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Canonical (Feynman Diagram) versus Non-Canonical (Stokes Expansion) Calculation of Resonant Interaction Between Surface Waves

Frank S. Henyey and Neil Pomphrey

Center For Studies of Nonlinear Dynamics

La Jolla Institute

P.O. Box 1434

La Jolla, Ca 92038

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Abstract

The resonant interaction of four surface gravity waves is calculated as a sum of seven "Feynman" diagrams. These diagrams are evaluated in a symmetric way to obtain an effective Hamiltonian. Longuet-Higgins has used a (non-canonical) Stokes expansion to calculate a special case of this resonant interaction. We calculate the same using the canonical description and obtain identical results. This illustrates a general principle, that the lowest order resonant interaction in any wave system is independent of representation, being the same for Eulerian, Lagrangian, canonical, non-canonical, etc., descriptions.

1. Introduction

A Hamiltonian description of the dynamics of a wavefield - specifically the gravity surface waves on a fluid for this paper - allows certain simplifications to be made in performing calculations. A particular simplification interests us in this paper, namely that perturbation theory can be organized as a sum of expressions obtained from diagrams which, when used for relativistic wave theories, are known as Feynman diagrams.

Feynman diagrams were introduced into oceanography by Hasselmann (1966), who emphasized their application to the statistical description of energy transport in random wavefields. In this paper, we use Feynman diagrams in the way they were originally intended, to solve strictly deterministic equations of motion. For this purpose, Hasselmann's rules are not sufficiently general; although a Hamiltonian was required to justify the diagrams; equations of motion were used to evaluate coefficients. As a result of not using canonical coordinates for the equations of motion, Hasselmann's diagrams do not exhibit the full symmetry which is possible for expansions which use these variables. In a non-dissipative system, the effects of an interaction on the time evolution of the various involved waves are related. The Feynman diagrams exploit this relation by having the same value no matter which of the involved waves is considered.

In Section 2, we present the Feynman rules in the general, symmetric form. We do not prove the rules, but merely state them with a heuristic indication of their origin. Their use is illustrated on a trivially soluble problem.

In Section 3, we apply the diagrams to compute the lowest order resonant interaction among surface gravity waves in terms of canonical field variables.

These variables are the surface displacement and velocity potential at the free surface (Broer (1974), Watson, et. al. (1976), Miles (1977), Milder (1977)).

Finally, in Section 4, the interaction is shown to be identical with a calculation of Longuet-Higgins (1962) which uses a non-canonical Stokes expansion. This application is of interest in itself: It demonstrates the general principle that the lowest order resonant interaction of any calculation is independent of representation since it can be given a definition in terms of experimental quantities. By contrast, non-resonant nonlinear interactions are representation dependent since the representations differ in nonlinear terms, even when they are made to agree in the linear approximation.

2. Rules for Drawing and Evaluating Feynman Diagrams

A Hamiltonian system is described by pairs of canonical variables p_j, q_j, a Hamiltonian function $H(p_j, q_j)$, and equations of motion

$$\frac{\partial p_j}{\partial t} = -\frac{\partial H}{\partial q_j} \quad , \qquad \frac{\partial q_j}{\partial t} = \frac{\partial H}{\partial p_j} \quad . \tag{1}$$

(For fluids, the index j represents spatial position as well as distinguishing different types of variables). If perturbation theory is meaningful, H must be an analytic function of its variables. It can, therefore, be expanded in a power series. The constant term of the series can be removed be re-defining the zero-point of energy. The linear terms can be removed by adding constants to the p's and q's to make $p = 0$, $q = 0$ the minimum energy configuration (assuming the energy achieves its greatest lower bound).

The lowest order nonremovable terms are the quadratic terms. In the usual case, the quadratic part of the Hamiltonian, H_2, is the sum of a positive quadratic form in the p's and a non-negative quadratic form in the q's. Then, by a linear coordinate transformation, H_2 can be put in normal form (Margenau and Murphy (1956), p. 326).

$$H_2 = \frac{1}{2}\sum_j (p_j^{\,2} + \omega_j^{\,2} q_j^{\,2}) \quad . \tag{2}$$

(For the more general case, see, e.g., Arnold (1978), appendix 6).

We define a dissipation - free wave system by the following requirements:

1) There exists a Hamiltonian H which is a non-negative analytic function of the p's and q's satisfying Hamilton's equations. The quadratic part, H_2, can be cast in the form of Eq. 2 by a canonical transformation.

2) The only zero values of ω_j occur as isolated points in the spectrum.

3) The p's and q's are functions of position which evolve in time, but there is no explicit space or time dependence in the Hamiltonian. (This condition can be relaxed at the expense of complicating the diagrams).

From Eq. 2, action amplitudes can be defined by

$$a_j = (\omega_j/2)^{1/2} q_j + i(2\omega_j)^{-1/2} p_j \tag{3}$$

so that

$$H = \sum_j \omega_j a_j^* a_j + \text{higher order terms} \quad , \tag{4}$$

where * denotes complex conjugation. The variables a_j and ia_j^* are canonically conjugate.

Since H has no explicit space dependence, the Fourier modes provide the a's which give the eigenfrequencies. The expansion for H is, therefore,

$$H = \int d\underset{\sim}{k}\, \omega_{\underset{\sim}{k}}\, a^*_{\underset{\sim}{k}}\, a_{\underset{\sim}{k}} + \sum_{\alpha=0}^{\infty} \sum_{\substack{\beta=0 \\ \alpha+\beta \geq 3}}^{\infty} \frac{1}{\alpha!\,\beta!} \int \prod_{\alpha'=1}^{\alpha} d\underset{\sim}{k}_{\alpha'}\, a_{\underset{\sim}{k}_{\alpha'}} \prod_{\beta'=1}^{\beta} d\underset{\sim}{k}'_{\beta'}\, a^*_{\underset{\sim}{k}'_{\beta'}}$$

$$\times \qquad V^{\underset{\sim}{k}'_1, \ldots, \underset{\sim}{k}'_\beta}_{\underset{\sim}{k}_1, \ldots, \underset{\sim}{k}_\alpha}\, \delta\left(\sum_{\alpha'=1}^{\alpha} \underset{\sim}{k}_{\alpha'} - \sum_{\beta'=1}^{\beta} \underset{\sim}{k}'_{\beta'}\right) \quad . \tag{5}$$

In this expression, the delta function comes from the spatial integral of $e^{ik.x}$ factors. By construction, the coupling coefficients, V, have various symmetries:

1) V is chosen to be completely symmetric under interchange of its subscripts.

2) V is chosen to be completely symmetric under interchange of its superscripts.

3) Since H is real, if the set of superscripts is interchanged with the set of subscripts, V becomes equal to its complex conjugate. E.g.,

$$V^{\underset{\sim}{k}_3, \underset{\sim}{k}_4}_{\underset{\sim}{k}_1, \underset{\sim}{k}_2} = \left(V^{\underset{\sim}{k}_1, \underset{\sim}{k}_2}_{\underset{\sim}{k}_3, \underset{\sim}{k}_4} \right)^* \tag{6}$$

In addition, for specific theories V has other symmetries. Thus, $V_{\underset{\sim}{k}_1, \underset{\sim}{k}_2, \underset{\sim}{k}_3}$ and $V^{-\underset{\sim}{k}_3}_{\underset{\sim}{k}_1, \underset{\sim}{k}_2}$ might be equal. In relativistic quantum field theories, such additional symmetries (known as crossing relations) follow from general principles, but for nonrelativistic theories they are "accidental." An example of such an accident would be the absence of any p's in the cubic terms of H(p,q).

A convenient way of doing perturbation theory on the Hamiltonian Eq. 5 is to use Feynman diagrams. To introduce this method, retain just one cubic term in Eq. 5 so that the equation of motion for amplitude a_k is

$$\frac{da_{\underset{\sim}{k}}}{dt} = -i \frac{\partial H}{\partial a^*_{\underset{\sim}{k}}}$$

$$= -i\omega_{\underset{\sim}{k}} a_{\underset{\sim}{k}} - i V^{\underset{\sim}{k}}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2} a_{\underset{\sim}{k}_1} a_{\underset{\sim}{k}_2} \delta_{\underset{\sim}{k}_1+\underset{\sim}{k}_2-\underset{\sim}{k}} \quad . \tag{7}$$

First order perturbation theory proceeds as follows: Write $a_{\underset{\sim}{k}}$ as an unperturbed part, $a^{(0)}_{\underset{\sim}{k}}$, satisfying

$$\frac{da^{(0)}_{\underset{\sim}{k}}}{dt} = -i\omega_{\underset{\sim}{k}} a^{(0)}_{\underset{\sim}{k}} \quad , \tag{8}$$

plus a perturbation, $\delta a_{\underset{\sim}{k}}$. Fourier transform Eq. 7 with respect to time, with frequency ω, obtaining (in lowest order)

$$-i\omega\, \delta a_{\underset{\sim}{k}}(\omega) = -i\omega_{\underset{\sim}{k}}\, \delta a_{\underset{\sim}{k}}(\omega)$$

$$-i\, V^{\underset{\sim}{k}}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}\, a^{(0)}_{\underset{\sim}{k}_1}(\omega_{\underset{\sim}{k}_1})\, a^{(0)}_{\underset{\sim}{k}_2}(\omega_{\underset{\sim}{k}_2})\, \delta_{\underset{\sim}{k}_1+\underset{\sim}{k}_2-\underset{\sim}{k}}\, \delta(\omega_{\underset{\sim}{k}_1}+\omega_{\underset{\sim}{k}_2}-\omega). \tag{9}$$

Now solve for $\delta a_{\underset{\sim}{k}}(\omega)$:

$$\delta a_{\underset{\sim}{k}}(\omega) = \left(\frac{i}{\omega-\omega_{\underset{\sim}{k}}}\right)\left(-iV^{\underset{\sim}{k}}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}\right) a^{(0)}_{\underset{\sim}{k}_1}(\omega_{\underset{\sim}{k}_1})\, a^{(0)}_{\underset{\sim}{k}_2}(\omega_{\underset{\sim}{k}_2})\, \delta_{\underset{\sim}{k}_1+\underset{\sim}{k}_2-\underset{\sim}{k}}\, \delta(\omega_{\underset{\sim}{k}_1}+\omega_{\underset{\sim}{k}_2}-\omega)\,. \tag{10}$$

The first factor is called the "propagator" of $a_{\underset{\sim}{k}}$; the second factor is the "vertex" for $\underset{\sim}{k} \rightarrow \underset{\sim}{k}_1 + \underset{\sim}{k}_2$; the $a^{(0)}_{\underset{\sim}{k}_j}$ are "external lines"; and the delta functions conserve wavenumber and frequency at the vertex. A diagram can be drawn which represents all these factors, and is shown in Fig. 1. It is an oriented figure made out of lines and vertices. External lines connect with a vertex at only one end. Propagators connect with vertices at both ends. If the free end of an external line lies below (or above) its vertex, the line represents action amplitude $a_{\underset{\sim}{k}}$ (or $a^*_{\underset{\sim}{k}}$, respectively). Vertices represent the V coefficients. Each line entering a vertex from below corresponds to one subscript of V, and each line leaving a vertex from above corresponds to one superscript. Each vertex conserves vector wavenumber and frequency. A propagator is associated with the inverse frequency mismatch $\frac{1}{\omega - \omega_{\underset{\sim}{k}}}$, where ω is the frequency flowing through the line and $\omega_{\underset{\sim}{k}}$ is the dispersion relation frequency corresponding to the wavenumber flowing through the line. It represents the frequency-space Green's function solution for free wave propagation due to Hamiltonian H_2.

"Bare" diagrams are the sole contribution to first order perturbation theory. Such diagrams correspond to individual terms of the Hamiltonian. Each consists of a single vertex and as many external lines below or above the vertex as there are subscripts or superscripts on the coefficient V of the term in the Hamiltonian from which the diagram is derived.

In higher order perturbation theory, additional diagrams appear as a "dressing" of the bare terms. N^{th} order perturbation theory consists of the set of bare diagrams, plus all possible connected diagrams with N or fewer propagators. For example, Fig. 2 shows the bare diagram plus dressing for the case of two external lines below and two external lines above the vertex. This second order perturbation theory expression has application to a surface gravity wave calculation described in Sec. 3.

The rules for evaluating Feynman diagrams are as follows:

1) Associate a factor $\left(\frac{i}{\omega - \omega_{\underset{\sim}{k}}}\right)$ with each propagator. ω is the frequency of the line while $\omega_{\underset{\sim}{k}}$ is the unperturbed frequency, given the wavenumber of the line. The same ω and $\underset{\sim}{k}$ are associated with both ends of the propagator.

2) Associate a factor (-iV) with each vertex. The sum of $\omega_{\underset{\sim}{k}}$'s and $\underset{\sim}{k}$'s are conserved at each vertex.

3) Associate a factor i, by convention, with the whole diagram. This factor causes the sum of diagrams to have the same symmetry properties as the Hamiltonian. When all diagrams with the same number, m, of external lines below the vertex, and the same number, n, of external lines above the vertex are summed, the result will be denoted as $C_{\underset{\sim}{k}_1,\ldots \underset{\sim}{k}_m}^{\underset{\sim}{k}'_1,\ldots \underset{\sim}{k}'_n}$. There are two more rules needed. They are:

4) Include overall delta functions for $\underset{\sim}{k}$ and ω conservation and integrate over the $\underset{\sim}{k}$'s.

5) Associate a factor a (or a*), with the correct momentum (and some specified frequency) for each external line below (or above, resp.) the vertex.

The function $C_{k_1,\ldots,k_m}^{k_1,\ldots,k_n'}$ is called the invariant m -to -n point function (we will often call it simply the (m + n) - point function), and plays an important role in Feynman diagram perturbation theory. When rules 4 and 5 are then applied to this function and all such quantities are summed (over m and n), the result is an "effective Hamiltonian," T. The effective Hamiltonian takes the place of the interaction Hamiltonian when the time ordering of the formal solution of the system is replaced by a normal ordering:

$$\mathscr{T} \exp \int L_H \, dt = \mathscr{N} \exp \int L_T \, dt \quad . \tag{11}$$

In these expression L refers to the Liouville operator associated with the quantity given in the subscript. The normal ordering, specified by $\mathscr{N}$, means that the derivatives in the Liouville operators act only on the initial state, while in the time ordering, specified by $\mathscr{T}$, they operate on the state which has dynamically evolved up to the time at which the Liouville operator acts. In this sense T solves the dynamics.

For the purposes of this paper, the full interpretation of T is not important. What is important is that it is an effective Hamiltonian. One uses it as if it were the true Hamiltonian, but only applies first order perturbation theory. (That is the interpretation of the normal ordering).

We close this section with a very simple application of the diagrams.

Consider the quadratic Hamiltonian

$$H_2 = \omega_{\underset{\sim}{k}} a_{\underset{\sim}{k}} a^*_{\underset{\sim}{k}} \quad , \tag{12}$$

and write $\omega_{\underset{\sim}{k}}$ as $\omega_{\underset{\sim}{k}} = \omega^{(0)}_{\underset{\sim}{k}} + \delta\omega_{\underset{\sim}{k}}$. Then we have an arbitrary decomposition of H_2 into an unperturbed part, $\omega^{(0)}_{\underset{\sim}{k}} a_{\underset{\sim}{k}} a^*_{\underset{\sim}{k}}$, and a perturbation, $\delta\omega_{\underset{\sim}{k}} a_{\underset{\sim}{k}} a^*_{\underset{\sim}{k}}$. Since the equations of motion are linear, they have an exact Green's function solution. We will calculate this solution in frequency space, i.e., the exact propagator.

The Feynman diagrams are shown in Fig. 3. The vertex (bare 1-to-1 point function) is denoted by X. Dots are placed at the free ends of the lines to indicate they are propagators rather than external lines. The sum of diagrams is

$$\left(\frac{i}{\omega - \omega^{(0)}_{\underset{\sim}{k}}}\right) + \left(\frac{i}{\omega - \omega^{(0)}_{\underset{\sim}{k}}}\right)^2 \left(-i\delta\omega_{\underset{\sim}{k}}\right) + \left(\frac{i}{\omega - \omega^{(0)}_{\underset{\sim}{k}}}\right)^3 \left(-i\delta\omega_{\underset{\sim}{k}}\right)^2 + \ldots \tag{13}$$

This infinite sum of diagrams is a geometrical series, with the sum $\left(\frac{i}{\omega - \omega^{(0)}_{\underset{\sim}{k}} - \delta\omega_{\underset{\sim}{k}}}\right)$, precisely the propagator we would have taken had we not put part of $\omega_{\underset{\sim}{k}}$ in the perturbation. The frequency shift is the bare 1-to-1 point function $\delta\omega_{\underset{\sim}{k}}$.

3. Application to Surface Gravity Waves

The Hamiltonian for surface gravity waves on an inviscid, irrotational ocean depth D and area A is (Miles, 1977)

$$H = \frac{1}{2} \int_A d^2\underset{\sim}{x} \left\{ \int_{-D}^{\zeta} (\nabla\phi)^2 \, dz + g\zeta^2 \right\} \quad . \tag{14}$$

The surface displacement, $\zeta(\underset{\sim}{x},t)$, and velocity potential at the free surface, $\phi_s(\underset{\sim}{x},t) \equiv \phi(\underset{\sim}{x}, z = \zeta, t)$, are canonical variables for this system. Eq. 14 is more useful in a strictly two-dimensional form. To eliminate the z-integral, integrate the kinetic energy term once by parts and use Laplace's equation $\nabla^2\phi = 0$. Then the equivalent Hamiltonian

$$H = \frac{1}{2} \int_A d^2\underset{\sim}{x} \left\{ \phi_s (w - \nabla\zeta\nabla\phi_s + w(\nabla\zeta)^2) + g\zeta^2 \right\} \tag{15}$$

is obtained, where $w \equiv \partial_z \phi \Big|_{z=\zeta}$.

To make use of Eq. 15, it is necessary to eliminate the variable w in favor of ϕ_s. Watson and West (1975) accomplish this by Taylor expanding w and ϕ_s in terms of ζ about $z = 0$. Then w is expressed in terms of ϕ_s by successive substitution. The resulting expression for the Hamiltonian is a power series expansion in the wave slope:

$$H = H_2 + H_3 + H_4 + \ldots . \tag{16}$$

Finally, the Hamiltonian is Fourier transformed to obtain the equivalent of Eq. 5. West (1981) has carried out the necessary algebra in the deep water limit and given explicit expressions for the first few terms in the Fourier expansion of H. In particular, he has supplied $V_{\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3}$, $V^{\underset{\sim}{k}_3}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}$, $V^{\underset{\sim}{k}_2,\underset{\sim}{k}_3}_{\underset{\sim}{k}_1}$, $V^{\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3}$, and $V^{\underset{\sim}{k}_3,\underset{\sim}{k}_4}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}$ †. The reader is referred to West's book for the evaluation of these couplings as well as for references to the original literature on surface wave Hamiltonians.

For the calculation presented in the next section, our interest is in lowest order resonant wave growth. The dispersion relation for deep water surface gravity waves, $\omega = (gk)^{1/2}$, does not allow resonant interaction between wave triads. The lowest possible resonant diagram is, therefore, given by the 4 - point function $C^{\underset{\sim}{k}_3,\underset{\sim}{k}_4}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}$. There are seven diagrams that sum to give this coefficient, and these are shown in Fig. 2. They are the bare diagram (which contains no propagator), and six dressing diagrams each of which contain one propagator connecting two 3-point vertices. The expressions for these diagrams are obtained using the rules given in Sec. 2. They are:

D1. $V^{\underset{\sim}{k}_3,\underset{\sim}{k}_4}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}$

D2. $V^{\underset{\sim}{k}_1+\underset{\sim}{k}_2}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2} \left(\dfrac{1}{\omega(\underset{\sim}{k}_1) + \omega(\underset{\sim}{k}_2) - \omega(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)}\right) V^{\underset{\sim}{k}_3,\underset{\sim}{k}_4}_{\underset{\sim}{k}_1+\underset{\sim}{k}_2}$

D3. $V_{\underset{\sim}{k}_1,\underset{\sim}{k}_2,-(\underset{\sim}{k}_1+\underset{\sim}{k}_2)} \left(\dfrac{1}{-\omega(\underset{\sim}{k}_1) - \omega(\underset{\sim}{k}_2) - \omega(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)}\right) V^{-(\underset{\sim}{k}_1+\underset{\sim}{k}_2),\underset{\sim}{k}_3,\underset{\sim}{k}_4}$

D4. $V^{\underset{\sim}{k}_3,\underset{\sim}{k}_1-\underset{\sim}{k}_3}_{\underset{\sim}{k}_1} \left(\dfrac{1}{\omega(\underset{\sim}{k}_1) - \omega(\underset{\sim}{k}_3) - \omega(\underset{\sim}{k}_1 - \underset{\sim}{k}_3)}\right) V^{\underset{\sim}{k}_4}_{\underset{\sim}{k}_1-\underset{\sim}{k}_3,\underset{\sim}{k}_2}$

† There is an error in Eq. (4.21) of West (1981). The numerical prefactor in the expression for $V^{\underset{\sim}{k}_3,\underset{\sim}{k}_4}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}$ should be 1/16, not 1/8.

D5. Same as D4 with $k_1 \leftrightarrow k_2$ and $k_3 \leftrightarrow k_4$

D6. Same as D4 with $k_3 \leftrightarrow k_4$

D7. Same as D5 with $k_3 \leftrightarrow k_4$. (17)

For the deep water limit, we take the explicit forms of the V coefficients from West (1981), and use $\omega(k) = (gk)^{1/2}$.

4. Stokes Versus Canonical Calculation of Resonant Growth Rates

Longuet-Higgins (1962) calculated the resonant interaction of two trains of surface gravity waves using a traditional Stokes expansion. In this non-canonical approach, field variables are expanded about the undisturbed surface z = 0 and substituted into the fluid equations. Terms of a given order in the expansion variable (wave slope) are collected together, and the resulting equations are solved. Resonant interaction can occur at third order, leading to a transfer of energy from three primary waves (of wavenumbers $\underset{\sim}{k}_1, \underset{\sim}{k}_2, \underset{\sim}{k}_3$, say) to a fourth wave (of wavenumber $\underset{\sim}{k}_4$). Longuet-Higgins calculated the tertiary wave growth rate for the specal case $\underset{\sim}{k}_1 = \underset{\sim}{k}_2$, $\underset{\sim}{k}_4 = 2\underset{\sim}{k}_1 - \underset{\sim}{k}_3$. In terms of the variable $\xi = \dfrac{\omega(k_1) - \omega(k_4)}{\omega(k_1)}$, $(|\xi| \leq \frac{1}{2})$ the growth rate of the wave action amplitude[†] was found to be

$$\frac{da_4}{dt} = \frac{-ik_1^{\,3}\,(2+\xi^2)^2\,(1-4\xi^2)}{4(1-\xi^2)^{1/2}} \left[1 + \frac{4\xi^2}{\xi^2 - |\xi|\,(6+\xi^2)^{1/2}}\right] a_1^{\,2}\, a_3^* \,. \tag{18}$$

We wish to compare this result with the Feynman diagram (canonical) calculation of the growth rate.

For surface gravity waves, the quantity

$$T_4 = \sum_{\substack{\underset{\sim}{k}_1,\underset{\sim}{k}_2,\\ \underset{\sim}{k}_3,\underset{\sim}{k}_4,}} C^{\underset{\sim}{k}_3,\underset{\sim}{k}_4}_{\underset{\sim}{k}_1,\underset{\sim}{k}_2}\; a_{\underset{\sim}{k}_1}\, a_{\underset{\sim}{k}_2}\, a^*_{\underset{\sim}{k}_3}\, a^*_{\underset{\sim}{k}_4}\, \delta_{\underset{\sim}{k}_1+\underset{\sim}{k}_2-\underset{\sim}{k}_3-\underset{\sim}{k}_4} \tag{19}$$

[†]Eq. (18) corresponds to Eq.'s (6.4) and (6.5) of Longuet-Higgins (1962) after reexpressing his amplitudes in terms of action amplitudes.

is the lowest order resonant contribution to the effective Hamiltonian. The corresponding contribution to the equation of motion of wave amplitude $a_{\underset{\sim}{k}_4}$ is

$$\frac{da_{\underset{\sim}{k}_4}}{dt} = -i \frac{\partial T_4}{\partial a^*_{\underset{\sim}{k}_4}}$$

$$= -2i \sum_{\underset{\sim}{k}_1, \underset{\sim}{k}_2, \underset{\sim}{k}_3} C^{\underset{\sim}{k}_3, \underset{\sim}{k}_4}_{\underset{\sim}{k}_1, \underset{\sim}{k}_2} \; a_{\underset{\sim}{k}_1} \; a_{\underset{\sim}{k}_2} \; a^*_{\underset{\sim}{k}_3} \; \delta_{\underset{\sim}{k}_1 + \underset{\sim}{k}_2 - \underset{\sim}{k}_3 - \underset{\sim}{k}_4} \quad . \qquad (20)$$

For the restriction $\underset{\sim}{k}_1 = \underset{\sim}{k}_2$, $\underset{\sim}{k}_4 = 2\underset{\sim}{k}_1 - \underset{\sim}{k}_3$ used by Longuet-Higgins, we therefore have

$$\frac{da_{\underset{\sim}{k}_4}}{dt} = -2i \; C^{\underset{\sim}{k}_3, 2\underset{\sim}{k}_1 - \underset{\sim}{k}_3}_{\underset{\sim}{k}_1, \underset{\sim}{k}_1} \; a^2_{\underset{\sim}{k}_1} \; a^*_{\underset{\sim}{k}_3} \quad . \qquad (21)$$

The coefficient $C^{\underset{\sim}{k}_3, 2\underset{\sim}{k}_1 - \underset{\sim}{k}_3}_{\underset{\sim}{k}_1, \underset{\sim}{k}_1}$ is obtained by summing the seven Feynman diagrams of Eq. 17. Quantities relevant to the evaluation of these diagrams are

$$\omega(k_2) = \omega(k_1) \equiv \omega_1$$

$$\omega(k_3) = (1 + \xi)\omega_1$$

$$\omega(k_4) = (1 - \xi)\omega_1$$

$$k_2 = k_1$$

$$k_3 = (1 + \xi)^2 k_1$$

$$k_4 = (1 - \xi)^2 k_1$$

$$|\underset{\sim}{k}_3 - \underset{\sim}{k}_1| = Qk_1, \; Q \equiv |\xi|(6 + \xi^2)^{1/2}$$

$$\underset{\sim}{k}_1 \cdot \underset{\sim}{k}_3 = (1 + 2\xi + 2\xi^3)k_1^{\ 2}$$

$$\underset{\sim}{k}_1 \cdot \underset{\sim}{k}_4 = (1 - 2\xi - 2\xi^3)k_1^{\ 2}$$

$$\underset{\sim}{k}_3 \cdot \underset{\sim}{k}_4 = (1 - 6\xi^2 - \xi^4)k_1^{\ 2} \quad . \qquad (22)$$

When these are substituted into the expression for the V's given in West (1981), and also into Eq. (17), we easily obtain

$$D1 = -\frac{1}{8} k_1^{\ 3} (1 - \xi^2)^{1/2} \left[-2 + Q^2 - Q\xi^2\right]$$

$$D2 = \frac{-(2^{1/2} + 1)\ k_1^{\ 3}}{8(1 - \xi^2)^{1/2}} \left[2^{1/2}\xi^2 (1 + 2\xi^2) + 4\xi^2 - 1\right]$$

$$D3 = \frac{-(2^{1/2} - 1)\ k_1^{\ 3}}{8(1 - \xi^2)^{1/2}} \left[2^{1/2}\xi^2 (1 + 2\xi^2) - 4\xi^2 + 1\right]$$

$$D4 = \frac{-\xi^2 k_1^{\ 3}}{32(1 - \xi^2)^{1/2}\ Q^{1/2}\ (\xi^2 - Q)} \Big[-\xi(6\xi^6 + 11\xi^4 + 14\xi^2 - 16)$$

$$+ Q^{1/2}(22\xi^6 + 77\xi^4 + 34\xi^2 - 16)$$

$$- Q\xi\ (10\xi^4 + 31\xi^2 + 16) - Q^{3/2}\ (6\xi^4 + 11\xi^2 - 8)\Big]$$

$$D5 = D4 \text{ with } \xi \to -\xi$$

$$D6 = D5$$

$$D7 = D4 \quad . \qquad (23)$$

One notices that the sums of D2 and D3, of D4 and D5, and of D6 and D7 are simpler than either one separately. Feynman noticed that such simplifications always occur in relativistic theories, but for us they are a consequence of the accidental crossing relation. The sum of expressions D1 - D7 gives

$$C^{\underset{\sim}{k}_3, 2\underset{\sim}{k}_1 - \underset{\sim}{k}_3}_{\underset{\sim}{k}_1, \underset{\sim}{k}_1} = \frac{k_1^{\,3}\,(2+\xi^2)^2\,(1-4\xi^2)}{8(1-\xi^2)^{1/2}}\left[1 + \frac{4\xi^2}{\xi^2 - |\xi|\,(6+\xi^2)^{1/2}}\right] . \qquad (24)$$

We now see that our Feynman diagram expression for $\frac{da_{\underset{\sim}{k}_4}}{dt}$ is identical to that of Longuet-Higgins. There is a difference between the two calculations however: we worked in the canonical framework whereas Longuet-Higgins worked with the Stokes expansion. These two Eulerian representations, as well as representations in the Lagrangian framework, always agree in lowest order but will usually disagree in higher order. Since we have calculated an effect which is two orders beyond the lowest, it may seem surprising to find agreement. In fact, the "bare" contribution to the tertiary wave growth rate does not agree in the different representations. The bare term in Longuet-Higgins calculation can be identified† as

$$\left(\frac{da_4}{dt}\right)_{\text{BARE}} = \frac{-ik_1^{\,3}\,(2+\xi^2)\,(1+2\xi)\,(2-3\xi^2)}{4(1-\xi^2)^{1/2}}\; a_1^{\,2}\, a_3^{*} , \qquad (25)$$

whereas the canonical description gives

$$\left(\frac{da_{\underset{\sim}{k}_4}}{dt}\right)_{\text{BARE}} = -2i\, V^{\underset{\sim}{k}_3, 2\underset{\sim}{k}_1 - \underset{\sim}{k}_3}_{\underset{\sim}{k}_1, \underset{\sim}{k}_1}\; a_{\underset{\sim}{k}_1}^{\,2}\, a_{\underset{\sim}{k}_3}^{*}$$
$$= \frac{-ik_1^{\,3}}{4}\,(1-\xi^2)^{1/2}\left[2 - Q^2 + Q\xi^2\right] a_{\underset{\sim}{k}_1}^{\,2}\, a_{\underset{\sim}{k}_3}^{*} \qquad (26)$$

(Expressions (25) and (26) are compared in Fig. 4 for the range $0 < \xi < 1/2$). There has to be agreement in the total rate, however, since this total is an experimentally accessible quantity. In lowest order, the values of the amplitudes agree, so one can ask without ambiguity what the lowest order

†The bare terms of Longuet-Higgins (1962) arise from those terms in his Eq. (3.11) which do not involve the subscript 11.

growth of the tertiary wave is. Any representation must give the same answer for this question. Non-resonant and higher order resonant terms, however, do differ between representations whether or not these representations are canonical. It is only when physical questions are asked that the answer is unambiguous.

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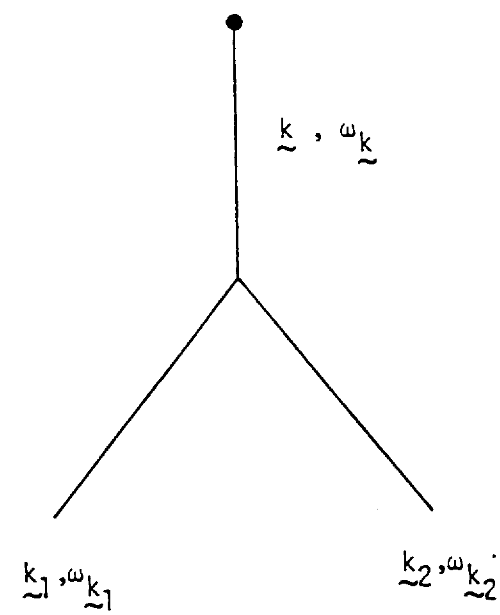
Figure Captions

Figure 1: Feynman diagram expressing the solution, Eq. 10. The dot has been placed at the free end of the line representing the propagator to distinguish it from an external line.

Figure 2: The seven Feynman diagrams which govern the lowest-order resonant interaction between surface gravity waves. Their sum yields the invariant 4-point function $C_{\underset{\sim}{k}_1, \underset{\sim}{k}_2}^{\underset{\sim}{k}_3, \underset{\sim}{k}_4}$. D1 is the "bare" diagram, and the other six "dressing" diagrams simplify when grouped as D2 + D3, D4 + D5, D6 + D7.

Figure 3: Feynman diagrams showing a calculation of the frequency-space Green's function solution for the quadratic Hamiltonian, Eq. 12. We have chosen $\omega_{\underset{\sim}{k}} = \omega_{\underset{\sim}{k}}^{(0)} + \delta\omega_{\underset{\sim}{k}}$.

Figure 4: Comparison between expressions 25 (---) and 26 (——). Expression 25 is the Stokes expansion bare contribution to the resonant growth rate of a tertiary wave, as calculated by Longuet-Higgins. It is different from expression 26 which was calculated using canonical variables. The total (bare + dressing) growth rates are the same, however.



$$\underset{\sim}{k} = \underset{\sim}{k}_1 + \underset{\sim}{k}_2$$

$$\omega = \omega_{\underset{\sim}{k}_1} + \omega_{\underset{\sim}{k}_2} \neq \omega_{\underset{\sim}{k}}$$

Figure 1.

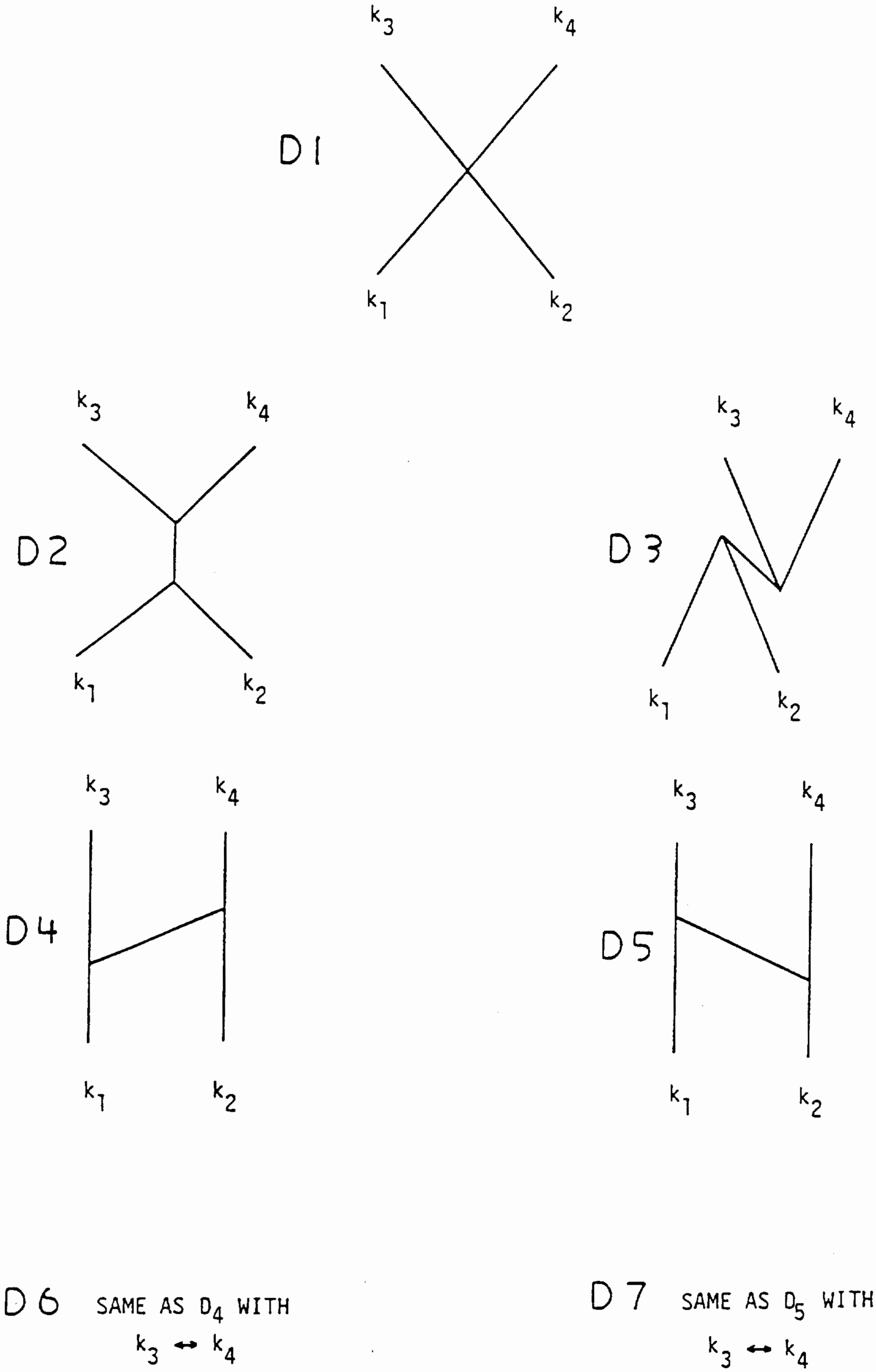
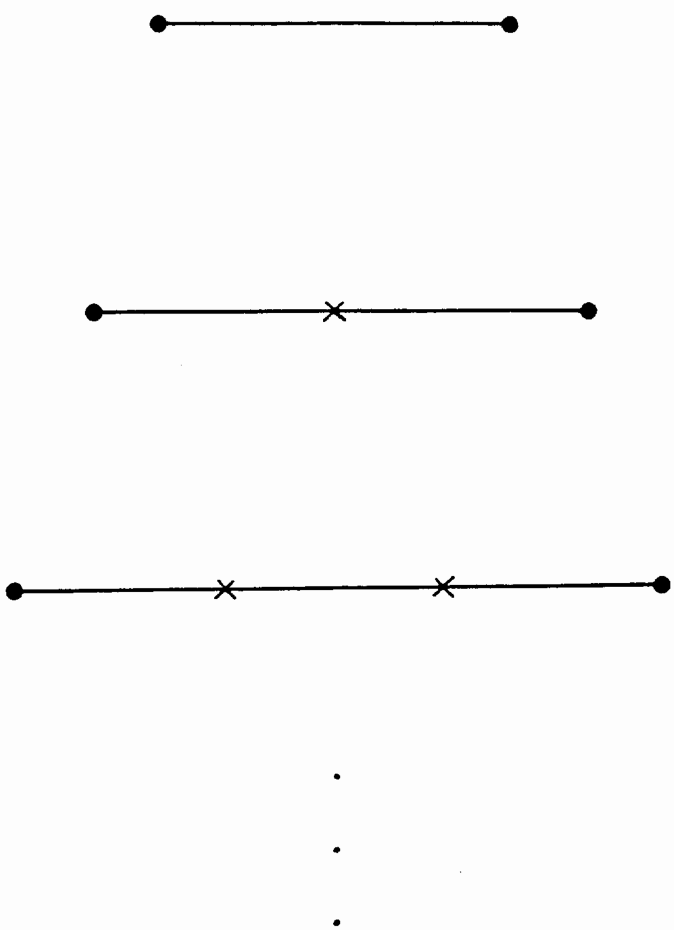

D1
k3
k4
k1
k2
D2
k3
k4
k1
k2
D3
k3
k4
k1
k2
D4
k3
k4
k1
k2
D5
k3
k4
k1
k2
D6 SAME AS D4 WITH
k3 ↔ k4
D7 SAME AS D5 WITH
k3 ↔ k4

Figure 2.



$$\left(\frac{i}{\omega - \omega_{\underset{\sim}{k}}^{(0)}}\right)$$

$$\left(\frac{i}{\omega - \omega_{\underset{\sim}{k}}^{(0)}}\right)^2 (-i\delta\omega_{\underset{\sim}{k}})$$

$$\left(\frac{i}{\omega - \omega_{\underset{\sim}{k}}^{(0)}}\right)^3 (-i\delta\omega_{\underset{\sim}{k}})^2$$

Figure 3.

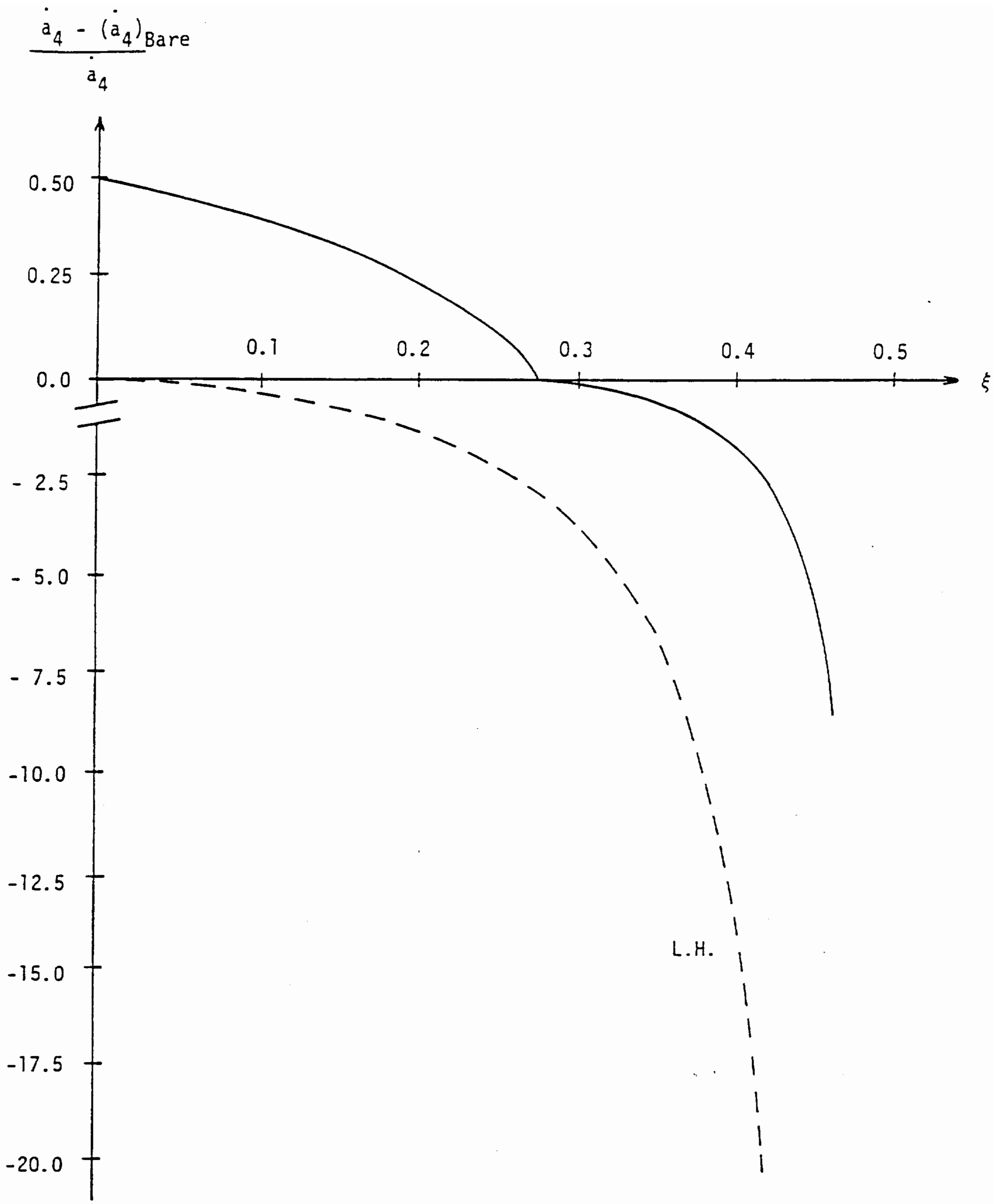

$\frac{\dot{a}_4 - (\dot{a}_4)_{Bare}}{\dot{a}_4}$
0.50
0.25
0.0
- 2.5
- 5.0
- 7.5
-10.0
-12.5
-15.0
-17.5
-20.0
0.1
0.2
0.3
0.4
0.5
ξ
L.H.

Figure 4.