RESISTIVE WALL MODE FEEDBACK STABILIZATION STUDIES USING A LUMPED-PARAMETER CIRCUIT EQUATION FORMULATION

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Abstract— **The role of the resistive wall mode in limiting tokamak plasma performance is well chronicled and is a central topic of the Feedback Stabilization Initiative (FSI). It is believed that stabilization of this mode, which is a converted branch of the ideal-MHD external kink mode, may lead to the design of devices capable of accessing higher performance advanced operating regimes. We have developed a formulation of the resistive wall mode, for the limiting case of infinite aspect ratio, using the elementary physical concepts of self and mutual inductance. This results in a set of coupled lumped-parameter circuit equations with the variables being the perturbed plasma current, the helical component of induced current in the resistive shell, and (with feedback) the current in the active coil. These equations, which describe plasma perturbations of $n \geq 1$, have a one to one correspondence with plasma vertical positional $n = 0$ control. Comparisons between the dispersion relations for the two cases show that the quantity that carries the strength of the instability for the resistive wall mode, equivalent to the negative decay index in vertical position control, is $L_{pl}(1 - f)$ where L_{pl} is the helical inductance of the perturbed plasma current and (1 - f) is related to the helicity of the ideal-MHD kink mode. This method has been applied successfully to describe the resistive wall mode in general terms and to describe analytically resistive wall mode feedback stabilization schemes. Formulation in this manner should facilitate numerical simulation of resistive wall mode feedback schemes. In this paper, we will describe the formulation in detail, show how the resulting circuit equations compare to the equations arrived at using traditional MHD analysis methods (particularly with the inclusion of feedback), and compare the resistive wall mode equations to those that describe the (n = 0) vertical instability.**

I. INTRODUCTION

The success achieved in controlling the n = 0 vertical instability in non-circular cross-section tokamaks has helped initiate a new era in tokamak fusion research with large tokamak devices that routinely operate in high temperature regimes. Key to the success of vertical position control was the integration of a passive stabilizing system which enabled the feedback control system to operate on a timescale much longer than the ideal-MHD time scale characteristic of the instability and maintain the plasma column at a desired vertical location.

With high temperature regimes achievable, current large-scale tokamak programs are confronting the next significant challenge, sustainment of a high β plasma near the ideal-MHD β limit. The ideal-MHD mode most often suspected of inducing high β disruption is the external kink. There are several experimental results which strongly indicate that a close-fitting passive shell (either installed conducting plates or the vacuum vessel) can reduce the growth rates of external kink modes and consequently these modes are modified into the resistive mode branch[1], [2], [3]. When the mode is converted into the resistive wall mode branch[4], [5], it can be controlled by a combination of a passive shell system and a feedback control system that operate on a slower time-scale in a manner similar to what has been successful for n = 0 vertical position control[6], [7].

Recently, several schemes for controlling the resistive wall mode have been proposed which utilize integration of both active and passive systems. One of these schemes is the intelligent shell, which was originally developed for the "thick" shell RFP devices to control locked modes caused by eddy currents near the gap of the conducting shell[8]. An alternative scheme is the fake rotating "thin" shell concept proposed for tokamaks, where the conducting wall becomes effectively thin because the period of operation is much longer than the magnetic field penetration time through the first wall or vacuum vessel[9]. This highly conducting plasma-facing component can also serve effectively as the thin shell needed in the rotating shell control concept.

Here, we present a formulation for resistive wall mode feedback control schemes utilizing concepts from electric circuit theory. Specifically we introduce the inductance matrix to describe the interactions between the plasma, passive shell, and active coil systems. In this formulation the off-diagonal terms of the inductance matrix (mutual inductance terms) are directly related to the geometry of the coupled components and should provide a means to readily evaluate the merits of proposed designs. This approach has proved useful in both the analysis and the design of power and control systems for the vertical position control scheme.

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To apply this technique to resistive wall mode stabilization, the definitions of several key elements of the feedback system must be redefined in a way that is consistent with the more complicated geometry needed to describe the resistive wall mode feedback stabilization system:

1. The resistive helical flux loss in the case of $n \geq 1$ resistive wall modes corresponds to the radial flux loss in the case of $n = 0$ vertically unstable modes.

2. The cause of the external kink mode, which in the presence of the conducting wall becomes a resistive wall mode, is related to the internal current or pressure profiles analogous to the manner in which the negative external field curvature (the magnetic decay index $n = -\frac{d}{dr}\ln(B_{ext}) < 0$, or more conveniently the second derivative of mutual inductance of the external field flux M''_{ext}) is the source term for the vertical position instability.

3. A helical current filament (or helical sheet current) must be employed as the trial function for the resistive wall mode in contrast to the rigid $n = 0$ plasma displacement represented by an axisymmetric current filament used to analyze the vertical position instability.

Results arrived at using this formulation will be useful to physicists and engineers tasked with the design of a system for controlling resistive wall modes who must deal with constraints imposed both by reasonable machine and control system design practices and the current state of the art. To facilitate the analysis the large aspect ratio limit is used in the formulation.

II. DERIVATION OF THE CIRCUIT EQUATIONS

The resistive wall mode has been studied by various authors[8], [11], [12]. The formalism used in analysis of the resistive tearing mode is also typically used to analyze the resistive wall mode as the eddy current on the passive shell, in the thin shell approximation, plays a role similar to Δ', the flux jump, in the tearing mode analysis.

Here the feedback control scheme is formulated from an electrical engineering point of view to describe explicitly the magnetic flux and its relationship to the active feedback coils and passive shell outside the plasma boundary. We describe the resistive wall mode in a cylindrical geometry in the infinite aspect ratio limit.

To derive the equation for the plasma perturbation we use the θ-component of the Momentum equation, the θ- and r-components of Faraday's Law, and an assumption of incompressibility[10]. Also, we will assume a uniform plasma current density profile throughout (for a more general derivation see [13]). Assuming that our solutions are of the form $e^{\gamma t} e^{i(m\theta - n\phi)}$, the θ-component of the linearized Momentum equation can be expressed as

$$\gamma \rho_0 v_\theta = -im\frac{X}{r} + iFB_\theta + J_z^0 B_r, \qquad (1)$$

where $X = p + \mathbf{B_0} \cdot \mathbf{B}$ is the perturbed pressure, $J_z^0(r) = (1/r)(\partial/\partial r)(rB_\theta^0)$ is the equilibrium axial current, and $F = (B_\theta^0/r)(m - nq)$ (note that unperturbed quantities carry the superscript "0"). The r- and θ-components of Faraday's Law (linearized) are:

$$\gamma B_r = iFv_r, \qquad (2)$$

and

$$\gamma B_\theta = ikB_z^0 v_\theta - ikB_\theta^0 v_z - \frac{\partial}{\partial r} v_r B_\theta^0 \qquad (3)$$

respectively. Finally incompressibility allows us to write

$$\frac{1}{r}\frac{\partial}{\partial r} r v_r + i\frac{m}{r} v_\theta + ikv_z = 0. \qquad (4)$$

Using 2, 3, and 4 to eliminate B_r, and B_θ from 1 we get

$$g\xi_\theta = -im\frac{X}{r} + 2\frac{B_\theta^0}{r} iF\xi_r, \qquad (5)$$

where $\vec{\xi} \equiv v/\gamma$, and $g \equiv \gamma^2\rho_0 + F^2$.

At the unperturbed plasma-vacuum interface, $r = a$,

$$X(a_+) = (\mathbf{B_0} \cdot \mathbf{B})^{vac} = F(a)\frac{a}{m} B_\theta^{vac}(a), \qquad (6)$$

where we have used $\hat{r} \cdot \nabla \times \mathbf{B}^{vac} = 0$ to write $B_z^{vac} = (kr/m)B_\theta^{vac}$. At the boundary $X(a_+) = X(a_-)$ so we can write 5 as

$$g\xi_\theta(a) = -iF_a B_\theta^{vac}(a) + 2\frac{B_\theta^0(a)}{a} iF_a \xi_r(a). \qquad (7)$$

The radial component of $\mathbf{B}$ is also continuous at $r = a$ which allows us to write $B_r^{vac}(a) = iF_a\xi_r(a)$. Substituting this into 7 gives

$$g\frac{\xi_\theta(a)}{\xi_r(a)} = F_a^2 \frac{B_\theta^{vac}(a)}{B_r^{vac}(a)} + 2B_\theta^0(a) iF_a. \qquad (8)$$

Because we are assuming a uniform current density equation ξ_θ/ξ_r can be replaced by i. In addition, by introducing $\tau_A = (\rho_0^{1/2} a)/B_\theta^0(a)$ (the edge poloidal Alfvén time) and also using $\mathbf{B} = \nabla \times \psi\hat{z}/(2\pi R_0)$, 8 simplifies to

$$(\gamma^2\tau_A^2 + f^2) = f^2\frac{a\psi'(a_+)}{m\psi(a_+)} + 2f, \qquad (9)$$

where $f \equiv (m - nq(a))$. Equation 9 is the form we will use in the circuit equation formulation.

The perturbed helical flux at $r = r_i$ is produced by perturbed helical currents in the plasma, passive shell eddy currents, and the active (feedback) coil current. In what follows we will consider each of these components as a current carrying circuit. The current path corresponding to the perturbed plasma current we denote as circuit "1" and the passive shell and active feedback coil circuits are denoted as circuits "2" and "3" respectively. The vacuum poloidal flux at r_i can be written as

$$\psi(r_i) = L_i I_i + M_{ij} I_j. \qquad (10)$$

The I_j are the currents in each circuit with the L_i and M_{ij} being the self and mutual inductances respectively. In general the inductance terms may be obtained numerically once the geometry is fixed and the perturbed plasma current path is known.

We can write an equation for the poloidal flux at the plasma boundary, $r = a_+$, using 10:

$$\psi(a_+) = L_1 I_1 + M_{12} I_2 + M_{13} I_3 \tag{11}$$

where L_1 is the self inductance of the perturbed plasma current "circuit" and the M_{1j} are the mutual inductances between the perturbed plasma current circuit and the passive shell and the active coil system. Taking the spatial derivative of 11 we obtain:

$$\psi'(a_+) = L_1' I_1 + M_{12}' I_2 + M_{13}' I_3. \tag{12}$$

Now substituting both 11 and 12 into 9 we get a circuit equation for the perturbed plasma current circuit

$$(\gamma^2 \tau_A^2 + f^2) - f^2 \frac{a}{m} \frac{L_1' I_1 + M_{12}' I_2 + M_{13}' I_3}{L_1 I_1 + M_{12} I_2 + M_{13} I_3} - 2f = 0. \tag{13}$$

We can write an equation for the flux at the passive shell similar to 11.

$$\psi(r_w) = M_{21} I_1 + L_2 I_2 + M_{23} I_3 \tag{14}$$

Differentiating 14 with respect to time and applying both Faraday's and Ohm's law at the wall to get $\partial\psi(r_w)/\partial t = -R_2 I_2$ we can write

$$\gamma M_{21} I_1 + (\gamma L_2 + R_2) I_2 + \gamma M_{23} I_3 = 0 \tag{15}$$

as the circuit equation for the passive shell. Similarly for the active coil, with the inclusion of a feedback voltage term, V_3, we get

$$\gamma M_{31} I_1 + \gamma M_{32} I_2 + (\gamma L_3 + R_3) I_3 = V_3 \tag{16}$$

where the form of V_3 depends of the details of the feedback.

III. DISPERSION RELATIONS

Equations 13, 15, and 16 can be used to derive dispersion relationships for various resistive wall mode feedback control schemes. First, consider a plasma that is not surrounded by a passive shell system and without active feedback system coils. In this instance we are left with 13 with both I_2 and I_3 set to zero. Inspection of the resulting equation reveals the usual dispersion relation for ideal external kinks:

$$\gamma_\infty^2 \tau_A^2 = 2f(1-f). \tag{17}$$

To complete the system we add a passive shell system at radius $r = r_2$ where r_2 is less than the critical radius where a perfectly conducting shell would stabilize ideal external kinks over the operating region considered (determined by the value of f in this model) and an active coil system at $r = r_3$ ($a < r_2 < r_3$). We note that $\gamma\tau_A << 1$ and that in the resistive wall mode limit we can drop $\gamma^2\tau_A^2$ in 13. In this limit 13 becomes

$$(\gamma_\infty^2 \tau_A^2) L_1 I_1 + (\gamma_\infty^2 \tau_A^2 + 2f^2)(M_{12} I_2 + M_{13} I_3) = 0. \tag{18}$$

Denoting the L/R times of the passive and active systems as τ_2 and τ_3 respectively we can rewrite 15 and 16 as

$$(\gamma\tau_2) M_{21} I_1 + (\gamma\tau_2 + 1) L_2 I_2 + (\gamma\tau_2) M_{23} I_3 = 0 \tag{19}$$

and

$$(\gamma\tau_3) M_{31} I_1 + (\gamma\tau_3) M_{32} I_2 + (\gamma\tau_3 + 1) L_3 I_3 = V_3 \tau_3 \tag{20}$$

respectively.

Next consider the case of a plasma surround by a passive shell. The equations of interest are 18 and 19 with terms having subscript 3 eliminated. If we solve these equations for $\gamma\tau_2$ and use $\gamma_\infty^2\tau_A^2 = 2f(1-f)$ we obtain

$$\gamma\tau_2 = -\frac{1}{1 - M_{12}^2/(L_1 L_2 (1-f))}. \tag{21}$$

It is convenient at this point to give explicit formulas for the self and mutual inductances and their derivatives for the limiting case of infinite aspect ratio. In this case the inductances are trivially given by:

$$\begin{aligned} L_i &= R_0, \\ M_{ij} &= R_0 (r_i/r_j)^m \qquad r_i < r_j, \text{ and} \\ &= R_0 (r_j/r_i)^m \qquad r_i > r_j. \end{aligned} \tag{22}$$

The formulas relating the self and mutual inductances to their spatial derivatives are also quite simple:

$$\begin{aligned} (r_i L_i')/(m L_i) &= -1, \text{ and} \\ (r_i M_{ij}')/(m M_{ij}) &= +1. \end{aligned} \tag{23}$$

When these substitutions are made, the quantity $M_{12}^2/(L_1 L_2(1-f))$ in 21 simplifies to $(a/r_2)^{2m}/(1-f)$ and the dispersion relation becomes

$$\gamma\tau_2 = -\frac{1}{1 - (a/r_2)^{2m}/(1-f)} \tag{24}$$

which is the familiar dispersion relation for the resistive wall mode.

It is interesting to compare 24 to the dispersion relation for the vertical positional instability of a filamentary plasma

$$\gamma\tau_2^{n=0} = -\frac{1}{1 - M_{12}'^2/L_2 M_{ext}''}. \tag{25}$$

Comparing 21 and 25 we see that whereas M_{ext}'' carries the strength of the vertical positional instability $L_1(1-f)$ plays that role in the resistive wall mode. Similarly the necessary condition for vertical stability, $L_2 M_{ext}'' - M_{12}'^2 < 0$, becomes $L_1 L_2 (1-f) - M_{12}^2$ in the resistive wall mode case. We see that the resistive wall mode can be thought of as a $n \geq 1$ version of the vertical positional instability.

IV. DISCUSSION

To put the equations in a form more suitable for simulation we differentiate 18 with respect to time to obtain.

$$\gamma L_1(1-f)I_1 + \gamma M_{12}I_2 + \gamma M_{13}I_3 = 0. \tag{26}$$

Equations 26, 15, and 16 comprise a complete set of circuit equations in a form suitable for simulation of the resistive wall mode. In matrix form we write

$$\mathbf{M}\dot{\mathbf{I}} + \mathbf{RI} = \mathbf{V} \tag{27}$$

with

$$\mathbf{M} = \begin{bmatrix} (1-f)L_1 & M_{12} & M_{13} \\ M_{21} & L_2 & M_{23} \\ M_{31} & M_{32} & L_3 \end{bmatrix}, \tag{28}$$

$$\mathbf{R} = \begin{bmatrix} 0 \\ R_2 \\ R_3 \end{bmatrix}, \tag{29}$$

and

$$\mathbf{V} = \begin{bmatrix} 0 \\ 0 \\ V_3 \end{bmatrix}. \tag{30}$$

We first note that in the equations that the L_1 term is multiplied by the quantity $(1-f)$ and we say that the effective inductance of the plasma is $L_1(1-f)$.

It is the term V_3 that produces the feedback action in the resistive wall mode feedback schemes. For example, in what is called "intelligent shell"[8] feedback we would replace the term V_3 by the quantity $G_{is}I_2$ where G_{is} is a gain applied to a measurement of the passive shell current to be applied in a manner so as to reduce the shell current to zero. In what is termed "fake rotation"[9] feedback V_3 is replaced by a quantity proportional to the flux measurement at radial position "0". In this scheme we replace V_3 by $imG_{fr}(M_{10}I_1 + M_{20}I_2 + M_{30}I_3)$.

It is clear from the above that this formulation results in equations suitable for numerical simulation studies of resistive wall mode feedback schemes. The simulations themselves should be straightforward once the particular geometry, operating regime, and feedback scheme are chosen. It is also apparent from the form of the equations that a hardware simulator could be constructed to allow realtime simulations of resistive wall mode stabilization schemes without an operating tokamak experiment. The main difficulty would be simulating in hardware the reduced plasma inductance $L_1(1-f)$. It is believed that this could be achieved by using a feedback system to produce a negative inductance in the hardware simulator.

V. SUMMARY

We have presented a formulation of the resistive wall mode instability that results in a set of linear lumped-parameter circuit equations. The set of equations yield dispersion relations identical to those derived using different means for the cases of the ideal external kink and the resistive wall mode without feedback. The set of equations are in form suitable for numerical simulation of resistive wall mode feedback schemes. The form of these equations should also facilitate resolution of engineering issues in an actual system design. These results also indicate that it should be possible to study resistive wall mode feedback stabilization schemes using a hardware simulator with $L_1(1-f)$ representing the plasma inductance and the strength of the kink mode. Further details will be presented in [13].

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