Determination of three-dimensional equilibria from flux surface knowledge only

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It is shown that the method of Christiansen and Taylor, from which complete tokamak equilibria can be determined given only knowledge of the shape of the flux surfaces, can be extended to three-dimensional equilibria, such as those of stellarators. As for the tokamak case, the given geometric knowledge has a high degree of redundancy, so that the full equilibrium can be obtained using only a small portion of that information.
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Christiansen and Taylor (CT) have shown[1] that complete magnetohydrodynamic (MHD) equilibria may be obtained for axisymmetric tokamaks with noncircular cross-sections, provided that one initially knows only the shapes of the flux surfaces, described by some flux surface label $\rho(\mathbf{x})$ over real-space $\mathbf{x}$. The demonstration is accomplished by taking appropriate flux–surface averages of the Grad–Shafranov (GS) equation[2,3] to obtain a simple radial second order ordinary differential equation (ODE) for the poloidal flux ψ as a function of ρ. This can be solved analytically, and, given $\psi(\rho)$, one can obtain expressions for the pressure gradient profile $p' \equiv dp/d\rho$ and poloidal current profile F which appear in the GS equation. Since $\rho(\mathbf{x})$ can be inferred from measurements of physical quantities which are also approximately flux functions (such as density, temperature, or pressure) the method has the potential to be an important diagnostic, and efforts have been made to apply this method to measuring the current and q profiles on the Joint European Torus (JET),[4] Alcator C-Mod,[5] and PEGASUS.[6]

It is natural to consider whether such a method also exists for three-dimensional (3D) toroidal equilibria such as stellarators. In this paper, we demonstrate that this is the case. The GS equation is an elliptic partial differential equation (PDE) in two dimensions, usually parametrized by the distance R from the major axis, and the vertical height Z above the midplane, independent of the geometrical toroidal azimuth ζ_g about that axis. Since the derivation of the GS equation makes use of axisymmetry, it is unclear that the method will generalize. However, Degtyarev *et al.*[7] have shown, through insightful choices of flux coordinate systems, that a 3D generalization of the GS equation for $\psi(\mathbf{x})$ exists, which we shall refer to as the 3D-GS equation. Here we show that this more complicated equation may also be subjected to a procedure like that in Ref. 1 to obtain a radial ODE for $\psi(\rho)$ of the same form as in the 2D case, but with more complicated coefficients. CT have pointed out that their procedure for tokamaks is more robust the more highly shaped the tokamak cross-section. One might conjecture that, because of the strong poloidal and toroidal shaping of typical stellarators, the CT procedure would in fact be more suited to stellarators than to tokamaks.

We briefly review the origin of the 3D-GS equation.[7] In a general flux coordinate system $\{q^i\} \equiv \{\rho,\theta,\zeta\}$ (for $i = 1,2,3$) parametrizing a torus, with poloidal angle θ, toroidal angle ζ, and flux-surface label ρ already introduced, one may represent the magnetic field in both the contravariant (Clebsch) representation,

$$\mathbf{B} = \nabla\psi\times\nabla\zeta + \nabla\Phi\times\nabla\theta = \mathbf{e}_\theta B^\theta + \mathbf{e}_\zeta B^\zeta, \tag{1}$$

and in the covariant representation

$$\mathbf{B} = J\nabla\theta + F\nabla\zeta - \nu\nabla\rho + \nabla\phi. \tag{2}$$

Here, the $\mathbf{e}_i \equiv \mathcal{J}\mathbf{e}^j\times\mathbf{e}^k$ are the contravariant basis vectors, where i, j, and k are cyclic. These are reciprocal to the covariant set $\mathbf{e}^i$, usually taken equal to ∇q^i, in which case $\mathbf{e}_i = \partial\mathbf{x}/\partial q^i$. $\mathcal{J} \equiv (\mathbf{e}^1\cdot\mathbf{e}^2\times\mathbf{e}^3)^{-1} = \mathbf{e}_1\cdot\mathbf{e}_2\times\mathbf{e}_3$ is the Jacobian. We assume good flux surfaces throughout the plasma. $\Phi(\rho)$ and $\psi(\rho)$ are, respectively, $1/2\pi$ times the toroidal magnetic flux inside, and the poloidal flux outside, flux surface ρ, and $J(\rho)$ and $F(\rho)$ are, respectively, $\mu_0/2\pi$ times the toroidal current inside, and the poloidal current outside, flux surface ρ. Using Eq. (2) in Ampère's law, one obtains the contravariant representation of the current,

$$\mu_0\mathbf{J} = \nabla\times\mathbf{B} = -(F' + \partial_\zeta\nu)\nabla\zeta\times\nabla\rho + (J' + \partial_\theta\nu)\nabla\rho\times\nabla\theta. \tag{3}$$

Equations (1) and (3) automatically satisfy $\nabla\cdot\mathbf{B} = 0$ and the steady-state condition $\nabla\cdot\mathbf{J} = 0$, as well as the equilibrium conditions $B^\rho \equiv \nabla\rho\cdot\mathbf{B} = 0$ and $J^\rho \equiv \nabla\rho\cdot\mathbf{J} = 0$ arising from dotting $\mathbf{B}$ and $\mathbf{J}$ into the force-balance equation $p'\nabla\rho = \mathbf{J}\times\mathbf{B}$. The final equilibrium condition is the radial component of this,

$$|\nabla\rho|^2 p' = \nabla\rho\cdot\mathbf{J}\times\mathbf{B}. \tag{4}$$

The standard (2D) GS equation uses neither the co- nor the contra-variant representations of $\mathbf{B}$ and $\mathbf{J}$ in Eq. (4), but the "mixed" representation

$$\mathbf{B} = \nabla\psi\times\mathbf{b} + \mathbf{b}F, \quad \mu_0\mathbf{J} = -\mathbf{b}\Delta^*\psi + \nabla F\times\mathbf{b}, \tag{5}$$

where $\mathbf{b} \equiv \nabla\zeta_g = \hat{\zeta}/R$, and $\mathbf{b}^2\Delta^*\psi \equiv \nabla\cdot(\mathbf{b}^2\nabla\psi)$. Using Eqs. (5) in Eq. (4), one obtains

$$\mathbf{b}^2\Delta^*\psi = -\mu_0 p'/\psi' - \mathbf{b}^2 FF'/\psi'. \quad (6)$$

Axisymmetry has been used in obtaining the simple forms for $\mathbf{J}$ and Eq. (6). Specializing ρ to ψ here so that $\psi'=1$, and noting that $\mathbf{b}^2=R^{-2}$ yields the GS equation.

In the fully 3D problem, Degtyarev *et al.*[7] have shown that a mixed representation may again be given, making use of two special flux coordinate systems, the "natural" and "conatural" systems (ρ,θ_n,ζ_n) and (ρ,θ_c,ζ_c), respectively, which become the same system in the 2D case. Demanding $\nabla\rho\cdot\nabla\times\mathbf{B}=0$ from representation Eq. (1) for $\mathbf{B}$, and further that this condition hold independent of the rotational transform $\iota\equiv-\psi'/\Phi'$ results in equations determining the angles for the natural coordinate system:

$$\nabla\cdot[\nabla\rho\times(\nabla\theta_n\times\nabla\rho)]=0, \quad \nabla\cdot[\nabla\rho\times(\nabla\zeta_n\times\nabla\rho)]=0, \quad (7)$$

and demanding $\nabla\cdot\mathbf{B}=0$ from representation Eq. (2) for the conatural system, and further that this hold independent of the ratio F/J results in similar conditions determining the angles for the conatural system:

$$\nabla\cdot[\nabla\rho\times(\nabla\theta_c\times\nabla\rho)/|\nabla\rho|^2]=0,$$
$$\nabla\cdot[\nabla\rho\times(\nabla\zeta_c\times\nabla\rho)/|\nabla\rho|^2]=0. \quad (8)$$

Each of Eqs. (7) and (8) have no radial derivatives, and so are 2D PDEs over a flux surface. Starting only with $\rho(\mathbf{x})$, one uses Eqs. (7) and (8) to obtain the full natural and conatural coordinate sets. Given these, the generalized mixed representation for $\mathbf{B}$ is shown in Ref. 7 to be

$$\mathbf{B}=\nabla\psi\times\mathbf{b}_c+\mathbf{b}_n F, \quad (9)$$

where $\mathbf{b}_c\equiv\mathbf{e}_{c3}/(\mathbf{e}_{c3}\cdot\mathbf{e}_{n3})$ and $\mathbf{b}_n\equiv\mathbf{e}_{n3}/(\mathbf{e}_{c3}\cdot\mathbf{e}_{n3})$. The current is then given by

$$\mu_0\mathbf{J}=\nabla\times(\nabla\psi\times\mathbf{b}_c)+\nabla F\times\mathbf{b}_n+F\nabla\times\mathbf{b}_n, \quad (10)$$

where the final term vanishes in the 2D case. Using Eqs. (9) and (10) in (4) yields the 3D-GS equation,

$$\mathbf{b}_c^2\Delta_c^*\psi=-\mu_0 p'/\psi'-\mathbf{b}_n^2 FF'/\psi'+F\mathbf{b}_c\cdot\nabla\times\mathbf{b}_n-F'\mathbf{b}_c\cdot(\mathbf{b}_n\times\nabla\rho)-(F/\psi'|\nabla\rho|^2)(\mathbf{b}_n\times\nabla\rho)\cdot\nabla\times(\mathbf{b}_c\times\nabla\psi)+(F^2/\psi'|\nabla\rho|^2)(\mathbf{b}_n\times\nabla\rho)\cdot(\nabla\times\mathbf{b}_n), \quad (11)$$

where the operator Δ_c^* generalizes Δ^* in the GS equation: $\mathbf{b}_c^2\Delta_c^*\psi\equiv-\mathbf{b}_c\cdot\nabla\times(\nabla\psi\times\mathbf{b}_c)=\nabla\cdot(\mathbf{b}_c^2\nabla\psi)-\nabla\psi\cdot\mathbf{b}_c\times(\nabla\times\mathbf{b}_c)$. Of the six terms on the right side of Eq. (11), all but the first two vanish in the axisymmetric case (6), as does the second term in the last form given for $\mathbf{b}_c^2\Delta_c^*\psi$.

We now show that Eq. (11) has a form amenable to the analysis to which CT subjected Eq. (6) in Ref. 1. Using ρ as the radial variable, we write each of the terms in $\psi'\times$ Eq. (11) as the product of some combination of the physics-related profile functions $\psi(\rho), p(\psi)$ and $F(\rho)$ and their derivatives, times a geometric coefficient $[A(\mathbf{x}),C(\mathbf{x}),D_{i=0-4}(\mathbf{x})]$ which varies over a flux surface:

$$\psi'\psi''A+\psi'^2C=-p'D_0-FF'D_1-F\psi'D_2-(F\psi')'D_3-F^2D_4, \quad (12)$$

where $A\equiv\mathbf{b}_c^2|\nabla\rho|^2$, $C\equiv\mathbf{b}_c^2\Delta_c^*\rho$, $D_0\equiv\mu_0$, $D_1\equiv\mathbf{b}_n^2$, $D_2\equiv[-\mathbf{b}_c\cdot\nabla\times\mathbf{b}_n+(\mathbf{b}_n\times\nabla\rho)\cdot\nabla\times(\mathbf{b}_c\times\nabla\rho)/|\nabla\rho|^2]$, $D_3\equiv\mathbf{b}_c\cdot(\mathbf{b}_n\times\nabla\rho)$, and $D_4\equiv-(\mathbf{b}_n\times\nabla\rho)\cdot(\nabla\times\mathbf{b}_n)/|\nabla\rho|^2$. Only the coefficients A, C, D_0 and D_1 are nonvanishing in the 2D case.

As in Ref. 1, we note that Eq. (12) has an immense amount of redundancy: an infinite number of coupled ODEs in ρ may be generated from it by taking different flux surface averages. For example, for any "test functions" $h_i(\mathbf{x})$, $(i=0-4)$ with vanishing flux surface average, $\langle h_i\rangle=0$, taking $\langle(12)\times h_i/D_i\rangle$ yields a 1D equation of the form of Eq. (12), but with the term in D_i annihilated. Generating such an ODE for each of the five D_i, one obtains a set of five linear equations in the seven "unknowns" $\{\psi'\psi'',\psi'^2;p',FF',F\psi',(F\psi')',F^2\}$. Thus, by taking linear combinations, one can eliminate the last five of these, and obtain an equation of the same form as found in Ref. 1,

$$\psi'\psi''A_T+\psi'^2C_T=0, \quad (13)$$

which is easily solved for $\psi(\rho)$. Calling the coefficient of the ith equation α_i, $(i=0-4)$, Eq. (13) is thus obtained by the flux-surface average $\langle(12)\times h_T\rangle$, with $h_T(\mathbf{x})\equiv\Sigma_{i=0}^4\alpha_i h_i(\mathbf{x})/D_i(\mathbf{x})$. Given ψ, one may obtain any of the other flux function unknowns, and thus p' and F, through other combinations of the five ODEs. A slightly different approach is to make contact with the starting point of CT for this part of the analysis, the GS equation written with ρ as the radial variable. As noted above, this is just Eq. (12) with vanishing D_2, D_3, and D_4. Using only the last three $(i=2,3,4)$ of the five averaged equations above, one can straightforwardly eliminate the unknowns $F\psi',(F\psi')'$, and F^2 from Eq. (12), obtaining

$$\psi'\psi''\tilde{\tilde{A}}+\psi'^2\tilde{\tilde{C}}=-p'\tilde{\tilde{D}}_0-FF'\tilde{\tilde{D}}_1, \quad (14)$$

where all four coefficients $\tilde{\tilde{X}}$ here are given by $\tilde{\tilde{X}}\equiv\tilde{X}-\langle\tilde{X}h_1/\tilde{D}_1\rangle\tilde{D}_4/\langle\tilde{D}_4h_1/\tilde{D}_1\rangle$, and $\tilde{X}\equiv X-\langle Xh_2/D_2\rangle D_3/\langle D_3h_2/D_2\rangle-\langle Xh_3/D_3\rangle D_2/\langle D_2h_3/D_3\rangle$. Equation (14) is of the same form as the GS equation, but with the replacements $\mathbf{b}^2|\nabla\rho|^2\to\tilde{\tilde{A}}$, $\mathbf{b}^2\Delta^*\rho\to\tilde{\tilde{C}}$, $\mu_0\to\tilde{\tilde{D}}_0$, and $\mathbf{b}^2\to\tilde{\tilde{D}}_1$. Thus, the same expressions given in Ref. 1 for ψ,p' and FF' apply here as well, with these replacements.

Summarizing, we have shown that the 2D result of Ref. 1, that knowing only the shape $\rho(\mathbf{x})$ of the flux surfaces in a toroidal MHD equilibrium is sufficient to determine the full equilibrium, can be extended to 3D equilibria, such as those of stellarators. This is achieved by building on the results of Ref. 7, which showed that a 3D analog for the GS equation exists, and by demonstrating that this 3D-GS equation retains the needed properties for the CT method to be applied. As noted in Ref. 1, the equilibrium equation has a great deal of redundancy, reflected in the great flexibility in the choice of the test functions $h_i(\mathbf{x})$. These may be chosen to be appreciable everywhere over each flux surface, or highly localized, depending, for example, on what type of data one has available to determine $\rho(\mathbf{x})$. The fact that $\rho(\mathbf{x})$ describes a 3D equilibrium assures that any choice will yield the same result. However, if the precision with which this information is known is limited, as will be the case if $\rho(\mathbf{x})$ is measured experimentally, a corresponding spread in the results for ψ, p' and FF' will arise for different choices of h_i. Study of this, and the practicality of the CT approach to profile determination in stellarators, are left to future work.

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