\mathrm{gy}}\cdot\nabla_\perp\boldsymbol{\rho}_{1\mathrm{gy}}\right]$$

- **Eikonal-averaged Gyrocenter Polarization**

$$\left\langle \boldsymbol{\pi}_{2\mathrm{gy}}\right\rangle_{\Theta} \;=\; e\,\mathbf{k}_\perp\times\left(i\,\overline{\boldsymbol{\rho}}_{1\mathrm{gy}}\times\overline{\boldsymbol{\rho}}^{*}_{1\mathrm{gy}}\right) \;\propto\; m^2$$

$\rightarrow$ Compare with Oscillation-center Polarization

$$\overline{\boldsymbol{\pi}}_{2\mathrm{oc}} \;=\; e\,\mathbf{k}\times\left(i\,\overline{\boldsymbol{\xi}}_1\times\overline{\boldsymbol{\xi}}^{*}_1\right) \;\propto\; m^{-2}$$

SUMMARY

- Polarization and magnetization effects arise as a result of dynamical reduction by phase-space transformation.
- Guiding-center polarization was calculated from higher-order guiding-center Hamiltonian theory.
- Nonlinear gyrocenter polarization is homologous to oscillation-center polarization.

- Polarization effects in turbulent tokamak plasmas (Wang & Hahm, 2009)