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by

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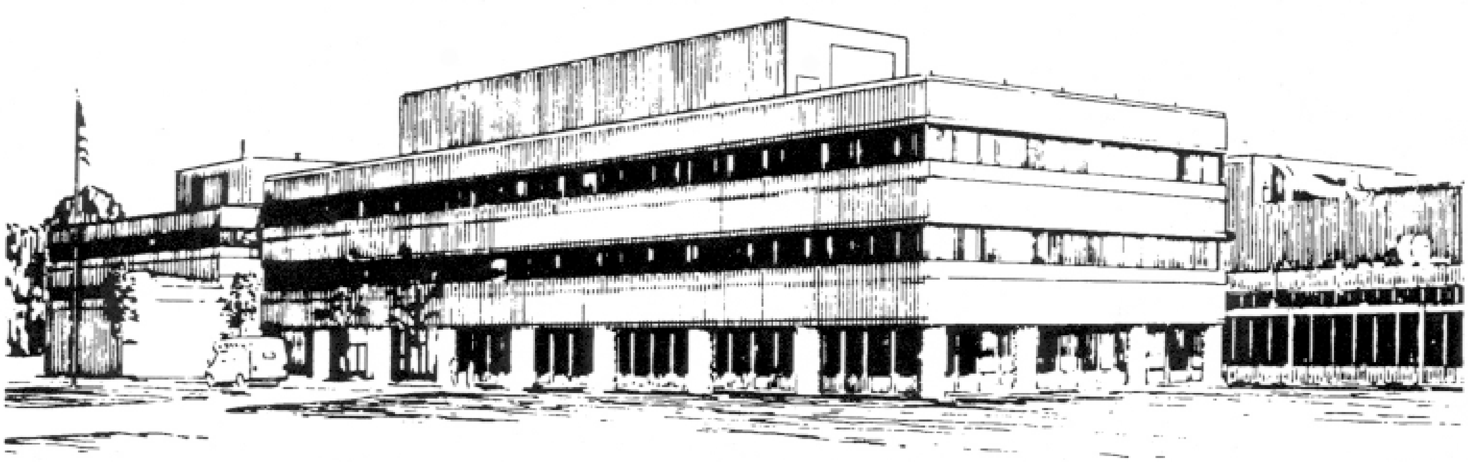

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Fundamental statistical descriptions of plasma turbulence in magnetic fields

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Abstract

A pedagogical review of the historical development and current status (as of early 2000) of systematic statistical theories of plasma turbulence is undertaken. Emphasis is on conceptual foundations and methodology, not practical applications. Particular attention is paid to equations and formalism appropriate to strongly magnetized, fully ionized plasmas. Extensive reference to the literature on neutral-fluid turbulence is made, but the unique properties and problems of plasmas are emphasized throughout. Discussions are given of quasilinear theory, weak-turbulence theory, resonance-broadening theory, and the clump algorithm. Those are developed independently, then shown to be special cases of the direct-interaction approximation (DIA), which provides a central focus for the article. Various methods of renormalized perturbation theory are described, then unified with the aid of the generating-functional formalism of Martin, Siggia, and Rose. A general expression for the renormalized dielectric function is deduced and discussed in detail. Modern approaches such as decimation and PDF methods are described. Derivations of DIA-based Markovian closures are discussed. The eddy-damped quasinormal Markovian closure is shown to be nonrealizable in the presence of waves, and a new realizable Markovian closure is presented. The test-field model and a realizable modification thereof are also summarized. Numerical solutions of various closures for some plasma-physics paradigms are reviewed. The variational approach to bounds on transport is developed. Miscellaneous topics include Onsager symmetries for turbulence, the interpretation of entropy balances for both kinetic and fluid descriptions, self-organized criticality, statistical interactions between disparate scales, and the roles of both mean and random shear. Appendices are provided on Fourier transform conventions, dimensional and scaling analysis, the derivations of nonlinear gyrokinetic and gyrofluid equations, stochasticity criteria for quasilinear theory, formal aspects of resonance-broadening theory, Novikov's theorem, the treatment of weak inhomogeneity, the derivation of the Vlasov weak-turbulence wave kinetic equation from a fully renormalized description, some features of a code for solving the direct-interaction approximation and related Markovian closures, the details of the solution of the EDQNM closure for a solvable three-wave model, and the notation used in the article.

Key words: plasma turbulence; statistical closure; direct-interaction approximation; MSR formalism; quasilinear theory; weak turbulence; resonance broadening; clumps; realizable Markovian closure; eddy viscosity; submarginal turbulence; bounds on transport

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1 INTRODUCTION

"It is open to every man to choose the direction of his striving; and also every man may draw comfort from Lessing's fine saying, that the search for truth is more precious than its possession." — *Einstein (1940).*

This article is devoted to a review of the conceptual foundations of statistical descriptions of turbulence in fully ionized, weakly coupled, classical plasmas as the discipline is understood at the end of the 20th century.[1] Particular attention is paid to plasmas in strong magnetic fields, both because of their importance to the magnetic confinement approach to thermonuclear fusion (Furth, 1975; Wesson, 1997) and because the physics is closely related to the behavior of neutral fluids. The goals are to unify disparate approaches, exhibit deep underlying connections to other fields (notably quantum field theory and neutral-fluid turbulence), and provide historical perspective. The emphasis is on philosophy and systematic mathematical techniques. It is hoped that awareness of unifying themes and techniques will aid the reader in organizing the abundance of information, clarify disagreements between competing theories, foster appropriate skepticism about new approaches, and give one a better appreciation of the roots of the back-of-the-envelope estimates that are frequently employed in practice. Most of the basic themes discussed in this article are not specific to plasma physics and should interest anyone intrigued by nonlinear behavior.

The article is not a review of most of the vast experimental, computational, and theoretical research that falls under the aegis of plasma turbulence. Motivations and a few highly idealized physical models that illustrate the formalism will by and large be distilled from applications arising in fusion research, but no attempt will be made to even-handedly survey the many fascinating and detailed practical applications that arise in astrophysics, fusion, or other places (National Research Council, 1986). For some of those, entry points to the literature can be found in the recent collection edited by Sudan and Cowley (1997), which records the activities of the most recent research workshop focused on plasma turbulence. A short review by the present author that addresses some of the topics elaborated below can also be found there (Krommes, 1997c). A useful introductory overview of selected issues in plasma turbulence was given by Similon and Sudan (1990). Some modern practical applications of turbulence theory to fusion were described in some detail by Itoh et al. (1999) and reviewed by Yoshizawa et al. (2001).

Given that turbulence is ubiquitous in many important plasma applications, including magnetic

[1] Work discussed in this article is limited to that published before 2000, except for a few selected papers related to themes developed in this review. Notably, (i) the pedagogical article by Grossmann (2000) describes current understanding of the submarginal onset of shear-flow turbulence [see Sec. 9 (p. 210)]; (ii) the work by Krommes and Kim (2000) serves as a minireview in its own right of some plasma-physics literature on zonal flows, the statistical interactions of disparate scales, and related issues [Secs. 7.3 (p. 189) and 12.7 (p. 248)]; and (iii) the review article by Yoshizawa et al. (2001) describes some of the more practical fluid and plasma turbulence issues, particularly those related to inhomogeneity, and is nicely complementary to the present article.

fusion, the need for a fundamental and systematic theory of plasma turbulence may appear to be self-evident. In reality, however, most theoretical "calculations" of turbulent plasma transport are still done at the level of primitive mixing-length estimates articulated 35 years ago (Kadomtsev, 1965), and actual research on systematic, mathematically justifiable approximations is almost nonexistent within the plasma community. To the extent that more quantitative results have been required, the overwhelming response has been to adopt the brute-force approach of direct numerical simulations (DNS[2]). That is both understandable and reasonable. Nevertheless, analytical theory has its place. Clever physics-based techniques are required even to derive equations that can be efficiently used in DNS [see the discussion of gyrokinetic and gyrofluid equations in Appendix C (p. 267)]. Moreover, it is difficult to make sense of the plethora of simulation data without a detailed and systematic theoretical superstructure, particularly in view of an abundance of theoretical misconceptions that have proliferated over the years. Formal methods are invaluable in deriving and justifying heuristic procedures. Finally, the turbulence problem presents one of the outstanding intellectual challenges of modern science; it is interesting in its own right.

The prospective student of plasma turbulence faces a daunting challenge. She is confronted with literally thousands of articles on nonlinear plasma behavior,[3] frequently in the context of extremely complicated confinement geometries and fusion scenarios. Focusing on papers of obviously conceptual nature helps to provide initial orientation, but those almost inevitably assume background in hydrodynamic (neutral-fluid) turbulence, a discipline under development since the early 1900s; contemporary literature in that field is quite sophisticated (HydrotConf, 2000). Mathematical turbulence theory was pioneered by workers in the field of neutral fluids, the incompressible Navier–Stokes equation (NSE) providing the fundamental mathematical model. Very much can be learned from that outstanding research, so there will be strong overlap between certain parts of the present article and discussions of statistical turbulence theory for neutral fluids. The collection edited by Frost and Moulden (1977) contains useful introductory articles on fluid turbulence and associated mathematical techniques. Some review articles on fluid turbulence are by Orszag (1977), Rose and Sulem (1978), and Kraichnan (1991); recent books[4] are by McComb (1990), Frisch (1995), and Lesieur (1997). There are also strong connections to quantum field theory and the theory of critical phenomena; some useful books are by Zinn-Justin (1996), Binney et al. (1992), and Goldenfeld (1992). Applications of dynamical systems analysis to turbulence problems were described by Bohr et al. (1998). Some important early books on plasma turbulence are by Kadomtsev (1965), Sagdeev and Galeev (1969), and Davidson (1972), but those do not achieve the breadth and unification for which the present article strives. A recent review article that discusses some of the more practical aspects of modeling turbulence in fluids and plasmas is by Yoshizawa et al. (2001); some of those topics were treated in more detail by Itoh et al. (1999).

Some of the implications of fluid-turbulence theory for plasmas were discussed by Montgomery

[2] The principal abbreviations used in this article are summarized in Appendix K.1 (p. 302).

[3] A useful survey of important topics in plasma physics can be found in volumes 1 and 2 of the Handbook of Plasma Physics, edited by Rosenbluth and Sagdeev (1984).

[4] The emphases in these books are loosely orthogonal. Frisch (1995) gives an elegant introduction to 3D turbulence, including an authoritative discussion of intermittency. McComb (1990) [a worthy successor to the earlier, now out-of-print, book by Leslie (1973b)] is strong on statistical and field-theoretic methods such as the direct-interaction approximation. Lesieur (1997) emphasizes Markovian statistical closures and much real-world phenomenology, including two-dimensional turbulence.

(1977) and Montgomery (1989). In Secs. 1.1 (p. 9) and 1.2 (p. 10) I shall expand on his remarks and briefly describe the important similarities and differences between turbulence in plasmas and in neutral fluids. That introductory discussion is not self-contained, being intended to quickly inform experts about the emphasis and special topics to be discussed here. The topics will all be revisited in a self-contained way in the body of the article.

1.1 Similarities between plasma and neutral-fluid turbulence

By definition, turbulence involves *random* motions of a physical system, so statistical descriptions are required at the outset.

1.1.1 The statistical closure problem

Turbulence is an intrinsically nonlinear phenomenon, so microscopically it exhibits extreme sensitivity to initial conditions or perturbations. To extract a macroscopic description that is robust under perturbations, it is useful to treat the turbulent fields as random variables and to introduce various statistical averaging procedures. The fundamental difficulty faced by any statistical description of turbulence is the *statistical closure problem*,[5] in which the time evolution of cumulants [defined in Sec. 3.5.2 (p. 59)] of order n for some field ψ is coupled by the nonlinearity to cumulants of order $n+1$. For example, if the equation of motion is schematically $\partial_t \psi = \frac{1}{2} M \psi^2$ and one writes $\psi = \langle \psi \rangle + \delta\psi$, where $\langle \ldots \rangle$ denotes either a time average (a concept well defined only in a statistically steady state) or an ensemble average over random initial conditions,[6] then

$$\partial_t \langle \psi \rangle = \tfrac{1}{2} M \langle \psi \rangle^2 + \tfrac{1}{2} M \langle \delta\psi^2 \rangle, \tag{1a}$$

$$\partial_t \langle \delta\psi(t) \delta\psi(t') \rangle = M \langle \psi(t) \rangle \langle \delta\psi(t) \delta\psi(t') \rangle + \tfrac{1}{2} M \langle \delta\psi(t) \delta\psi(t) \delta\psi(t') \rangle, \tag{1b}$$

and so on. A moment-based statistical closure (approximation) provides a way of expressing a cumulant of some order (say, 3) in terms of lower-order cumulants, thereby closing the chain of coupled equations of which Eqs. (1) are the first two members. Representative general discussions of the statistical closure problem include those by Kraichnan (1962a,b, 1966a, 1972, 1975a, 1988a), Rose and Sulem (1978), Krommes (1984a,b), McComb (1990), and Frisch (1995).

The statistical closure problem is common to both plasmas and neutral fluids, and indeed to any nonlinear system that exhibits random behavior. Most work has been done on quadratic nonlinearity. The familiar Navier–Stokes (NS) equation (NSE) of neutral fluids (Sec. 2.1.1, p. 23) is quadratically nonlinear, as is the Klimontovich kinetic equation of plasma physics (Sec. 2.2.2, p. 27). Furthermore, in strongly magnetized plasmas an $\boldsymbol{x}$-space fluid description is often adequate, giving rise to various multifield, quadratically nonlinear generalizations of the NSE [see, for example, Sec. 2.4.3 (p. 34)]. The dominant plasma nonlinearity describes advection by $\boldsymbol{E} \times \boldsymbol{B}$ drift motions across the magnetic field $\boldsymbol{B}$. A very early but clear paper that discussed the importance of $\boldsymbol{E} \times \boldsymbol{B}$ drifts to turbulent plasma transport was by Spitzer (1960).

[5] The discussion here relates to *moment-based* statistical closures. More generally, one can consider approximations to entire probability density functions (PDF's). PDF-based closure methods are discussed in Sec. 10.4 (p. 224).

[6] For some discussion of the differences between various averaging procedures, see Appendix A.7 of Balescu (1975).

Although much neutral-fluid research has been on three-dimensional (3D) turbulence, certain important applications drawn from geophysics (Holloway, 1986) can profitably be described as 2D; there are profound differences between 2D and 3D turbulence. In plasmas the presence of a strong magnetic field introduces a fundamental anisotropy between the directions perpendicular and parallel to $\boldsymbol{B}$; various 2D or quasi-2D fluid models result (Sec. 2.4, p. 33), and their physical behavior has much in common with the 2D neutral fluid. An important review article on 2D turbulence is by Kraichnan and Montgomery (1980).

1.1.2 Fundamental physics vs practical engineering

Just as in neutral fluids, a dichotomy has arisen in plasma research between the "engineering" and "fundamental-physics" approaches to the description and calculation of turbulence. To date, the overwhelmingly dominant practical application of plasma physics has been controlled thermonuclear fusion research (CTR). The extreme practical complexity of magnetic confinement devices such as the tokamak (Sheffield, 1994) has inevitably led to a predominance of engineering ("mixing-length") estimates of "anomalous" [7] transport, confinement scaling laws based on statistical analyses of experimental data (Kaye, 1985; Kaye et al., 1990), *etc.* Many analogies can be drawn to practical applications of neutral-fluid turbulence—statistical closure theory is hopelessly inadequate for the detailed quantitative design of aircraft, for example. Nevertheless, although the engineering approach is clearly necessary, the need for systematic physics- and mathematics-based foundations exists in plasma physics just as it does in neutral fluids.

1.1.3 The role of computing

Finally, the continuing development of ever faster, ever cheaper computing power (Orszag and Zabusky, 1993) has influenced the fields of plasma as well as neutral-fluid turbulence. In both fields this shows up most clearly in engineering-type modeling of real physical situations. In fusion research, for example, a Grand-Challenge "Numerical Tokamak" project (Cohen et al., 1995) attempts to employ state-of-the-art 3D simulations to address practical issues of turbulence and transport in tokamaks. (An already enormous and rapidly evolving literature on those topics is not reviewed here.) One may hope that this aggregation of sophisticated codes will eventually shed light on fundamental and conceptual issues of plasma turbulence just as recent numerical studies of intermittency have done in neutral fluids [see, for example, Chen et al. (1993)].

1.2 Differences between plasma and neutral-fluid turbulence

The differences between this review and a conventional review of neutral-fluid turbulence theory are more numerous than the similarities.

[7] *Anomalous* refers to effects beyond those due to Coulomb scattering involving discrete particles. It is an unfortunate and somewhat outmoded term that suggests profound mystery and ignorance. In fact, it is the premise of this article that much of the mystery has now been dispelled.

1.2.1 Heuristics vs systematics

Philosophically, there has been in plasma physics a disturbing emphasis on heuristics rather than systematics. The need for the engineering approach to CTR has already been stated and is not in question; however, there has been great confusion over what constitutes a properly *systematic* theory of plasma turbulence. One goal of this article is to draw this line more clearly. In particular, it must be emphasized that in the analytical theory of turbulence described here the starting point is a *given* nonlinear equation [representative ones are described in Sec. 2 (p. 22)], and the goal is to deduce by mathematically defensible operations physically measurable properties (usually statistical) of that equation. This differs substantially from the problem faced by an experimentalist, who can perform direct measurements but is typically not confident of the (appropriately simple) underlying equations. Nevertheless, it is worth repeating that a sound theoretical framework can greatly enhance one's ability to interpret experimental data.

1.2.2 Rich linear theory of plasmas

Mathematically, profound differences between the NSE and plasmas show up already in linear theory through distinctive forms of the linear Green's functions g_0:

$$g_{0,\mathrm{NSE}} = [-\mathrm{i}(\omega + \mathrm{i}k^2\mu_{\mathrm{cl}})]^{-1}, \tag{2a}$$

$$g_{0,\mathrm{plasma}} = [-\mathrm{i}(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + \mathrm{i}\epsilon)]^{-1}\delta(\boldsymbol{v}-\boldsymbol{v}') = [\mathrm{i}\,\mathrm{P}(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-1} + \pi\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})]\delta(\boldsymbol{v}-\boldsymbol{v}'), \tag{2b,c}$$

where μ_{cl} is the classical kinematic molecular viscosity, ϵ is a positive infinitesimal that ensures causality, P denotes principal value, and the unmagnetized, collisionless form of the plasma function is displayed. Linearized Navier–Stokes dynamics describe (in the absence of mean fields) viscous dissipation. In unmagnetized plasmas, on the other hand, particles free-stream in linear order, and the Landau resonance $\omega - \boldsymbol{k}\cdot\boldsymbol{v} = 0$ [captured by the delta function in Eq. (2c)] gives rise to collisionless dissipation (Landau damping). It is important to note that $g_{0,\mathrm{plasma}}$ is the *particle* Green's function; the full Green's function R_0 of the linearized Vlasov equation contains an important extra term relating to self-consistent dielectric response [see Eq. (36a)]. (For the NSE, $R_0 = g_0$.) Plasmas can support a rich abundance of linear motions, including both ballistic particle streaming and collective wave effects; the waves are supported by the nonresonant particles [described by the principal-value term in Eq. (2c)]. The dominance of linear phenomena can in some circumstances lead to the possibility of a weak- rather than strong-turbulence description. (One interpretation of the linear fluid propagator for small μ_{cl} is that all particles are resonant.)

Of course, waves are present in certain descriptions of neutral fluids as well, and weak-turbulence descriptions have been profitably employed in that context. A frequently cited example is gravity waves in the ocean (Phillips, 1977). However, it is easier for plasmas to support linear collective oscillations because of the self-consistent coupling between the particles and Maxwell's equations in the presence of the long-range Coulomb interaction. What results is an easily polarizable dielectric medium that exhibits rich collective behavior, particularly in the presence of a magnetic field. [8]

The most fundamental description of plasmas is at the level of a *phase-space* (position $\boldsymbol{x}$

[8] The literature on microturbulence in magnetized, especially toroidal plasmas comprises many thousands of papers. One early review article is by Tang (1978).

and velocity $\boldsymbol{v}$) fluid.[9] In a useful ordering appropriate for laboratory fusion plasmas, the plasma discreteness parameter[10] ϵ_p is very small (the plasma is "weakly coupled"), so to zeroth order the many-particle plasma can be idealized as a continuum phase-space fluid. Even in the limit $\epsilon_p \to 0$, however, single-particle (ballistic) effects can be important, so it is often natural to view the plasma as a collection of interacting waves and particles. The simultaneous presence of both waves and particles means that plasmas have more and different linear decorrelation mechanisms than does the neutral fluid. Dupree (1969) argued that the presence of such mechanisms actually simplifies the description of plasma turbulence; certainly they have consequences for the transport of macroscopic quantities such as particles or heat. Unfortunately, because waves are collective oscillations involving an infinite number of particles, questions of double counting can arise [see the discussion of clump formalisms in Sec. 4.4 (p. 119)], a difficulty not present in the neutral fluid.

1.2.3 A nonlinear dielectric medium

Although introductory plasma-physics texts [e.g., Krall and Trivelpiece (1973), Chen (1983), Nicholson (1983), Stix (1992), Nishikawa and Wakatani (1994), Goldston and Rutherford (1996), or Hazeltine and Waelbroeck (1998)] stress the calculation and role of the *linear* dielectric function $\mathcal{D}^{\text{lin}}$, the true plasma is a *nonlinear* dielectric medium. The form of the completely nonlinear dielectric function $\mathcal{D}$ is not well understood (or discussed) even for neutral fluids; it is much more complicated for the plasma. Great confusion has arisen on this point, some of which I hope to dispel in this article. A thorough discussion of the nonlinear dielectric function for turbulent mediums is given in Sec. 6.5 (p. 170).

1.2.4 Spectral paradigms

Hand-waving discussions of the behavior of turbulent fluids and plasmas often invoke what I shall call *spectral paradigms*—scenarios by which energylike quantities are injected, transferred, and dissipated in the medium. Those for the NSE are quite different from those for typical fusion-plasma applications. The well-known Kolmogorov scenario for 3D NS turbulence [see, for example, Landau and Lifshitz (1987), Hunt et al. (1991), and Frisch (1995)] involves energy injection by random (typically Gaussian) forcing at long wavelengths, nonlinear and semilocal *cascade* of energy through a well-defined *inertial range* of intermediate scales, and viscous dissipation at very short scales. In this fluid scenario the macroscale Reynolds number is very large and the inertial range is asymptotically infinite in extent. The detailed physics of the inertial and dissipation ranges, especially higher-order statistics and intermittency, is the subject of intense current research. For some references, see Sec. 10 (p. 220) and HydrotConf (2000).

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9 In plasma-physics jargon, descriptions and/or phenomena that require velocity space are called *kinetic* whereas solely $\boldsymbol{x}$-space descriptions are called *fluid*. In linear plasma kinetic theory, the *fluid limit* means phase velocities greater than the characteristic particle velocity. Compare the adiabatic limit, discussed in footnote 48 (p. 35).

10 ϵ_p is the inverse of the number of particles in a Debye sphere of radius λ_D: $\epsilon_p \doteq (n\lambda_D^3)^{-1}$, where $\lambda_D^{-2} \equiv k_D^2 \doteq \sum_s (4\pi n q^2/T)_s$, n_s is the density, T_s is the temperature, and q_s is the charge of species s. For strongly coupled plasmas (Ichimaru, 1992), one frequently uses the alternate parameter $\Gamma \doteq e^2/aT$, where $a \doteq (3/4\pi n)^{1/3}$ measures the interparticle spacing. These parameters are related by $\Gamma = O(\epsilon_p^{2/3})$. The weak-coupling regime is $\epsilon_p \ll 1$ or $\Gamma \ll 1$.

In plasmas, in contrast, two factors conspire against the development of a well-developed inertial range: the abundance of linear dissipation mechanisms (Landau damping, in particular), which limit the minimum excitable scale; and the nature of the forcing, which is often better modeled by a self-limiting linear growth-rate term than by an external random forcing that can be made arbitrarily large.[11] This situation unfortunately limits the amount of practically useful asymptotics that can be done, but fortunately focuses the plasma-turbulence problem more toward the relatively well-understood issue of *transport due to the energy-containing modes*. Thus in the present article relatively little is said about intermittency and higher-order statistics. That is probably the most important way in which this review fails to adequately describe state-of-the-art research activities in fundamental neutral-fluid turbulence. Nevertheless, I shall briefly introduce in Sec. 10 (p. 220) a few relevant topics, including some recent results on the probability density function (PDF) for forced Burgers turbulence, which both serves as an illustrative mathematical example and arises naturally in a variety of physical applications.

1.2.5 Interesting mean-field dynamics

In the absence of boundaries or with periodic boundary conditions, the NSE admits solutions with zero mean velocity. The Vlasov description of plasmas, however, is based on a particle PDF, which like all PDF's can be considered (Sec. 3.5.1, p. 59) to be the nonvanishing mean of a singular microdensity (Sec. 2.2.2, p. 27). At its core the plasma requires (at least) a mean-field theory.

Even when the fluid description of plasmas is adequate (as in fusion contexts it frequently is), the electromagnetic nature of plasmas guarantees a mean-field dynamics richer than that of the neutral fluid. Magnetic-field geometries, mean electric fields, background profiles (of density, temperature, and flow velocity), microturbulence, transport, and extremely complicated boundary conditions are intricately coupled. Bifurcation scenarios have been identified—first experimentally, then analytically—in which abrupt transitions between regimes of low (L) and high (H) energy confinement times occur because of subtle changes in macroscopic conditions. The details are so complex and applications-specific that those scenarios are largely beyond the scope of this article in spite of their great practical importance. Nevertheless, a few words on mean-field dynamics can be found in Sec. 12.6 (p. 245). Additional discussion about the importance and modeling of inhomogeneity effects in plasmas is given by Yoshizawa et al. (2001).

1.3 Why a theory of plasma turbulence is needed

Although this article is primarily about the systematic mathematical description of plasma turbulence, it is useful to be aware of the practical applications, experimental data, and intuitive dynamical considerations that motivate the development of more formal theories.

1.3.1 Transport in plasmas

From the point of view of practical applications such as fusion, the most important output of a theory of turbulence is the rate of turbulence- or fluctuation-induced *transport* of some macroscopic

[11] However, Rose (2000) has pointed out that in the context of laser–plasma interactions a useful (non-self-consistent) model is to prescribe the rate of injection of energy into Langmuir waves in a spatially coherent, temporally incoherent manner.

quantity such as density n, momentum $\boldsymbol{p}$, or temperature T. Typically these quantities are assumed to have macroscopic variations in the x direction,[12] such as $-\partial_x \ln\langle n\rangle \doteq L_n^{-1} \doteq \kappa_n$. (The mean values of those fields are said to be the *background profiles*; L_n is called the *density scale length*.) The advective contribution to the spatial particle flux $\mathbf{\Gamma}$ in the x direction is

$$\Gamma_x = \langle \delta V_x(\boldsymbol{x},t)\delta n(\boldsymbol{x},t)\rangle = \sum_{\boldsymbol{k}} \langle \delta V_{x,\boldsymbol{k}}(t)\delta n^*_{\boldsymbol{k}}(t)\rangle, \tag{3a,b}$$

where δV_x is the fluctuation in the x component of the fluid velocity (usually the $\boldsymbol{E}\times\boldsymbol{B}$ drift $\boldsymbol{V}_E \doteq c\boldsymbol{E}\times\widehat{\boldsymbol{b}}/B$, where $\widehat{\boldsymbol{b}}$ is the unit vector in the direction of the magnetic field) and a discrete spatial Fourier transform was introduced [see Appendix A (p. 262) for conventions]. Formula (3a) shows that the flux can be evaluated from an equal-time Eulerian[13] cross correlation function. The form (3b) shows that transport is determined by the following properties of the turbulent (usually steady-state) fluctuations: (i) *wave-number spectra* (including both shape and overall intensity); and (ii) the *phase shift* between the advecting velocity and the advected quantity. It is important to note that although turbulent flux is directly specified by a *second-order* correlation function, higher-order correlations enter indirectly in determining the actual spectra and phase shift, according to the discussion of the statistical closure problem in Sec. 1.1 (p. 9).

If the transport is sufficiently local in space and time [a question still actively debated; see Sec. 12.4 (p. 241)], it can be described by a Fick's law[14] such as $\Gamma_x = -D\,\partial_x\langle n\rangle$. The particle transport coefficient D can be estimated from the random-walk formula (Uhlenbeck and Ornstein, 1930; Chandrasekhar, 1943) $D \sim \Delta x^2/2\Delta t$, where Δt is a characteristic time and Δx is the typical spatial step taken during that time. More precisely, when classical dissipation is very small D can in principle be calculated from the pioneering formula of Taylor (1921):

$$D = \int_0^\infty \mathrm{d}\tau\,\langle \delta V_x(\tau)\delta V_x(0)\rangle, \tag{4}$$

where $\delta V(\tau)$ denotes the Lagrangian dependence $\delta V(\tau) \doteq \delta V(\boldsymbol{x}(\tau),\tau)$, $\boldsymbol{x}(\tau)$ being the actual turbulent trajectory of a fluid element. Formula (4) can be estimated as $D \sim \overline{V}^2\tau_{\mathrm{ac}}$, where $\overline{V}$ is the rms velocity fluctuation and τ_{ac} is a Lagrangian correlation time. Alternatively, a Lagrangian *mixing length* ℓ can be defined by $\ell \doteq \overline{V}\tau_{\mathrm{ac}}$; then $D \sim \overline{V}\ell$. Frequently $\overline{V}$ and ℓ can be estimated by dimensional considerations [Appendix B (p. 264), but see the warning in footnote 175 (p. 141)].

[12] Unfortunately, this convention, universally adopted in plasma-physics research, differs from the one used for geophysical and laboratory flows, for which macroscopic variations are taken to be in the y direction. The coordinate system usually used for neutral-fluid shear flows—namely, $x \equiv$ streamwise, $y \equiv$ inhomogeneity, $z \equiv$ spanwise—translates *via* a cyclic permutation of -1 to $z \equiv$ magnetic-field direction, $x \equiv$ inhomogeneity, $y \equiv$ orthogonal or poloidal direction.

[13] By definition, in an *Eulerian correlation function* the space and time coordinates are specified independently. Eulerian correlations are (in principal) easy to measure in the laboratory, for example by inserting probes. In contrast, in a *Lagrangian correlation function* the spatial variable is evaluated along the time-dependent trajectory of a fluid element. Lagrangian correlations are very difficult to measure experimentally. One technique is optical tagging (Skiff et al., 1989); the state of the art was reviewed by Skiff (1997). Unfortunately, that is ineffective in the very hot cores of large experiments; appropriate measurement and visualization techniques are still under development.

[14] When self-consistency effects are important, this statement is an oversimplification; see Sec. 6.5 (p. 170).

However, the difficulties of precisely calculating such Lagrangian quantities are severe in general (Lumley, 1962; Kraichnan, 1964a; Weinstock, 1976). Much of the statistical theory describeqd in this article can be viewed as addressing this point in one way or another.

There are both "classical" and turbulent contributions to transport. For plasmas, classical contributions are defined to be those stemming from Coulomb collisions between two discrete particles (in the presence of dielectric shielding) and are well understood; see Sec. 2.3.2 (p. 30). They are formally described by the Balescu–Lenard collision operator [Eq. (32)]; the Landau (1936) approximation to that operator [Eq. (34)] is usually used in practice. Classical transport was reviewed by Braginskii (1965).

For transport across a strong magnetic field, the particle transport [15] is intrinsically ambipolar (a consequence of momentum conservation): $D_\perp = \rho_e^2 \nu_{ei} = \rho_i^2 \nu_{ie}$, where $\rho_s \doteq v_{ts}/\omega_{cs}$ is the gyroradius of species s and $\nu_{ss'}$ is the collision frequency for momentum exchange between species s and s'. Here $v_{ts} \doteq (T_s/m_s)^{1/2}$ and $\omega_{cs} \doteq q_s B/m_s c$ are the thermal velocity and gyrofrequency, respectively.

Classical discreteness effects are very small in hot plasmas. A representative value for the neoclassical ion thermal diffusivity in the experimental Tokamak Fusion Test Reactor (TFTR; Grove and Meade, 1985) was [16] 5 cm^2/s; observed values were several orders of magnitude greater. Such findings suggest, in agreement with the theory to be introduced in Sec. 1.3.3 (p. 16), that even in the limit $\epsilon_p \to 0$ collective effects remain, involving a turbulent mixture of waves and eddies that usually strongly dominate the transport.

1.3.2 Selected observational data on plasma turbulence

In magnetized plasmas *Bohm diffusion*, with coefficient

$$D_B \doteq cT_e/eB, \tag{5}$$

provides an important reference level for turbulent transport across a strong magnetic field. In describing experiments on arc plasma discharges, Bohm (1949) actually wrote $D = 10^4 T/B$ (T in eV and B in kG), which is usually quoted as $D = \frac{1}{16} D_B$. He did not provide a theoretical justification for the factor of $\frac{1}{16}$; see Spitzer (1960) for some related discussion. For parameters typical of envisaged fusion reactors, D_B is unacceptably large. Early experiments on the stellarator configuration (Miyamoto, 1978) were plagued by diffusion with the Bohm level and magnetic-field scaling. Taylor (1961) proved that Bohm's formula represents the maximum value that cross-field diffusion can attain, and it was feared that turbulence-induced transport in all magnetic confinement devices might achieve that maximum. Fortunately, later and more detailed experimental and theoretical research revealed that the situation need not be so dire. In particular, the magnitude of transport in modern tokamak configurations is several orders of magnitude below the Bohm level. As we will see in the next section,

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[15] In the toroidal field configurations characteristic of magnetic confinement, the classical formulas must be corrected to include *neoclassical* enhancements (Galeev and Sagdeev, 1967; Rosenbluth et al., 1972; Hinton and Hazeltine, 1976) due to magnetically trapped particles that execute so-called "banana" orbits (Furth, 1975).

[16] Representative TFTR parameter values as quoted by Redi et al. (1995) for shot #67 241 were major radius $R = 2.6$ m, minor radius $a = 0.96$ m, toroidal magnetic field $B = 4.5$ T, central electron density $n_{e0} = 3.8 \times 10^{13}\ \text{cm}^{-3}$, and central electron temperature $T_{e0} = 6.0$ keV.

simple theory typically predicts[17] a *gyro-Bohm* scaling [Eq. (6)], although controversy remains about the proper form of the experimentally observed scaling with B (Perkins et al., 1993).

Recent experiments have shown that it is possible to dramatically suppress microturbulence in the core of TFTR-scale plasmas by working in so-called *enhanced reversed shear* operating regimes (Levinton et al., 1995). Although extremely important from a practical point of view, such details are beyond the scope of this article, which addresses how to quantitatively calculate what happens when turbulence is actually present.

1.3.3 Introduction to drift-wave transport

Elementary considerations can be used to illustrate formula (3) and motivate the need for a theory of strong plasma turbulence. If $\boldsymbol{E} \times \boldsymbol{B}$ advection occurs in a region with density scale length L_n, it is reasonable to believe that self-limiting density fluctuations δn can grow no larger than $\nabla_\perp \delta n \sim \nabla_\perp \langle n \rangle$—i.e., if $k_\perp$ is a typical fluctuation wave number, then[18] $\delta n / \langle n \rangle \lesssim (k_\perp L_n)^{-1}$. Because electrons stream rapidly along field lines, they tend to establish a perturbed Boltzmann response [frequently called *adiabatic*; see footnote 48 (p. 35)]: $\delta n / \langle n \rangle \approx e\, \delta\varphi / T_e$, where φ is the electrostatic potential. An estimate of the fluctuating $\boldsymbol{E} \times \boldsymbol{B}$ velocity then leads to $\delta V_E \lesssim V_*$, where $V_* \doteq (cT_e/eB) L_n^{-1}$ is the *diamagnetic velocity.* Alternatively, the characteristic nonlinear advection frequency $\delta\omega \sim k_\perp \delta V_E$ is of the order of the *diamagnetic frequency* $\omega_* \doteq k_y V_*$: $\delta\omega \lesssim \omega_*$. Drift waves (Krall, 1968) with characteristic frequencies $\omega \lesssim \omega_*$ are ubiquitous in confined plasmas with profile gradients (Horton, 1984, 1999). Order-unity line broadening, $\delta\omega/\omega = O(1)$, is one characteristic of a strong-turbulence regime.

One can now estimate the flux (3a) by (temporarily and incorrectly) assuming maximal correlation between δV_E and δn; then $\Gamma \sim \delta V_E\, \delta n \sim V_* (k_\perp L_n)^{-1} \langle n \rangle \sim (V_*/k_\perp)(\langle n \rangle / L_n) = -D\, \partial_x \langle n \rangle$, where $D \doteq (k_\perp \rho_\mathrm{s})^{-1} (\rho_\mathrm{s}/L_n) D_B$. Here $\rho_\mathrm{s} \doteq c_\mathrm{s}/\omega_{ci}$, where $c_\mathrm{s} \doteq (ZT_e/m_i)^{1/2}$ is the *sound speed* and Z is the atomic number, is called the *sound radius* (the ion gyroradius computed with the electron temperature); it was introduced here as a convenient normalization, but appears naturally in more elaborate theories involving the ion polarization drift [see the derivation of the Hasegawa–Mima equation in Sec. 2.4.3 (p. 34)]. Theory (essentially dimensional analysis) suggests that $k_\perp \rho_\mathrm{s}$ is characteristically of order unity, in which case

$$D \sim (\rho_\mathrm{s}/L_n) D_B \tag{6}$$

—so-called *gyro-Bohm scaling.* In practice $\rho_\mathrm{s}/L_n \ll 1$ so $D \ll D_B$, in agreement with recent experimental observations.

The result (6) also follows from random-walk considerations if one estimates $\Delta x \sim k_\perp^{-1}$ and $\Delta t \sim (k_\perp V_*)^{-1} \sim \omega_*^{-1}$. But this approach highlights some uncertainties. It is not clear why the real mode frequency should determine the autocorrelation time. If an oscillator of frequency Ω is damped at the rate η, the area under the response curve, a measure of τ_ac, scales with[19] $(\eta/\Omega)\Omega^{-1}$ for $\eta/\Omega \ll 1$,

[17] More elaborate theories can be compatible with a variety of scalings; see the concluding paragraph of Appendix B (p. 266).

[18] This estimate is called the *mixing-length level* and is frequently attributed to Kadomtsev (1965). The overly simplified discussion does not distinguish between electron and ion densities, which in the absence of polarization-drift and finite-Larmor-radius effects are approximately equal by quasineutrality.

[19] Specifically, $\mathrm{Re} \int_0^\infty \mathrm{d}\tau\, \exp(-\mathrm{i}\Omega\tau - \eta\tau) = \eta/(\Omega^2 + \eta^2)$.

introducing the possibility of additional parametric dependence [frequently η, which ultimately arises from nonlinear effects, is proportional to the linear growth rate[20] γ, since it is γ that excites the turbulence] and suggesting that the estimate (6) is merely an upper bound. One of the goals of a proper theory of strong turbulence is to systematically determine such nonlinear damping rates and thus to quantify the proper autocorrelation time to be used in calculations of transport coefficients.

1.3.4 An extended quote from Kadomtsev (1965)

Most of the points made so far in this introductory discussion are far from new. Consider, for example, the following extended quotation from the Introduction to the seminal, 35-year-old monograph on plasma turbulence by Kadomtsev (1965)[21] (italics added):

"It is well known that a real plasma is rarely quiescent; as a rule many forms of noise and oscillation arise spontaneously in the plasma. Langmuir pointed out that these fluctuations represent more than just harmless oscillations about an equilibrium position and often wholly determine the character of the phenomena occurring in the plasma. ...

"Experiments with plasmas in a magnetic field and in particular experiments on magnetic containment of a high temperature plasma in connection with controlled thermonuclear reactions have revealed further unexpected phenomena essentially connected with oscillations in the plasma. Prominent amongst these is the 'anomalous' diffusion of a plasma across a magnetic field. ...

"Following the work of Bohm [(1949)], who suggested that *the enhanced diffusion of a plasma is due to random oscillations of the electric field set up by an instability*, the term 'turbulence' has been increasingly applied to this process. ... [W]hen applying the term 'turbulence' to a plasma, it is used in a broader sense than in conventional hydrodynamics. If hydrodynamic turbulence represents a system made up of a large number of mutually interacting eddies, then *in a plasma we have together with the eddies (or instead of them), also the possible excitation of a great variety of oscillations.* ...

"During the eddy motion of an ordinary fluid the separate eddies, in the absence of their mutual interaction, do not propagate in space. When their interaction is included the eddies 'spread out' in space with time, though the corresponding velocity is not large and therefore each separate eddy has a considerable time available to interact with its neighbours. In this case we are faced with a strong interaction of excitations and correspondingly with a *strong turbulence*. On the other hand, during a wave motion the separate wave packets can separate from one another over large distances. In this case the interaction of separate wave packets with one another is weak, and we can therefore refer to a *weak turbulence*. The motion of the plasma in the weakly turbulent state, constituting a system of weakly correlated waves, shows greater similarity to the motion of the wavy surface of the sea or the oscillations of a crystal lattice than to the turbulent motion of an ordinary fluid.

"The theoretical consideration of a weakly turbulent state is considerably facilitated by the possibility of applying perturbation theory, i.e., *an expansion in terms of a small parameter such as the ratio between the energy of interaction between the waves and their total energy.* ...

"For the case of very small amplitude, when the interaction between the oscillations can be neglected, one can use the so-called *quasi-linear approximation in which only the reaction of the oscillations on the average velocity distribution function of the particles is considered.* ...

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[20] It is worth emphasizing that calculations of linear growth rates in realistic confinement geometries are far from trivial, sometimes requiring large-scale codes (Rewoldt et al., 1982) and hours of supercomputer time.
[21] Kadomtsev's review was first published in Russian in 1964. I cite the 1965 English translation because it was that book that strongly influenced the predominantly Western plasma-physics research reviewed in this article.

"Unfortunately the quasi-linear method has only a fairly narrow field of application, since non-linear interaction of the oscillations already begins to play a considerable part at not very large amplitudes. ...

"In the simplest variant of the *kinetic wave equation*, only three-wave processes are considered, namely, the decay of the wave $\boldsymbol{k}, \omega$ into two waves $\boldsymbol{k}', \omega'$ and $\boldsymbol{k}'', \omega''$, and the merging of two waves into one. Such processes are important only for dispersion relations $\omega_{\boldsymbol{k}} = \omega(\boldsymbol{k})$ for which it is possible to satisfy simultaneously the laws of conservation of energy and of momentum: $\boldsymbol{k}'' = \boldsymbol{k} - \boldsymbol{k}'$, $\omega_{\boldsymbol{k}''} = \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'}$. When these conditions are not satisfied, scattering of the waves by the particles is a more important process and can be taken into account only on the basis of a full kinetic theory. ...

"Unfortunately, in numerous practical cases one is faced not by weak but by strong turbulence. In particular, *strong turbulence is related to an anomalous diffusion of the plasma across the magnetic field. To determine the fluctuation spectrum in a strongly turbulent plasma and the effect of these fluctuations on the averaged quantities, it is sometimes possible to use the analogy with ordinary hydrodynamics and, in particular, to apply a phenomenological [mixing-length] description of the turbulent motion.* ...

"However, in a plasma other strongly turbulent motions which are different from the eddy motion of an ordinary fluid may develop. *It is therefore desirable to have available more systematic methods for describing strong turbulence. In our view, such a method may be the weak coupling approximation*[22] .... In this approximation, ... the turbulent motion is described by a system of non-linear integral equations for the spectral density $I_{\boldsymbol{k}\omega}$ and the Green's function $\Gamma_{\boldsymbol{k}\omega}$ describing the response of the system to an external force. As the coupling between the oscillations decreases, this system of equations goes over into the kinetic wave equation.

"In conventional hydrodynamics, the weak coupling equations have been obtained by Kraichnan [(1959b)] who showed that in their simplest form the weak coupling equations lead to a spectrum which is different from Kolmogorov's spectrum in the region of large k. As will be shown ..., the reason is that *in Kraichnan's equations the adiabatic character of the interaction of the short wave with the long wave pulsations is not taken into account.* The consideration of this adiabatic interaction makes it possible to obtain improved weak coupling equations. ...

"As we have mentioned earlier, the turbulent diffusion problem goes back to Bohm [(1949)], who put forward the hypothesis that *an inhomogeneous plasma in a magnetic field must always be unstable because of the presence of a drift current of the electrons relative to the ions.* If this be in fact so, the corresponding instability must lead to *a turbulent ejection of the plasma with a velocity of the order of the drift velocity.* According to Bohm, this process can be considered phenomenologically as a diffusion with coefficient of diffusion of the order $D_B = 10^4 T/H$, where T is the electron temperature in electron volts and H the magnetic field in kilogauss.

"Bohm's argument gave rise to the illusion of a universal validity for this coefficient and as a result attempts to obtain Bohm's coefficient from more general considerations have continued to this day.[23] *It has now become evident, however, that the coefficient of turbulent diffusion cannot be obtained without a detailed investigation of the instability of an inhomogeneous plasma and in particular of its drift instability.* ..."

In large measure the research reviewed in the present article provides a mathematical systematization of the physical concepts emphasized by Kadomtsev, particularly for the description of strong plasma turbulence. I discuss quasilinear theory in Sec. 4.1 (p. 90), weak-turbulence theory in Sec. 4.2 (p. 98), and the direct-interaction approximation (Kadomtsev's weak-coupling approximation)

[22] Kadomtsev is referring to the direct-interaction approximation of Kraichnan (1959b); see Sec. 5 (p. 126) for detailed discussion.

[23] One model that leads unambiguously to Bohm diffusion is the 2D thermal-equilibrium *guiding-center plasma* (Taylor and McNamara, 1971). Of course, a turbulent plasma is not in thermal equilibrium, and the true physics is three dimensional.

in Sec. 5 (p. 126). Facets of drift-wave turbulence are discussed throughout the article: for example, fundamental equations in Sec. 2.4 (p. 33); basic nonlinear dynamics and statistical closures in Sec. 8.4 (p. 206); submarginal turbulence mechanisms in Sec. 9 (p. 210); and the wave kinetic equation in Appendix G.3 (p. 291). I shall use the previous quotation as a focus for my concluding remarks in Sec. 13 (p. 251).

1.4 Nonlinear dynamics and statistical descriptions

Experimental observations make it abundantly clear that plasmas can be turbulent. Some indication of the physical and mathematical problems that must be described and solved by a satisfactory theory of plasma turbulence may be found by surveying the many years of research on neutral fluids (Monin and Yaglom, 1971; Frisch, 1995) and recent advances in the theory of nonlinear dynamical systems (Lanford, 1982; Lichtenberg and Lieberman, 1992; Meiss, 1992; Ott, 1993). It is now well known that such systems can exhibit extreme sensitivity to small changes in initial conditions; that observation is sometimes used to justify various statistical assumptions such as mixing or ergodicity (Zaslavskiĭ and Chirikov, 1972). Gibbsian thermal-equilibrium solutions can be found for the Euler equation [24] truncated to a finite number of Fourier modes. Although extensive research on the NSE in the presence of forcing and dissipation shows that actual turbulent steady states are far from equilibrium, statistical methods have made substantial inroads. A fundamental difficulty with the statistical approach is that nonlinear systems can also display a tendency toward self-organization (Hasegawa, 1985). Certain fluid equations admit the possibility of soliton solutions; in plasmas, the Vlasov equation can support Bernstein–Greene–Kruskal (BGK) modes (Bernstein et al., 1957). Nonlinear systems can be intermittent; i.e., fluctuations can be distributed sparsely in space, with turbulent patches intermixed with laminar ones. This observation argues against theories pinned too closely to Gaussian Ansätze. On the other hand, pronounced intermittency does not seem to dominate most laboratory plasmas. [25] Extensive analyses of diagnostic and simulation data seem to support the basic ideas of random-walk processes, so one anticipates that reasonable quantitative predictions of transport coefficients should be possible at least for idealized situations. Some general remarks on the relationships between statistical closures and nonlinear dynamics were made by Krommes (1984b). Many of the topics mentioned in this paragraph have been discussed in the context of plasmas by Horton and Ichikawa (1996).

Ruelle (1976) stated that "[i]t would be a miracle if the usual procedure of imposing stationarity . . . and looking for a Gaussian solution would lead to results much related to physics." Indeed, a simple argument given in Sec. 3.8.1 (p. 71) proves that the fluctuations in forced, dissipative steady states cannot be exactly Gaussian. Nevertheless, that does not preclude the possibility that some sort of quasi-Gaussian hypothesis may be useful if it is applied in an appropriate way. In fact, although the renormalized statistical moment closures described later retain selected terms of all orders in non-Gaussian statistics, their structure can both technically and heuristically be understood in terms of a perturbation theory based on statistics that are Gaussian at lowest order. It is true that in workable closures the vast majority of all terms is omitted, so it is not surprising that success depends on the question. Experience shows that conventional low-order moment-based closures can be strikingly

[24] The *Euler equation* is defined to be the NSE with the dissipation set to zero.

[25] Nevertheless, experimental observations clearly show that some intermittency is present. For more discussion, see Sec. 12.4.3 (p. 244).

successful for predictions of fluxes or spectra even for quite intermittent fluctuations [see Fig. 1 (p. 41) and Sec. 8.5 (p. 208)]. Nevertheless, predictions of entire PDF's require more sophisticated treatments; see, for example, Sec. 10.4.3 (p. 227).

So far the discussion has focused on states of fully developed turbulence. The theory of nonlinear dynamics is of particular importance in discussing the *transition to turbulence*; for some general references, see Eckmann (1981) and Ott (1981). Although transition is largely outside the scope of this article, the discussion I give in Sec. 9 (p. 210) of mechanisms for *submarginal turbulence* is best understood in that context. A somewhat related although much less detailed scenario was proposed by Manheimer et al. (1976) and Manheimer and Boris (1977); they suggested that information about turbulent transport can be obtained by assuming that the steady-state profiles sit at *linear marginal stability*. Further discussions of that hypothesis and its relation to nonlinear dynamics are given in Secs. 9 (p. 210) and 12.4 (p. 241).

Clearly, the ultimate theory of turbulence will unify robust statistical predictions with detailed understanding of the underlying nonlinear dynamics. Some of the useful techniques and results in this direction were discussed by Bohr et al. (1998).

1.5 Resonance, nonresonance, averaging procedures, and renormalized statistical dynamics

From the outset, plasma physics must face head-on the linear particle propagator g_0 [Eq. (2c)] and especially its nonlinear generalizations. Distinctive formalisms and lines of research can be classified by how nonlinear corrections to the free-streaming motion are treated. In *weak-turbulence theory* (WTT) waves are the central entities. In the presence of a wave-induced potential, nonresonant particles nonsecularly oscillate around a streaming *oscillation center*, which can be defined with the aid of perturbative transformations that remove interaction terms in the Hamiltonian to desired order. Those averaging procedures can be viewed as special cases of general ensemble-averaging techniques and thus nominally fall within the purview of this article. Although a complete account cannot be given here, modern averaging techniques in the form of Lie transforms are reviewed briefly in Appendix C (p. 267); they have important applications to the derivations of the gyrokinetic and gyrofluid equations that underpin the modern description of strongly magnetized plasmas. The role of the oscillation center in quasilinear theory is discussed in Sec. 4.1.3 (p. 95).

A particle resonant with a single wave can be trapped; if multiple waves are present, stochasticity is likely to ensue and the resonant particles are likely to diffuse. A description of resonant diffusion is not accessible from the nonresonant averaging procedures. Furthermore, in the presence of stochastic diffusion, an intrinsically nonlinear phenomenon, the Landau resonance is broadened, rendering for sufficiently strong turbulence the clean distinction (2c) between resonant and nonresonant particles useless; resonant and nonresonant effects are inevitably mixed together in nonlinear regimes. The formal renormalized statistical methods to be described later handle those general regimes naturally; they are a significant advance over the approaches based on nonresonant perturbations. With the aid of such formulations one can smoothly pass from a fully kinetic formalism to a strongly turbulent fluid description, displaying at least at the formal level a beautiful unification of the statistical dynamics of the physically disparate plasma and neutral fluid. Some of the material related to this topic [Secs. 6.5.4 (p. 176) and 6.5.5 (p. 178)] is published here for the first time.

1.6 Outline of the article

In the present article I shall describe the progress that has been made on systematic statistical formalisms for plasma turbulence, with much reference to earlier corresponding work on neutral fluids. No claim is made that statistical theories as they presently exist are adequate to describe the full range of random and coherent nonlinear phenomena known to be exhibited by plasmas, or that such theories can usefully be applied to the calculations of turbulent transport in complicated practical situations. It is clear that large-scale direct numerical simulations will continue to play an important role in quantifying turbulence phenomenology, as will various nonstatistical analytical theories; however, those are subjects for other review articles.

The subsequent discussion is organized as follows. I begin in Sec. 2 (p. 22) by introducing a representative sampling of the fundamental equations that are generally used for descriptions of plasma turbulence. In Sec. 3 (p. 46) I give an elementary introduction to the statistical theory of turbulence, including both rigorous statistical mechanics and phenomenology [Secs. 3.1 (p. 46) through 3.8 (p. 71)] as well as statistical closure techniques (Sec. 3.9, p. 76). Section 4 (p. 90) is devoted to a review of the historical development of statistical theories of turbulent plasmas, including quasilinear theory (QLT; Sec. 4.1, p. 90), weak-turbulence theory (WTT; Sec. 4.2, p. 98), resonance-broadening theory (RBT; Sec. 4.3, p. 108), and the clump formalism (Sec. 4.4, p. 119); conceptual difficulties with the last approach are stressed. In Sec. 5 (p. 126) Kraichnan's seminal direct-interaction approximation (DIA), a theory of strong turbulence, is discussed in detail. I turn in Sec. 6 (p. 146) to the generating-functional formalism of Martin, Siggia, and Rose (MSR; 1973), which provides a most elegant unification of much earlier research and many clumsy techniques. A brief history of renormalization techniques is given in Sec. 6.1 (p. 147). The functional apparatus for closure is described in Sec. 6.2 (p. 153); the DIA emerges as the natural[26] lowest-order MSR renormalization. Non-Gaussian effects are discussed in Sec. 6.3 (p. 165). A path-integral representation of MSR theory is given in Sec. 6.4 (p. 166). In Sec. 6.5 (p. 170) the MSR formalism is used to derive a formally exact expression for the nonlinear plasma dielectric function. The reductions of the renormalized equations to QLT, WTT, and RBT are indicated, and the important role of self-consistency between the particles and the fields is stressed. In Sec. 7 (p. 181) various approaches alternative to that of MSR are described briefly, including Lagrangian methods (Sec. 7.1, p. 181), Markovian approximations (Sec. 7.2, p. 182), the use of eddy viscosity and the statistical description of interactions of disparate scales (Sec. 7.3, p. 189), renormalization-group techniques (Sec. 7.4, p. 196), and statistical decimation (Sec. 7.5, p. 197). Section 8 (p. 199) is devoted to modern developments in the statistical description of plasmas, including the development and numerical solutions of a new realizable Markovian statistical closure, the RMC. An introduction to the as yet poorly understood topic of submarginal turbulence is given in Sec. 9 (p. 210). Some topics relating to higher-order statistics, intermittency, and coherent structures are discussed in Sec. 10 (p. 220). Variational methods for bounds on transport are described in Sec. 11 (p. 230). A variety of miscellaneous topics are considered in Sec. 12 (p. 235), including Onsager symmetries for turbulence (Sec. 12.1, p. 235), the interpretation of various entropy balances (Sec. 12.2, p. 238), statistical determination of mode-coupling coefficients (Sec. 12.3, p. 241), self-organized criticality (Sec. 12.4, p. 241), percolation theory (Sec. 12.5, p. 245), some discussion of the role and dynamics of mean fields (Sec. 12.6, p. 245), and brief remarks on zonal flows and other long-wavelength fluctuations (Sec. 12.7, p. 248). The present state of affairs and prospects

[26] As Martin (1976) has stressed, one should not infer that the formalism is a panacea.

for the future are assessed in the concluding Sec. 13 (p. 251). Fourier-transform conventions are recorded in Appendix A (p. 262). A brief review of dimensional and scaling analysis is given in Appendix B (p. 264). In Appendix C (p. 267) aspects of the derivations of nonlinear gyrokinetic and gyrofluid equations (central in modern plasma-turbulence research) are described, with particular focus on methods based on modern Hamiltonian dynamics and Lie perturbation theory. In Appendix D (p. 279) Chirikov criteria for stochasticity are discussed. Some formal aspects of RBT are considered in Appendix E (p. 281). The generalization of spectral balance equations to include weak inhomogeneity is described in Appendix F (p. 286). Details of the derivation of the Vlasov weak-turbulence wave kinetic equation from the renormalized spectral balance equation are provided in Appendix G (p. 288). Some miscellaneous discussion of Gaussian functionals is given in Appendix H (p. 293). Salient features of the author's `DIA` code, which has been used to study various problems in statistical plasma physics, are described in Sec. I (p. 295). Steady-state solutions to a pedagogical three-mode version of the EDQNM closure are derived in Appendix J (p. 297). Finally, notation is summarized in Appendix K (p. 302).

An overview of selected papers discussed in this review is presented in Figs. 34 (p. 260) through 36 (p. 262) as a chronological time line that attempts to put into perspective the relatively recent plasma-physics research, frequently derivative with respect to seminal work in other fields. The reader may wish to scan those figures now, then return to them at the conclusion of the article. One cannot help but be impressed by the broad scope and very high quality of the pioneering research on neutral fluids, to which plasma physics owes a very large debt.

2 FUNDAMENTAL EQUATIONS AND MODELS FOR PLASMA DYNAMICS

"Unaware of the scope of simple equations, man has often concluded that nothing short of God, not mere equations, is required to explain the complexities of the world." — *Feynman et al. (1964).*

Turbulence is intrinsically a nonlinear phenomenon. Although much of the formalism to be discussed below will be quite general, it must ultimately be applied to specific nonlinear equations. Therefore I discuss in the present section some of the important equations and models of plasma dynamics.[27] First, however, it is useful to introduce the fundamental equations of neutral-fluid turbulence.

2.1 Fundamental equations for neutral fluids

I shall consider two extreme paradigms for the descriptions of neutral fluids: the incompressible Navier–Stokes equation (Sec. 2.1.1, p. 23); and the infinitely compressible Burgers

[27] The emphasis on the analysis of specific nonlinear equations already distinguishes the thrust of this article from much of the experimental focus in fusion physics, in which the goal, not yet entirely successful, is to *deduce* the operative model from appropriate diagnostics. [A somewhat related technique, in which the effective mode-coupling coefficients of the turbulent plasma are inferred from experimental statistics, is described in Sec. 12.3 (p. 241).]

equation (Sec. 2.1.2, p. 25).

2.1.1 The Navier–Stokes equation

The continuity equation for mass density ρ_m is $\partial_t \rho_m + \boldsymbol{\nabla}\cdot(\rho_m \boldsymbol{u}) = 0$, where $\boldsymbol{u}$ is the fluid velocity. I consider incompressible flows, $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$; that constraint permits a constant-density solution.[28] The incompressible *Navier–Stokes equation* (NSE) is then

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\rho_m^{-1}\boldsymbol{\nabla}p + \mu_{\rm cl}\nabla^2\boldsymbol{u}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{7a,b}$$

Here p is the pressure and $\mu_{\rm cl}$ is the classical kinematic viscosity (taken to be constant). Given the incompressibility constraint (7b), one can determine p as a functional of $\boldsymbol{u}$ by taking the divergence of Eq. (7a) and solving the resulting Poisson equation[29]

$$\rho_m^{-1}\nabla^2 p = -\boldsymbol{\nabla}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}). \tag{8}$$

In the absence of dissipation ($\mu_{\rm cl} \equiv 0$), Eq. (7a) is called the *Euler equation*.

If in Eq. (7a) the velocity is scaled to a typical velocity $\overline{u}$ and lengths are measured with respect to a macroscopic scale L, then $\mu_{\rm cl}$ is replaced by $\mathcal{R}^{-1}$, where the unique dimensionless parameter $\mathcal{R}$ (frequently written as Re) is called the *Reynolds number*:

$$\mathcal{R} \doteq \overline{u}L/\mu_{\rm cl}. \tag{9}$$

Further discussion of $\mathcal{R}$ is given in Sec. 3.4.2 (p. 57).

When Eq. (7a) is subjected to nontrivial boundary conditions, turbulence can arise for sufficiently large $\mathcal{R}$; see many interesting visualizations in van Dyke (1982) and discussions by Lesieur (1997). Upon averaging Eq. (7a) over many realizations of the turbulence, one finds the equation for the mean velocity $\boldsymbol{U} \doteq \langle\boldsymbol{u}\rangle$ to be $\partial_t\boldsymbol{U} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{U} = \rho_m^{-1}\boldsymbol{\nabla}\cdot\mathsf{T}$, where $\mathsf{T} \doteq -P\mathsf{I} + 2\rho_m\mu_{\rm cl}\mathsf{S} + \boldsymbol{\tau}$ and

$$\mathsf{S} \doteq \tfrac{1}{2}[(\boldsymbol{\nabla}\boldsymbol{U}) + (\boldsymbol{\nabla}\boldsymbol{U})^T], \quad \boldsymbol{\tau} \doteq -\rho_m\langle\delta\boldsymbol{u}\,\delta\boldsymbol{u}\rangle. \tag{10a,b}$$

S is called the *rate-of-strain tensor*. The very important nonlinear term $\boldsymbol{\tau}$ is called the *Reynolds stress*[30] (Reynolds, 1895). It describes the effects of the fluctuations on the mean flow (through turbulent transport of momentum, or turbulent viscosity); a good discussion was given by Tennekes and Lumley (1972). In general, if one is to understand the macroscopic flow, one must either compute or approximate $\boldsymbol{\tau}$. There are analogs of $\boldsymbol{\tau}$ in all of the plasma equations to be described shortly. For some discussion of nontrivial mean fields, see Sec. 12.6 (p. 245).

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[28] In theoretical discussions of incompressible turbulence, the constant ρ_m is frequently taken to be 1. Although I shall usually retain ρ_m in the subsequent formulas for dimensional purposes, the equations will be generally correct only for the case of constant ρ_m.

[29] Note that the solution of Eq. (8) gives p as a *spatially nonlocal* functional of $\boldsymbol{u}$. The effects of boundaries are felt everywhere in the fluid; if the boundary is slightly changed, the effect is instantaneous. Physically, such influences are transmitted by sound waves that propagate with infinite phase velocity in the incompressible limit (which corresponds to zero Mach number).

[30] $\boldsymbol{\tau}$ is frequently defined without the minus sign. However, the present convention is correct for a positive stress.

To learn about properties of the fluctuations, it is useful to study energetics. Let the total velocity be broken into mean and fluctuating parts, $\boldsymbol{u} = \boldsymbol{U} + \delta\boldsymbol{u}$. Then the fluctuations obey

$$\partial_t \delta\boldsymbol{u} + \delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{U} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\delta\boldsymbol{u} + \delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\delta\boldsymbol{u} - \langle\delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\delta\boldsymbol{u}\rangle = \rho_m^{-1}\boldsymbol{\nabla}\cdot(-\delta p\,\mathsf{I} + 2\rho_m\mu_{\rm cl}\delta\mathsf{s}), \qquad (11)$$

where $\delta\mathsf{s}$ is given by formula (10a) with $\boldsymbol{U}$ replaced by $\delta\boldsymbol{u}$. Upon dotting Eq. (11) with $\delta\boldsymbol{u}$ and averaging the result, one obtains [see, for example, Tennekes and Lumley (1972)] a balance equation for the turbulent energy density $\mathcal{E} \doteq \frac{1}{2}\rho_m\langle|\delta\boldsymbol{u}|^2\rangle$:

$$\partial_t\mathcal{E} = -\boldsymbol{\nabla}\cdot\boldsymbol{W} + \mathcal{P} - \mathcal{D}, \qquad (12)$$

where

$$\boldsymbol{W} \doteq \langle\delta\boldsymbol{u}(\tfrac{1}{2}\rho_m|\delta\boldsymbol{u}|^2 + \delta p)\rangle - 2\rho_m\mu_{\rm cl}\langle\delta\mathsf{s}\cdot\delta\boldsymbol{u}\rangle, \quad \mathcal{P} \doteq \boldsymbol{\tau}{:}\,\mathsf{S}, \quad \mathcal{D} \doteq 2\rho_m\mu_{\rm cl}\langle\delta\mathsf{s}{:}\,\delta\mathsf{s}\rangle. \qquad (13a,b,c)$$

$\boldsymbol{W}$ is the *energy flux*, $\mathcal{P}$ is called the *production* (of fluctuations), and $\mathcal{D}$ (clearly positive definite) is called the *dissipation*. ($\mathcal{D}$ is a generic symbol applicable to all such balance equations; for the NSE, the dissipation is conventionally represented by ε.) Production involves the interaction of the Reynolds stresses with the mean strain; it is clearly a property of the large scales ($\mathcal{P} \sim U^3/L$).[31] Dissipation, however, is negligible at the large scales. Furthermore, under a spatial average with reasonable boundary conditions the divergence of the energy flux vanishes, leading to the space-averaged balance equation[32]

$$\partial_t\overline{\mathcal{E}} = \overline{\mathcal{P}} - \overline{\mathcal{D}}. \qquad (14)$$

These observations strongly suggest that steady turbulent states are achieved by a flow of energy from the large scales to the small ones. [This conclusion is correct for 3D, but must be modified for 2D; see Sec. 3.8.3 (p. 74).]

For situations with nontrivial boundary conditions that can support a mean strain, $\mathcal{P}$ serves as the driving term whereby turbulent fluctuations are excited. Analysis of this drive requires detailed study of the turbulent dynamics of the *energy-containing fluctuations*. Sometimes, however, as when one is studying properties of the very small scales of turbulence, such large-scale details are of no concern. In those cases it is common to impose periodic boundary conditions on Eq. (7a) (so that the mean field vanishes), but then to include a random, solenoidal Gaussian forcing $\boldsymbol{f}^{\rm ext}(\boldsymbol{x}, t)$ on the right-hand side of the NSE (7a) in order to ensure that a steady state is maintained against the viscous damping. If $\boldsymbol{f}^{\rm ext}$ is intended to model the production terms, then its spectral support ought to be at long wavelengths at least in 3D.

It is frequently useful to introduce the *vorticity*[33] $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$, a measure of the local circulation of the flow. An alternate representation of Eq. (7a) is

$$\partial_t\boldsymbol{u} = \boldsymbol{u}\times\boldsymbol{\omega} - \rho_m^{-1}\boldsymbol{\nabla}(p + \tfrac{1}{2}\rho_m u^2) + \mu_{\rm cl}\nabla^2\boldsymbol{u}, \qquad (15)$$

[31] One estimates $\overline{u} \doteq \langle\delta u^2\rangle^{1/2} \sim U$, a dimensional result that assumes fluctuations are excited by interactions with the macroscopic flow. More generally, it is better to write $\mathcal{P} \sim \overline{u}^3/L$, which embraces the case (discussed in the next paragraph) in which the mean field vanishes and the system is stirred externally.

[32] Such spatially averaged balance equations form the starting point for variational approaches to rigorous upper bounds on transport; see Sec. 11 (p. 230).

[33] Some remarks on the significance of vorticity were made by Saffman (1981).

where the identity $\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \boldsymbol{\nabla}(\frac{1}{2}u^2)-\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ was used. The vorticity equation follows immediately as

$$\partial_t\boldsymbol{\omega} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega}) + \mu_{\rm cl}\nabla^2\boldsymbol{\omega}, \tag{16a}$$

which with the identity $\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}\boldsymbol{\nabla}\cdot\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{A} - (\boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}\boldsymbol{B})$ together with $\boldsymbol{\nabla}\cdot\boldsymbol{u} = \boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$ reduces to

$$\partial_t\boldsymbol{\omega} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \mu_{\rm cl}\nabla^2\boldsymbol{\omega}. \tag{16b}$$

In a 2D approximation in which $\partial_z = 0$, so $\boldsymbol{\omega} = \omega\,\widehat{\boldsymbol{z}}$, the vortex-stretching term $\boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ on the right-hand side of Eq. (16b) vanishes and one arrives at the 2D vorticity equation

$$\partial_t\omega + \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = \mu_{\rm cl}\nabla^2\omega. \tag{17}$$

The equations of strongly magnetized plasma are closely related to Eq. (17) because of the 2D nature of the $\boldsymbol{E}\times\boldsymbol{B}$ velocity. Because that equation conserves all powers of vorticity on the average in the absence of dissipation whereas Eq. (16b) does not, there are important differences between the dynamics of two- and three-dimensional turbulence.

It is natural to formulate a balance equation for energy $\mathcal{E}$ because the nonlinear terms in the NSE conserve $\mathcal{E}$ under spatial averaging; one says that $\mathcal{E}$ is a quadratic *nonlinear invariant.* A second quadratic invariant is the *fluid helicity* [34]

$$\mathcal{H} \doteq \rho_m\overline{\boldsymbol{u}\cdot\boldsymbol{\omega}}, \tag{18}$$

as can be shown from Eqs. (15) and (16a). This quantity vanishes identically in 2D; in 3D it vanishes for homogeneous, isotropic turbulence with mirror symmetry. That $\mathcal{H}$ does not vanish for isotropic turbulence without mirror symmetry is important in the theory of the magnetic dynamo (Sec. 2.4.8, p. 43).

2.1.2 Burgers equation

It is also useful to mention Burgers's nonlinear diffusion equation (Burgers, 1974)

$$\partial_t u(x,t) + u\,u_x - \mu_{\rm cl}\,u_{xx} = f^{\rm ext}(x,t), \tag{19}$$

written here with external forcing. Although this equation is reminiscent of the NSE, it is really very different since it is missing the nonlocal pressure term; the Burgers equation describes infinitely compressible turbulence and is local. It is generally studied in 1D, [35] but can be formulated in higher dimensions as well. It is sometimes used to test statistical approximations to the NSE in a simpler context, is of interest in its own right (it displays a tendency to form shocks and highly intermittent

[34] Helicity is related to the degree of linkage or knottedness of vortex lines, as discussed, for example, by Moffatt (1969).

[35] The 1D initial-value problem for the Burgers equation can be solved exactly by means of the *Cole–Hopf transformation* $u = -2\mu_{\rm cl}\,\partial_x\ln v$, which reduces Eq. (19) to the *linear* diffusion equation for v. Nevertheless, the form of the solution is not useful for performing statistical averages over random forcing or initial conditions.

states), and also arises in certain plasma applications (Sec. 2.4.7, p. 42) and in modern work on self-organized criticality (Sec. 12.4, p. 241). It is the first spatial derivative of the Kardar–Parisi–Zhang (KPZ) equation (Kardar et al., 1986)

$$\partial_t h + \tfrac{1}{2}h_x^2 - \mu_{\mathrm{cl}} h_{xx} = f_h^{\mathrm{ext}}, \tag{20}$$

where h is the fluctuation in the height of an interface: $u = -\partial_x h$. Some general discussion of the Burgers equation, interface dynamics, and their relation to turbulence was given by Bohr et al. (1998). Some recent results on Burgers intermittency are described in Sec. 10.4.3 (p. 227).

2.2 Exact dynamical equations of classical plasma physics

In the classical approximation, which is adequate for a wide variety of physical applications, the N-particle plasma can be described exactly by the set of $2Nd$ coupled (scalar components of) Newton's laws. In rare situations those can actually be numerically integrated as they stand [the so-called molecular-dynamics approach; see, for example, Verlet (1967), Hansen et al. (1975), and Evans and Morris (1984)], but the N^2 scaling of the operation count makes this direct approach impractical for large numbers of particles[36]; in a modern tokamak such as TFTR (footnote 16, p. 15), N might approach Avogadro's number (approximately 6×10^{23}). Clever particle simulation techniques for weakly coupled plasmas (Birdsall and Langdon, 1985) achieve an $O(N)$ scaling, but cannot be discussed here. An alternate analytical approach is to consider first exact, then approximate equations for low-order PDF's of the particles. Two equivalent and formally exact descriptions are used: the *Liouville equation*, for the smooth N-particle PDF in the so-called Γ space of all $2Nd$ coordinates and momenta; and the *Klimontovich equation*, for the singular density in the so-called μ space of a typical particle whose coordinates are $\boldsymbol{x}$ and $\boldsymbol{v}$ and whose species is s. For some early textbook discussions of plasma kinetic theory, see Montgomery and Tidman (1964) and Montgomery (1971b). A recent book that includes some related material is by Balescu (1997).

2.2.1 Liouville equation

The *Liouville equation* has been described in many textbooks [for example, Hoover (1991) or Evans and Morris (1990)]; it has the general form

$$\frac{\partial}{\partial t} f_N(\underline{1}, \underline{2}, \ldots, \underline{N}, t) + \sum_{i=1}^{N} \frac{\partial}{\partial z_i}(\dot{z}_i f_N) = 0, \tag{21}$$

where f_N is the N-particle PDF, z stands for $\{\boldsymbol{x}, \boldsymbol{v}, s\}$, and $\underline{1} \equiv z_1$. (The set $\{z_1, t_1\}$ is frequently denoted just by 1; the underline denotes all coordinates except time.) Upon successively integrating over all but 1, 2, ..., $N-1$ coordinates, one is led to the conventional Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy that links the one-particle PDF f, the pair correlation function g, the triplet correlation function h, and so on; a particularly elegant discussion was given by Ecker (1972).[37] As the BBGKY form of the statistical hierarchy will not be used explicitly in the

[36] Tractable situations arise, among other places, in the theory and application of non-neutral plasmas; see, for example, Dubin and O'Neil (1988a) and Dubin and Schiffer (1996).
[37] A BBGKY formalism for fluid turbulence was given by Montgomery (1976).

following analysis, I shall not write it here. Nevertheless, it is well worth noting that the Liouville equation is *linear*, and that both it and the derived BBGKY hierarchy are *time reversible*. It may seem remarkable that such equations can adequately describe dissipative turbulence; the issue has been discussed by Orszag (1977). The resolution is that the characteristics of the Liouville equation (Newton's second laws of motion) are highly nonlinear and can exhibit stochasticity; if the hierarchy is closed by any reasonable kind of coarse-graining or statistical averaging procedure, nonlinear and dissipative equations result.

2.2.2 Klimontovich equation

The *Klimontovich equation* (Klimontovich, 1967)[38] evolves the microscopic μ-space density

$$\widetilde{N}_s(\boldsymbol{x},\boldsymbol{v},t) \doteq \frac{1}{\overline{n}_s}\sum_{i=1}^{N_s}\delta(\boldsymbol{x}-\widetilde{\boldsymbol{x}}_i(t))\delta(\boldsymbol{v}-\widetilde{\boldsymbol{v}}_i(t)), \tag{22}$$

where tildes denote random variables, s is a species label, N_s is the number of particles of species s, and $\overline{n}_s \doteq N_s/V$ is the mean density of species-s particles in the volume V. By time-differentiating Eq. (22), one finds

$$[\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla} + (\widetilde{\boldsymbol{E}} + c^{-1}\boldsymbol{v}\times\widetilde{\boldsymbol{B}})\cdot\boldsymbol{\partial}_s]\widetilde{N}_s = 0, \tag{23}$$

where $\boldsymbol{\partial} \equiv (q/m)\partial_{\boldsymbol{v}}$ and $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{B}}$ are the microscopic electric and magnetic fields. Self-energy interactions [terms $i = j$ in the product $\widetilde{N}(\boldsymbol{x},\boldsymbol{v},t)\widetilde{N}(\boldsymbol{x}',\boldsymbol{v}',t) \sim \sum_i\sum_j$] are to be discarded in the nonlinear term. This equation is appealing for several reasons: its mathematical characteristics are the equations of motion, so it is closely related to numerical "particle-pushing" schemes (Hu and Krommes, 1994); and it is quadratically nonlinear (since $\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{B}} \propto \widetilde{N}$ according to Maxwell's equations), so it fits neatly into conventional analytical theories of turbulence (Rose, 1979).

For simplicity I shall mostly ignore magnetic fluctuations,[39] so $\widetilde{\boldsymbol{B}} = \boldsymbol{B}$ is an externally specified background magnetic field. Effectively, we will work in the electrostatic approximation.

The Klimontovich density is normalized[40] such that

$$\langle\widetilde{N}(\underline{1},t)\rangle = f(\underline{1},t), \quad \langle\delta\widetilde{N}(\underline{1},t)\,\delta\widetilde{N}(\underline{1}',t)\rangle = \overline{n}_1^{-1}\delta(\underline{1},\underline{1}')f(1) + g(\underline{1},\underline{1}',t), \tag{24a,b}$$

etc. Here $\delta(\underline{1},\underline{1}')$ is the product of a Kronecker delta function in the species indices and a Dirac delta function in the other phase-space coordinates. The average of Eq. (23) then reproduces the first

[38] Although plasma physicists tend to cite Klimontovich (1967), the equation is well known to kinetic theorists generally. Martin et al. (1973) confusingly called it the "Liouville equation."

[39] This is done solely for pedagogical purposes. In no way should this be taken to imply that magnetic fluctuations are never important (Mackay, 1841, p. 304). They are obviously important in radiation phenomena (Dupree, 1964), but can also play a role in transport by altering the dispersion properties of the linear waves ("finite–β effects") or by inducing stochastic diffusion of magnetic field lines (Rechester and Rosenbluth, 1978; Krommes et al., 1983).

[40] In finite volumes there are normalization subtleties, relating to the distinction between "generic" and "specific" distribution functions, that cannot be discussed here; see, for example, Schram (1966) and Ecker (1972).

member of the BBGKY hierarchy,

$$\mathrm{D}f/\mathrm{D}t = -\boldsymbol{\partial}\cdot\langle\delta\boldsymbol{E}\,\delta N\rangle = -\boldsymbol{\partial}\cdot\widehat{\boldsymbol{\mathcal{E}}}g, \tag{25}$$

where the Vlasov operator is

$$\mathrm{D}/\mathrm{D}t \doteq \partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla} + (\boldsymbol{E} + c^{-1}\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{\partial} \tag{26}$$

and $\widehat{\boldsymbol{\mathcal{E}}}$ is the linear operator that determines $\widetilde{\boldsymbol{E}}$ from $\widetilde{N}$. In more detail, $\widetilde{\boldsymbol{E}}$ is determined from Poisson's equation $\boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{E}} = 4\pi\widetilde{\rho}$. The solution can thus be represented by $\widetilde{\boldsymbol{E}} = \widehat{\boldsymbol{\mathcal{E}}}\widetilde{N} \equiv \int \mathrm{d}t' \sum_{s'} \int \mathrm{d}\boldsymbol{x}' \int \mathrm{d}\boldsymbol{v}' \times \widehat{\boldsymbol{\mathcal{E}}}_{ss'}(\boldsymbol{x},\boldsymbol{v},t;\boldsymbol{x}',\boldsymbol{v}',t')\widetilde{N}_{s'}(\boldsymbol{x}',\boldsymbol{v}',t')$. Explicitly, the Fourier transform of $\widehat{\boldsymbol{\mathcal{E}}}$ with respect to $\boldsymbol{x}-\boldsymbol{x}'$ is the non-Hermitian kernel

$$\widehat{\boldsymbol{\mathcal{E}}}_{ss',\boldsymbol{k}}(\boldsymbol{v},t;\boldsymbol{v}',t') = \boldsymbol{\epsilon}_{\boldsymbol{k}}(\overline{n}q)_{s'}\delta(t-t'), \tag{27}$$

where $\boldsymbol{\epsilon}_{\boldsymbol{k}} \doteq -4\pi\mathrm{i}\boldsymbol{k}/k^2$ is the Fourier transform of the field of a unit point particle.

It must be stressed that the g term on the right-hand side of Eq. (25) contains *all* fluctuation-related effects, both those related to discrete-particle effects (classical n-body collisions) and all possible turbulence-related nonlinear collective processes.

2.3 Nonlinear kinetic equations for plasmas

In many laboratory situations one has $\epsilon_p \ll 1$; such plasmas are almost collisionless. The continuum limit $\epsilon_p \to 0$ can be thought of as arising from a *chopping process* (Rostoker and Rosenbluth, 1960) in which particles are successively divided in two, doubling the density n and halving the particle charge q and mass m at each step. This preserves the charge density nq and the charge-to-mass ratio q/m, so the plasma frequency $\omega_p \doteq (4\pi nq^2/m)^{1/2}$ is invariant under the rescaling. In order to also preserve the thermal velocity v_t, a natural statistical measure that remains relevant in the collisionless limit, the temperature must be halved. Then the Debye length λ_D is also invariant; note $\lambda_D\omega_p = v_t$. The chopping process is thus consistent with the orderings $n = O(\epsilon_p^{-1})$, $\{q,m,T\} = O(\epsilon_p)$, $\{\omega_p, v_t, \lambda_D\} = O(1)$. For example, one quickly finds that the minimum impact parameter $b_0 \doteq e^2/T$ is $O(\epsilon_p^2/\epsilon_p) = O(\epsilon_p)$; the previously mentioned result (footnote 10, p. 12) that the strong-coupling parameter Γ is $O(\epsilon_p^{2/3})$ also follows immediately.

2.3.1 Collisionless kinetic equations

In the limit $\epsilon_p \to 0$ explicit discreteness terms, such as those involving b_0, can be dropped from the BBGKY hierarchy, giving the *Vlasov cumulant hierarchy* (Davidson, 1967, 1972). That is an infinite coupled hierarchy of n-point cumulants in which the first member retains the form (25) (it is *not* the Vlasov equation), but the evolution equations for g and higher cumulants do not contain ϵ_p. Although it is common in the literature to initiate a theory of "Vlasov turbulence" by writing $f = \langle f\rangle + \delta f$, that makes no sense in the context of the hierarchy because f is a PDF, hence is already averaged.[41] What is intended is to write the *Klimontovich* density as the sum of mean and fluctuating parts,

[41] Rose (2000) has stressed that a different point of view may be useful. Because Vlasov dynamics are highly nonlinear, they may be susceptible to symmetry-breaking perturbations that lead to solutions that are not asymptotically time-translation invariant in a strong sense; i.e., they are turbulent. If the symmetry is broken

$\widetilde{N} = f + \delta N$, then to work in the continuum limit $\epsilon_p \to 0$. When fluctuations arising from both particle discreteness and collective effects are completely negligible, g can be ignored and one recovers the *Vlasov equation* $\mathrm{D}f/\mathrm{D}t = 0$, a mean-field theory.

The Vlasov equation is necessary for the description of collisionless phenomena with frequencies comparable to or higher than the gyrofrequency $\omega_c \doteq qB/mc$. For lower-frequency phenomena, however, the Vlasov equation is unwieldy, both analytically and numerically, and it is better to turn to a *gyrokinetic* (GK) description in terms of the particle *gyrocenters*, which move slowly across the magnetic field with the $\boldsymbol{E} \times \boldsymbol{B}$ drift $\boldsymbol{V}_E$ and the magnetic (gradient and curvature) drifts (Chandrasekhar, 1960; Spitzer, 1962; Northrop, 1963b; Bernstein, 1971)

$$\boldsymbol{V}_d \doteq \omega_c^{-1}[\tfrac{1}{2}v_\perp^2 \widehat{\boldsymbol{b}} \times \boldsymbol{\nabla} \ln B + v_\parallel^2 \widehat{\boldsymbol{b}} \times (\widehat{\boldsymbol{b}} \cdot \boldsymbol{\nabla})\widehat{\boldsymbol{b}}]. \tag{28}$$

In a straight, constant magnetic field $\boldsymbol{B} = B\widehat{\boldsymbol{z}}$, for which the magnetic drifts vanish, the simplest collisionless *nonlinear gyrokinetic equation* (GKE; Frieman and Chen, 1982; Lee, 1983; Dubin et al., 1983) is (Appendix C, p. 267)

$$\frac{\partial F}{\partial t} + v_\parallel \frac{\partial F}{\partial z} + \overline{\boldsymbol{V}}_E \cdot \boldsymbol{\nabla}_\perp F + \left(\frac{q}{m}\right) \overline{E}_\parallel \frac{\partial F}{\partial v_\parallel} = 0, \tag{29a}$$

where $F(\boldsymbol{x}, \mu, v_\parallel, t)$ is the PDF of gyrocenters, μ is the magnetic moment [an adiabatic invariant (Appendix C.1.1, p. 267) that is conserved in the GK approximation], and the overlines indicate the effective (gyration-averaged) fields seen by the gyrocenters. In the Fourier representation $\overline{\varphi}_{\boldsymbol{k}} \doteq J_0(k_\perp v_\perp/\omega_{ci})\varphi_{\boldsymbol{k}}$; to the extent that $k_\perp v_\perp/\omega_{ci} \neq 0$ ($J_0 \neq 1$), one refers to finite-Larmor-radius (FLR) effects. To Eq. (29a) must be adjoined the *gyrokinetic Poisson equation* (Appendix C.1.6, p. 274)

$$\nabla^2\varphi + \epsilon_\perp \nabla_\perp^2 \varphi = -4\pi e(Zn_i^{\mathrm{G}} - n_e) \tag{29b}$$

(for overall charge neutrality, $\overline{n}_e = Z\overline{n}_i$), where $\epsilon_\perp \approx \omega_{pi}^2/\omega_{ci}^2$ is the perpendicular dielectric constant (Chandrasekhar, 1960) of the so-called *gyrokinetic vacuum* (Krommes, 1993a, 1993c) that describes the effects of the ion polarization drift velocity (Chandrasekhar, 1960)

$$\boldsymbol{V}_i^{\mathrm{pol}} \doteq \omega_{ci}^{-1}\partial_t(c\boldsymbol{E}_\perp/B). \tag{30}$$

The importance of the GK formulation cannot be overstated; it provides the basis for much modern analytical theory and a huge body of numerical simulations. Derivations of the GKE are reviewed in Appendix C.1 (p. 267).

Typically $\epsilon_\perp$ is large; the condition $\omega_{pi}^2/\omega_{ci}^2 \gg 1$ defines the so-called *gyrokinetic regime* (Krommes et al., 1986) that is relevant for fusion plasmas. The opposite regime $\omega_{pi}^2/\omega_{ci}^2 \ll 1$ is called the *drift-kinetic regime*. It is described by the *drift-kinetic equation*, in which $J_0 \to 1$ and $\nabla^2\varphi = -4\pi e(Zn_i - n_e)$. Finally, when parallel motion is completely ignored one obtains the *guiding-center plasma model*

$$\partial_t F + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}_\perp F = 0. \tag{31}$$

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by external noise, it may be possible to let the strength of the noise approach zero and to recover a fluctuating distribution with nonzero variance. By analogy with Ising and other models that undergo symmetry-breaking transitions to ordered states (Forster, 1990), it may be simpler to consider symmetry-breaking perturbations to Vlasov dynamics than to begin with the full N-body ensemble.

That model [closely related to the dynamics of point vortices moving in two dimensions (Kraichnan, 1975b)] prominently figured in early attempts to understand the nonlinear behavior of plasma dynamics in strong magnetic fields (Taylor and McNamara, 1971; Vahala and Montgomery, 1971).[42]

For general magnetic fields the operator ∂_z in Eq. (29a) must be generalized to $\widehat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}$. One can then analyze several important situations: (i) the effects of (background) *magnetic shear*; (ii) self-consistent *magnetic perturbations*; and (iii) spatially and/or temporally *random magnetic fields*. Magnetic shear has important practical consequences that are largely beyond the scope of this review, although see Sec. 12.6.3 (p. 246). Magnetic fluctuations arise from currents according to Maxwell's equations. For sufficiently small β (the ratio of plasma pressure to magnetic-field pressure), the currents are parallel to $\boldsymbol{B}$ and it is sufficient to derive the perturbed magnetic field from just the parallel component of the vector potential $\boldsymbol{A}$: $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{\nabla} \times (A_\parallel \widehat{\boldsymbol{z}})$. The resulting approximation describes *field-line bending* and the evolution of *shear-Alfvén waves*, but ignores compressional Alfvén waves. The assumption of a *random* perturbed magnetic field is a useful device[43] that enables one to assess the consequences of particle transport in systems with broken flux surfaces (Krommes et al., 1983; vanden Eijnden and Balescu, 1996).

2.3.2 Collisional kinetic equations

In classical, weakly coupled ($0 < \epsilon_p \ll 1$) plasma kinetic theory (Montgomery and Tidman, 1964), a kinetic equation analogous to the Boltzmann equation can be derived from various physical and/or formal points of view. For simplicity I assume $\boldsymbol{B} = 0$. Fokker–Planck techniques (Chandrasekhar, 1943) that take into account the physics of the motion of shielded test particles[44] lead (Ichimaru, 1973) to a physically clear derivation of the *Balescu–Lenard* (BL) *equation* (Balescu, 1960; Lenard, 1960) $Df_s/Dt = -C_s[f]$, where

$$C_s[f] \doteq -\frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\pi \left(\frac{q^2}{m} \right)_s \sum_{\overline{s}} (\overline{n} q^2)_{\overline{s}} \int \mathrm{d}\overline{\boldsymbol{v}} \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3} \frac{\boldsymbol{\epsilon}_{\boldsymbol{k}} \boldsymbol{\epsilon}_{\boldsymbol{k}}^*}{|\mathcal{D}^{\mathrm{lin}}(\boldsymbol{k}, \boldsymbol{k}\cdot\boldsymbol{v})|^2} \delta(\boldsymbol{k}\cdot(\boldsymbol{v} - \overline{\boldsymbol{v}})) \right.$$

[42] The guiding-center model [see, for example, Joyce and Montgomery (1973) and Kraichnan (1975b)] is an interesting dynamical system worthy of a review of its own. Nevertheless, because of space constraints and the practical importance of parallel motion, discussion of this model is limited in the present article to only a few scattered remarks and references.

[43] In the context of particle transport in stochastic magnetic fields, the application of standard statistical techniques to passive-advection kinetic equations with random magnetic fields was first done by Krommes et al. (1983). See also Rosenbluth et al. (1966) for a seminal paper on the quasilinear description of stochastic magnetic fields.

[44] The concept of a shielded test particle is one of the central ideas of plasma kinetic theory. Formally, it arises by expanding the BBGKY hierarchy or the equivalent Klimontovich equation in the small plasma discreteness parameter ϵ_p and proving that to lowest order the natural entity is a discrete test particle surrounded by its (Vlasov-continuum) shielding cloud. Early work was by Rostoker and Rosenbluth (1960). The results—notably, the *Test Particle Superposition Principle* (Rostoker, 1964a, 1964b)—are described both in formal treatises on plasma kinetic theory (Montgomery and Tidman, 1964; Montgomery, 1971b) and in elementary textbooks (Krall and Trivelpiece, 1973). More modern proofs of the Superposition Principle, based on the two-time statistical hierarchy discussed by Krommes and Oberman (1976a), were given by Krommes (1976). Turbulent generalizations are discussed in Sec. 6.5.3 (p. 173).

$$\cdot \left(f_{\overline{s}} \frac{1}{m_s} \frac{\partial f_s}{\partial \boldsymbol{v}} - \frac{1}{m_{\overline{s}}} \frac{\partial f_{\overline{s}}}{\partial \overline{\boldsymbol{v}}} f_s \right) \Bigg] \tag{32}$$

and

$$\mathcal{D}^{\text{lin}}(\boldsymbol{k}, \omega) \doteq 1 + \sum_{\overline{s}} \frac{\omega_{\text{p}\overline{s}}^2}{k^2} \int \mathrm{d}\overline{\boldsymbol{v}} \, \frac{\boldsymbol{k} \cdot \partial \bar{f}_{\overline{s}} / \partial \overline{\boldsymbol{v}}}{\omega - \boldsymbol{k} \cdot \overline{\boldsymbol{v}} + \mathrm{i}\epsilon} \tag{33}$$

is the linear dielectric function. Although the present article is not a review of classical kinetic theory (Frieman, 1967, 1969; Montgomery, 1967, 1971b), it is nevertheless useful to recall the interpretation of the various parts of Eq. (32), as such understanding has provided key insights and motivations (some misguided) for various developments of plasma turbulence theory. Dimensional analysis reveals that $C \sim \sigma \overline{n} v$, where $\sigma \doteq b_0^2$ is the classical collisional cross section, $\overline{n}$ is the mean density of scatterers, and v is a characteristic relative velocity. The term in $\overline{f}(\partial_{\boldsymbol{v}} f)$ describes *velocity-space diffusion*; the term in $(\partial_{\overline{\boldsymbol{v}}} \overline{f}) f$ represents the *polarization drag* or *self-consistent backreaction* of the test-particle-induced shielding cloud on the test particle; it is responsible for conservation of momentum and kinetic energy. (The proper treatment of self-consistency in plasma turbulence theory is a recurring theme throughout this article.) The spatial and dynamical structure of the total shielded test particle is represented by $\mathcal{D}^{\text{lin}}(\boldsymbol{k}, \omega)$, with ω evaluated at the characteristic transit frequency $\boldsymbol{k} \cdot \boldsymbol{v}$ of a particle moving with velocity $\boldsymbol{v}$ and impact parameter $b = k^{-1}$. The $\delta(\boldsymbol{k} \cdot (\boldsymbol{v} - \overline{\boldsymbol{v}}))$ arises because the scattering is computed perturbatively (quasilinear approximation) using straight-line orbits as the zeroth-order approximation. The $\boldsymbol{k}$ integration describes the accumulated effect of particles with a distribution of impact parameters $b \sim k^{-1}$.

Equation (32) does not contain the effects of large-angle two-body collisions (Boltzmann's contribution), so a large-wave-number cutoff at b_0^{-1} is required.[45] However, because of the presence of dielectric shielding, the wave-number integral can be shown to possess a natural infrared wave-number cutoff at $k \approx k_D$. One then recovers the *Landau collision operator* (Landau, 1936)

$$C_{s,\overline{s}}[f] \doteq -2\pi \left(\frac{e^2}{m} \right)_s (\overline{n} e^2)_{\overline{s}} \ln \Lambda \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}\overline{\boldsymbol{v}} \, \mathsf{U}(\boldsymbol{v} - \overline{\boldsymbol{v}}) \cdot \left(\frac{1}{m_s} \frac{\partial}{\partial \boldsymbol{v}} - \frac{1}{m_{\overline{s}}} \frac{\partial}{\partial \overline{\boldsymbol{v}}} \right) f_s(\boldsymbol{v}) f_{\overline{s}}(\overline{\boldsymbol{v}}), \tag{34}$$

where[46] $\Lambda \doteq \lambda_D / b_0$ and $\mathsf{U}(\boldsymbol{u}) \doteq (\mathsf{I} - \widehat{\boldsymbol{u}}\widehat{\boldsymbol{u}})/|\boldsymbol{u}|$. This is the form that is generally used in most theoretical and computational analyses of classical (Braginskii, 1965) and neoclassical (Rosenbluth et al., 1972; Hinton and Hazeltine, 1976) transport.

More formally, the BL operator arises by inserting the *linearized* solution of the equation for the Klimontovich fluctuation $\delta \widetilde{N}$ into the right-hand side of Eq. (25) (Klimontovich, 1967; Wu, 1967; Montgomery, 1971b). The Green's function R_0 for the linearized Klimontovich equation obeys

$$g_0^{-1} R_0 + \boldsymbol{\partial} f \cdot \widehat{\boldsymbol{\mathcal{E}}} R_0 = 1, \tag{35}$$

[45] Various authors have derived uniformly valid collision operators that do not require a wave-number cutoff; see, for example, Frieman and Book (1963).

[46] The form $\Lambda = \lambda_D / b_0$ must be modified in the presence of large magnetic fields B and, at sufficiently high energies, to satisfy the Heisenberg uncertainty principle; the details are not important here. For very large B, so that $\rho \ll \lambda_D$, one must reconsider the derivation of the collision operator to take account of nuances of guiding-center transport. Representative calculations are by Dubin and O'Neil (1988b, 1997).

where $g_0 \doteq (\partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla})^{-1}$ is the free-particle Green's function (propagator) whose Fourier transform is Eq. (2c); the last term on the left-hand side of Eq. (35) describes the self-consistent polarization effect of the particles on the fields. Comparison with Eq. (26) reveals that R_0 is also Green's function for the linearized *Vlasov* equation, which explains why Vlasov dynamics figure so prominently in many-particle plasma kinetic theory. One can verify by direct calculation that

$$R_0 = g_0 - g_0 \boldsymbol{\partial} f \cdot (\mathcal{D}^{\mathrm{lin}})^{-1} \widehat{\boldsymbol{\mathcal{E}}} g_0, \quad \widehat{\boldsymbol{\mathcal{E}}} R_0 = (\mathcal{D}^{\mathrm{lin}})^{-1} \widehat{\boldsymbol{\mathcal{E}}} g_0. \tag{36a,b}$$

These are the formal statements that to lowest order in ϵ_p bare test particles [the first term on the right-hand side of Eq. (36a)] carry their shielding clouds (the second term) along with them. It is remarkable that Eqs. (36) generalize without change in form to fully renormalized turbulence theory. This crucial result is derived in some detail in Sec. 6.5 (p. 170).

The linearization implies that a vast host of nonlinear processes are ignored. First, of course, discrete n-particle collisions are ignored for $n \geq 3$, but those are very small for $\epsilon_p \ll 1$. More significantly, nonlinear collective phenomena, which exist even for $\epsilon_p \to 0$, are ignored; those comprise all of the turbulence effects (both weak and strong), including n–wave and wave–wave–particle interactions, resonance broadening, trapping, *etc.*

An alternate derivation (Frieman, 1967; Montgomery, 1967) of the BL operator proceeds from the BBGKY hierarchy by dropping the triplet correlation function in the equation for g. With the further neglect of the bare two-particle interaction term (large-angle scattering), the Green's function for the resulting left-hand side factors into the product of two one-particle (Vlasov) Green's functions, showing the equivalence to the linearized Klimontovich solution. [47]

In all approaches it is the full Vlasov response function R_0, including [Eq. (36a)] both free-streaming motion and self-consistent dielectric shielding, that underlies the Test Particle Superposition Principle (footnote 44, p. 30). A renormalized response function R prominently figures in the formal turbulence theories described in Secs. 5 (p. 126) and 6 (p. 146).

The gyrokinetic equation with collisions can be obtained by adding the appropriately gyro-averaged collision operator to the right-hand side of Eq. (29a).

The forms (32) and (34) of the classical collision operator provide initial insights about the problems to be faced by a theory of plasma turbulence. As in classical theory, one must deal with not only turbulent diffusion but also self-consistent backreaction. One may expect some sort of nonlinearly modified dielectric function to appear. Indeed, Mynick (1988) has advanced a "generalized Balescu–Lenard" (gBL) operator for Vlasov turbulence by asserting a precise analogy to Eq. (32). Nevertheless, one must be very cautious. For turbulent situations particles are scattered away from their free trajectories, collective effects dominate over discrete ones [calling into question the specific form of the shielding term in Eq. (32)], the actual form of the nonlinear dielectric function is very involved, and a formal description based on velocity-space effects may not be appropriate or practically useful when turbulent motions on hydrodynamic scales (long wavelengths and times) dominate the physics. All of these problems will be addressed and at least partially resolved later in the article.

In fact, a more appropriate transition to the formalism of strong plasma turbulence might be had by reviewing selected results from the theory of *strongly* coupled many-particle plasmas ($\epsilon_p \gtrsim 1$). Unfortunately, space limitations preclude such a discussion here; see, for example, Ichimaru (1992).

[47] A renormalized version of this factorization was used by Krommes and Oberman (1976a) to derive a renormalized plasma collision operator that included the effects of convective cells; see Sec. 5.10.1 (p. 143).

2.4 Nonlinear fluid models for plasmas

In many important situations the details of the kinetic (velocity-space) effects are unimportant, so fluid models are appropriate. I briefly describe some of the more important ones here, and shall return to some of them later in the article.

2.4.1 Introduction to the fluid closure problem

I have already mentioned the statistical closure problem, the central difficulty of turbulence theory. Interestingly, in attempting to derive fluid equations from the more fundamental kinetic equations, one encounters a *fluid closure problem* that is closely related to the statistical one. It has prominently figured in recent derivations of simulation models for tokamak turbulence and in a variety of other areas. I introduce the fluid closure problem in this section, deferring a more detailed discussion to Appendix C.2 (p. 276).

By definition, a fluid moment (such as density or momentum) is a weighted velocity integral of the kinetic PDF. The difficulty with obtaining closed fluid equations arises most fundamentally from the streaming term $\boldsymbol{v}\cdot\boldsymbol{\nabla}$ (Vlasov theory) or $v_\parallel\nabla_\parallel$ (gyrokinetic theory); in gyrokinetics additional closure problems arise from the dependence of the magnetic drifts and effective potential $\overline{\varphi}_{\boldsymbol{k}}$ on $\boldsymbol{v}$. The nth velocity moment of a kinetic equation such as $\partial_t F + v_\parallel \nabla_\parallel F + \cdots = 0$ is coupled to the moment of order $n+1$ by the streaming term. If only the explicitly shown terms are retained, a simple device makes this problem mathematically identical to the statistical closure problem for passive advection. Namely, write $F = (1+\chi)F_0$, where F_0 is a time-independent PDF that is often taken to be the Maxwellian distribution F_M. Then use Dirac notation to write $\chi F_0 \equiv |\,\chi\,\rangle$; also introduce a corresponding bra such that $\langle\,\psi\,| = \psi$. It is now natural to define a scalar product such that $\langle\psi \mid \chi\rangle \doteq \sum_{\overline{s}} \int \mathrm{d}\overline{\boldsymbol{v}}\, \psi_{\overline{s}}(\overline{\boldsymbol{v}})\chi_{\overline{s}}(\overline{\boldsymbol{v}})F_{0,\overline{s}}(\overline{\boldsymbol{v}})$. This scalar product is equivalent to an ensemble average taken with probability measure F_0; the fluctuating density is $\delta n_s = \langle \overline{n}_s \delta_{s,\overline{s}} \mid \chi\rangle$. When F_0 is Maxwellian, the velocity is Gaussianly distributed. The fluid closure problem for $\partial_t|\,\chi\,\rangle + v_\parallel\nabla_\parallel|\,\chi\,\rangle = 0$ with Maxwellian background is then formally equivalent to a problem of passive advection by a *time-independent* Gaussian velocity. As I shall describe in Sec. 3.3 (p. 52), exactly this model has previously been discussed in detail as a paradigm for the difficulties of statistical closure for strong turbulence (Kraichnan, 1961), so many results on strong-turbulence closures have immediate applicability to the fluid closure problem.

In classical transport theory the fluid closure problem is dealt with by the procedure of Chapman and Enskog (Chapman and Cowling, 1952), who exploited an asymptotic ordering in the inverse of the collision frequency to obtain rigorously closed equations (Braginskii, 1965) valid for timescales much longer than the collision time and wavelengths much longer than the collisional mean free path. This procedure cannot be justified for high-temperature plasmas, for which the collision frequency is very small. Chang and Callen (1992a,b) discussed a formally exact generalization. Hammett and Perkins (1990) advocated a more pragmatic approach in which unknown cumulants are parametrized in terms of known ones in such a way that the *linear* fluid response well matches the *linear* kinetic response. Because the collisionless dissipation mechanism is Landau damping [well understood (van Kampen and Felderhof, 1967) as a phase-mixing phenomenon], such closures are sometimes called *Landau-fluid closures*. There are relations to the theory of Padé approximants and orthogonal polynomial expansions, as discussed by Smith (1997). See Appendix C.2 (p. 276) for further discussion.

2.4.2 Zakharov equation

Zakharov (1972) derived the following system of equations for the slowly varying envelope ψ of nonlinear Langmuir oscillations and ion density n:

$$\nabla^2[\mathrm{i}(\partial_t + \nu_e) + \nabla^2]\psi = -\boldsymbol{\nabla} \cdot (n\,\boldsymbol{E}), \quad (\partial_t^2 + 2\nu_i\,\partial_t - \nabla^2)n = \nabla^2|\boldsymbol{E}|^2, \tag{37a,b}$$

where $\boldsymbol{E} \doteq -\boldsymbol{\nabla}\psi$ and ν_e and ν_i are linear damping terms. In the limit in which the time derivatives in Eq. (37b) may be neglected, this system reduces to the nonlinear Schrödinger equation (Benney and Newell, 1967), which in 1D and in the absence of linear damping is

$$\mathrm{i}\,\partial_t E + \nabla^2 E + (|E|^2 - \langle |E|^2\rangle)E = 0. \tag{38}$$

Such equations have been the subject of extensive investigation; generically, they exhibit strongly nonlinear, often coherent behavior including collapse, the formation of solitary solutions, *etc.* Unfortunately, much of that important research cannot be discussed here because of space limitations and a lack of perceived relevance to practical problems of strongly magnetized plasmas. [For more information and references, see the early reviews of Thornhill and ter Harr (1978) and Rudakov and Tsytovich (1978) as well as the more recent work of Dyachenko et al. (1992).] The equations do prominently figure in various problems of laser–plasma interactions, and important statistical analyses have been done of them; see, for example, DuBois and Rose (1981), Sun et al. (1985), and DuBois et al. (1988). A separate review article on related topics is warranted. A review of Langmuir turbulence was given by Goldman (1984); see also the short introduction to that subject by Similon and Sudan (1990).

2.4.3 Hasegawa–Mima and Terry–Horton equations

Equations of central importance to the fundamental theory of microturbulence in tokamaks and other systems with fluctuations driven by gradients of macroscopic parameters are the Hasegawa–Mima (HM) equation (HME) and its generalization, the Terry–Horton (TH) equation (THE). The HME is arguably the simplest generic description of the nonlinear behavior of drift waves. It is used in several places in this article to illustrate general theory.

The original derivation of the HME (Hasegawa and Mima, 1978) proceeded from fluid equations expressed in particle coordinates [often referred to in plasma physics as the equations of Braginskii (1965)]. However, it is far more concise and elegant to proceed from the GKE (29a) (Dubin et al., 1983). Temporarily, let us ignore particle discreteness (classical dissipation). For simplicity, consider the fluid limit $T_i \to 0$ (i.e., ignore FLR effects). Then, upon integrating Eq. (29a) over the velocity coordinates (μ and $v_\parallel$), one arrives at the continuity equation for the gyrocenter density:

$$\partial_t n^{\mathrm{G}} + \boldsymbol{\nabla} \cdot (\boldsymbol{V}_E\, n^{\mathrm{G}}) + \nabla_\parallel (u_\parallel^{\mathrm{G}} n^{\mathrm{G}}) = 0. \tag{39}$$

For the ions, large inertia suggests that $u_{\parallel i}^{\mathrm{G}}$ is negligible (this approximation is relaxed in more sophisticated descriptions that include ion sound propagation). One also assumes a frozen-in-time background (mean) density profile, varying in the x direction such that $-\partial_x \ln\langle n\rangle \doteq L_n^{-1} \equiv \kappa = \text{const}$, and recalls the definition of the diamagnetic velocity V_* given in Sec. 1.3.3 (p. 16). Then the ions obey $\partial_t(\delta n_i^{\mathrm{G}}/\langle n\rangle_i) + V_*\partial_y(e\delta\varphi/T_e) + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}(\delta n_i^{\mathrm{G}}/\langle n\rangle_i) = 0$. The electron response could also be analyzed from appropriate gyrofluid equations, as in Sec. 2.4.5 (p. 38). For present purposes, however, one simply asserts a linear, almost adiabatic response [48]:

$$(\delta n_e/\langle n \rangle_e)_{\boldsymbol{k},\omega} \approx (1 - \mathrm{i}\delta_{\boldsymbol{k}})(e\delta\varphi/T_e)_{\boldsymbol{k},\omega}. \tag{40}$$

(Electron polarization is negligible, so $n_e^{\mathrm{G}} \approx n_e$.) The last two equations can be combined with the GK Poisson equation (29b) to yield a closed equation for the electrostatic potential. Before doing so, however, it is usual to introduce a convenient set of normalized variables [called *gyro-Bohm normalization* after the scaling (6)] in which one normalizes velocities to c_{s}, perpendicular lengths to ρ_{s}, and times to ω_{ci}; this makes the normalized V_* equal to ρ_{s}/L_n. It is then reasonable to normalize $\delta\varphi$ to T_e/e (the natural units for a perturbed Boltzmann distribution) and δn to $\langle n \rangle$. This normalization is frequently used in gyrokinetic particle simulations. Nevertheless, an alternate normalization (also called gyro-Bohm) is possible. [49] If one anticipates that saturated fluctuation levels will scale with κ, it is reasonable to normalize $\delta\varphi$ to $(T_e/e)\kappa\rho_{\mathrm{s}}$; correspondingly, one normalizes δn to $\langle n \rangle\kappa\rho_{\mathrm{s}}$. In the dimensionless time $(\kappa\rho_{\mathrm{s}})\omega_{ci}t$, the equations are unchanged in form (with V_* equal to unity). In either normalization the resulting equations for the potential and density fluctuations are

$$\nabla_\perp^2 \delta\varphi = -(\delta n_i^{\mathrm{G}} - \delta n_e), \tag{41a}$$

$$\partial_t \delta n_i^{\mathrm{G}} + V_* \partial_y \delta\varphi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} \delta n_i^{\mathrm{G}} = 0, \tag{41b}$$

$$\delta n_e = (1 - \mathrm{i}\widehat{\delta})\delta\varphi. \tag{41c}$$

Upon inserting Eq. (41c) into Eq. (41a), one obtains

$$\delta n_i^{\mathrm{G}} = (1 + \widehat{\chi})\delta\varphi, \quad \text{where} \quad \widehat{\chi} \doteq -\nabla_\perp^2 - \mathrm{i}\widehat{\delta}. \tag{42a,b}$$

Finally, upon inserting Eq. (42a) into Eq. (41b), one obtains the equation of Terry and Horton (1982) [see also Horton and Ichikawa (1996)]:

$$(1 + \widehat{\chi})\partial_t \delta\varphi + V_* \partial_y \delta\varphi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}(\widehat{\chi}\, \delta\varphi) = 0. \tag{43}$$

(One noted that $\boldsymbol{V_E} \cdot \boldsymbol{\nabla} \delta\varphi = 0$.) In $\boldsymbol{x}$ space $\widehat{\delta}$ is a generally nonlocal operator (denoted by the caret), so $\widehat{\chi}$ is as well; Eq. (43) is more tractable in $\boldsymbol{k}$ space, where $\widehat{\delta}$ and $\widehat{\chi}$ become purely multiplicative.

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[48] "Adiabatic" is used here not in the thermodynamic sense but rather as "very slowly varying." In plasma kinetic theory *adiabatic response* refers to fluctuations whose frequencies obey $\omega \ll k_\parallel v_t$. In this limit particle density perturbations approach the Boltzmann distribution $\delta n/\langle n \rangle \approx q\, \delta\varphi/T$ by rapidly streaming along the magnetic field lines. Typically, electrons obey the adiabatic ordering $\omega \ll k_\parallel v_{te}$ whereas ions obey the opposite "fluid" ordering $\omega \gg k_\parallel v_{ti}$. In linear theory the small nonadiabatic correction $-\mathrm{i}\widehat{\delta}$ can be calculated by detailed solution of the GKE. Dimensionally $\widehat{\delta} = O(\gamma^{\mathrm{lin}}/\Omega)$, where γ^{lin} is the linear growth rate and Ω is the real frequency. In nonlinear theory the iδ *model* (40) [sometimes attributed to Waltz (1983)] is an approximation that has been criticized (Gang et al., 1991; Liang et al., 1993) on the reasonable grounds that nonlinear corrections to γ^{lin} are missing.

[49] For the general theory of dimensional analysis and scaling, see Appendix B (p. 264).

In Eq. (42b) the $\nabla_\perp^2$, inherited from the GK Poisson equation, describes the effect of the ion polarization drift. That is *not* an ion FLR effect, contrary to frequent assertions, as it survives in the limit $T_i \to 0$. Furthermore, it is not small. [50]

The statistical dynamics of such equations are strongly constrained by invariants of the nonlinear terms; see Sec. 3.7.2 (p. 68). The THE has the single nonlinear invariant

$$\widetilde{\mathcal{Z}} = \sum_{\boldsymbol{k}} \widetilde{\mathcal{Z}}_{\boldsymbol{k}}, \quad \widetilde{\mathcal{Z}}_{\boldsymbol{k}} \doteq \frac{1}{2}|1+\chi_{\boldsymbol{k}}|^2 |\varphi_{\boldsymbol{k}}|^2. \tag{44a,b}$$

(The tilde denotes a random variable, a property of a particular realization; statistically averaged quantities will be written without tildes, e.g., $\mathcal{Z} \doteq \langle \widetilde{\mathcal{Z}} \rangle$.) One has

$$\partial_t \widetilde{\mathcal{Z}} = \kappa \widetilde{\Gamma}, \tag{45}$$

where

$$\widetilde{\Gamma} \doteq \operatorname{Re} \sum_{\boldsymbol{k}} \delta V_{Ex,\boldsymbol{k}} \delta n^*_{s,\boldsymbol{k}} = \sum_{\boldsymbol{k}} k_y \delta_{\boldsymbol{k}} |\varphi_{\boldsymbol{k}}|^2 \tag{46a,b}$$

($s = e$ or i) is the species-independent (intrinsically ambipolar) value of the random gyrocenter flux. [As noted in Sec. 1.3.1 (p. 13), only the nonadiabatic correction $\delta_{\boldsymbol{k}}$ enters in expressions for turbulent fluxes.] This result demonstrates a significant deficiency of the THE. The average of Eq. (45), $\partial_t \mathcal{Z} = \kappa \Gamma$, is a degenerate form of the general balance equation

$$\partial_t \overline{\mathcal{I}} = \overline{\mathcal{P}} - \overline{\mathcal{D}} \tag{47}$$

[cf. Eq. (14)], where $\mathcal{I}$ is a nonlinear invariant and the overline implies both spatial and statistical averaging. In the present case the production term $\overline{\mathcal{P}}$ is proportional to the flux Γ, but the dissipation term $\overline{\mathcal{D}}$ is absent, implying that either the turbulent flux vanishes if a statistical steady state is achieved or, more likely (Krommes and Hu, 1994), the system does not saturate in the absence of dissipation. This is a deep result that is discussed further in Secs. 3 (p. 46) and 12.2 (p. 238). The cure is to insert, either by hand or systematically, dissipation into the ion equation (41b). One example is provideded by the Hasegawa–Wakatani equations discussed below in Sec. 2.4.5 (p. 38).

The THE is complicated by the presence of nonadiabatic electron response in two places: linear theory; and the nonlinear term. According to Eq. (42b), the full TH nonlinearity is composed of two parts: the *polarization-drift nonlinearity* $\boldsymbol{V}_E \cdot \boldsymbol{\nabla}(-\nabla_\perp^2 \delta\varphi)$; and the *$\boldsymbol{E} \times \boldsymbol{B}$ nonlinearity* $\boldsymbol{V}_E \cdot \boldsymbol{\nabla}[\mathrm{i}\widehat{\delta}(\delta\varphi)]$. Properties of Eq. (43) with a particular model of dissipative effects were discussed by Liang et al. (1993). At the other extreme, if the nonadiabatic contributions are neglected both linearly and

[50] Stix (1992, p. 402) might appear to suggest that polarization provides a correction of higher order in ω/ω_{ci} to the basic drift-wave dispersion relation. Nevertheless, that remark is misleading; the correction is order unity.

nonlinearly,[51] one arrives at the equation of Hasegawa and Mima (1978)[52]:

$$(1-\nabla_\perp^2)\partial_t\delta\varphi + V_*\partial_y\delta\varphi + \boldsymbol{V}_E\cdot\boldsymbol{\nabla}(-\nabla_\perp^2\delta\varphi) = 0. \tag{48}$$

The HME is conservative; it contains neither growth nor dissipation and is a generalization of the 2D Euler equation. (The absence of dissipative effects arises, of course, from the neglect of contributions from a collision operator $\mathcal{C}$; those will be reinstated in the subsequent discussion of the Hasegawa–Wakatani equation.) Indeed, the term $\nabla_\perp^2\delta\varphi$ is just the z component $\delta\omega$ of the vorticity fluctuation due to the $\boldsymbol{E}\times\boldsymbol{B}$ motion: $\boldsymbol{\nabla}\times\boldsymbol{V}_E[\varphi] = \boldsymbol{\nabla}\times(\widehat{\boldsymbol{z}}\times\boldsymbol{\nabla}\varphi) = \nabla_\perp^2\varphi \doteq \omega$. Thus the HME can be written

$$\partial_t(\delta\omega - \delta\varphi) - V_*\partial_y\delta\varphi + \boldsymbol{V}_E[\delta\varphi]\cdot\boldsymbol{\nabla}\delta\omega = 0. \tag{49}$$

This equation possesses two quadratic invariants,[53] the energy $\widetilde{\mathcal{E}}$ and the (potential) enstrophy $\widetilde{\mathcal{W}}$:

$$\begin{pmatrix}\widetilde{\mathcal{E}}\\ \widetilde{\mathcal{W}}\end{pmatrix} \doteq \sum_{\boldsymbol{k}}\begin{pmatrix}1\\ k^2\end{pmatrix}\widetilde{\mathcal{E}}_{\boldsymbol{k}}, \quad \widetilde{\mathcal{E}}_{\boldsymbol{k}} \doteq \frac{1}{2}(1+k^2)|\delta\varphi_{\boldsymbol{k}}|^2. \tag{50a,b}$$

These conserved quantities play important roles in the nonlinear statistical dynamics of the equation, as discussed in Sec. 8.4 (p. 206).

If the terms explicitly involving $\delta\varphi$ are neglected, Eq. (49) becomes the 2D neutral-fluid Euler equation in the vorticity representation [Eq. (17) with $\mu_{\mathrm{cl}} = 0$; see also the guiding-center model (31)], which has been studied extensively. However, although this observation is instructive, such neglect is physically unjustified. The first $\delta\varphi$ term in Eq. (49) arises from the nearly adiabatic response of the electrons, which rapidly stream along the magnetic field lines and exhibit a nearly Boltzmann response. This behavior is essentially three dimensional and cannot be ignored [at least for long-wavelength fluctuations ($k_\perp\rho_{\mathrm{s}} \ll 1$), which, as one will see, are the important ones]. The second $\delta\varphi$ term represents the presence of a background density gradient, imparts a crucial wavelike component to the dynamics, and [in more complete descriptions; cf. Sec. 2.4.5 (p. 38)] is ultimately responsible for a variety of instabilities.

As it stands, the conservative HME ($\widehat{\delta} = 0$) predicts vanishing particle transport because there is no phase shift between density and potential, although dimensional analysis (Appendix B, p. 264) predicts gyro-Bohm scaling. It is conventional to insert linear dissipative effects by hand; such models are called *forced HM equations* and have entirely nontrivial statistical dynamics (Ottaviani and

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[51] If, on the other hand, one neglects the $\nabla_\perp^2$ in Eq. (42b)—i.e., ignores the polarization-drift nonlinearity—one arrives at an equation studied earlier by Horton (1976). The neglect is justifiable for very long wavelengths, $k_\perp^2\rho_{\mathrm{s}}^2 \ll \gamma/\omega$, but note that it is $k_\perp\rho_{\mathrm{s}}$ that enters, not $k_\perp\rho_i$. That is, this $k_\perp^2$ term does not describe an FLR effect; it remains finite as $T_i \to 0$ (footnote 50, p. 36). In gyrokinetics one nominally orders $k_\perp\rho_{\mathrm{s}} = O(1)$, so neglect of this term is dubious.

[52] The HME is intimately related to the nonlinear equation for Rossby waves (Charney and Stern, 1962; Dickinson, 1978) in geophysics; it is frequently called the *Charney–Hasegawa–Mima equation.*

[53] These conservation laws can be proved in either $\boldsymbol{k}$ space or $\boldsymbol{x}$ space. The latter proof proceeds by multiplying Eq. (49) by $(\varphi,\, \omega)^T$, integrating over $\boldsymbol{x}$, integrating by parts, and recalling that $\omega = \nabla_\perp^2\varphi$. The diamagnetic term can be formed into a perfect y derivative whose integral vanishes under periodic boundary conditions. For the nonlinear term, (i) $\int\mathrm{d}\boldsymbol{x}\,\varphi\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\omega = \int\mathrm{d}\boldsymbol{x}\,\boldsymbol{E}\cdot\boldsymbol{V}_E\,\omega = 0$ (the $\boldsymbol{E}\times\boldsymbol{B}$ drift does no work), and (ii) $\int\mathrm{d}\boldsymbol{x}\,\omega\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\omega = \int\mathrm{d}\boldsymbol{x}\,\boldsymbol{\nabla}\cdot(\boldsymbol{V}_E\frac{1}{2}\omega^2) = 0$. Note that in the limit of small $\widehat{\delta}$ the TH invariant is $\widetilde{Z} \approx \widetilde{\mathcal{E}} + \widetilde{\mathcal{W}}$.

Krommes, 1992). The real utility of the HME lies in its clean description of the polarization-drift nonlinearity; it is an important limit to which more complete, possibly kinetic theories should reduce.

Direct numerical simulations and statistical theories of the HM and TH equations are described in Sec. 8.4 (p. 206).

2.4.4 Generalized Hasegawa–Mima dynamics

The HME is a reasonable model provided that $k_\parallel \neq 0$. It must be modified for fluctuations with $k_\parallel = 0$ (sometimes called *convective cells* [54]), whose response is strongly nonadiabatic. Although the precise response can be obtained from the gyrokinetic equation, a common approximation (Dorland, 1993; Hammett et al., 1993) is to constrain the electrons to not respond at all for $k_\parallel = 0$. This requirement changes the 1 in Eqs. (41c) and (42a) to the operator [55] $\widehat{\beta}$, where $\widehat{\beta}$ vanishes for $k_\parallel = 0$ and is the identity operator otherwise (i.e., $\widehat{\beta}$ projects onto the $k_\parallel \neq 0$ subspace), and leads for $\widehat{\delta} = 0$ to the *generalized Hasegawa–Mima equation*

$$(\widehat{\beta} - \nabla_\perp^2)\partial_t \delta\varphi + V_* \partial_y \delta\varphi + \boldsymbol{V}_E[\delta\varphi] \cdot \boldsymbol{\nabla}[(\widehat{\beta} - \nabla^2)\delta\varphi] = 0. \tag{51}$$

Like the THE, Eq. (51) possesses only the single invariant $\mathcal{Z}_{\widehat{\delta}=0}$, where $\mathcal{Z}$ is defined by Eq. (44) (Lebedev et al., 1995; Smolyakov and Diamond, 1999). It figures importantly in the theory of zonal flows and other long-wavelength fluctuations; see Sec. 12.7 (p. 248).

2.4.5 Hasegawa–Wakatani equations

The TH description is fundamentally incomplete in that the nonadiabatic (dissipative) electron response ($\propto \mathrm{i}\delta_{\boldsymbol{k}}$) is simply specified (it must ultimately be computed from a subsidiary kinetic theory), not determined self-consistently. In more sophisticated fluid models, dissipative processes enter more naturally. One such model was developed by Hasegawa and Wakatani (1983) and Wakatani and Hasegawa (1984), who considered nonadiabatic response due to electron–ion collisions.

To derive the Hasegawa–Wakatani (HW) model from the GKE, one rejects Eq. (40) in favor of an explicit calculation beginning from Eq. (39) for the electrons. Since electron inertia is negligible, a simplified electron momentum equation is $0 \approx -n_e e E_\parallel - \nabla_\parallel P_e - n_e m_e \nu_{ei} u_{\parallel e}$, where $P \doteq nT$; temperature fluctuations and ion parallel motion are neglected for simplicity. This can be rewritten as $u_{\parallel e} = D_\parallel \nabla_\parallel (\varphi - n_e)$, where $D_\parallel \doteq v_{te}^2/\nu_{ei}$ is the classical parallel diffusion coefficient. Upon substituting for $u_{\parallel e}$, one can reduce the electron continuity equation in the usual gyro-Bohm units to $\mathrm{d}\delta n_e/\mathrm{d}t = \widehat{\alpha}(\delta\varphi - \delta n_e) - \kappa\,\partial_y \delta\varphi$, where $\widehat{\alpha} \doteq -\omega_{ci}^{-1} D_\parallel \nabla_\parallel^2$. Upon recalling the GK Poisson equation (29b), one can replace the δn_i equation by $\mathrm{d}\omega/\mathrm{d}t = \widehat{\alpha}(\delta\varphi - \delta n_e)$. The latter two equations are the HW equations in the absence of perpendicular dissipation. Dissipative effects may be added by hand, but also follow systematically from a careful treatment of collisional gyrokinetics (not discussed

[54] Frequently *convective cell* is used to refer to any fluctuation with $k_\parallel = 0$. More specifically, it is also the name of a particular linear normal mode of 2D magnetized plasmas, as discussed by Krommes and Oberman (1976b); see Sec. 5.10.1 (p. 143).

[55] Krommes and Kim (2000) used the notation $\widehat{\alpha}$ instead of $\widehat{\beta}$, but that conflicts with the operator defined in Sec. 2.4.5 (p. 38) in conjunction with the Hasegawa–Wakatani equations.

here). One may summarize the HW system in the form usually used for computation[56] as

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \widehat{\alpha}(\varphi - n) + \mu_{\mathrm{cl}}\nabla_\perp^2\omega, \quad \frac{\mathrm{d}n}{\mathrm{d}t} = \widehat{\alpha}(\varphi - n) - \kappa\frac{\partial\varphi}{\partial y} + D_{\mathrm{cl}}\nabla_\perp^2 n, \tag{52a,b}$$

with $\omega \doteq \nabla_\perp^2\varphi$. All variables represent fluctuations in these equations.

It is straightforward to show that the HW system possesses the four nonlinear invariants

$$\mathcal{V} \doteq \tfrac{1}{2}\langle|\boldsymbol{\nabla}\varphi|^2\rangle, \quad \mathcal{W} \doteq \tfrac{1}{2}\langle\omega^2\rangle, \quad \mathcal{N}_e \doteq \tfrac{1}{2}\langle n^2\rangle, \quad \mathcal{X} \doteq \langle\omega\, n\rangle = -\langle\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}n\rangle. \tag{53a,b,c,d}$$

Therefore the combination

$$\mathcal{N}_i \doteq \tfrac{1}{2}\langle(n-\omega)^2\rangle = \tfrac{1}{2}\langle\delta n_i^2\rangle, \tag{54a,b}$$

where the gyrokinetic Poisson equation (41a) was used in obtaining Eq. (54b), is also an invariant. Because $\mathcal{N}_i$ is a physically interesting quantity, one may use it in place of the cross correlation $\mathcal{X}$. One may also use the fluid energy $\mathcal{E} \doteq \mathcal{V} + \mathcal{N}_e$ in place of $\mathcal{N}_e$. Straightforward algebra shows that these quantities evolve according to

$$\partial_t\mathcal{V} = -\langle\varphi \mid \widehat{\alpha} \mid (\varphi - n)\rangle - \mu_{\mathrm{cl}}\langle\omega^2\rangle, \tag{55a}$$
$$\partial_t\mathcal{W} = \langle\omega \mid \widehat{\alpha} \mid (\varphi - n)\rangle - \mu_{\mathrm{cl}}\langle|\boldsymbol{\nabla}\omega|^2\rangle, \tag{55b}$$
$$\partial_t\mathcal{E} = \kappa\Gamma - \langle(\varphi - n) \mid \widehat{\alpha} \mid (\varphi - n)\rangle - \mu_{\mathrm{cl}}\langle\omega^2\rangle - D_{\mathrm{cl}}\langle|\boldsymbol{\nabla}n|^2\rangle, \tag{55c}$$
$$\partial_t\mathcal{N}_i = \kappa\Gamma - \mathcal{D}, \tag{55d}$$

where

$$\mathcal{D} \doteq \mu_{\mathrm{cl}}\langle|\boldsymbol{\nabla}\omega|^2\rangle - (\mu_{\mathrm{cl}} + D_{\mathrm{cl}})\langle\boldsymbol{\nabla}n\cdot\boldsymbol{\nabla}\omega\rangle + D_{\mathrm{cl}}\langle|\boldsymbol{\nabla}n|^2\rangle \ge 0. \tag{56}$$

The proof that $\mathcal{D} \ge 0$ follows from a Schwartz inequality.[57] Because $\widehat{\alpha} \propto -\partial_z^2$ is a positive-semidefinite operator, the dissipation of $\mathcal{E}$ is also positive semidefinite. The parallel dissipations of $\mathcal{V}$ and $\mathcal{W}$ are of indefinite sign. Note the absence of (explicit) parallel dissipation in Eq. (55d).

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[56] Often one or more of the dissipative terms like $\mu_{\mathrm{cl}}\nabla^2 \to -\mu_{\mathrm{cl}}k^2$ are generalized to $-\mu_{\mathrm{cl}}(k)k^2$, where $\mu_{\mathrm{cl}}(k)$ contains terms of positive order in k^2 (hyperviscosity).

[57] The spatial average together with the Cartesian dot product can be interpreted as a scalar product. Then (writing μ and D instead of μ_{cl} and D_{cl} to avoid clutter)

$$\mathcal{D} = (\sqrt{\mu}\,\|\boldsymbol{\nabla}\omega\|)^2 - (\mu + D)\langle\boldsymbol{\nabla}n \mid \boldsymbol{\nabla}\omega\rangle + (\sqrt{D}\,\|\boldsymbol{\nabla}n\|)^2. \tag{f-1}$$

By a Schwartz inequality, $|\langle\boldsymbol{\nabla}n \mid \boldsymbol{\nabla}\omega\rangle| \le \|\boldsymbol{\nabla}n\|\,\|\boldsymbol{\nabla}\omega\|$. Thus

$$\mathcal{D} \ge (\sqrt{\mu}\,\|\boldsymbol{\nabla}\omega\|)^2 - (\mu + D)\|\boldsymbol{\nabla}n\|\,\|\boldsymbol{\nabla}\omega\| + (\sqrt{D}\,\|\boldsymbol{\nabla}n\|)^2. \tag{f-2}$$

From $(\sqrt{\mu} - \sqrt{D})^2 = \mu - 2\sqrt{\mu D} + D \ge 0$, one proves that $\mu + D \ge 2\sqrt{\mu D}$. Thus

$$\mathcal{D} \ge \|\sqrt{\mu}\,\boldsymbol{\nabla}\omega - \sqrt{D}\,\boldsymbol{\nabla}n\|^2 \ge 0. \tag{f-3}$$

Equations (52) are intrinsically three dimensional since $\widehat{\alpha}$ is an operator. 3D simulations of Eqs. (52) and similar equations are feasible and have been done both with and without magnetic shear [see, for example, Guzdar et al. (1993)]; the results are essential to the theory of submarginal turbulence (Sec. 9, p. 210). However, in the simplest model $\widehat{\alpha}$ is replaced by the constant parameter $\alpha \doteq \omega_{ci}^{-1} k_{\parallel}^2 D_{\parallel}$ for constant $k_{\parallel} \neq 0$; in this approximation Eqs. (52) form a 2D system that is amenable to rapid computation. That system can be viewed as a *paradigm* useful for demonstrating and understanding certain features of plasma turbulence. The equations may have some relevance to fluctuations in the cold edges of tokamaks, but exploring the practical ramifications or deficiencies of such models is not the focus of this article.

The 2D HW system with constant α exhibits several important conceptual features [for a review, see Koniges and Craddock (1994)]: (i) The linear theory exhibits a density-gradient-driven instability that is determined from the equations themselves,[58] not inserted *ad hoc*. The eigenvalues of the linear matrix coupling ω and n describe two modes, one unstable (for some wave numbers) and propagating in the electron diamagnetic direction, the other always stable and propagating in the ion diamagnetic direction. (ii) The system contains a single parameter α that can be varied to exhibit different physical regimes. (a) For $\alpha \gg 1$ consistent balance requires that $n = \varphi$ to lowest order in α^{-1}. This becomes exact as $\alpha \to \infty$; then subtracting Eq. (52b) from Eq. (52a) yields the HME in the limit of zero dissipation. For finite $\alpha \gg 1$ Eqs. (52) behave as a forced, dissipative HME with small growth rate $\gamma/\omega \ll 1$; this regime is called the *adiabatic regime*. (b) For $\alpha \ll 1$ the vorticity equation almost decouples from the density equation and reduces to the 2D Euler equation. The density is almost passively advected by the $\boldsymbol{E} \times \boldsymbol{B}$ velocity. [The diamagnetic term in Eq. (52b) can be considered to be a random forcing.] Because of the analogy to the 2D Euler equation, $\alpha \ll 1$ is called the *hydrodynamic regime*. Here the growth rate is of the order of the real frequency: $\gamma/\omega \sim 1$.

It is interesting to see how the HW invariants reduce to those of TH and HM. $\mathcal{N}_i$ reduces to the TH invariant U for $\widehat{\delta} = 0$. $\mathcal{E}$ reduces to the HM $\mathcal{E}$ [Eqs. (50)] as $\widehat{\alpha} \to \infty$, for then one has $n_e \to \varphi$; similarly, $\mathcal{V} + \mathcal{N}_e \to \mathcal{W}$, the potential enstrophy.

Equations (55d) and (55c) show that statistically steady states with positive flux can be achieved, production of turbulent fluctuations (the $\kappa\Gamma$ term) being balanced by positive definite dissipation. Aspects of the transition to turbulence of the HW system (52) were considered by Vasil'ev et al. (1990). The equations have been simulated by several groups (Koniges et al., 1992; Biskamp et al., 1994; Hu et al., 1995), and there is general agreement on the principle features. The particle transport $\Gamma(\alpha)$ that follows from DNS is shown by the solid line in Fig. 1 (p. 41). A distinctive scaling $\Gamma \sim \alpha^{-1/3}$ is exhibited for $\alpha \ll 1$. This is consistent with the prediction of scaling analysis (Appendix B, p. 264) applied (LoDestro et al., 1991) to the simplified system arising by dropping the α term in Eq. (52b). For large α the ordering $\gamma/\omega \ll 1$ suggests a weak-turbulence treatment [see Sec. 4.2 (p. 98)]; this leads (Hu et al., 1995, 1997) to the scaling[59] $\Gamma \sim \alpha^{-2}$. The other curves in Fig. 1 (p. 41) will be discussed in Sec. 8.5 (p. 208). In particular, the lowest curve, which closely tracks the exact solution, is the prediction of the so-called *Realizable Markovian Closure* (RMC). Much of this article will be devoted

[58] For the physics of the instability, see Hu et al. (1997).

[59] Interestingly, Connor–Taylor analysis applied to the large-α regime predicts (LoDestro et al., 1991) the scaling $\Gamma \sim \alpha^{-1}$, which is actually the mixing-length scaling and disagrees with the weak-turbulence analysis. This paradox can be resolved by arguing (Hu et al., 1997) that the numerical coefficient of the dominant Connor–Taylor scaling vanishes in this regime, the scaling theory not taking account of the rapid variation of the linear waves.

to a development of the analytical techniques that underlie that clearly successful approximation. The RMC is described in Sec. 8.2.3 (p. 203).

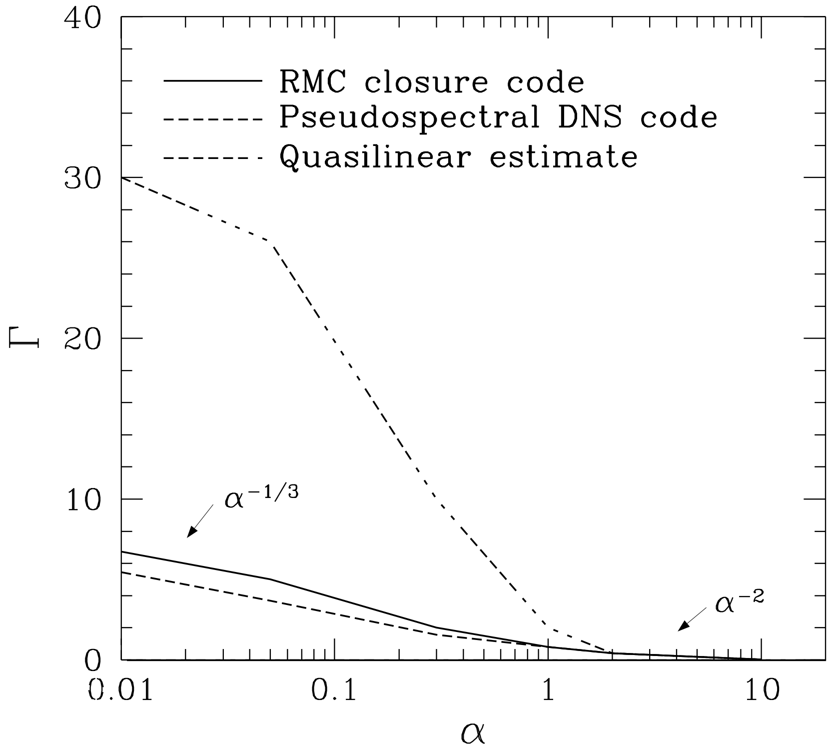

Fig. 1. Particle flux Γ vs adiabaticity parameter α for the 2D Hasegawa–Wakatani equations. Solid line, direct numerical simulation; dashed line, Realizable Markovian Closure; dotted line, quasilinear prediction. After Fig. 10 of Hu et al. (1997), used with permission.

Time snapshots of typical $\boldsymbol{x}$-space vorticity fields for $\alpha \ll 1$ show (Koniges et al., 1992) well-defined vortices amidst a sea of random turbulence. The tendency to form such vortices is well known from extensive studies of decaying 2D Navier–Stokes turbulence (McWilliams, 1984; Benzi et al., 1988). However, whereas in strictly decaying turbulence the vortex coalescence proceeds indefinitely, generating larger and larger scales as $t \to \infty$ and an ever-increasing kurtosis [a fourth-order statistic defined by Eq. (96b)], the forced, dissipative 2D NSE that Eq. (52a) becomes for $\alpha \ll 1$ achieves a balance between the nonlinear advection in the presence of dissipation, which favors the coalescence, and the linear forcing, which tends to destroy the vortices. The actual steady-state kurtosis measured for representative parameters was 12. Such a highly non-Gaussian kurtosis presents a challenge for analytical theory; see Secs. 8.5 (p. 208) and 10.4.2 (p. 225).

In 3D the physics of systems like that of HW become more interesting because of nonlinear coupling between planes of $k_\parallel = 0$ [*convective cells*; see footnote 54 (p. 38)] and $k_\parallel \neq 0$. Although such systems can frequently be linearly stabilized by magnetic shear,[60] nonlinear instability remains a possibility (Biskamp and Zeiler, 1995; Drake et al., 1995). The *submarginal turbulence* that can result is discussed in Sec. 9 (p. 210).

[60] Magnetic shear is defined in Sec. 12.6.3 (p. 247). A detailed discussion of the effects of magnetic shear on linear stability would carry us too far afield. Early work was cited by Antonsen (1978), who proved that the collisionless universal instability possesses only stable eigenmodes in the presence of shear.

2.4.6 *Equations with ion temperature gradients*

The HM, TH, and HW equations all describe variants of the universal drift wave (Krall, 1968; Horton, 1984) driven by gradients in the mean density profile. In the presence of intense ion heating, as is typical for modern tokamaks, ion-temperature-gradient-driven (ITG) modes (Kadomtsev and Pogutse, 1970b; Horton, 1984; Cowley et al., 1991) are of considerable interest (Ottaviani et al., 1997). ITG fluid equations can systematically be derived by taking moments of the GKE in the presence of mean temperature gradients, then invoking a Landau-fluid closure (Appendix C.2, p. 276). The equations actually used for modern simulations are substantially too complicated to be recorded here; see recent representative works such as those of Beer (1995) or Snyder (1999). However, when FLR effects (Dorland and Hammett, 1993) and the effects of nonconstant magnetic fields are ignored, one is led to a relatively simple set of fluid equations for a slab ITG mode:

$$\partial_t \varphi + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} n = -\nabla_\parallel u_\parallel - \partial_y \varphi, \tag{57a}$$

$$\partial_t u_\parallel + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} u_\parallel = -\nabla_\parallel (T + 2\varphi), \tag{57b}$$

$$\partial_t T + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} T = -2\nabla_\parallel u - \eta_i \, \partial_y \varphi - \widehat{\nu}\, T, \tag{57c}$$

where $\eta_i \doteq \mathrm{d}\ln T_i / \mathrm{d}\ln n_i = L_T/L_n$ and $\widehat{\nu} \propto |k_\parallel|$ arises from a Landau-fluid closure. In the limit $\omega_*^T \to \infty$ the linear dispersion relation for this system is approximately $\omega_{\boldsymbol{k}} = 1^{1/3}(k_\parallel^2 c_\mathrm{s}^2 \omega_*^T)^{1/3}$, where $1^{1/3} = \exp(2\pi \mathrm{i} n/3)$ with $n = 0, 1, 2$. When ω_*^n is kept nonzero, the root $n = 0$ reduces to the universal drift wave as $\omega_*^T \to 0$. The root $n = 1$ is the ITG mode, a nonresonant instability that is driven unstable by negative compressibility; a clear physical picture was given by Cowley et al. (1991). The root $n = 2$ is a stable branch of the ITG mode. For tokamak geometries it is important to retain the effects of magnetic curvature; those can approximately be accounted for (Ottaviani et al., 1997) by adding a term $\omega_R \, \partial_y T$ to the left-hand side of Eq. (57a). As ω_R is raised from 0, the roots migrate such that as $\omega_R \to \infty$ the unstable root becomes $\omega_{\boldsymbol{k}} \approx \mathrm{i}(\omega_R \, \omega_*^T)^{1/2}$. This *curvature-driven ITG mode* [61] is the one considered to be important for experiments.

2.4.7 *Nonlinear equations for trapped-ion modes*

I have already noted that in toroidal magnetic configurations magnetically trapped particles may be important. In addition to their contribution to classical transport, trapped populations may lead to new classes of microinstabilities. Those may be important in a variety of contexts, including space plasma physics (Cheng and Qian, 1994); the most detailed work has been done on fusion plasma, a subject by and large too arcane and practically detailed to be described here. Nevertheless, it is worth mentioning model equations for the trapped-ion mode, as there are interesting links to both nonlinear dynamics and the theory of 2D turbulence.

The importance of trapped-ion modes was suggested early on in various works by Kadomtsev and Pogutse [see, for example, Kadomtsev and Pogutse (1970b)]. They considered a two-field set of coupled equations for the trapped-electron and trapped-ion densities. LaQuey et al. (1975) derived a one-field model and attempted to consider its nonlinear saturation. Cohen et al. (1976) gave a more extensive discussion of the one-field equation in 1D. In appropriately dimensionless units the equation

[61] In reality the proper treatment of curvature is more complicated than is suggested here. These equations are presented only for purposes of illustration.

is

$$\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial y^2} + \alpha \frac{\partial^4 \varphi}{\partial y^4} + \nu\varphi + \frac{\partial(\varphi^2)}{\partial y} = 0. \tag{58}$$

An equation of this form was also derived by Kuramoto (1978) in the context of chemical reactions and by Sivashinsky (1977) for the description of flame fronts; it is now known as the *Kuramoto–Sivashinsky* (KS) *equation*. The second-derivative term in y is *anti*-diffusive; it describes a linear instability driven by dissipation. The fourth-derivative term, which is stabilizing, arises in this context from an approximation to ion Landau damping. The ion collision term $\nu\varphi$ is also stabilizing. The last term is a Burgers-like nonlinearity [cf. Eq. (19)] that leads to the formation of shocks. Cohen et al. (1976) found chaotic solutions of Eq. (58) and calculated some of their properties analytically; the rich nonlinear dynamical behavior of the equation has subsequently been studied in considerable detail [Bohr et al. (1998), Wittenberg (1998), Wittenberg and Holmes (1999), and references therein].

For the fusion application Cohen et al. (1976) emphasized that the 1D approximation was severe and that detailed predictions from the model should not be believed. Kadomtsev and Pogutse (1970b) had earlier derived the 2D, one-field equation

$$\partial_t n + \tfrac{1}{2}\widehat{V}_* \partial_y n + (\widehat{V}_*^2/4\nu) n_{yy} + (V_*/4\nu)\widehat{\boldsymbol{z}} \times \boldsymbol{\nabla}(\partial_y n) \cdot \boldsymbol{\nabla} n = 0, \tag{59}$$

where $\widehat{V}_*$ is the diamagnetic velocity of the trapped ions. Diamond and Biglari (1990) argued that the 2D nature of this equation was important, as it could lead to broadband strong turbulence. For further remarks on Eq. (59), see Sec. 3.8.4 (p. 76).

2.4.8 Equations for magnetohydrodynamic turbulence

The plasma equations introduced so far have mostly assumed a spatially and temporally constant magnetic field. It is not difficult to generalize them to include spatial variations, which introduces among other things the effect of *magnetic shear*. The physical effects of magnetic shear are mostly beyond the scope of this article, but one should appreciate that spatial dependence of $\boldsymbol{B}$ arises from nontrivial current distributions and boundary conditions. The associated theory of magnetohydrodynamic (MHD) equilibria is very well developed (Freidberg, 1982, 1987). For turbulence, however, one is concerned as well with nontrivial time dependence, so one must consider time-dependent MHD. This subject is vast, and even the part of it that overlaps basic turbulence theory cannot properly be treated in this article because of space constraints; see Biskamp (1993). Nevertheless, because MHD not only provides interesting illustrations of some of the fundamental turbulence concepts but is also useful in many important practical applications, I shall briefly introduce the basic MHD equations.

The evolution of electromagnetic fields is described by Maxwell's equations

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho. \tag{60a,b,c,d}$$

For low-frequency motions the $\partial_t \boldsymbol{E}$ term in Eq. (60a) is omitted (giving the pre-Maxwell equations). For consistency one must then require the quasineutrality condition $\rho = 0$. The simplest Ohm's law in a frame moving with velocity $\boldsymbol{u}$ is

$$\boldsymbol{E} + c^{-1}\boldsymbol{u} \times \boldsymbol{B} = \eta_{\text{cl}}\boldsymbol{j}, \tag{61}$$

where a scalar resistivity $\eta_{\rm cl}$ is assumed. Straightforward vector algebra leads one to the equivalent representations

$$\partial_t \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \mu_{m,\rm cl}\nabla^2 \boldsymbol{B} \quad \text{or} \quad \frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{u} + \mu_{m,\rm cl}\nabla^2 \boldsymbol{B}, \tag{62a,b}$$

where $\mu_{m,\rm cl} \doteq \eta_{\rm cl}c^2/4\pi$. The unique dimensionless parameter of Eqs. (62) is the *magnetic Reynolds number* [62] $\mathcal{R}_m \doteq uL/\mu_{m,\rm cl}$.

With $\boldsymbol{A}$ being the vector potential, it is useful to note the analogies $\boldsymbol{A} \sim \boldsymbol{u}$, $\boldsymbol{B} \sim \boldsymbol{\omega}$ ($\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$). When $\boldsymbol{u}$ is a specified function (possibly stochastic), Eqs. (62a) or (62b) define the *kinematic dynamo* problem, which describes the amplification of magnetic fields because of line stretching. [63] [Note that although Eqs. (62) are analogous to Eqs. (16) for the fluid vorticity $\boldsymbol{\omega}$, the latter have no analog of the kinematic dynamo because $\boldsymbol{\omega}$ is intrinsically linked to $\boldsymbol{u}$.] If $\boldsymbol{u}$ is instead allowed to evolve, one must adjoin to Eqs. (62) the generalization of the NSE to include magnetic forces:

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{1}{\rho_m}(-\boldsymbol{\nabla}p + c^{-1}\boldsymbol{j} \times \boldsymbol{B}) + \mu_{\rm cl}\nabla^2\boldsymbol{u} = -\frac{1}{\rho_m}\boldsymbol{\nabla}(p + \frac{1}{2}B^2) + \frac{1}{4\pi\rho_m}\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B} + \mu_{\rm cl}\nabla^2\boldsymbol{u}. \tag{63a,b}$$

The resulting self-consistent dynamo problem is of great current interest, but will mostly not be discussed in this article because of lack of space.

It is not difficult to show that in 3D the nonlinear terms of Eqs. (62) and (63) conserve [64]

$$\mathcal{E} \doteq \frac{1}{2}\rho_m\overline{u^2} + \frac{1}{8\pi}\overline{B^2} \quad \text{(total energy)}, \quad \mathcal{H} \doteq \rho_m\overline{\boldsymbol{u} \cdot \boldsymbol{\omega}} \quad \text{(fluid helicity)}, \tag{64a,b}$$

$$\mathcal{H}_c \doteq \overline{\boldsymbol{u} \cdot \boldsymbol{B}} \quad \text{(cross helicity)}, \quad \mathcal{H}_m \doteq \overline{\boldsymbol{A} \cdot \boldsymbol{B}} \quad \text{(magnetic helicity)} \tag{64c,d}$$

when integrated over a volume on whose boundary (with unit normal $\widehat{\boldsymbol{n}}$) $\widehat{\boldsymbol{n}} \cdot \boldsymbol{\omega} = 0$ and $\widehat{\boldsymbol{n}} \cdot \boldsymbol{B} = 0$. The interpretation of $\mathcal{H}_m$ as the degree of knottedness of magnetic field lines was discussed by Moffatt (1969).

Consider a system for which the fluid energy is negligible ($u \approx 0$). One is left with the two magnetic invariants $\mathcal{E} \approx \overline{B^2}/8\pi$ and $\mathcal{H}_m = \overline{\boldsymbol{A} \cdot \boldsymbol{B}}$. In a highly conductive fluid ($\mathcal{R}_m \gg 1$) both $\mathcal{E}$ and $\mathcal{H}_m$ are approximately conserved. However, when small-scale fluctuations are excited $\mathcal{E}$ is dissipated more rapidly, since one can verify that

$$\mathrm{d}\mathcal{E}/\mathrm{d}t = -\eta_{\rm cl}\overline{|\boldsymbol{j}|^2}, \quad \mathrm{d}\mathcal{H}_m/\mathrm{d}t = -2\eta_{\rm cl}c\,\overline{\boldsymbol{j} \cdot \boldsymbol{B}} \tag{65a,b}$$

and $\dot{\mathcal{E}}$ contains a higher spatial derivative because $\boldsymbol{j} = c\boldsymbol{\nabla} \times \boldsymbol{B}/4\pi$. [That invariants dissipate at different rates was called *selective decay* by Matthaeus and Montgomery (1980).] This faster dissipation of energy suggested to Taylor (1974b) that the variational principle *Minimize* $\mathcal{E}$ *subject to constant* $\mathcal{H}_m$ might be used to determine the ultimate magnetic field that results from turbulent relaxation in

[62] When u is scaled to the Alfvén velocity, one refers to the *Lundquist number*.
[63] Considerable progress has been made in the dynamical and statistical description of the kinematic dynamo. For recent work and prior references, see Boldyrev and Schekochihin (2000) and Schekochihin (2001).
[64] In 2D the fluid helicity $\mathcal{H}$ and the magnetic helicity $\mathcal{H}_m$ vanish because $\boldsymbol{\omega}$ and $\boldsymbol{A}$ are in the z direction whereas $\boldsymbol{u}$ and $\boldsymbol{B}$ lie in the x–y plane. However, $\mathcal{A} \doteq \langle A_z^2 \rangle$ is then an independent inviscid invariant (Fyfe and Montgomery, 1976).

pinch experiments. The resulting Euler–Lagrange equation is $\boldsymbol{\nabla} \times \boldsymbol{B} = \lambda \boldsymbol{B}$, where λ is a Lagrange multiplier; i.e., the relaxed states are *force-free*, $\boldsymbol{j} \times \boldsymbol{B} = \boldsymbol{0}$. The consequences of this prediction have been explored in depth. As reviewed by Taylor (1986) and Taylor (1999), the theory has been spectacularly successful in predicting quantitative features of magnetic pinches, including the onset condition for spontaneous magnetic-field reversal and the shapes of the radial field profiles.

2.4.9 Other nonlinear equations

A variety of other nonlinear dynamical equations are important in the literature; some are cited and/or discussed by Yoshizawa et al. (2001). Conceptually, they mostly present more or less detailed variations on the themes introduced above, perhaps by providing more elaborate descriptions of the evolution of the temperature and/or parallel current. A Liouville equation for magnetic field lines was posed and analyzed by Rosenbluth et al. (1966); a related model was used in the statistical description of particle transport in magnetic fields by Krommes et al. (1983) and by vanden Eijnden and Balescu (1996). One should also mention the four-field equations of Hazeltine et al. (1985) and Hazeltine et al. (1987), which bridge between electrostatic and MHD equations. Very detailed equations for drift-Alfvén microturbulence have been discussed by Scott (1997).

2.4.10 The essence of the nonlinear plasma equations

To summarize the general properties of all of the equations mentioned so far, they (i) are *nonlinear* [65] (typically quadratic, although the nonlinear Schrödinger equation is cubic); (ii) possess, in the linear approximation, intrinsic sources of free energy (e.g., profile gradients or linear growth rates), so are self-forced (leading to *intrinsic stochasticity* excited by either linear or nonlinear instabilities); (iii) include linear dissipation that can balance the forcing and permit statistically steady states; (iv) often involve multiple coupled fields; and (v) contain an advecting velocity field, most often the $\boldsymbol{E} \times \boldsymbol{B}$ velocity, that is usually *self-consistently* determined from one or more of the advected fields. These properties help to focus and guide the development of appropriate analytical theories of plasma turbulence.

3 INTRODUCTION TO THE STATISTICAL THEORY OF TURBULENCE

"The essential difficulties of the turbulence problem arise from the strongly dissipative character of the dynamical system and the non-linearity of the equations of motion. The first of these two characteristics effectively precludes treatment by conventional methods of statistical mechanics. The second is responsible for the fact that the Navier–Stokes equation does not yield closed differential equations for the velocity covariance, the statistical quantity of principal interest. The equations of motion for this covariance contain third-order moments of the velocity field, the equations of motion for the third-order moments contain fourth-order moments, and so forth, *ad infinitum*. A central goal of turbulence theory is the closing of this

[65] The kinematic dynamo problem [Eqs. (62) with statistically specified $\boldsymbol{u}$] is dynamically linear, but is quadratically nonlinear in random variables. For more discussion of the statistical closure problem for such cases, see Sec. 3.3 (p. 52).

infinite chain of coupled equations into a determinate set containing only moments below some finite order." — *Kraichnan (1959b).*

In this and the next several sections I discuss topics in the statistical theory of turbulence. The present section is introductory: I describe the basic philosophy and goals, present several solvable models, introduce important dimensionless parameters, give simple random-walk estimates of transport, define key statistical measures, and survey various strategies that can be used to develop analytical approximations. In subsequent sections I develop some of those in detail.

3.1 Philosophy and goals

Why does one need a "fundamental" (systematic) theory of turbulence? Useful analogies can be drawn to Maxwell's equations of electromagnetism and to the Vlasov equation. Maxwell's equations provide the foundation underpinning all of electromagnetic theory and experiment, even though in many practical electrical-engineering applications they are not solved explicitly. The Vlasov equation underlies a vast field of linear and nonlinear wave and other collisionless phenomena. In the same spirit it is important to understand the most general analytical foundations of turbulence even though in practice much rougher engineering estimates may often be employed.

As I have remarked, turbulence has various facets, both statistical and coherent. In focusing on statistical approaches rather than descriptions of coherent phenomena, one commits to a basic philosophy. Upon recalling the goal of calculating transport, one notes that the simplest diffusive transport problem, a discrete random walk in one dimension, is inherently statistical, so it seems natural to generalize such ideas as much as possible. Coherent structures may be important, but in many circumstances they are either subdominant or embedded in a sea of random motions. Because in some ways statistical theories are simpler than coherent ones (they discard various phase correlations, for example), it seems reasonable and prudent to develop those first.

This is certainly not to say that statistical methods are either the last word or even at all useful for certain important problems of nonlinear physics. Some progress in elucidating the dynamical underpinnings of turbulence was described by Bohr et al. (1998). What needs to be avoided, however, is the tendency of some workers to dismiss statistical approaches out of hand as being *never* useful. In fact, particular statistical theories perform very well indeed for significant questions (e.g., transport) of central importance to modern applications (e.g., fusion). It is important to understand the reasons for this success, and we will see that some answers are known. Furthermore, whatever ingredients the ultimate theory of turbulent phenomena will involve, it seems unreasonable to believe that insights gained from the statistical approaches will be useless if there is a random aspect of the dynamics at all.

The goals of a statistical theory of turbulence are both qualitative and quantitative. It is important to note that if a transport coefficient D is assumed to exist [66] in a particular model, dimensional and scaling analysis (Appendix B, p. 264) of the primitive equations already determines D to have the form $D/D_0 = C\mathcal{F}(\epsilon_1, \epsilon_2, \ldots)$, where D_0 is a combination of appropriate dimensional quantities such as T or B [cf. the Bohm diffusion coefficient D_B, Eq. (5)], C is a constant, and $\mathcal{F}$ is a dimensionless function of various dimensionless parameters ϵ_i. Qualitative understanding of the turbulent physics

[66] This assumption is far from trivial: D may be infinite or may vanish, signifying the need for a more refined model of the physics. See the last paragraph of Appendix B (p. 266).

helps one to understand the functional form of $\mathcal{F}$, and most research on practical applications involving turbulence is devoted to this end. However, *quantitative* solution of adetailed statistical approximation is needed to determine the value of C and pin down the precise form of $\mathcal{F}$.[67] Additionally, quantitative analysis is needed to predict detailed wave-number and frequency spectra, rates of energy transfer between modes, and other quantities such as higher-order statistics or PDF's that can be compared with experiment. Of course, such quantities can also be computed (at least in principle) by diagnosing numerical simulations. The analytical and computational approaches are complementary. Note that the mere numerical computation of a constant coefficient does not explain why it has that value. Analytical theories tend to be couched in terms of physically intuitive concepts, such as diffusion coefficients or mean damping rates, that facilitate back-of-the-envelope estimates and heuristic explanations. Furthermore, the conceptual formulation of the analytical methods may suggest specific diagnostics to be employed in the analysis of the numerical data. To date, large-scale numerical simulations in plasma physics have by and large not lived up to their promise of clarifying in detail the nonlinear dynamics leading to the measured transport.

Another role of an analytical statistical theory of turbulence is to predict general *qualitative* properties that are at least in principle amenable to experimental verification. An important example is the Onsager symmetries. Those are known to hold for small perturbations of thermal equilibrium, but their status for general, far-from-equilibrium classes of turbulence has been highly confused in the literature. A discussion is given in Sec. 12.1 (p. 235).

In this article I concentrate on the systematic analysis of well-specified yet tractable nonlinear equations such as those described in Sec. 2.4 (p. 33). (The Liouville and Klimontovich equations are well specified, but are intractable in general.) That is not (nor should it be) the principle focus of current research in plasma confinement, in which many diverse effects operate simultaneously and an appropriately simple model that describes an entire device may not exist and has certainly not yet been found. Sufficient motivation is the intellectual challenge of understanding the nonlinear plasma state. Furthermore, there exists more than one instance of a dramatic qualitative conclusion drawn from some heuristic statistical turbulence analysis that turns out upon closer inspection to be erroneous. Some of the issues are quite subtle and demand a robust and systematic analytical framework for their resolution.

3.2 Classical Brownian motion and the Langevin equations

"Une particule comme celle que nous considérons, grande par rapport à la distance moyenne des molécules du liquide, et se mouvant par rapport à celui-ci avec la vitesse $\boldsymbol{\xi}$ subit une résistance visqueuse égale à $-6\pi\mu a\xi$ d'après la formule de Stokes. En réalité, cette valeur n'est qu'une moyenne, et en raison de l'irrégularité des chocs des molécules environnantes, l'action du fluide sur la particule oscille autour de la valeur précédente, de sorte que l'équation de mouvement est, dans la direction x, $m\,\mathrm{d}^2x/\mathrm{d}t^2 = -6\pi\mu a\,\mathrm{d}x/\mathrm{d}t + X$. Sur la force complémentaire X nous savons qu'elle est indifféremment positive et négative, et sa grandeur est telle qu'elle maintient l'agitation de la particule que, sans elle, la résistance visqueuse finirait par arrêter."[68] —

[67] Important issues such as the possibility of thermonuclear breakeven or the ultimate economy of fusion reactors sometimes come down to numbers.

[68] "A particle such as the one we consider, large in comparison with the mean distance between the molecules of the fluid, and moving with respect to the latter with velocity ξ, is subject to a viscous resistance equal to

Langevin (1908).

As I will discuss in Sec. 3.3 (p. 52), the formal structure of the turbulence problem involves a *multiplicatively nonlinear* random coefficient. Let us begin, however, by examining equations with *additive* random forcing. Those are much simpler to analyze and the results have wide applications, including certain Langevin representations of statistical closures [see Secs. 5.3 (p. 132) and 8.2.2 (p. 201)].

3.2.1 Statement of the classical Langevin equations

The key example, to which I will refer a number of times in this article, is the system of classical *Langevin equations* (Langevin, 1908; Uhlenbeck and Ornstein, 1930) for the Brownian motion of a large test particle (e.g., pollen or a hydrogen ion) of mass M randomly kicked by a sea of much smaller particles (e.g., air molecules or electrons) in thermal equilibrium at temperature T:

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \quad \dot{\boldsymbol{v}} + \nu \boldsymbol{v} = \widetilde{\boldsymbol{a}}(t). \tag{66a,b}$$

The damping term ν describes the mean frictional drag imparted by the medium to the test particle whereas the random acceleration $\widetilde{\boldsymbol{a}}$ describes the random excitation due to the individual kicks; it is usually taken to be Gaussian white noise with $\langle \widetilde{\boldsymbol{a}} \rangle = \mathbf{0}$ and

$$F(t, t') \doteq \langle \delta\widetilde{\boldsymbol{a}}(t) \delta\widetilde{\boldsymbol{a}}(t') \rangle = 2D_v \delta(\tau) \mathsf{I}, \tag{67a,b}$$

where $\tau \doteq t - t'$ and D_v is a constant (the short-time velocity-space diffusion coefficient, as will be seen shortly). The *Einstein relation* (Einstein, 1905) $D_v = T/M\nu$ is a statement of energy conservation and the *fluctuation–dissipation theorem* (Martin, 1968). Montgomery (1971a) discussed the relation of the Langevin equations to Boltzmann's equation. In plasmas such equations can be justified [and formally derived from the Landau collision operator (34)] for the classical motion of a heavy ion due to collisions with light electrons (Braginskii, 1965). However, their intuitive content is valuable quite generally.

Several features of this Langevin system are important for the discussion of turbulence theory to follow. (i) The delta function on the right-hand side of Eq. (67b) reflects a *coarse-graining of the time scale*; the microscopic events (Coulomb collisions, in classical kinetic theory) occur on a timescale much shorter than the time interval with which the motion of the test particle is resolved. It can more revealingly be written $\delta(\tau) = \tau_{\text{ac}}^{-1} \delta(\tau/\tau_{\text{ac}})$, where τ_{ac} is a microscopic autocorrelation time. In near-equilibrium discrete plasmas $\tau_{\text{ac}} \sim \omega_p^{-1} = \lambda_D/v_t$, the time for a thermal particle to traverse a Debye cloud. Turbulent plasmas may support a variety of autocorrelation times; cf. the quasilinear τ_{ac}, Eq. (152). (ii) The Gaussian assumption reflects a belief that the individual microscopic interactions are statistically independent. This is a Markov approximation that can usually be justified with the aid

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$-6\pi\mu a\xi$ according to the Stokes formula. In reality, this value is but an average, and due to the irregularity of collisions with the neighboring molecules, the action of the fluid on the particle oscillates around the aforementioned value, so that the equation of motion in the x direction is $m\, \mathrm{d}^2x/\mathrm{d}t^2 = -6\pi\mu a\, \mathrm{d}x/\mathrm{d}t + X$. As regards the additional force X, we know that it may equally well be positive or negative, and that its magnitude is such that it maintains the motion of the particle, which would otherwise be stopped by the viscous resistance." I am grateful to A. Schekochihin for this translation.

of the *central limit theorem*[69] under an appropriate coarse-graining in time; see the next paragraph. (iii) The drag coefficient ν is a *statistical* property of the underlying microscopic fluctuations. This is evident both on physical grounds and from the Einstein relation relating it to D_v.

The relationship between the coarse-graining of the timescale and the Gaussian assumption requires further discussion. The statistics at the end of one microscopic interaction of duration τ_{ac} are not Gaussian even if they were at the beginning of the interval, because nonlinearity induces non-Gaussian effects. (This is the issue surrounding the justification of Boltzmann's *Stosszahlansatz.*) However, consider a time interval $\Delta t \gg \tau_{\text{ac}}$. During such an interval, *many* microscopic interactions will occur. Let the ith kick, of duration τ_{ac}, be δv_i. The associated acceleration is $\delta v_i/\tau_{\text{ac}}$, and the total acceleration during Δt is $\delta a(\Delta t) = \sum_{i=1}^{\Delta t/\tau_{\text{ac}}} \delta v_i/\tau_{\text{ac}}$. If the kicks are independent (a reasonable lowest-order idealization), then to the extent that $\Delta t/\tau_{\text{ac}} \gg 1$, $\delta a(\Delta t)$ is the sum of many independent random variables and one can appeal to the central limit theorem to conclude that $\delta a(\Delta t)$ is essentially Gaussian and independent of the actual statistics of the microscopic accelerations. If Δt is taken to *scale* with τ_{ac} (for example, $\Delta t = 5\tau_{\text{ac}}$), then in the limit $\tau_{\text{ac}} \to 0$ Δt also shrinks to 0 and one recovers the Gaussian white-noise approximation.

Equation (67b) shows that $D_v = \int_0^\infty \mathrm{d}\tau\, \langle \delta a_x(t+\tau)\delta a_x(t)\rangle$, a special case of Taylor's formula (4). By describing the properties of the Langevin equations in terms of microscopic fluctuations, one needs to rely less on their classical interpretation. When classical weakly coupled kinetic theory is appropriate, ν and D_v can be computed simply; for plasmas, the calculation reduces to the solution of the linearized Klimontovich equation (Sec. 2.2.2, p. 27). When, on the other hand, the microscopic events (those on the shortest dynamical timescale of interest) are highly nonlinear, other techniques must be employed. Those are just the statistical closure approximations for turbulence; they will be discussed later.

3.2.2 Solution of the classical Langevin equations

Now consider the solution of the Langevin equations (66) for times longer than the microscopic correlation time τ_{ac} (in the classical Langevin theory, $\tau_{\text{ac}} \to 0$). Because Eqs. (66) are linear, the quantities $\boldsymbol{x}(t)$ and $\boldsymbol{v}(t)$ can be found by a straightforward Green's-function approach. For example, the velocity fluctuation is the time convolution of Green's function for Eq. (66b) with the random acceleration. Because integration, a linear operation, can be represented as the limit of a discrete Riemann sum, one may appeal to the theorem that any sum of Gaussian variables is again Gaussian. Hence $\boldsymbol{x}$ and $\boldsymbol{v}$ are (jointly) Gaussian, and the entire probability density functional (Sec. 3.5.1, p. 59) is specified by the two-time correlation matrix of those variables. The calculations are straightforward. As a special case, the first- and second-order equal-time moments (in 1D for simplicity) conditional on initial conditions (x_0, v_0) at $t = 0$ are (Uhlenbeck and Ornstein, 1930; Wang and Uhlenbeck, 1945)

$$\langle v \mid x_0, v_0\rangle = \mathrm{e}^{-\nu t} v_0, \qquad \langle x \mid x_0, v_0\rangle = x_0 + (1 - \mathrm{e}^{-\nu t})\lambda_{\text{mfp},0}, \tag{68a,b}$$
$$\langle \delta v^2\rangle = V_t^2(1 - \mathrm{e}^{-2\nu t}), \qquad \langle \delta x\, \delta v\rangle = \lambda_{\text{mfp}} V_t (1 - \mathrm{e}^{-\nu t})^2, \tag{68c,d}$$
$$\langle \delta x^2\rangle = \lambda_{\text{mfp}}^2 (2\nu t - 3 + 4\mathrm{e}^{-\nu t} - \mathrm{e}^{-2\nu t}), \tag{68e}$$

[69] The central limit theorem (Feller, 1967) states that under certain restrictions the sum of n independent random variables becomes Gaussian as $n \to \infty$.

where $V_t \doteq (T/M)^{1/2}$, $\lambda_{\text{mfp},0} \doteq v_0/\nu$, and $\lambda_{\text{mfp}} \doteq V_t/\nu$. The short- and long-time limits of these results are collected in Table 1 (p. 50). In particular, one has v-space diffusion with diffusion coefficient[70] $D_v = V_t^2\nu$ for $0 = \tau_{\text{ac}} < t < \nu^{-1}$. For $\nu t \gg 1$ the velocity of the test particle thermalizes to V_t and x-space diffusion ensues with diffusion coefficient $D_x = V_t^2/\nu = \lambda_{\text{mfp}}^2\nu$. The latter result is the usual random-walk formula for a diffusion process with step size λ_{mfp} and step time ν^{-1}.[71]

Table 1
Limits of the classical Langevin statistics. Here $D_v \doteq V_t^2\nu$ and $D_\parallel \doteq V_t^2/\nu$. $\lambda_{\text{mfp},0}$ and V_t are defined after Eq. (68).

	$\nu t \ll 1$ (*short times*)	$\nu t \gg 1$ (*long times*)
$\langle v \mid x_0, v_0\rangle$	$(1-\nu t)v_0$ (collisional slowing down)	0 (randomization of directed velocity)
$\langle x \mid x_0, v_0\rangle$	$x_0 + v_0 t$ (free streaming)	$x_0 + \lambda_{\text{mfp},0}$ (randomized in a mean free path)
$\langle \delta v^2\rangle$	$2D_v t$ (v-space diffusion)	V_t^2 (thermalization)
$\langle \delta x\, \delta v\rangle$	$D_v\, t^2$ (integral of v-space diffusion)	$D_\parallel$ (parallel transport)
$\langle \delta x^2\rangle$	$\frac{2}{3}D_v\, t^3$ (double integral of v-space diffusion)	$2D_\parallel t$ (x-space diffusion)

In the collisionless limit the spatial dispersion $\langle \delta x\rangle^2 = \frac{2}{3}D_v t^3$ figures importantly in the justification of quasilinear theory [Sec. 4.1.2 (p. 91) and Appendix D (p. 279)] and in Dupree's resonance-broadening theory (Sec. 4.3, p. 108). Aspects of the collisionless equal-time PDF corresponding to these Langevin dynamics are discussed in Sec. 12.6.3 (p. 246) and Appendix E.1.2 (p. 284).

In magnetized plasmas an appropriate Langevin model comprises Eqs. (66) (for motion along $\boldsymbol{B}$) plus $\dot{\boldsymbol{x}}_\perp = \boldsymbol{V}_\perp(t)$ with $\langle \delta\boldsymbol{V}_\perp(\tau)\delta\boldsymbol{V}_\perp(0)\rangle = 2D_\perp\delta(\tau)\mathsf{I}$. Such models have figured in discussions of transport in stochastic magnetic fields (Krommes et al., 1983; Balescu et al., 1995; vanden Eijnden and Balescu, 1996).

[70] According to the theory of classical random walks, a velocity-space diffusion coefficient scales as $D_v \sim \Delta v^2/\Delta t$. However, one must *not* infer from the result $D_v = V_t^2\nu$ that $\Delta v \sim V_t$ and $\Delta t \sim \nu^{-1}$, which would mean one huge kick in one collision time. Instead, the duration of a kick is $\Delta t = \tau_{\text{ac}} = \epsilon\nu^{-1}$, where ϵ is a small parameter that is taken to zero in the classical Langevin problem. (Physically, ϵ is the plasma discreteness parameter ϵ_p.) The size of a kick is $\Delta v = \sqrt{\epsilon}\, V_t$. [This is consistent with the representation $F(\tau) = 2D_v\tau_{\text{ac}}^{-1}\delta(\tau/\tau_{\text{ac}})$. The coefficient of the dimensionless delta function is $(\Delta v/\Delta t)^2$, or $\Delta v = (\tau_{\text{ac}}D_v)^{1/2} = (\nu\tau_{\text{ac}})^{1/2}V_t$.] Then $D_v = \lim_{\epsilon\to 0}(\sqrt{\epsilon}\, V_t)^2/(\epsilon\nu^{-1}) = V_t^2\nu$. When one coarse-grains over a microscopic timescale, one must take that scale to zero as the last limiting operation.

[71] In the presence of a background magnetic field, the Langevin calculation generalizes in a straightforward way; one merely solves the full Lorentz equations of motion, including gyrospiraling, at the cost of possibly tedious algebra (Kursunoglu, 1962). As a special case, one obtains for $\nu t \gg 1$ the familiar result $\langle \delta x^2\rangle = 2D_\perp t$, where for $\omega_c \gg \nu$ one finds $D_\perp \approx \rho^2\nu$, ρ being the gyroradius.

The form of the two-time velocity correlation function $C(t,t') \doteq \langle \delta v(t) \delta v(t') \rangle$ is also instructive. For a thermalized particle ($\nu t,\ \nu t' \to \infty$), the statistics become stationary and C is found to depend on only $\tau \doteq t - t'$ according to

$$C(\tau) = V_t^2 \mathrm{e}^{-\nu|\tau|}. \tag{69}$$

This result holds even for motion across a magnetic field, but consider parallel motion for simplicity. Then for $\tau \gtrless 0$ Eq. (69) is seen to involve $R(\pm\tau)$, where $R(\tau) \doteq H(\tau)\mathrm{e}^{-\tau}$ is Green's function for the left-hand side of Eq. (66b). [$H(\tau)$ is the Heaviside unit step function defined in Appendix K.2 (p. 304).] It is revealing to consider the temporal Fourier transform:

$$C(\omega) = \frac{2\nu V_t^2}{\omega^2 + \nu^2} = |R(\omega)|^2 F(\omega), \tag{70}$$

where

$$F(\omega) = 2D_v \quad \text{or} \quad F(\tau) = 2D_v \delta(\tau) \tag{71a,b}$$

is the covariance of the forcing $\delta\tilde{a}$. Equation (70) provides a clean statement of the steady-state balance between forcing (F) and dissipation (ν, encapsulated in R); see related discussion in the vicinity of Eq. (12) of Fox and Uhlenbeck (1970). By inverse Fourier transformation, it leads to the interesting alternate representation

$$C(\tau) = R(\tau) \star F(\tau) \star R^\dagger(\tau), \tag{72}$$

where $R^\dagger(\tau) \doteq R(-\tau)$ and $\star$ denotes convolution. It can be shown that the form (72) transcends its derivation and is retained in general turbulence theory; see the discussion at the end of the next paragraph.

It is impossible to overstate the conceptual importance of these physically elementary and mathematically straightforward results. Without a firm grasp of the heuristic content of the classical Langevin problem at hand (especially the roles of the various timescales, the random-walk scalings of diffusion coefficients, and the concept of a balance between forcing and dissipation), attacks on the turbulence problem will likely degenerate into a morass of unrecognizable (and probably incorrect) mathematics. Several of the important statistical closures to be derived, including the DIA, can be developed in terms of rigorous Langevin representations, the mere existence of which guarantees important realizability properties. Furthermore, the general form of the equations provides a welcome unification. Indeed, the Langevin balance between random acceleration and coherent drag, particularly in the form (72), generalizes to a highly nonlinear *spectral balance equation* for turbulent fluctuations, as discussed in Secs. 5.4 (p. 133) and 6.2.2 (p. 155). One important difference between classical and turbulence theory is that whereas in classical theory the autocorrelation time is taken to be vanishingly small [reflected by the delta function in Eq. (71b)], so that microscopic, sub-τ_{ac} dynamics are not seen, in general turbulence theory τ_{ac} must be obtained as a self-consistent property of the fluctuations. In some cases a separation of timescales need not exist; in strong turbulence both τ_{ac}^{-1} and ν meld into a single, nonlinearly determined damping rate Σ. It can be said that the essence of the turbulence problem is the "opening up" and the self-consistent determination of the microscopic dynamics. Of course, that is just what is done in the derivation of the Balescu–Lenard collision operator. However, both that operator as well as the previous Langevin model benefit from properties of thermal equilibrium. (In the Langevin calculation the fluctuation level is known once the background temperature is specified. At the microscopic level the result $V_t^2 = T/M$ is ultimately a consequence of a Gibbs distribution for the combined system of test particle plus background.) The extra difficulty of a theory of turbulence is that the statistical distribution is far from equilibrium, so its form is not known explicitly. In the course of solving the balance equation, both the fluctuation level and autocorrelation time are obtained simultaneously. In general, one must deal with the coupling of multiple spatial scales as well. Thus with I being intensity, the classical form $C(\omega) = 2\nu I/(\omega^2 + \nu^2)$ generalizes to the transcendental equation

$$C_{\boldsymbol{k}}(\omega) = |R_{\boldsymbol{k}}(\omega; \Sigma_{\boldsymbol{k},\omega}[I])|^2 F_{\boldsymbol{k}}(\omega; \Sigma_{\boldsymbol{k},\omega}[I]), \tag{73}$$

where $I_{\boldsymbol{k}} \doteq (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{d}\omega\, C_{\boldsymbol{k}}(\omega)$ and an appropriate form is given for $\Sigma[I]$. The bracket notation is used to indicate that Σ may depend *functionally* on I. Powerful ways of determining that functional dependence are discussed in Sec. 6 (p. 146).

The linear, additively forced Langevin example leads directly to the concept of the spectral balance equation. However, the classical problem has nothing to say about the determination of ν, a property of *nonlinear* microscopic dynamics. To provide insights into that problem, I discuss in Sec. 3.3 (p. 52) a solvable problem with multiplicative statistics. First, however, a few words on nonclassical random walks are in order.

3.2.3 Generalized Brownian motion; Lévy flights

It must be emphasized that the classical diffusion law $\langle \delta x^2 \rangle \propto t$ is a very special case. For arbitrary nonlinear physical processes, there is no reason why $\lim_{\Delta x,\, \Delta t \to 0} \Delta x^2/\Delta t$ should be finite and nonzero. More generally, $\langle \delta x^2 \rangle \propto t^\alpha$ is possible, with $\alpha = 2$ being called *ballistic*, $1 < \alpha < 2$ being called *superdiffusion*, and $\alpha < 1$ being called *subdiffusion*. Such processes arise from various kinds of accelerated or "sticky" motion as particles execute their random walks. A short and readable introduction to such *Lévy flights* was given by Klafter et al. (1996); see also Zumofen et al. (1999). More information and references can be found in Balescu (1997). By generalizing the classical Langevin theory sketched above, Mandelbrot (1982) showed how to construct *fractional Brownian motion* that possesses an $\alpha \neq 1$; see, for example, the review by Feder (1988). Obviously, a complete turbulence theory should be able to cope with such unusual processes. At this point one should simply appreciate that although the structure of the classical Langevin problem is enormously instructive, it does not capture all possibilities.

3.3 The stochastic oscillator: A solvable example with multiplicative statistics

"There arise from the dynamical equations an infinite hierarchy of coupled equations which relate given ensemble averages to successively more complicated ones. . . . This situation, which commonly is called the closure problem, arises even when the nonlinear stochastic terms are linear in the dynamic variables." — *Kraichnan (1961).*

The simplicity of the Langevin systems discussed in the last section arose because the random forcing entered *additively*. In practice, however, *multiplicative* statistics are more common [cf. the advective nonlinearities in the Navier–Stokes, MHD, gyrokinetic, and other equations discussed in Sec. 2 (p. 22)]. In the present section I discuss an extremely instructive solvable model, the so-called *stochastic oscillator* (SO).[72] The model can be derived from radical simplifications of the nonlinear terms of the NS or GK equations [see point (ii) below]; it also arises naturally in various physical applications such as stochastic line broadening in magnetic spin resonance (Kubo, 1959). Variants of this model have been frequently used to illustrate the merits and deficiencies of various attacks on the statistical closure problem.[73] The most important reference in this context is by Kraichnan (1961); see also the earlier work by Kraichnan (1958a) and the review by Krommes (1984a). A generalization was used by Kraichnan (1976a) in his treatment of the role of helicity fluctuations on magnetic-field diffusion; see Sec. 10.3 (p. 223) for further discussion. For modern plasma-physics applications, see Krommes and Hu (1994) and Krommes (2000b). A special limit of the model also describes exactly the linear part of the Landau-fluid closure problem introduced in Sec. 2.4.1 (p. 33); see Appendix C.2.2 (p. 278).

The model is the following primitive equation for a random variable ψ:

$$\partial_t \psi(t) + \mathrm{i}\widetilde{\omega}(t)\psi = 0. \tag{74}$$

Here $\widetilde{\omega}(t)$ is a Gaussian random variable with zero mean and specified, stationary covariance $\Upsilon(t, t') = \Upsilon(\tau)$ characterized[74] by an autocorrelation time $\tau_{\mathrm{ac}}^{\mathrm{lin}}$. The reason for the superscript lin is described in point (iii) of the next paragraph. For the initial conditions on ψ, see the last paragraph of this section.

The model is intended to capture several important features of the typical quadratically nonlinear primitive equations that arise in practice (Sec. 2, p. 22): (i) As emphasized by Kraichnan (1961), although Eq. (74) is linear in the dynamical variable ψ it is *quadratically nonlinear* in *random* variables. It thus displays the same closure problem that plagues more complicated equations: the equation for a cumulant of order n involves a cumulant of order $n + 1$. (ii) The form $\mathrm{i}\widetilde{\omega}$ of the random coefficient echoes the structure of the spatial Fourier transform of an advective nonlinearity $\widetilde{\boldsymbol{V}} \cdot \boldsymbol{\nabla}$; in the limit that the advecting field has infinite wavelength, the correspondence is exact with $\widetilde{\omega} \doteq \boldsymbol{k} \cdot \widetilde{\boldsymbol{V}}$. [For some related discussion, see Krommes (2000b).] (iii) The presence of a characteristic autocorrelation time $\tau_{\mathrm{ac}}^{\mathrm{lin}}$ in $\Upsilon(\tau)$ suggests the linear-theory-induced decorrelation mechanisms of steady-state turbulence such as wave dynamics or particle streaming; see Sec. 4.1 (p. 90). [For physical problems $\Upsilon(\tau)$ is best interpreted as a Lagrangian correlation function. Thus the effective $\tau_{\mathrm{ac}}^{\mathrm{lin}}$ can be finite even for a static Eulerian correlation function provided that the latter has nontrivial spatial variation.]

One property that Eq. (74) does not share with the equations of Sec. 2.4 (p. 33) is that it models *passive* advection (the statistics of $\widetilde{\omega}$ are specified and are independent of ψ; cf. the kinematic dynamo problem) whereas in most physical situations the advecting velocity is determined self-consistently, being a (usually linear) functional of ψ itself [cf. the Vlasov equation ($\boldsymbol{E} = \boldsymbol{E}[f]$) or GF models involving $\boldsymbol{E} \times \boldsymbol{B}$ advection ($\boldsymbol{V}_E = \boldsymbol{V}_E[\varphi]$)]. There are important differences between the statistical descriptions of problems with passive and self-consistent advection. (In particular, self-consistent problems possess symmetries and conservation laws that are not shared by passive ones.) Another simplification is the absence in Eq. (74) of a nontrivial linear response or dependence on independent variables such as $\boldsymbol{x}$ or $\boldsymbol{v}$. For example, the characteristic linear streaming term $\boldsymbol{v} \cdot \boldsymbol{\nabla}$ of Vlasov theory

72 See van Kampen (1976) for discussion of more complicated second-order oscillator models.
73 Feynman strongly believed in the utility of "toy models" (Mehra, 1994).
74 For definiteness one usually takes $\Upsilon(\tau) = \beta^2 \exp(-|\tau|/\tau_{\mathrm{ac}}^{\mathrm{lin}})$. This exponential form is demanded by Doob's theorem (Wang and Uhlenbeck, 1945; Papoulis, 1991) if the processes giving rise to $\widetilde{\omega}$ are Markov.

is absent. Such linear effects provide important decorrelation mechanisms; in the stochastic oscillator, those are encapsulated in the prescribed $\tau_{\rm ac}^{\rm lin}$.

For initial conditions one usually takes either (Kraichnan, 1961) $\psi(0) = 1$ or asserts a centered Gaussian distribution for $\psi_0 \doteq \psi(0)$. The latter choice (Krommes, 1984a) is closer in spirit to the behavior of Navier–Stokes-like equations, so I follow it here; dynamical linearity permits one to take $\langle|\psi_0|^2\rangle = 1$ without loss of generality. If $\widetilde{\omega}$ and ψ_0 are statistically independent, it is easy to show that $\langle\psi_0\rangle = 0$ implies $\langle\psi(t)\rangle = 0$.

3.3.1 Response function for the stochastic oscillator

The great merit of Eq. (74) as a pedagogical example is that the primitive dynamics can be solved explicitly (Kubo, 1962b, 1963), whereas the dynamically nonlinear equations that arise in practice cannot. Thus cumulants of any order can be calculated from the exact solution and compared with various closure approximations. Introduce the unit step function $H(\tau)$ that ensures causality, and define $\psi_+(t) \doteq H(t)\psi(t)$. One then finds $\psi_+(t) = \widetilde{R}(t;0)\psi_0$, where

$$\widetilde{R}(t;t') \doteq H(\tau)\exp\left(-\mathrm{i}\int_{t'}^{t}\mathrm{d}t''\,\widetilde{\omega}(t'')\right) \tag{75}$$

is the random *infinitesimal response function* or random Green's function that obeys[75]

$$\partial_t\widetilde{R}(t;t') + \mathrm{i}\widetilde{\omega}(t)\widetilde{R} = \delta(t-t'). \tag{76}$$

In the present dynamically linear problem there is no difference between infinitesimal and finite response; more generally, though, it turns out to be the *infinitesimal* response function that is most useful, a possibly counterintuitive result. In problems of self-consistent advection the equation for the response function is more complicated because the random frequency must be perturbed as well; see further discussion in Sec. 3.9.1 (p. 77). In all cases the mean infinitesimal response function $R \doteq \langle\widetilde{R}\rangle$ is of great importance, as we will see in Secs. 5 (p. 126) and 6 (p. 146). In the present model the significance of R is emphasized by the easy-to-prove fact

$$C_+(\tau) = R(\tau)\langle|\delta\psi_0|^2\rangle. \tag{77}$$

Only one dimensionless parameter can be built from the two dimensional parameters β and $\tau_{\rm ac}^{\rm lin}$ remaining in the problem, the *Kubo number*(Kubo, 1959, 1962b; Toda et al., 1995; van Kampen, 1976)

$$\mathcal{K} \doteq \beta\tau_{\rm ac}^{\rm lin}. \tag{78}$$

$\mathcal{K}$ is a normalized measure of the linear autocorrelation time, and the size of $\mathcal{K}$ relative to unity controls the behavior of the time correlations and infinitesimal response. For example, the exact mean

[75] The semicolon between t and t' is intended to remind one that the function is causal; a comma is used for the two-sided correlation function $C(t,t')$. More generally, the arguments to the right of the semicolon denote when infinitesimal perturbations were applied, and the arguments to the left denote when response was measured—for example, the "two-in, one-out" response function is $R(t;t',t'')$. For more discussion of response functions, see Sec. 3.5.4 (p. 64).

infinitesimal response function $R(t;t')$ can be calculated from Eq. (75) and the Gaussian property to be

$$R(\tau) = H(\tau)\exp[-(\beta\tau_{\mathrm{ac}}^{\mathrm{lin}})^2(\tau/\tau_{\mathrm{ac}}^{\mathrm{lin}} - 1 + \mathrm{e}^{-\tau/\tau_{\mathrm{ac}}^{\mathrm{lin}}})], \tag{79}$$

or in terms of the dimensionless time $\overline{\tau} \doteq \beta\tau$, $R(\tau) = H(\tau)\exp[-\mathcal{K}^2(\mathcal{K}^{-1}\overline{\tau} - 1 + \mathrm{e}^{-\mathcal{K}^{-1}\overline{\tau}})]$. One identifies two regimes: short-time ($\tau < \tau_{\mathrm{ac}}^{\mathrm{lin}}$ or $\overline{\tau} < \mathcal{K}$),

$$R(\tau) \approx H(\tau)\exp(-\tfrac{1}{2}\overline{\tau}^2) = H(\tau)\exp(-\tfrac{1}{2}\beta^2\tau^2); \tag{80}$$

and long-time (with the inequalities reversed),

$$R(\tau) \approx H(\tau)\exp(-\mathcal{K}\overline{\tau}) = H(\tau)\exp(-\beta^2\tau_{\mathrm{ac}}^{\mathrm{lin}}\tau). \tag{81}$$

The area under $R(\tau)$ is thus controlled by the size of $\mathcal{K}$. For $\mathcal{K} > 1$ the short-time regime dominates and the area is $O(\beta^{-1})$; for $\mathcal{K} < 1$ times longer than $\tau_{\mathrm{ac}}^{\mathrm{lin}}$ dominate and the area is $O(\beta^{-1}\mathcal{K}^{-1})$. That the short-time R does not decay as a simple exponential is a signature that the statistics of ψ are not Gaussian-Markov in that regime, according to Doob's theorem. Note that for $\mathcal{K} = \infty$ R lies entirely in the short-time regime, whose strongly non-Gaussian nature makes it difficult to treat.

3.3.2 Transport estimates

The behavior of $R(\tau)$ can be seen to be in accord with simple estimates of diffusion coefficients if one makes the correspondence $\beta = \overline{k}\,\overline{V}$, where $\overline{k}$ is a characteristic wave number and $\overline{V}$ is the rms level of a very-long-wavelength advecting velocity. The spatial Fourier transform of the Green's function for a diffusion equation is $G_k(\tau) = H(\tau)\exp(-k^2D\tau)$, having area $(k^2D)^{-1}$. Upon comparing this area with the above results for $R(\tau)$ and identifying k with $\overline{k}$ at this crude level of analysis, one finds

$$D \sim \overline{V}^2\tau_{\mathrm{ac}}^{\mathrm{lin}} \quad (\mathcal{K} < 1) \quad \text{or} \quad D \sim \overline{V}/\overline{k} \quad (\mathcal{K} > 1). \tag{82a,b}$$

Equation (82a) is sometimes called the *quasilinear* or *weak-turbulence* form of the diffusion coefficient ($D \sim \overline{V}^2$) while Eq. (82b) is called the *strong-turbulence* form ($D \sim \overline{V}^1$). It is said that the quasilinear scaling possesses the *classical exponent* 2 whereas the strong-turbulence scaling possesses the *anomalous exponent* 1. [More refined considerations of transport in strong-turbulence regimes lead to an exponent that differs slightly from 1; see Sec. 12.5 (p. 245).] The appearance of anomalous exponents is discussed from a deeper perspective in Sec. 6.1.2 (p. 150) and Appendix B (p. 264). Both of the results (82) follow from the general random-walk estimate $D \sim \overline{V}^2\tau_{\mathrm{ac}}$, where the true autocorrelation time for the random process is

$$\tau_{\mathrm{ac}} = \tau_{\mathrm{ac}}^{\mathrm{lin}} \quad (\mathcal{K} < 1) \quad \text{or} \quad \tau_{\mathrm{ac}} = (\overline{k}\,\overline{V})^{-1} = \beta^{-1} \quad (\mathcal{K} > 1). \tag{83a,b}$$

In other words, $\mathcal{K} = \tau_{\mathrm{ac}}^{\mathrm{lin}}/\beta^{-1}$ is the ratio of the linear autocorrelation time to the nonlinear one, and it is always the shorter of those that controls the random walk. Of course, as a function of $\mathcal{K}$ the weak- and strong-turbulence regimes are smoothly connected. Note that one unrealistic feature of such simple models is that they exhibit no stochasticity threshold; transport exists for any nonzero fluctuation level.

For cross-field transport in strong magnetic fields, the characteristic velocity is the $\boldsymbol{E} \times \boldsymbol{B}$ velocity: $\overline{V} \sim V_E \propto B^{-1}$. One is thus led to anticipate a crossover from a weak-turbulence scaling $D \sim B^{-2}$ to a strong-turbulence, Bohm-like scaling $D \sim B^{-1}$ when the fluctuations grow so large that $(\overline{k}V_E)^{-1} < \tau_{\text{ac}}^{\text{lin}}$, *provided that neither* $\tau_{\text{ac}}^{\text{lin}}$ *nor* $\overline{k}$ *depend on* B. B-independent $\overline{k}$ amounts to $\overline{k}L = O(1)$, where L is a macroscopic length; Eq. (82b) can be written $D \sim (\overline{k}L)^{-1}(\overline{V}L)$. If, on the other hand, $\overline{k}$ is determined by microscopic physics, $\overline{k}\rho_{\text{s}} = O(1)$, then one recovers gyro-Bohm scaling from Eq. (82b): $D = (\overline{k}\rho_{\text{s}})^{-1}(\rho_{\text{s}}/L)(\overline{V}L)$.

The properties (82) and (83) are general features of transport problems that any sensible statistical closure theory should be expected to reproduce. The stochastic-oscillator results will serve as a very useful guide as one proceeds to develop various approximations in Sec. 3.9 (p. 76).

The oscillator model discussed here contains neither forcing nor dissipation. If those are added, the final fluctuation level depends on the balance between forcing and dissipation. Although that provides a more faithful representation of the structure of realistic turbulence problems, I shall not pursue it here but instead refer the reader to the closely related discussion by Krommes (2000b).

3.3.3 Random oscillator with nondecaying response function

For later discussion of the fidelity of statistical closures, it is useful to introduce a slightly more complicated variant of the simple oscillator:

$$\partial_t \psi(t) + \mathrm{i}a\cos(t + \widetilde{\theta})\psi = 0, \tag{84}$$

where $\widetilde{\theta}$ is a random phase distributed uniformly on the interval $[0, 2\pi)$ and a is a constant. The significant feature of this model, which possesses a *non-Gaussian* random multiplicative coefficient, is that its mean response function, $R(\tau) = H(\tau)J_0\big(2a\sin(\tfrac{1}{2}\tau)\big)$, does not approach zero as $\tau \to \infty$. This behavior is intended to model various features of integrable or coherent phenomena that present significant challenges for statistical theories. As we will see, those challenges have not been fully met to date. For example, the response function of the DIA (Sec. 5, p. 126) for this model incorrectly decays to 0 as $\tau \to \infty$ (Sec. 5.6.1, p. 137).

3.4 Dimensionless parameters for turbulence

The two most important dimensionless parameters for turbulence problems are the Kubo number $\mathcal{K}$ and the Reynolds number $\mathcal{R}$. In many heuristic discussions of plasma turbulence, these parameters are often not distinguished clearly or are confused.

3.4.1 Kubo number $\mathcal{K}$

In principle, some sort of Kubo number—a property of the advecting velocity field—can be defined for any kind of problem involving passive advection that evinces an autocorrelation time τ_{ac}. (In this section, to avoid clutter and to permit a later generalization I omit the superscript lin from τ_{ac}.) Specifically, introduce the macroscopic eddy turnover or circulation time $\tau_L \doteq L/\overline{u}$, where $\overline{u}$ is the characteristic rms velocity and L is the system size. τ_L is the time for the macroscopic flow to advect a perturbation across the entire system, or for an eddy of the order of the system size to turn over once. Then

$$\mathcal{K} \doteq \frac{\text{autocorrelation time}}{\text{eddy turnover time}} = \frac{\tau_{\text{ac}}}{\tau_L} = \frac{\overline{u}\tau_{\text{ac}}}{L}. \tag{85}$$

The principal difficulty is that usually the advecting field is a function of both $\boldsymbol{x}$ and t, so the appropriate τ_{ac} to be used in the definition (85) should be a Lagrangian autocorrelation time, which is difficult to compute precisely. This point was discussed in the context of the Vlasov stochastic-acceleration problem by Dimits and Krommes (1986).

3.4.2 Reynolds number $\mathcal{R}$

Additionally, if the problem involves linear dissipation, another dimensionless parameter, the *Reynolds number* $\mathcal{R}$ (often denoted as Re in fluid problems), can be constructed (Rott, 1990). For an equation such as that of Navier and Stokes (either passive or self-consistent), one introduces the hydrodynamic or classical diffusion time $\tau_h \doteq L^2/\mu_{\text{cl}}$. This is the time for a perturbation to diffuse by microscopic classical processes across the entire system. Then

$$\mathcal{R} \doteq \frac{\text{classical diffusion time}}{\text{eddy turnover time}} = \frac{\tau_h}{\tau_L} = \frac{\overline{u}L}{\mu_{\text{cl}}}. \tag{86}$$

For example, a simple dissipative and passive model that generalizes the stochastic oscillator to include dissipation[76] in a finite-sized domain and thus contains both $\mathcal{R}$ and $\mathcal{K}$ is the *generalized reference model* of Krommes and Smith (1987) [see also Krommes and Ottaviani (1999)]:

$$\partial_t T(x,t) + \widetilde{u}(t)T_x - \mu_{\text{cl}}T_{xx} = 0, \tag{87a}$$

with boundary conditions

$$T(0,t) = \Delta T, \quad T(L,t) = 0. \tag{87b}$$

3.4.3 The $\mathcal{R}$–$\mathcal{K}$ parameter space

For passive problems $\mathcal{K}$ and $\mathcal{R}$ are independent parameters, so the dynamical behavior must be classified in terms of an $\mathcal{R}$–$\mathcal{K}$ parameter space, displayed in Fig. 2 (p. 58). Various regimes are evident.[77] Let a dimensional flux be called $\overline{\Gamma}$ [the bar stands for volume average, an operation that is explicitly exploited in Sec. 11 (p. 230)] and let its dimensionless version be called $\overline{\gamma} \doteq \overline{\Gamma}/(\overline{u}|\Delta T|)$. Also introduce an effective diffusivity $\overline{D}$ such that $\overline{\Gamma} = -\overline{D}(\langle\Delta T\rangle/L)$. Of course, dimensionally

$$\overline{D} \sim \Delta x^2/\Delta t \sim \Delta v^2 \Delta t. \tag{88a,b}$$

One has $\overline{D} = \overline{u}L\overline{\gamma}$. It is clear from this latter relation that $\overline{D}$ does not necessarily describe local transport on scales much smaller than L; in general, it is a global property of the entire slab.

[76] For general passive advection it is not conventional to call the dissipation coefficient μ_{cl}. That is done in this pedagogical discussion in order that a uniform notation involving a (generalized) Reynolds number can be used.
[77] For a more thorough version of the following discussion, see Krommes and Smith (1987).

In the model (87) there always flows at least the *classical flux* $\gamma_{\rm cl} = \mu_{\rm cl}/(\overline{u}L)$; according to Eq. (86), $\gamma_{\rm cl} = \mathcal{R}^{-1}$. To estimate advective contributions, it is convenient to use Eq. (88b) with $\Delta v \sim \overline{u}$ and $\Delta t = \min(\tau_{\rm ac}, \tau_h, \tau_L)$. For $\tau_{\rm ac} < \tau_h$ $(\mathcal{K} < \mathcal{R})$ and $\tau_{\rm ac} < \tau_L$ $(\mathcal{K} < 1)$, one obtains the usual quasilinear result $\overline{D}_q \sim \overline{u}^2\tau_{\rm ac}$ or $\overline{\gamma}_q \sim \mathcal{K}$ $(\mathcal{K} < \mathcal{R},\ \mathcal{K} < 1)$. Krommes and Smith (1987) called this regime the *kinetic-quasilinear regime* on the grounds that it is usually kinetic processes (e.g., free streaming) that determine $\tau_{\rm ac}$, hence $\mathcal{K}$. For $\tau_h < \tau_{\rm ac}$ $(\mathcal{K} > \mathcal{R})$ and $\tau_h < \tau_L$ $(\mathcal{R} < 1)$, a *hydrodynamic-quasilinear regime*, one obtains $\overline{D}_h \sim \overline{u}^2\tau_h$ or $\overline{\gamma}_h \sim \mathcal{R}$ $(\mathcal{R} < \mathcal{K},\ \mathcal{R} < 1)$. Finally, when $\tau_L < \tau_{\rm ac}$ $(\mathcal{K} > 1)$ and $\tau_L < \tau_h$ $(\mathcal{R} > 1)$, one obtains a *strong-turbulence regime* with $\overline{D} \sim \overline{u}L$ or $\overline{\gamma} \sim 1$ $(\mathcal{R} > 1,\ \mathcal{K} < 1)$. The three major regimes are delimited by the solid lines OX, XB, and XC in Fig. 2 (p. 58). It is straightforward to deduce that the advective contribution to transport dominates to the right of line AXC, which is thus the most interesting part of parameter space. The original SO model corresponds to $\mathcal{R} = \infty$, so sits either in the kinetic-quasilinear regime $(\mathcal{K} < 1)$ or the strong-turbulence regime $(\mathcal{K} > 1)$.

For self-consistent fluid problems the autocorrelation time is not an independent parameter; one must take $\tau_{\rm ac} = \min(\tau_h, \tau_L)$, which corresponds to the curve OXB in Fig. 2. In particular, in a strongly turbulent fluid problem the effective $\mathcal{K}$ is $O(1)$. However, in self-consistent kinetic (Vlasov-like) problems $\tau_{\rm ac}$ can persist as an independent parameter related to the streaming motion of the particles through the wave packets; see the discussion of the quasilinear autocorrelation time in Sec. 4.1.2 (p. 91).

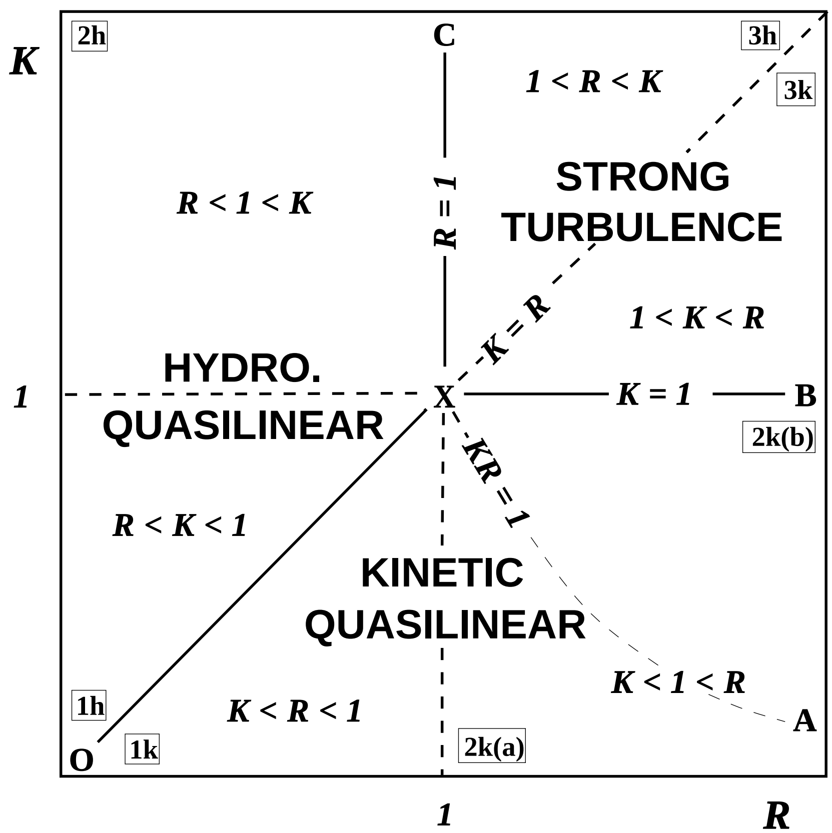

Fig. 2. The parameter space of Reynolds number $\mathcal{R}$ and Kubo number $\mathcal{K}$ [after Fig. 2 of Krommes and Smith (1987), used with permission].

3.5 Key statistical measures

Having demonstrated some simple solvable models that display important features of the statistical closure problem, I turn in the next several subsections to a survey of the formal techniques that can be brought to bear on the statistics of such models as well as on the much more difficult PDE's of plasma physics, for which useful exact solutions do not exist in general. I begin with the key measures that can be used to quantify the statistical behavior of such systems.

3.5.1 Probability density functions

The most complete description of a continuous random variable $\widetilde{x}$ (sometimes called X) is provided by the PDF [78] $P_X(x) \equiv P(x)$; the probability that $\widetilde{x}$ takes on the value x in an interval $\mathrm{d}x$ is $P(x)\mathrm{d}x$. The nonrandom variable x is called the *observer coordinate*. A useful identity is [79] $P(x) = \langle \delta(x - \widetilde{x}) \rangle$. A good introduction to probability theory can be found in Papoulis (1991); see also Feller (1967). A review of various properties of and methods related to PDF's was given by Pope (1985); see also Haken (1975).

In practice one deals more frequently with random fields $\psi(x, t)$ that are parametrically dependent on space and time. Then one must consider most generally the fully multivariate (including the entire continuum of space and time points) probability density *functional* $P[\psi]$; the brackets indicate functional dependence. The meaning of and manipulations with such functionals can be understood (Beran, 1968; Zinn-Justin, 1996) by discretizing the space and time axes and considering ordinary functions of the very large number of variables representing the values of the fields at each of the discrete points in space-time. Seminal discussion in the physics literature was given by Feynman (1948b), whose work is further reviewed in Sec. 6.1 (p. 147); see also Mehra (1994, Chap. 10.4). Because probability density functionals are central to the later discussion of renormalized field theory in Sec. 6 (p. 146), some of the details are elaborated in Appendix H (p. 293).

Because of the wealth of information contained in even a 1D PDF, PDF methods are still in their infancy [see Sec. 10 (p. 220) for further discussion]. Much better developed are moment-based approximations.

3.5.2 Moments and cumulants

The nth moments of the PDF $P(x)$ are defined by $M_n \doteq \langle x^n \rangle$ (these may be infinite). The Fourier transform of $P(x)$, $Z(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}kx} P(x)$, can be written

$$Z(k) = \langle \exp(-\mathrm{i}k\widetilde{x}) \rangle. \tag{89}$$

---

[78] The PDF is called P in order to distinguish it from the one-particle "distribution function" f of kinetic theory, which is normalized differently: $\int_{-\infty}^{\infty} \mathrm{d}x\, P(x) = 1$ whereas $\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{x}, \boldsymbol{v}) = V$, where V is the volume. Note that this usage of the phrase "distribution function" is confusing because in standard probability theory a distribution function $F_X(x)$ is the probability that X achieves a value less than x; i.e., $F_X(x) = \int_{-\infty}^{x} \mathrm{d}\overline{x}\, P_X(\overline{x})$.

[79] This identity is nontrivial if $\widetilde{x}$ depends on another random variable $\widetilde{z}$ and the average is performed with $P_Z(z)$. It lies at the heart of the path-integral representation of renormalized field theory, as discussed in Sec. 6.4 (p. 166).

$Z(k)$ is called the *characteristic function.* The characteristic function of the Gaussian PDF $P(x) = (2\pi\sigma^2)^{-1/2}\exp[-(x-\overline{x})^2/2\sigma^2]$ is $Z_G(k) = \exp(-\mathrm{i}k\overline{x} - \frac{1}{2}k^2\sigma^2)$. $Z(k)$ is also the *moment generating function*, since a formal Taylor expansion of the exponential leads to

$$Z(k) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}k)^n}{n!} M_n, \quad M_n = \left.\frac{\partial^n Z(k)}{\partial(-\mathrm{i}k)^n}\right|_{k=0}. \tag{90a,b}$$

[$M_0 = 1$ because $P(x)$ is normalized.] For the use of moment generating *functionals* in statistical field theory, see Sec. 6.2 (p. 153).

It is not required that moments of all orders exist. A simple counterexample is the Cauchy or Lorentzian PDF $P(x) = \pi^{-1}a/(x^2+a^2)$, for which even-integer moments for $n \geq 2$ are infinite. In this case the difficulty is manifested in Fourier space by the appearance of a branch point at the origin for the characteristic function $Z(k) = \mathrm{e}^{-|k|a}$. Such PDF's with infinite variance arise in the context of violations of the central limit theorem. Lévy (1937) inquired about the class of PDF's of the sum $c_0 z = \sum_{i=1}^{n} c_i x_i$ that obeyed the scaling relation $\phi(x) = n^{1/\gamma}P_n(n^{1/\gamma}x)$ subject to the constraint $c_0^\gamma = \sum_{i=1}^{n} c_i^\gamma$. He proved that the characteristic function obeyed $Z(k) \propto \exp(-|k|^\gamma)$ for $\gamma \leq 2$. The special value $\gamma = 2$ recovers the Gaussian PDF; processes with $\gamma < 2$ have infinite variance. However, Lévy flights with finite variance can be constructed (Zumofen et al., 1999). Further discussion and references can be found in Balescu (1997).

Even when they exist, moments need not uniquely determine a PDF. *Carleman's criterion* (Carleman, 1922; Wall, 1948) states that the PDF is determined if $\sum_n (1/M_{2n})^{1/2n}$ diverges.[80] Kraichnan (1985) discussed appropriate procedures for dealing with PDF's that violate Carleman's criterion. For some early related discussion, see Orszag (1970b).

Moments form a poor basis for statistical approximations since they typically grow at least exponentially rapidly with order; for example, a centered Gaussian distribution with unit variance has $M_{2n} = (2n-1)!!$. Another, usually undesirable property is that truncation of Eq. (90a) leads to a singular description of $P(x)$ in terms of derivatives of delta functions. For example, if Eq. (90a) is truncated at second order one finds $P(x) \approx \delta(x) - M_1\delta'(x) + \frac{1}{2}M_2\delta''(x)$.

Cumulants (Kubo, 1962a) usually provide a better description than do raw moments. Formally, cumulants $C_n \equiv \langle\langle x^n \rangle\rangle$ are generated from the logarithm of the characteristic function: with $W(k) \doteq \ln Z(k)$, one writes by definition

$$W(k) = \sum_{n=1}^{\infty} \frac{(-\mathrm{i}k)^n}{n!} C_n, \quad C_n = \left.\frac{\partial^n W(k)}{\partial(-\mathrm{i}k)^n}\right|_{k=0}. \tag{91a,b}$$

Combinatoric properties of the logarithm then lead to a *cluster expansion* that relates the moments to the cumulants. For several random variables denoted by 1, 2, ..., the first few members of the cluster expansion are

$$M(1) = C(1), \tag{92a}$$

[80] That is not true if the moments increase more rapidly than exponentially with order. An example of a PDF that violates Carleman's criterion is the log-normal PDF for a variable ε defined such that $z \doteq \ln\varepsilon$ is Gaussian with mean $\overline{z}$ and variance σ^2. One has $P(\varepsilon) = (2\pi\varepsilon^2\sigma^2)^{-1/2}\exp[-(\ln\varepsilon - \overline{z})^2/2\sigma^2]$ and $M_n = \exp(n\overline{z} + \frac{1}{2}n^2\sigma^2)$. This distribution figured prominently in early discussions of intermittency (Kolmogorov, 1962; Frisch, 1995) and also arises in the theory of passive advection (Schekochihin, 2001).

$$M(1,2) = C(1)C(2) + C(1,2), \tag{92b}$$
$$M(1,2,3) = C(1)C(2)C(3) + [C(1)C(2,3) + 2 \text{ terms}] + C(1,2,3), \tag{92c}$$
$$\begin{aligned} M(1,2,3,4) = {}& C(1)C(2)C(3)C(4) \\ &+ [C(1)C(2)C(3,4) + 5 \text{ terms}] + [C(1)C(2,3,4) + 3 \text{ terms}] \\ &+ [C(1,2)C(3,4) + 2 \text{ terms}] + C(1,2,3,4). \end{aligned} \tag{92d}$$

For random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with jointly Gaussian statistics, only the first two cumulants $C(\boldsymbol{x}) \equiv \langle\langle \boldsymbol{x} \rangle\rangle = \langle \boldsymbol{x} \rangle$ and $C(\boldsymbol{x}, \boldsymbol{y}) \equiv \langle\langle \boldsymbol{x}\, \boldsymbol{y}^T \rangle\rangle = \langle \delta\boldsymbol{x}\, \delta\boldsymbol{y}^T \rangle$ survive.

Upon combining Eqs. (89) and (91a), one obtains a rule for interchanging averaging and exponentiation that is often useful in practice:

$$\langle \exp(-\mathrm{i}k\widetilde{x}) \rangle = \exp\left(\sum_{n=1}^{\infty} \frac{(-\mathrm{i}k)^n}{n!} C_n \right) \to \exp(-\mathrm{i}k\langle \widetilde{x} \rangle - \tfrac{1}{2}k^2 \langle \delta\widetilde{x}^2 \rangle) \quad \text{(Gaussian)}. \tag{93a,b}$$

The latter result is often employed in Dupree's resonance-broadening theory (Sec. 4.3, p. 108).

Simple manipulations with the multivariate form of Eq. (93a) can be used to prove that the cumulant of any two statistically independent variables vanishes (Kubo, 1962a). This result provides physical insight to the cluster expansion, which is well known as the *Mayer cluster expansion* (Mayer, 1950) in many-body kinetic theory and as *Wick's theorem* in quantum field theory (Wick, 1950; Zinn-Justin, 1996). In the context of Coulomb interactions between charged particles, the term $C(1)C(2,3)$ in Eq. (92c) is illustrated in Fig. 3.

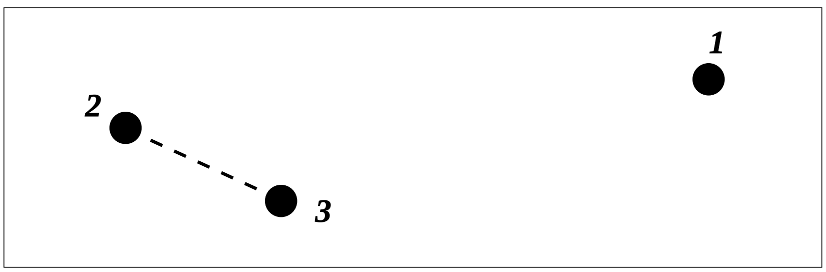

Fig. 3. Illustration of the Mayer cluster (cumulant) expansion. Imagine three charged particles interacting with the Coulomb potential (denoted by a dashed line). The various terms of the cumulant expansion (92c) are generated by turning off the interactions between one or more of the particles in all possible ways. Even when particle 1 does not interact with particles 2 and 3, as depicted in the figure, there is still a probability of finding the particles somewhere in phase space; the drawing corresponds to the term $C(1)C(2,3)$ in Eq. (92c).

Since all cumulants of order higher than 2 vanish for a Gaussian, it might be hoped that PDF's that are "nearly" Gaussian will possess higher-order cumulants that are small. Unfortunately, "nearly" is ill defined. If the cumulants grow smaller with order sufficiently rapidly, then reasonable approximations can sometimes be obtained by truncating the cluster expansion at some order.[81] Nevertheless, even if a particular cumulant is small, if it has the wrong sign and the cumulant expansion is truncated inappropriately, the resulting PDF may not be normalizable. Furthermore, innocent-looking PDF's can have cumulants that grow with order. Kraichnan (1985) pointed out the simple PDF

$$P(x) = (2\pi)^{-1/2} x^2 \exp(-\tfrac{1}{2}x^2), \tag{94}$$

[81] This is the scheme used in classical plasma kinetic theory.

for which[82] $C_0 = 1$, $C_{2n+1} = 0$, $C_2 = 3$, and $C_{2n} = -(-1)^n[(2n)!/n]$ for $n > 1$. In such cases superior and more versatile representations may be found by expanding the PDF in orthogonal polynomials (Kraichnan, 1985); see Sec. 3.9.12 (p. 89).

Typical *moment-based* (more properly, *cumulant-based*) statistical approximations provide closed equations for the first few cumulants. The most important cumulants have special names. If the *mean field* $\langle\psi\rangle$ is nonvanishing, it is crucial to retain it. An example is the Klimontovich density $\widetilde{N}$, whose mean is the one-particle distribution function $f = \langle\widetilde{N}\rangle$; the Vlasov equation (closed in terms of f) is a mean-field theory. However, descriptions based solely on the mean fail to capture a great deal of physical information. For example, it has been remarked[83] that if male $\Leftrightarrow$ 1 and female $\Leftrightarrow -1$, the mean value $\langle\text{gender}\rangle = 0$ does not quite capture the essence of the problem (Gray, 1994).

The most common moment-based approximations are second order, based on the *covariance* $C(1,1') \doteq \langle\delta\psi(t)\delta\psi(1')\rangle$. In the moment hierarchy C is driven by a multipoint generalization of the *skewness parameter* S, a normalized triplet correlation function:

$$S \doteq \langle\delta\psi^3\rangle/\langle\delta\psi^2\rangle^{3/2}; \tag{95}$$

for a Gaussian, $S = 0$. Because the three-point correlation function is related to the rate of energy transfer between modes (Sec. 3.8.1, p. 71), which does not vanish for forced, dissipative turbulence, *such turbulence cannot be Gaussian.* This observation is fundamental in statistical turbulence theory.

Typical fourth-order statistics are the *flatness* F or the *kurtosis* K:

$$F \doteq \langle\delta\psi^4\rangle/\langle\delta\psi^2\rangle^2, \quad K \doteq F - 3; \tag{96a,b}$$

for a Gaussian,[84] $F = 3$ and $K = 0$. It is not hard to find non-Gaussian PDF's. Although the sum of two Gaussian variables is Gaussian, their product is not. An explicit example (a bilinear random flux) is discussed in Sec. 10.1 (p. 221).

The kurtosis is frequently said to be a measure of *intermittency*, although that is not entirely in accord with the refined definition of intermittency given by Frisch (1995).[85] A simple definition of an intermittent flow is one in which laminar and turbulent regions are intermixed. Let the fractional area occupied by turbulence be ϵ, and consider a fluctuating field $\delta\psi$ that vanishes in the laminar region and is approximately Gaussian in the turbulent region. Then $K = \epsilon(3\langle\delta\psi^2\rangle^2)/(\epsilon\langle\delta\psi^2\rangle)^2 - 3 = 3(1-\epsilon)/\epsilon$. For this simple model, a 50% mixture of laminar and turbulent regions has $K = 3$; a flow that is 20% turbulent has $K = 12$. In Sec. 2.4.5 (p. 41) I mentioned simulations of the HW model, which

[82] The result $C_2 = 3$ corrects a misprint in Kraichnan (1985). The characteristic function is $Z(k) = (1-k^2)\mathrm{e}^{-k^2/2}$.

[83] Unfortunately and with apologies, the identity of the original author of this incisive observation is lost among the more than 800 references on statistics in the library system of Princeton University, illustrating that a good idea is much more memorable than even the most elegant formal mathematics.

[84] The flatness of a scalar function of a Gaussian *vector* need not equal 3. For example, for a Gaussian vector $\boldsymbol{u}$ in 3D, $\langle|\boldsymbol{u}|^4\rangle/\langle|\boldsymbol{u}|^2\rangle^2 = \frac{5}{3}$.

[85] According to Frisch, a random signal is intermittent at small scales if the flatness of the high-pass filtered signal grows without bound with the filter frequency. That includes the simple definition involving intermixed laminar and turbulent regions given in the text immediately following this footnote reference, but it excludes such self-similar functions as fractional Brownian motion. For more discussion of intermittency, see Frisch and Morf (1981).

leads to mixtures of coherent vortices and turbulence, for which such a large value of kurtosis was actually measured.

The values of S, K, and other similar statistics are not entirely arbitrary; they must obey *realizability inequalities*, as described next.

3.5.3 Realizability constraints

As we have seen, in a moment-based closure a moment (or cumulant) of some order is approximated in one way or another by terms of lower-order quantities. That leads to an economy of description; however, there is no guarantee that the resulting equations are well behaved. In particular, it is not assured that the infinity of *realizability constraints* associated with the very existence of a PDF are preserved.

Realizability constraints stem from the intrinsic non-negativity of a PDF. A trivial example of such a constraint is that the mean square of a random variable x is non-negative ("positive semidefinite"). This result follows immediately from the definition: $\langle x^2 \rangle \doteq \int_{-\infty}^{\infty} \mathrm{d}x\, P(x) x^2 \geq 0$. Less trivially, this conclusion also applies to the mean-square fluctuation from the mean (the second cumulant), $\langle \delta x^2 \rangle \geq 0$, which implies the constraint (97a) below. When x is a physical variable such as an electric field or fluid velocity, the theorem states that quadratic energylike quantities must be non-negative.

This result may appear to be obvious. Nevertheless, for any particular moment-based closure, which provides approximate *time-evolution equations* for such energylike quantities, there is no guarantee that the constraint is preserved in the course of time, and it is easy to demonstrate approximations for which it is not (Kraichnan, 1961). Typically, if an energylike quantity goes negative, it does so catastrophically and diverges to $\pm\infty$ in a finite time. A telling example of this behavior that is highly relevant to the theory of drift waves in plasma is described in Sec. 8.2.1 (p. 201).

The condition $\langle \delta x^2 \rangle \geq 0$ is but one example of an infinite number of constraints that can be deduced (Wall, 1948; Kraichnan, 1979, 1980, 1985) by asserting the positive-semidefiniteness of $\langle F \rangle$, where F is any non-negative function. That is equivalent to requiring that $\langle Q_r^2(x) \rangle > 0$, where Q_r is the rth orthogonal polynomial.[86] The first two of the resulting constraints are

$$\langle x^2 \rangle - \langle x \rangle^2 \geq 0, \quad \langle x^4 \rangle - \langle x^2 \rangle^2 \geq (\langle x^3 \rangle - \langle x \rangle \langle x^2 \rangle)^2 / (\langle x^2 \rangle - \langle x \rangle^2). \tag{97a,b}$$

The latter equation constrains the relationship between the kurtosis and skewness statistics: For a centered distribution, Eq. (97b) reduces with the aid of the definitions (95) and (96) to $K \geq S^2 - 2$. For example, this constraint is satisfied by the example (94), for which $S = 0$ and $K = -\frac{4}{3}$.

An excellent introductory discussion of realizability inequalities, including additional difficulties that arise when the random variables have compact support, was given by Dubin (1984b). He illustrated some of the issues by describing applications to the *logistic map*

$$x_{n+1} = \lambda x_n (1 - x_n) \quad (0 \leq \lambda \leq 4), \tag{98}$$

properties of which have been intensively studied in the literature (May, 1976).

There are appropriate generalizations of the realizability constraints for several random variables. When the set of variables is the infinitely multivariate collection of all $\boldsymbol{x}$'s and all t's, the constraints become quite difficult to work with explicitly; however, their very existence is profound.

[86] It is not necessary to introduce the orthogonal polynomials; see Theorem 86.1 of Wall (1948) and the last exercise on p. 5 of van Kampen (1981).

The importance of realizability constraints for turbulence was recognized quite early (Kraichnan, 1959b, 1961; Orszag and Kruskal, 1968). Kraichnan (1979, 1980) used them to suggest a computational scheme in which moments of order higher than fourth never appeared. They have also figured in crucial ways in several recent developments, both theoretical and computational, of interest to plasma physics. Those are described in some detail in Secs. 8.2 (p. 201) through 10 (p. 220).

Realizability constraints are necessary but not sufficient. For example, as defined above they place no constraint on the mean of a random variable. Suppose that variable is the Klimontovich microdensity $\widetilde{N}$. Then one knows (Sec. 2.2.2, p. 27) that $\langle \widetilde{N} \rangle = f$, which is proportional to the one-particle PDF. The fundamental constraint that $f \geq 0$ is *not* guaranteed by the theory of realizability inequalities.

3.5.4 Response functions

The ψ cumulants (multipoint correlation functions) are basically measures of fluctuations. Because a fluctuation, once arisen, must decay in a statistical steady state, it is also useful, both heuristically and technically, to introduce independent measures of dissipation. Those are provided by the so-called *response functions*. Consider a nonlinear field equation of the form

$$\partial_t \psi(1) + \mathrm{i}\widehat{\mathcal{L}}\psi + \mathcal{N}[\psi] = \widehat{\eta}(1), \tag{99}$$

where $\widehat{\mathcal{L}}$ is a linear operator, $\mathcal{N}$ is a nonlinear functional, and $\widehat{\eta}$ is an arbitrary source field.[87] The random infinitesimal response function $\widetilde{R}$ is defined by $\widetilde{R}(1;1') \doteq \delta\psi(1)/\delta\widehat{\eta}(1')|_{\widehat{\eta}=0}$; it obeys

$$\frac{\partial}{\partial t}\widetilde{R}(1;1') + \mathrm{i}\widehat{\mathcal{L}}\widetilde{R} + \int \mathrm{d}\overline{1} \left(\frac{\delta \mathcal{N}(1)}{\delta \psi(\overline{1})} \right) \widetilde{R}(\overline{1};1') = \delta(1-1'), \tag{100}$$

where the functional chain rule was used.[88] The mean infinitesimal response function R (*response function*, in brief) is then $R(1;1') \doteq \langle \widetilde{R}(1;1') \rangle$. It is not easy to find closed equations for R because the coefficient $\delta\mathcal{N}/\delta\psi$ in Eq. (100) depends on the random variable ψ; i.e., Eq. (100) has a multiplicative random coefficient and thus possesses the closure problem.

Higher-order response functions such as

$$R(1;1',1'') \doteq \left. \left\langle \frac{\delta^2 \psi(1)}{\delta\widehat{\eta}(1')\delta\widehat{\eta}(1'')} \right\rangle \right|_{\widehat{\eta}=0} \tag{101}$$

can be similarly defined in terms of the functional Taylor coefficients of $\psi[\widehat{\eta}]$, which in general depends on $\widehat{\eta}$ through all orders because of the nonlinearity.

Equation (100) can be used to succinctly describe the difference between self-consistent and passive problems. Let $\mathcal{N}[\psi] \sim \mathcal{V}\psi$. (Physically, $\mathcal{V}$ represents an advecting velocity; I ignore details such as gradient operators for this general discussion.) In a self-consistent problem $\mathcal{V} = \mathcal{V}[\psi]$ depends

[87] The hat notation maintains consistency with the general MSR formalism of Sec. 6 (p. 146).

[88] For an introduction to functional derivatives, see Beran (1968) or any textbook on variational calculus. $\widetilde{R}$ is nothing but the linearization of Eq. (99) with respect to an infinitesimal source, subsequently rescaled to unit amplitude at $t = t' + \epsilon$.

functionally on ψ; in a passive problem it does not. Thus

$$\frac{\delta \mathcal{N}}{\delta \widehat{\eta}} = \begin{cases} \mathcal{V}\widetilde{R} & \text{(passive)} \\ \mathcal{V}\widetilde{R} + \left(\dfrac{\delta \mathcal{V}}{\delta \psi}\widetilde{R}\right)\psi & \text{(self-consistent).} \end{cases} \tag{102}$$

The extra term in the self-consistent response describes the *backreaction* of the advected field on the advecting velocity; it leads, for example, to an energy-conservation law absent from the passive problem.

For two-point statistics one thus has available the covariance $C(1, 1')$ as a measure of fluctuations and the response function $R(1; 1')$ as a measure of dissipation. [89] In turbulence theory the development of coupled and closed nonperturbative equations for $\langle\psi\rangle$, C, and R was pioneered by Kraichnan with his famous DIA, to be described in Sec. 5 (p. 126). An elegant generalization of the cumulant formalism was found by Martin et al. (1973); their work will be described in Sec. 6 (p. 146).

3.6 Alternate representations and properties of second-order spectra

The correlation function $C(1, 1')$ can be discussed in either $\boldsymbol{x}$ space, $\boldsymbol{k}$ space, or other bases such as wavelets (Farge, 1992). For homogeneous turbulence $\boldsymbol{k}$ space is particularly suitable. Thus one may consider $C(\boldsymbol{k})$, the Fourier transform with respect to $\boldsymbol{x} - \boldsymbol{x}'$. However, certain superior alternate representations are also in common use.

3.6.1 Energy spectral density

Instead of $C(\boldsymbol{k})$ one frequently considers the *energy spectral function* $E(k)$, from which the total energy $\mathcal{E}$ can be calculated according to $\mathcal{E} = \int_0^\infty \mathrm{d}k\, E(k)$, where $k \doteq |\boldsymbol{k}|$. Thus the volume element in $\boldsymbol{k}$ space is included in $E(k)$, so the physical dimensions of $E(k)$ are independent of spatial dimensionality d. No assumption about isotropy is made at this point; if the turbulence is anisotropic, $E(k)$ includes a nontrivial average over solid angle in $\boldsymbol{k}$ space and does not provide a complete description of the wave-number spectrum. Although use of $E(k)$ is completely standardized in neutral-fluid theory, in plasma physics mean-squared fields [*sans* volume element, i.e., $C(\boldsymbol{k})$] are often plotted instead.

The energy spectrum is the derivative $E(k) = \mathrm{d}\mathcal{E}(k)/\mathrm{d}k$ of the cumulative energy spectrum $\mathcal{E}(k) \doteq C^{<}(k)$, where $C^{<}(k)$ is a low-pass-filtered version of $C(\boldsymbol{x})$ that contains wave-number components less than k in magnitude. A detailed discussion of the filtering procedure was given by Frisch (1995).

3.6.2 Structure functions

Instead of the two-point correlation function $C(1, 1')$, it is often useful to consider the *structure function*

$$S(1, 1') \doteq \langle[\delta\psi(1) - \delta\psi(1')]^2\rangle = 2[C(1, 1) - C(1, 1')]. \tag{103a,b}$$

[89] In thermal equilibrium C and R are related *via* the fluctuation–dissipation theorem; see Sec. 3.7.1 (p. 67).

A lucid discussion was given by Frisch (1995), who emphasized the importance of such functions for describing random processes with stationary increments.[90] Spatial structure functions are often used in analyses of the statistics of the small scales of turbulence.[91] They have been seldom employed in plasma physics to date, but see the application discussed by Krommes (1997a) and in Sec. 4.4 (p. 119).

3.6.3 The Taylor microscale

The *Taylor microscale* λ_T (Taylor, 1935) is a measure of the second derivative of the correlation function at the origin. Specifically, for a homogeneous, isotropic function $C(\boldsymbol{x}, \boldsymbol{x}') = C(\rho)$, where $\rho \doteq |\boldsymbol{\rho}|$ and $\boldsymbol{\rho} \doteq \boldsymbol{x} - \boldsymbol{x}'$, λ_T is defined by

$$C(\rho)/C(0) = 1 - \rho^2/\lambda_T^2 + \cdots \quad (\rho \to 0). \tag{104}$$

Alternatively,

$$\lambda_T \doteq \left(\frac{-C''(0)}{2C(0)}\right)^{-1/2} = \left(\frac{\int_0^\infty \mathrm{d}k\, k^2 E(k)}{2\int_0^\infty \mathrm{d}k\, E(k)}\right)^{-1/2}. \tag{105a,b}$$

Note that the numerator of Eq. (105b) is the enstrophy or mean-squared velocity shear:

$$\int_0^\infty \mathrm{d}k\, k^2 E(k) = \langle|\boldsymbol{\omega}|^2\rangle, \tag{106}$$

where $\boldsymbol{\omega} \doteq \boldsymbol{\nabla} \times \boldsymbol{u}$ is the vorticity. Thus $k_T \doteq \lambda_T^{-1}$ is a normalized measure of the rms velocity shear (Krommes, 1997a, 2000b). For further discussion of the role of turbulent velocity shear, see Sec. 12.7 (p. 248).

Another interpretation of λ_T arises by evaluating the Navier–Stokes energy dissipation ε, Eq. (13c), for homogeneous, isotropic turbulence. One finds (Taylor, 1935)

$$\varepsilon = 15\mu_{\mathrm{cl}}\,\overline{u}^2/\lambda_T^2 = \frac{15}{2}\mu_{\mathrm{cl}}\int_0^\infty \mathrm{d}k\, k^2 E(k), \tag{107a,b}$$

where $\overline{u}$ is the rms level of any Cartesian velocity component. The result (107a) might lead one by dimensional reasoning to associate λ_T with the characteristic scale at which viscous dissipation occurs. However, *that conclusion is false for high Reynolds-number turbulence.* It would be correct if $E(k)$ decayed exponentially rapidly [consider inserting a Gaussian shape into Eq. (107b)]. However, as will be discussed in Sec. 3.8.2 (p. 73), the actual (Kolmogorov inertial-range) spectrum is algebraic, $E(k) \propto \varepsilon^{2/3}k^{-5/3}$, out to a Kolmogorov dissipation wave number $k_d \gg k_T$ (after which it falls off rapidly). Indeed, if that spectrum is inserted into Eq. (107b) and the definition (9) of the Reynolds number $\mathcal{R}$ is used, one finds

$$\varepsilon^{1/3} \propto \mathcal{R}^{-1}\int_0^{k_d} \mathrm{d}k\, k^{1/3} \propto \mathcal{R}^{-1}k_d^{4/3}, \tag{108}$$

[90] An example of a nonstationary random process with stationary increments is the Brownian path $\widetilde{x}(t)$, the solution of $\dot{\widetilde{x}} = \widetilde{v}(t)$ for delta-correlated $\widetilde{v}$.

[91] Note that any $\boldsymbol{k} = \boldsymbol{0}$ component of the fluctuations cancels out in the definition of the structure function. That is, Eq. (103) is invariant under the addition of a constant ψ_0 to $\delta\psi$.

so the dominant contribution to dissipation comes from the vicinity of the upper limit. If ε is to be $O(1)$ (production at the large scales), it then follows that $1 \sim \mathcal{R}^{-1} k_T^2 \sim \mathcal{R}^{-1} k_d^{4/3}$, or

$$k_T = O(\mathcal{R}^{1/2}), \quad k_d = O(\mathcal{R}^{3/4}). \tag{109a,b}$$

Thus $k_d \gg \kappa_T$ as claimed and merits its interpretation as the dissipation wave number. k_d^{-1} is called the *Kolmogorov microscale*.

The Taylor microscale figures crucially in the theory of the so-called *clump algorithm*, one crude procedure for estimating saturation levels in plasma turbulence that is critiqued in Sec. 4.4 (p. 119) and shown to suffer from misapprehensions about the interpretation of λ_T. Those difficulties were explained by Krommes (1997a, 2000b).

3.7 Statistical dynamics of thermal equilibrium

Turbulence, being intrinsically forced and dissipative, represents a state that is far from thermal equilibrium. Nevertheless, particular properties of thermal-equilibrium solutions obtained in the absence of forcing and dissipation importantly figure in the qualitative description of nonequilibrium steady states.

3.7.1 Fluctuation–dissipation theorems

One of the most profound results of equilibrium statistical mechanics is the *fluctuation–dissipation theorem* (FDT; Martin, 1968; Toda et al., 1995), which states that the equilibrium fluctuation spectrum C and a particular linear response function K (different from R) are proportional. This result is by no means intuitively obvious; indeed, it is remarkable since C describes *finite*-amplitude fluctuations whereas K describes the response to an *infinitesimal* perturbation of the Hamiltonian.

A well-known consequence of the classical FDT for discrete many-particle systems is that for weakly coupled plasmas the thermal-equilibrium fluctuation spectrum is $\langle \delta E^2 \rangle(\boldsymbol{k})/8\pi = \frac{1}{2}T/(1+k^2\lambda_D^2)$ [see, for example, Ecker (1972)]; this result is frequently used to test particle simulation codes (Lee, 2000). The generalization of this result to gyrokinetic plasmas is both important and subtle (Krommes, 1993c). For further discussion, see Appendix C.1.7 (p. 276).

Kraichnan (1959a) proved an FDT that directly links C and R and is of more direct relevance to the theory of turbulence. By considering a hypothetical weak coupling of two initially isolated systems in thermal equilibrium, he showed [see also Orszag (1977)] that in thermal equilibrium

$$C_+(\boldsymbol{k}, \omega) = R(\boldsymbol{k}, \omega) C(\boldsymbol{k}), \tag{110}$$

where $C_+(\omega)$ is the temporal Fourier transform of the one-sided function $H(\tau)C(\tau)$ and $C(\boldsymbol{k})$ is the equal-time wave-number spectrum (independent of time in the steady state). The physical distinction between R and the K of the original FDT is that R describes the response to infinitesimal perturbations *additive* to the equations of motion whereas K describes the response to perturbations additive to the Hamiltonian, which become *multiplicative* perturbations to the equations of motion. The specific mathematical relations between R and K were discussed by Krommes (1993b), who illustrated some of the formulas with the guiding-center model (31) of cross-field transport. See also the discussion of dielectric response in Sec. 6.5 (p. 170).

The importance of an FDT like Eq. (110) for a theory of turbulence is that it provides a powerful constraint that any statistical theory relating R and C should satisfy in the limit of thermal equilibrium. Because that limit is achieved by removing forcing and dissipation from the equations of motion, the constraint is on the *nonlinear* structure of the theory and is entirely nontrivial. As we will see in Sec. 8.2 (p. 201), the fluctuation–dissipation relation (110) also serves as a plausible *Ansatz* that can be employed (with varying degrees of fidelity) even in highly nonequilibrium situations. The practical advantage of such a relation is that the two-time dependence (or the frequency spectrum) can be described by one independent function rather than two.

3.7.2 Gibbs ensembles for turbulence

It is well known that the existence of Liouville's theorem $\sum_i \partial(\dot{z}_i)/\partial z_i = 0$, where the z_i's are the phase-space coordinates, permits an equilibrium statistical mechanics (Tolman, 1938). Let ϕ_n be one of a complete set of real orthonormal eigenfunctions such that a general field ψ can be expanded as $\psi(\boldsymbol{x},t) = \sum_n \psi_n(t)\phi_n(\boldsymbol{x})$. For homogeneous turbulence and periodic boundary conditions, plane waves are appropriate eigenfunctions. Assume that when the equation of motion is truncated to a finite number of those eigenfunctions, an inviscid constant of motion exists of the form $\widetilde{\mathcal{I}} = \sum_n \psi_n^2(t)$. (The tilde denotes a quantity that is in principle a random variable although here it is actually constant in each realization; the corresponding mean quantity is denoted $\mathcal{I} \doteq \langle\widetilde{\mathcal{I}}\rangle$.) It can then readily be shown (Kraichnan, 1965a; Kraichnan and Montgomery, 1980) that the ψ_n obey Liouville's theorem with $\psi_n = z_n$. Accordingly, a *Gibbs distribution* $P[\psi] \propto \exp(-\alpha\widetilde{\mathcal{I}}[\psi])$ is an equilibrium (and stable) solution of Liouville's equation. This predicts an equipartition spectrum for the ψ_n, as was first shown for some important special cases in the pioneering paper by Lee (1952). If there are several constants of motion $\widetilde{\mathcal{I}}_i$, one is led to multiparameter Gibbs distributions, $P[\psi] \propto \exp(-\sum_i \alpha_i\widetilde{\mathcal{I}}_i[\psi])$, and nontrivial generalizations of the equipartition spectrum.

The absolute equilibrium distributions do not describe turbulence, which is a forced, dissipative state with nonvanishing net energy flow from mode to mode.[92] Nevertheless, they are important in several ways: they can be used to partially test numerical simulations and statistical closures; and they also suggest that the nonlinear terms will transfer energy (or other invariants in some cases) in the direction that would tend to bring the wave numbers to their thermal equilibrium level. Of course, for actual forced, dissipative turbulence that attempt is defeated for wave numbers in the dissipation range, where the steady-state spectrum must lie far below the equilibrium prediction. See further discussion in Sec. 3.8 (p. 71).

A difficulty with Gibbs's procedure as applied to the equations of turbulence is that the full set of constants of motion may not be known or may be infinite. Fortunately, usually the conservation of an infinite set of quantities does not survive truncation to a finite $\boldsymbol{k}$ space or other bases (Kraichnan and Montgomery, 1980); frequently only the quadratic quantities survive. If one were to fail to recognize one or more of the quadratic invariants, qualitatively incorrect results could arise[93]; see the following discussion of the two-parameter Gibbs distribution.

Another issue is that Gibbs's form is not the unique stable solution of the Liouville equation.

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[92] Some general aspects of the distinctions between equilibrium and turbulence were discussed by Kraichnan (1958b).

[93] Higher-order invariants may be important as well; see the discussion in Secs. 5.10.3 (p. 144) and 7.2.1 (p. 183) of the problem of three interacting modes, in which in addition to energy and enstrophy a cubic Hamiltonian invariant is also nonlinearly conserved.

Strictly speaking, if the $\{\widetilde{\mathcal{I}}_i\}$ have the same values in all realizations, then the distribution should be *microcanonical*, $P[\psi] \propto \prod_i \delta(\mathcal{I}_i - \widetilde{\mathcal{I}}_i[\psi])$. The usual justification for Gibbs's form is to evaluate averages by a saddle-point integration that exploits the number of modes as a large parameter. The differences between the microcanonical and canonical distributions were discussed by Kells and Orszag (1978), who performed numerical simulations of systems with small number of degrees of freedom and compared the results with the theoretical predictions of the various equilibrium ensembles.

The (inviscid) HM equation (48) provides an important and nontrivial example of these ideas. As discussed in Sec. 2.4.3 (p. 37), the HME conserves two quadratic invariants, the energy $\widetilde{\mathcal{E}}$ and the enstrophy $\widetilde{\mathcal{W}}$ [defined by Eqs. (50)]. Those quantities remain invariant if one removes from the spectrum all triad interactions with the magnitude of any leg larger than some $k_{\max}$, which I shall subsequently assume has been done. (In the following discussion, all $\boldsymbol{k}$'s are really $\boldsymbol{k}_\perp$'s.) One is then led to the two-parameter Gibbs distribution

$$P[\varphi] \propto \exp[-\textstyle\sum_{\boldsymbol{k}}(\alpha + \beta k^2)\widetilde{E}_{\boldsymbol{k}}] \tag{111}$$

(the sum is over the *independent* Fourier components; $\varphi_{\boldsymbol{k}}$ and $\varphi_{-\boldsymbol{k}}$ are not independent because of the reality condition $\varphi_{-\boldsymbol{k}} = \varphi^*_{\boldsymbol{k}}$), which in turn leads to mean values $\mathcal{E} \doteq \langle\widetilde{\mathcal{E}}\rangle$ and $\mathcal{W} \doteq \langle\widetilde{\mathcal{W}}\rangle$ with the equilibrium wave-number spectrum $\mathcal{E}_{\boldsymbol{k}} \doteq \langle\widetilde{\mathcal{E}}_{\boldsymbol{k}}\rangle = (\alpha + \beta k^2)^{-1}$.

As was pointed out in Sec. 2.4.3 (p. 37), if the 1 is neglected in the factor $(1 + k^2)$, the HM description reduces to that of the 2D Euler equation (Onsager, 1949; Joyce and Montgomery, 1973). Kraichnan (1975b) has given an extensive discussion of the two-parameter Gibbs distribution for that case; his results can be taken over directly. Let M be the number of modes remaining in the truncated spectrum and introduce the mean energy and enstrophy per mode, respectively, $\overline{\mathcal{E}} \doteq \mathcal{E}/M$ and $\overline{\mathcal{W}} \doteq \mathcal{W}/M$. Also define the dimensionless parameters

$$\overline{\alpha} \doteq \overline{\mathcal{E}}\,\alpha, \quad \overline{\beta} \doteq \overline{\mathcal{E}}\,\beta, \quad \widehat{\alpha} \doteq \alpha/\beta = \overline{\alpha}/\overline{\beta}. \tag{112a,b,c}$$

Specifying $(\overline{\alpha}, \overline{\beta})$ then determines $(\overline{\mathcal{E}}, \overline{\mathcal{W}})$ and *vice versa*:

$$\begin{pmatrix} \overline{\mathcal{E}} \\ \overline{\mathcal{W}} \end{pmatrix} = \frac{1}{2\beta}\left\langle \begin{pmatrix} 1 \\ k^2 \end{pmatrix} \left(\frac{1}{\widehat{\alpha} + k^2}\right)\right\rangle_{\boldsymbol{k}}, \tag{113}$$

where $\langle\ldots\rangle_{\boldsymbol{k}}$ denotes the wave-number average over the discrete spectrum: $\langle A\rangle_{\boldsymbol{k}} \doteq M^{-1}\sum_{\boldsymbol{k}} A_{\boldsymbol{k}}$. Now introduce the ratio of enstrophy to energy, which is the square of a dimensionless wave number κ: $\kappa^2 \doteq \mathcal{W}/\mathcal{E} = \overline{\mathcal{W}}/\overline{\mathcal{E}}$. Then

$$\kappa^2 = (2\overline{\beta})^{-1} - \widehat{\alpha}, \quad \overline{\beta} \doteq \tfrac{1}{2}\langle(\widehat{\alpha} + k^2)^{-1}\rangle_{\boldsymbol{k}}. \tag{114a,b}$$

The usual situation is that one is given $\{\overline{\mathcal{E}}, \overline{\mathcal{W}}\}$ (prescribed, say, as initial conditions). The associated α and β can then be found as follows. Replace the set $\{\overline{\mathcal{E}}, \overline{\mathcal{W}}\}$ by $\{\overline{\mathcal{E}}, \kappa^2\}$. Invert Eq. (114a) with the aid of Eq. (114b) to give $\widehat{\alpha}(\kappa^2)$. Then $\overline{\beta}$ is known from Eq. (114b) and $\overline{\alpha}$ follows from Eq. (112c). The actual α and β can then be obtained from Eqs. (112).

Not all combinations of $\{\overline{\mathcal{E}}, \overline{\mathcal{W}}\}$ or $\{\overline{\alpha}, \overline{\beta}\}$ are accessible. A striking result is that in a discrete spectrum ($k_{\min} > 0$) one or the other of α or β may be *negative*, corresponding to *negative-temperature equilibrium states*. The parameter space can be analyzed by demanding that $\overline{\mathcal{E}}$, $\overline{\mathcal{W}}$, and $E_{\boldsymbol{k}}$ be non-negative,[94] and one can identify three regimes:

I: $\alpha < 0$	**II:** $\alpha, \beta > 0$	**III:** $\beta < 0$
$k^2_{\min} \le \kappa^2 \le k^2_a,$	$k^2_a \le \kappa^2 \le k^2_b,$	$k^2_b \le \kappa^2 \le k^2_{\max},$
$-k^2_{\min} \le \widehat{\alpha} \le 0,$	$0 \le \widehat{\alpha} < \infty,$	$-\infty < \widehat{\alpha} \le -k^2_{\max},$
$-\infty \le \overline{\alpha} \le 0,$	$0 \le \overline{\alpha} \le 1,$	$1 \le \overline{\alpha} \le \infty,$
$\infty \ge \overline{\beta} \ge k_a^{-1};$	$k_a^{-1} \ge \overline{\beta} \ge 0;$	$0 \ge \overline{\beta} \ge -\infty.$

The qualitative features of this behavior are summarized in Fig. 4 (p. 71), which plots the approximation obtained by assuming that the spectrum is dense and spherically truncated (Kraichnan, 1975b): $M \approx \pi(k^2_{\max} - k^2_{\min})$,

$$\overline{\beta} \approx \frac{1}{2} \ln\left(\frac{\widehat{\alpha} + k^2_{\max}}{\overline{\alpha} + k^2_{\min}}\right) / (k^2_{\max} - k^2_{\min}), \quad k^2_a \approx \frac{k^2_{\max} - k^2_{\min}}{\ln(k^2_{\max}/k^2_{\min})}, \quad k^2_b \approx \frac{1}{2}(k^2_{\min} + k^2_{\max}). \quad \text{(115a,b,c)}$$

Regime I corresponds to negative-α states; symmetrically, regime III corresponds to negative-β states. States with highly negative α have the longest-wavelength modes excited to very high levels and have small ratios of enstrophy to energy. For states with highly negative β, the excitation is concentrated at the shortest wavelengths and the ratio of enstrophy to energy is large.

A frequent argument is that the equilibrium wave-number distribution of the invariants provides a clue about the behavior of nonequilibrium dynamics; the nonlinear terms strive to relax the system to equilibrium, but are thwarted by forcing and dissipation. For example, the existence of the negative α (large-energy, long-wavelength) and negative β (large-enstrophy, short-wavelength) equilibrium regimes suggests that a nonequilibrium HM system forced at intermediate wavelengths may exhibit a *dual cascade* in which energy is transferred to long wavelengths while simultaneously enstrophy is transferred to short wavelengths. This point is pursued in Sec. 3.8.3 (p. 74). Dual cascades exist in nonhelical MHD [see Eqs. (64)] as well; a readable introduction was given by Montgomery (1989).

Although two-parameter Gibbs distributions are frequently discussed because of the practical importance of 2D turbulence, they are not the only possibility. For example, it was noted in Sec. 2.4.5 (p. 39) that the nonlinear terms of the HW equations conserve four quadratic invariants; the

[94] Considered as a function of $\widehat{\alpha}$, $E_{\boldsymbol{k}}$ is singular at $\widehat{\alpha} = -k^2_{\min}$ and $\widehat{\alpha} = -k^2_{\max}$, and one can verify that the region $-k^2_{\max} < \widehat{\alpha} < -k^2_{\min}$ is forbidden since one or more of the $E_{\boldsymbol{k}}$ would be negative. To analyze the behavior in the vicinity of $\widehat{\alpha} = -k^2_{\min}$, write $\widehat{\alpha} = -k^2_{\min} + \epsilon/M$. Then one can see that $\overline{\beta} \sim \epsilon^{-1} \to +\infty$ as $\epsilon \to 0_+$. For fixed $\overline{\mathcal{E}}$, which will always be assumed in considering the various limiting cases, one can see that also $\beta \to +\infty$. The behavior of $\overline{\alpha} \doteq \alpha\overline{\mathcal{E}}$ follows from $\overline{\alpha} = \widehat{\alpha}\overline{\beta} = (-k^2_{\min} + \epsilon/M)\overline{\beta} \approx -k^2_{\min}\overline{\beta} \to -\infty$. One also has $\kappa^2 \to -\widehat{\alpha} \to k^2_{\min}$. Symmetrical behavior ensues in the vicinity of $\widehat{\alpha} = -k^2_{\max} - \epsilon/M$, where the roles of α and β as well as $k_{\min}$ and $k_{\max}$ are reversed. The other interesting points are $\widehat{\alpha} = 0$ and $\widehat{\alpha} = \pm\infty$. Define the special wave numbers k_a and k_b according to $k^2_a \doteq \langle k^{-2} \rangle^{-1}_{\boldsymbol{k}}$, $k^2_b = \langle k^2 \rangle_{\boldsymbol{k}}$. [That $k_b \ge k_a$ is a consequence of the Schwartz inequality applied to the identity $\langle (k^2)(k^{-2}) \rangle = 1$.] Then at $\widehat{\alpha} = 0$ one finds $\overline{\mathcal{E}} = (2\beta k^2_a)^{-1}$, $\overline{\mathcal{W}} = (2\beta)^{-1}$, and $\kappa^2 = k^2_a$. As $\widehat{\alpha} \to +\infty$ one has $\overline{\mathcal{E}} = (2\alpha)^{-1}$, $\overline{\mathcal{W}} = (2\alpha)^{-1}k^2_b$, and $\kappa^2 = k^2_b$. Since for fixed $\overline{\mathcal{E}}$ α remains finite, one can see that $\beta \to 0_+$ as $\widehat{\alpha} \to +\infty$. The point $\widehat{\alpha} = -\infty$ is obtained continuously from $\widehat{\alpha} = +\infty$ as β passes continuously through 0 from above.

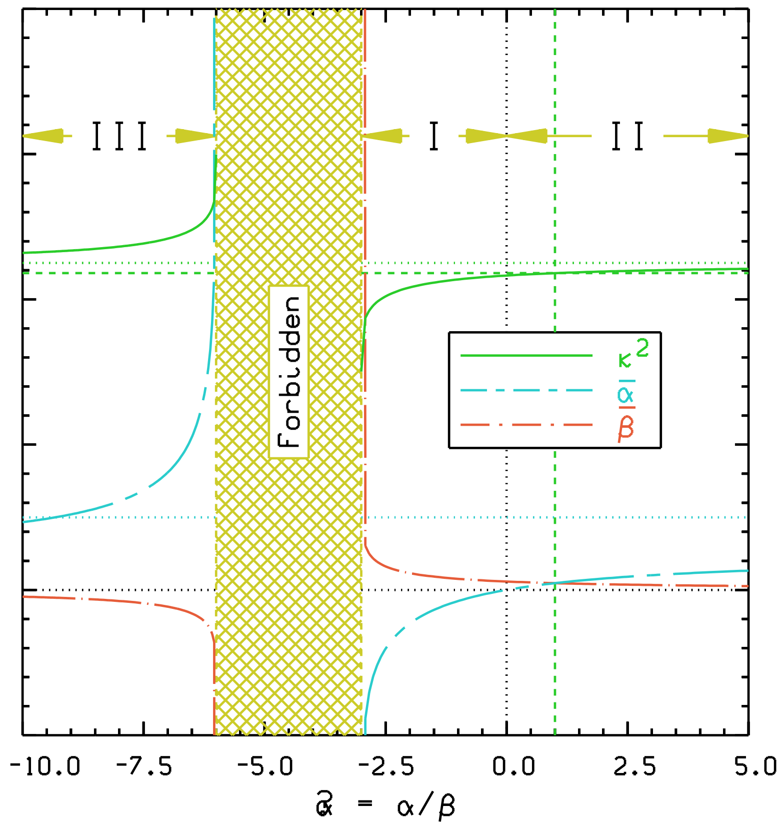

Fig. 4. Parameter space for a two-parameter Gibbs distribution, displayed as a function of $\widehat{\alpha} \doteq \alpha/\beta$. The solid curves plot κ^2, the ratio of enstrophy to energy. The chain-dashed curves plot $\overline{\alpha}$; the chain-dotted curves plot $\overline{\beta}$. The inverse temperatures $\overline{\alpha}$ and $\overline{\beta}$ are seen to pass continuously through 0 as functions of κ^2.

corresponding four-parameter Gibbs distribution was discussed by Koniges et al. (1991) and used by Hu et al. (1995) to partially verify numerical solutions of statistical closures [see Sec. 8.5 (p. 208)]. A three-parameter Gibbs equilibrium for a model of electromagnetic turbulence was discussed by Craddock (1990).

3.8 Spectral paradigms

Since transport is fully determined by two-point spectral functions [see Eq. (3b)], it is important to have an intuitive understanding of the characteristic shapes of wave-number spectra and of the directions of flow of energy and other nonlinear invariants in $\boldsymbol{k}$ space. Various scenarios can arise; they are called *spectral paradigms*. The characteristic spectral paradigm for the quasi-2D turbulent fluctuations of strongly magnetized plasma is quite different from the standard one for the 3D NSE.

3.8.1 Definition of transfer

Transfer [95] is defined as the net amount of a nonlinear invariant leaving a particular region of $\boldsymbol{k}$ space; a seminal reference is by Kraichnan (1959b). Consider a primitive amplitude equation of the

[95] The following discussion of invariant transfer is a slight paraphrasing of Sec. II A of Krommes (1997c).

form $\partial_t\psi_{\boldsymbol{k}} + \mathrm{i}\mathcal{L}_{\boldsymbol{k}}\psi_{\boldsymbol{k}} = \frac{1}{2}\mathcal{N}_{\boldsymbol{k}}[\psi]$. For $\langle\psi\rangle = 0$ this leads to the spectral evolution equation

$$\partial_t\widetilde{C}_{\boldsymbol{k}} = 2\gamma_{\boldsymbol{k}}\widetilde{C}_{\boldsymbol{k}} + \mathrm{Re}(\mathcal{N}_{\boldsymbol{k}}[\delta\psi]\delta\psi^*_{\boldsymbol{k}}), \tag{116}$$

where $\widetilde{C}_{\boldsymbol{k}} \doteq |\delta\psi_{\boldsymbol{k}}|^2$ and $\gamma_{\boldsymbol{k}} \doteq \mathrm{Im}\,\mathcal{L}_{\boldsymbol{k}}$. Let a (nonlinearly conserved) invariant $\widetilde{\mathcal{I}}$ be defined by the appropriately weighted sum over all $\boldsymbol{k}$'s: $\widetilde{\mathcal{I}} \doteq \sum_{\boldsymbol{k}}\sigma_{\boldsymbol{k}}\widetilde{C}_{\boldsymbol{k}}$. A generalization is to define a partial sum over only a particular $\boldsymbol{k}$-space region $\Delta_{\boldsymbol{k}}$: $\widetilde{\mathcal{I}}(\Delta_{\boldsymbol{k}}) \doteq \sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}}\sigma_{\boldsymbol{k}}\widetilde{C}_{\boldsymbol{k}}$, with $\widetilde{\mathcal{I}}(\Delta_\infty) = \widetilde{\mathcal{I}}$ (Δ_∞ denotes the whole of the $\boldsymbol{k}$ space). Upon summing Eq. (116) over $\Delta_{\boldsymbol{k}}$, one finds the fundamental transfer equation

$$\tfrac{1}{2}\partial_t\widetilde{\mathcal{I}}(\Delta_{\boldsymbol{k}}) = \widetilde{\mathcal{P}}(\Delta_{\boldsymbol{k}}) - \widetilde{\mathcal{D}}(\Delta_{\boldsymbol{k}}) - \widetilde{\mathcal{T}}(\Delta_{\boldsymbol{k}}), \tag{117}$$

where the forcing or *production* is $\widetilde{\mathcal{P}}(\Delta_{\boldsymbol{k}}) \doteq \sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}} H(\gamma_{\boldsymbol{k}})\sigma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}\widetilde{C}_{\boldsymbol{k}}$, the *dissipation* is $\widetilde{\mathcal{D}}(\Delta_{\boldsymbol{k}}) \doteq -\sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}} H(-\gamma_{\boldsymbol{k}})\sigma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}\widetilde{C}_{\boldsymbol{k}}$, and the nonlinear *transfer* is $\widetilde{\mathcal{T}}(\Delta_{\boldsymbol{k}}) \doteq -\frac{1}{2}\,\mathrm{Re}\sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}}\sigma_{\boldsymbol{k}}\mathcal{N}_{\boldsymbol{k}}\delta\psi^*_{\boldsymbol{k}}$. On the average, Eq. (117) holds without tildes; in steady state the averaged transfer $\mathcal{T}(\Delta_{\boldsymbol{k}})$ equals the net forcing $\mathcal{P}(\Delta_{\boldsymbol{k}}) - \mathcal{D}(\Delta_{\boldsymbol{k}})$. For quadratic nonlinearity $\mathcal{T}$ is a weighted sum of the triplet correlation function $T_{\boldsymbol{kpq}} \doteq \langle\delta\psi_{\boldsymbol{k}}\delta\psi_{\boldsymbol{p}}\delta\psi_{\boldsymbol{q}}\rangle$:

$$\mathcal{T}(\Delta_{\boldsymbol{k}}) = -\frac{1}{2}\,\mathrm{Re}\sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}}\sum_{\Delta}\sigma_{\boldsymbol{k}}M_{\boldsymbol{kpq}}T^*_{\boldsymbol{pqk}}, \tag{118}$$

where $\sum_\Delta$ is defined in Appendix A (p. 262). $\mathcal{T}$ is thus proportional to the skewness of the fluctuations [Eq. (95)], a non-Gaussian effect. For forced, dissipative steady-state turbulence, the transfer cannot vanish identically because the forcing and dissipation occur in different regions of $\boldsymbol{k}$ space, so *the fluctuations are necessarily skewed and cannot be Gaussian.* The sign and magnitude of the transfer from various finite regions of $\boldsymbol{k}$ space provide important insights into the nonlinear dynamics, and verifying that $\widetilde{\mathcal{T}}(\Delta_\infty) = 0$ for each quadratic nonlinear invariant is a powerful test of a simulation code.

For homogeneous, isotropic turbulence all statistical quantities depend only on $k \doteq |\boldsymbol{k}|$, so it makes sense to define $\Delta_{\boldsymbol{k}}$ as a spherical (3D) or circular (2D) region centered on the origin; this defines the conventional isotropic function $\mathcal{T}(k)$. Let us pass to a continuum of wave numbers. As a consequence of isotropy, both $M_{\boldsymbol{kpq}}$ and $T_{\boldsymbol{kpq}}$ depend on just wave-number magnitudes. The angular integrations in $\sum_\Delta$ can then be performed (Appendix A, p. 263), and Eq. (118) becomes $\mathcal{T}(k) = -\frac{1}{2}\int_0^k \mathrm{d}\overline{k}\int_\Delta \mathrm{d}\overline{p}\,\mathrm{d}\overline{q}\,\mathcal{T}(\overline{k},\overline{p},\overline{q})$, where for the important 2D case

$$\mathcal{T}(k,p,q) \doteq 2\pi k[2/|\sin(\boldsymbol{p},\boldsymbol{q})|]\sigma_{\boldsymbol{k}}M_{\boldsymbol{kpq}}(\delta k^{-2d}T^*_{\boldsymbol{kpq}}) \tag{119}$$

(δk being the mode spacing). Because $\mathcal{T}(\infty) = 0$, one can alternatively write $\mathcal{T}(k) = \frac{1}{2}\int_k^\infty \mathrm{d}\overline{k}\,\times \int_0^\infty \mathrm{d}\overline{p}\int_0^\infty \mathrm{d}\overline{q}\,\mathcal{T}(\overline{k},\overline{p},\overline{q})$, where $\mathcal{T}(k,p,q)$ is assumed to vanish outside of the domain $\Delta(k,p,q)$. Simple manipulations using the detailed conservation property $\sigma_{\boldsymbol{k}}\mathcal{T}(k,p,q) + \mathrm{c.p.} = 0$ (c.p. means the cyclic permutation $\boldsymbol{k} \to \boldsymbol{p} \to \boldsymbol{q}$) then lead to the form given by Kraichnan (1959b),

$$\mathcal{T}(k) = \frac{1}{2}\left(\int_k^\infty \mathrm{d}\overline{k}\int_0^k \mathrm{d}\overline{p}\int_0^k \mathrm{d}\overline{q} - \int_0^k \mathrm{d}\overline{k}\int_k^\infty \mathrm{d}\overline{p}\int_k^\infty \mathrm{d}\overline{q}\right)\mathcal{T}(\overline{k},\overline{p},\overline{q}). \tag{120}$$

For a graphical illustration, see Kraichnan's Fig. 1.

Studying merely the angle-averaged $\mathcal{T}(k)$ can be misleading for anisotropic situations such as those characteristic of drift-wave problems in fusion plasmas, in which various important frequencies are proportional to k_y. So that one does not overlook unexpected physics, one should also study appropriately defined functions $\mathcal{T}(k_x)$ and $\mathcal{T}(k_y)$ [and $\mathcal{T}(k_\parallel)$ in 3D]; unfortunately, this is virtually never done. Nevertheless, the tendency of $\boldsymbol{E} \times \boldsymbol{B}$ advection to isotropize fluctuations means that $\mathcal{T}(k)$ is a useful and simple diagnostic for many situations of 2D plasma turbulence.

It should be pointed out that transfer is a necessary but not sufficient diagnostic. By itself transfer does not provide information about locality of the interactions (Waleffe, 1992). Furthermore, steady-state transfer measurements provide no direct information about timescales.

In the next several sections I shall discuss various distinctive qualitative scenarios of k-space transfer that have been identified. To describe all of the various cases within a common framework, I will employ the energy spectrum $E(k)$ defined and discussed in Sec. 3.6.1 (p. 65).

3.8.2 Direct cascade

For the 3D NSE [Eq. (7a)] the conventional picture [see, for example, Landau and Lifshitz (1987) and Frisch (1995)] is as follows. For homogeneous, isotropic, mirror-symmetric [96] turbulence the nonlinear terms conserve the single quadratic invariant [97] $\widetilde{\mathcal{E}} \doteq \frac{1}{2}\sum_{\boldsymbol{k}} |\delta \boldsymbol{u}_{\boldsymbol{k}}|^2$. A Gibbsian thermal equilibrium is thus $\langle |\delta \boldsymbol{u}_{\boldsymbol{k}}|^2 \rangle \propto \alpha^{-1}$ or $E(k) \propto k^2 \alpha^{-1}$, where α is the inverse temperature and is determined by the initial value of the mean energy $\mathcal{E}$. Because this equipartition solution weights the large k's most heavily, it is expected that the tendency of the nonlinear terms is to transfer energy to the large k's (on the average). In the presence of viscous dissipation, energy will therefore be absorbed at the large k's; thermal equilibrium will not be achieved. Instead, energy will enter the spectrum at the forcing wave numbers (assumed to be at large scales, or small k's), flow in a *direct cascade* through the spectrum, and be dissipated at large k's. This strong statistical disequilibrium is the crux of the difficulty of developing a satisfactory theory of forced, dissipative turbulence.

The famous theory of Kolmogorov (1941), generally referred to as *K41*, makes a definite prediction for the shape of the spectrum in the *inertial range*—the range of wave numbers intermediate between the *energy-containing range* of small, directly forced k's and the *dissipation range* where viscous dissipation is important and the energy flow is absorbed. The argument, dimensional in nature, was presented elegantly by Frisch (1995); see also Landau and Lifshitz (1987). In its simplest heuristic form it states that in the inertial range $E(k)$ should depend on just k and the rate of energy transfer [98] ε. Dimensional or similarity analysis [see, for example, Tennekes and Lumley (1972)] then leads uniquely to the *Kolmogorov spectrum*

$$E(k) = C_E \varepsilon^{2/3} k^{-5/3} \quad \text{(energy cascade)}, \tag{121}$$

where the *Kolmogorov constant* C_E is undetermined. The $k^{-5/3}$ scaling was first verified experimentally in the tidal-channel experiments of Grant et al. (1962); one finds $C_E \approx 1.5$.

A physical interpretation of Kolmogorov's result (121) is as follows. Divide the k axis into logarithmically spaced bands. [99] The energy content within a band is then $kE(k)$. The square root

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[96] In the absence of mirror symmetry, the fluid helicity $\mathcal{H}$ [Eq. (18)] is a second nontrivial (in 3D) quadratic invariant. Equilibrium ensembles for this case were considered by Kraichnan (1973a).

[97] In this and similar discussions I set the mass density ρ_m to one.

[98] In steady state ε is also the value of the energy production [$\mathcal{P}$ in Eq. (12)] as well as the value of the energy dissipation [$\mathcal{D}$ in Eq. (12)].

[99] The arguments can also be couched in terms of length scales ℓ, as advocated by Frisch (1995). That is

of this expression should be interpreted as a characteristic velocity *difference* Δu across an *eddy*, a fluctuation of characteristic dimension k^{-1}. [A rigorous definition of an eddy can be found in Lumley (1970).] The *eddy turnover time*, the k-dependent time for the eddy to be substantially sheared and thus to change its wave-number content by transferring its energy to neighboring wave-number bands, is then

$$\tau_{\text{eddy}} \sim (k\Delta u)^{-1} = k^{-3/2}E(k)^{-1/2}. \tag{122}$$

The associated rate of energy transfer, assumed to be constant for each band in the inertial range, is $\varepsilon \sim kE(k)/\tau_{\text{eddy}} = k^{5/2}E(k)^{3/2}$. Upon solving this equation for $E(k)$, one recovers Eq. (121). It should be noted that the argument assumes that only eddies, not waves, are present and that the transfer is local in k space. One or both of these assumptions are often violated in plasmas.

If an independent, noneddylike process decorrelates the spectral transfer in a time τ_{ac}, then the previous transfer rate must be reduced by $\tau_{\text{ac}}/\tau_{\text{eddy}}$. If τ_{ac} is independent of $E(k)$, as it would be for a linear mechanism, then from $\varepsilon \sim [kE(k)/\tau_{\text{eddy}}](\tau_{\text{ac}}/\tau_{\text{eddy}})$ and formula (122), one finds $E(k) \sim [\varepsilon/\tau_{\text{ac}}(k)]^{1/2}k^{-2}$. If there exists a $\overline{u}$ for which $\tau_{\text{ac}}^{-1}(k) \sim k\overline{u}$, then

$$E(k) \sim (\varepsilon\overline{u})^{1/2}k^{-3/2}. \tag{123}$$

Kraichnan (1965b) predicted this spectrum for isotropic[100] hydromagnetic turbulence, for which $\overline{u}$ is an Alfvén velocity; see also Orszag and Kruskal (1968).

According to the arguments in Sec. 3.6.3 (p. 66), the inertial range should terminate at a large k that scales with the Kolmogorov dissipation wave number $k_d = \lambda_d^{-1} = O(\mathcal{R}^{3/4})$. λ_d is called the *Kolmogorov microscale*. It can be estimated directly from the qualitative arguments sketched above by equating the eddy turnover frequency $\tau_{\text{eddy}}^{-1}(k_d)$ with the dissipation rate $k_d^2\mu_{\text{cl}}$. If one estimates the energy production rate as $\varepsilon \sim U^3/L$ [see Eq. (13b)], notes the Kolmogorov result $\tau_{\text{eddy}}(k) \sim k^{-2/3}\varepsilon^{-1/3}$ [which follows from Eq. (122)], and recalls Eq. (9), one is led immediately to $k_d L \sim \mathcal{R}^{3/4}$. As discussed in Sec. 3.6.3 (p. 66), one must carefully distinguish the Kolmogorov dissipation microscale $\lambda_d = O(\mathcal{R}^{-3/4})$ from the Taylor microscale $\lambda_T = O(\mathcal{R}^{-1/2})$ [see Eqs. (109)].

3.8.3 Dual cascade

Kraichnan (1967) argued that in 2D turbulence the picture of direct energy cascade must be profoundly modified because, in addition to energy, enstrophy is also conserved[101] by the nonlinear

particularly useful for discussions of intermittency.

[100] In the presence of a mean magnetic field (appropriate for applications to astrophysics), the spectrum is anisotropic and the subject of active research. An incomplete list of recent references is Sridhar and Goldreich (1994), Goldreich and Sridhar (1995), Ng and Bhattacharjee (1996; this includes a critique of the work by Sridhar and Goldreich), and Ng and Bhattacharjee (1997).

[101] Strict enstrophy conservation is not the whole story behind the inverse energy cascade. Fournier and Frisch (1978) have considered the predictions of the EDQNM statistical closure (Sec. 7.2.1, p. 183) analytically continued to *noninteger* spatial dimensionality d. Enstrophy conservation is broken for $d \neq 2$ and apparently does not generalize to another conservation law. A critical dimension of $d_c \approx 2.05$ is obtained such that for $d < d_c$ the energy cascade is inverse whereas for $d > d_c$ the cascade is direct. However, it may be that enstrophy conservation need only be approximately satisfied over some relevant dynamical time in order to significantly constrain the dynamics. I am indebted to H. Rose (2000) for emphasizing these points.

interactions; the same logic applies to the HME (48). Simultaneous direct transfer of those two invariants, which have differing k weightings, would appear to be inconsistent with steady states of both invariants. This implies the possibility of a *dual cascade*, [102] with enstrophy cascading to the right and energy cascading to the left in an *inverse cascade* from an intermediate forcing wave number. The argument is consistent with the existence of the negative-temperature regimes of HM equilibrium statistics; see Fig. 1 of Kraichnan (1967) for a revealing diagram of triadic energy and enstrophy transfer. Because the K41 arguments are insensitive to the sign of the transfer, an inverse energy cascade should still exhibit the $k^{-5/3}$ scaling. For the direct cascade, however, the arguments must be repeated using enstrophy rather than energy as the fundamental quantity that is being transferred. This leads readily to

$$E(k) = C_W \eta^{2/3} k^{-3} \quad \text{(enstrophy cascade)}, \tag{124}$$

where η is the rate of enstrophy injection. [103] However, because Eq. (124) predicts that every octave below a given wave number contributes equally to the mean-square shear [Eq. (106)], which is infrared-divergent in an infinite inertial range, Kraichnan (1967) argued for the likelihood of logarithmic corrections to Eq. (124): $E(k) \to C_W \eta^{2/3} k^{-3} [\ln(k/k_1)]^{-1/3}$, where k_1 is characteristic of the low wave numbers. For detailed discussion and a practical correction to Kraichnan's form, see Bowman (1996a).

Very high computer resolution is needed to verify a dual cascade. Early numerical work was by Lilly (1969). Fyfe et al. (1977) considered 2D MHD; Fyfe and Montgomery (1979) studied the HM equation. For more recent, high-resolution studies, see Brachet et al. (1988).

The predictions of dual cascade strictly hold only when the inertial ranges are taken to be asymptotically infinite in extent. Terry and Newman (1993) discussed modifications for spectra of finite width. The long-time fate of the steady-state inverse cascade depends on the nature of the long-wavelength dissipation. For some discussion, see Hossain et al. (1974). Recent high-quality numerical calculations by Montgomery et al. (1992) show that *decaying* 2D NS turbulence approaches

[102] It is frequently said [see, for example, Diamond and Biglari (1990)] that it is the number of invariants rather than the spatial dimensionality that determines the nature of nonequilibrium cascades. Although there is considerable truth to this, tensorial properties of the invariants also play a role. For example, Kraichnan (1973a) considered 3D helical turbulence, which (Sec. 2.1.1, p. 23) possesses both energy and fluid helicity as inviscid invariants. He showed that there are no analogs to either the negative-temperature states of 2D equilibria or the 2D inverse energy cascade, and that strong helicity should inhibit energy transfer to longer wavelengths. The distinction between helicity and enstrophy is that whereas enstrophy is fully determined by the energy spectrum, the helicity and energy spectra are independent.

[103] Kraichnan (1967) gave an elegant formal similarity analysis of the dual cascade. If inertial-range similarity solutions are sought such that $E(ak)/E(k) = a^{-n}$ and consistently $\mathcal{T}(ak, ap, aq)/\mathcal{T}(k, p, q) = a^{-(1+3n)/2}$, then rescaling manipulations of the form (120) lead to

$$\begin{pmatrix} \mathcal{T}_E(k) \\ \mathcal{T}_W(k) \end{pmatrix} = \begin{pmatrix} k^{(5-3n)/2} \\ k^{(9-3n)/2} \end{pmatrix} \int_0^1 \mathrm{d}v \int_1^\infty \mathrm{d}w \begin{pmatrix} W_E(v, w; n) \\ W_W(v, w; n) \end{pmatrix} \mathcal{T}(1, v, w), \tag{f-4}$$

expressed as a (v, w) integration over triangle shapes. Kraichnan found explicit forms for the W's, which express the roles of different triangle sizes. Note that the values $n_E = \frac{5}{3}$ and $n_W = 3$ produce k-independent spectra. One finds $W_E(v, w; 3) = 0$ (no enstrophy transfer in an energy cascade), $W_W(v, w; \frac{5}{3}) = 0$ (no energy transfer in an enstrophy cascade), $W_E(v, w; \frac{5}{3}) > 0$, and $W_W(v, w; 3) < 0$. Kraichnan argued in detail that $\mathcal{T}(1, v, w)$ should be negative, implying inverse energy cascade and direct enstrophy cascade.

a particular maximum-entropy state; however, a complete theoretical justification is lacking.

3.8.4 Saturated spectra in plasma physics

The previous discussions implicitly assume that (i) forcing is concentrated in a very narrow band of wave numbers, (ii) the Reynolds number is very large (so a well-defined inertial range exists), and (iii) the statistics are isotropic. Unfortunately, none of these is true for a wide class of problems of interest to contemporary plasma applications. Because of the highly dispersive nature of the plasma medium, linear growth rates are typically broadly distributed in wave number. In addition to collisional dissipation, strong kinetic (Landau) damping processes also arise at even moderate scales (in magnetized plasmas, for $k_\perp \rho_\mathrm{s} \gtrsim 1$) and greatly limit the width of the excited spectrum. These observations imply that well-formed inertial ranges will be relatively rare in laboratory plasma problems. Finally, because instabilities are driven typically by profile gradients, which introduce some sort of diamagnetic frequency $\omega_* \propto k_y$, spectra are naturally anisotropic even in the $\boldsymbol{k}_\perp$ plane. Krommes (1997c) discussed a possible generalization of $\mathcal{T}(k)$ that may aid in quantifying the anisotropy. Furthermore, the presence of a background magnetic field introduces a strong anisotropy between the parallel and perpendicular directions. Albert et al. (1990) showed that HM fluctuations excited with a single $k_\parallel$ are unstable to the development of a broad $k_\parallel$ spectrum. If each $k_\parallel$ labels a wave-number plane, one must now consider interplane as well as intraplane transfers. The steady-state dynamics can be complex (Biskamp and Zeiler, 1995).

Although the general plasma problem is complicated, specific circumstances yield to useful modeling. For example, Diamond and Biglari (1990) used Eq. (59) as a description of trapped-ion modes. They noted that because it conserves the single quadratic invariant $\langle \delta n^2 \rangle$, it should display a direct cascade. Thus it is not fundamentally the spatial dimensionality that controls the direction of cascade but rather the number of invariants [but see footnote 102 (p. 75) for a caveat].

This concludes the discussion of key qualitative and exact properties of equilibrium and nonequilibrium turbulent systems. A successful statistical approximation should be compatible with those properties if at all possible. In the next section I begin the discussion of such approximate analytical techniques.

3.9 Introduction to formal closure techniques

A variety of statistical closure techniques have been developed over many years of research. I give an introductory survey of some of them here; much more detailed development of selected ones will be given later. Good introductions to some of these techniques (as applied mostly to passive problems with coefficients depending solely on time) were given by Brissaud and Frisch (1974) and van Kampen (1976); see also van Kampen (1981). An important fluid-oriented review is by Orszag (1977).

3.9.1 Formal integral equation

Consider quadratically nonlinear equations of the form [104]

$$\partial_t\psi + \mathrm{i}\widehat{\mathcal{L}}\psi = \widehat{M}\begin{pmatrix}\chi\\ \frac{1}{2}\psi\end{pmatrix}\psi, \tag{125a}$$

where $\widehat{\mathcal{L}}$ is a linear operator and $\widehat{M}$ is a bilinear operator; both $\widehat{\mathcal{L}}$ and $\widehat{M}$ are assumed to be local in time. The upper and lower choices in the column vector refer, respectively, to a passive problem (χ being a specified random variable) or a self-consistent problem. $\widehat{M}$ may implicitly integrate over other variables such as velocity. It should really carry a passive or self-consistent label as well; only in the latter case may $\widehat{M}$ be taken to be symmetric. [105] After Fourier transformation in $\boldsymbol{x}$, Eq. (125a) can be written

$$\partial_t\psi_{\boldsymbol{k}} + \mathrm{i}\mathcal{L}_{\boldsymbol{k}}\psi_{\boldsymbol{k}} = \sum_{\Delta} M_{\boldsymbol{kpq}}\begin{pmatrix}\chi_{\boldsymbol{p}}\\ \frac{1}{2}\psi_{\boldsymbol{p}}\end{pmatrix}^{*}\psi^{*}_{\boldsymbol{q}}. \tag{125b}$$

Notice that (i) the exact random infinitesimal response function $\widetilde{R}$ obeys

$$(\partial_t + \mathrm{i}\widehat{\mathcal{L}})\widetilde{R}(t;t') - \widehat{M}\begin{pmatrix}\chi\\ \psi\end{pmatrix}\widetilde{R} = \delta(t-t'); \tag{126}$$

and (ii) for passive problems, the formal solution of Eq. (125a) is

$$\psi(t) = \widetilde{R}(t;0)\psi(0). \tag{127}$$

[Recall the discussion of the stochastic oscillator in Sec. 3.3.1 (p. 54).]

Let the goal be to find a closed equation for $\langle\psi\rangle$; this is a restricted case of the general closure problem (of determining all cumulants of ψ) that is particularly appropriate for a pedagogical introduction. The exact equation for $\langle\psi\rangle$ is

$$(\partial_t + \mathrm{i}\widehat{\mathcal{L}})\langle\psi\rangle - \widehat{M}\begin{pmatrix}\langle\chi\rangle\\ \frac{1}{2}\langle\psi\rangle\end{pmatrix}\langle\psi\rangle = \widehat{M}\begin{pmatrix}\langle\delta\chi\,\delta\psi\rangle\\ \frac{1}{2}\langle\delta\psi\,\delta\psi\rangle\end{pmatrix}. \tag{128}$$

Setting the left-hand side to zero defines a mean-field theory, closed in terms of $\langle\psi\rangle$; an example is the Vlasov equation, for which $\langle\psi\rangle = \langle\widetilde{N}\rangle = f$, $\widetilde{N}$ being the Klimontovich microdensity (22). The right-hand side specifies the effects of fluctuations. The exact equation for the fluctuations is obtained by subtracting Eq. (128) from Eq. (125a) and is

$$(\partial_t + \mathrm{i}\widehat{\mathcal{L}})\delta\psi - \widehat{M}\begin{pmatrix}\langle\chi\rangle\,\delta\psi + \delta\chi\,\langle\psi\rangle\\ \langle\psi\rangle\,\delta\psi\end{pmatrix} = \widehat{M}\begin{pmatrix}\delta\chi\,\delta\psi - \langle\delta\chi\,\delta\psi\rangle\\ \frac{1}{2}(\delta\psi\,\delta\psi - \langle\delta\psi\,\delta\psi\rangle)\end{pmatrix}. \tag{129}$$

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[104] In Eq. (125a) the factor of $\frac{1}{2}$ is introduced so that the linearization of the self-consistent problem does not contain a factor of 2; cf. Eq. (130).

[105] For example, consider the HM polarization-drift nonlinearity $(1-\nabla^2)^{-1}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}(-\nabla^2\varphi)$ with $\psi = \varphi$. If $\boldsymbol{V}_E$ is passively determined from an external potential φ^{ext}, then one chooses $\chi = \varphi^{\mathrm{ext}}$ and readily finds that $M^{\mathrm{pas}}_{\boldsymbol{kpq}} = (1+k^2)^{-1}\widehat{\boldsymbol{b}}\cdot\boldsymbol{p}\times\boldsymbol{q}\,q^2$. In a self-consistent problem (conventional HM), $\varphi^{\mathrm{ext}}\to\varphi$ and $M^{\mathrm{self}}_{\boldsymbol{kpq}} = M^{\mathrm{pas}}_{\boldsymbol{kpq}} + M^{\mathrm{pas}}_{\boldsymbol{kqp}} = (1+k^2)^{-1}\widehat{\boldsymbol{b}}\cdot\boldsymbol{p}\times\boldsymbol{q}(q^2-p^2)$.

A formal integral equation for $\delta\psi$ can be formed by introducing the zeroth-order infinitesimal response function R_0, which obeys the linearization of the left-hand side of Eq. (128):

$$(\partial_t + \mathrm{i}\widehat{\mathcal{L}})R_0(t;t') - \widehat{M}\begin{pmatrix}\langle\chi\rangle\\ \langle\psi\rangle\end{pmatrix} R_0 = \delta(t-t'). \tag{130}$$

(For the self-consistent case the symmetry of $\widehat{M}$ was used.) Then the formal solution of Eq. (129) is

$$\begin{aligned}\delta\psi(t) = R_0(t;0)\delta\psi(0) &+ \int_0^t \mathrm{d}\bar{t}\, R_0(t;\bar{t})\widehat{M}\begin{pmatrix}\delta\chi\,\langle\psi\rangle\\ 0\end{pmatrix}\\ &+ \int_0^t \mathrm{d}\bar{t}\, R_0(t;\bar{t})\widehat{M}\begin{pmatrix}\delta\chi\,\delta\psi - \langle\delta\chi\,\delta\psi\rangle\\ \frac{1}{2}(\delta\psi\,\delta\psi - \langle\delta\psi\,\delta\psi\rangle)\end{pmatrix}(\bar{t}).\end{aligned} \tag{131}$$

Notice that for self-consistent problems the symmetry of $\widehat{M}$ means that the term in $\widehat{M}$ in Eq. (130) contains two physically distinct pieces. For example, for the self-consistent Vlasov equation, for which $\widehat{M}(1,2,3) = \widehat{\boldsymbol{\mathcal{E}}}(1,2)\cdot\boldsymbol{\partial}_1\delta(1-3) + (2\leftrightarrow 3)$, Eq. (130) would explicitly read

$$(\partial_t + \underbrace{\boldsymbol{v}\cdot\boldsymbol{\nabla}}_{\mathrm{i}\widehat{\mathcal{L}}})R_0 + \underbrace{\langle\boldsymbol{E}\rangle\cdot\boldsymbol{\partial}R_0 + \underline{\boldsymbol{\partial}f\cdot\widehat{\boldsymbol{\mathcal{E}}}R_0}}_{-\widehat{M}\langle\psi\rangle} = \delta(\boldsymbol{x}-\boldsymbol{x}')\delta(\boldsymbol{v}-\boldsymbol{v}')\delta_{s,s'}\delta(t-t'). \tag{132}$$

The presence of the underlined term, which is responsible for nontrivial dielectric and collective effects (Sec. 6.5, p. 170), distinguishes R_0 from the single-particle propagator g_0, which obeys Eq. (132) in the absence of the underlined term [cf. Eq. (2b)]. For passive problems g_0 and R_0 are identical, since for such cases the analog of the underlined term in Eq. (132) appears as the second term on the right-hand side of Eq. (131).

3.9.2 The Bourret approximation and quasilinear theory

In general, Eq. (131) may appear to be rather useless since the nonlinear terms on the right-hand side may be large. However, note that the nonlinear terms always enter in conjunction with a time integral, so their size must be interpreted as an appropriately nondimensionalized autocorrelation time such as the Kubo number discussed in Secs. 3.3.1 (p. 54) and 3.4 (p. 56). If the nonlinear terms are small, the integral equations provide viable starting points for further approximations. In particular, to lowest order one may neglect the nonlinear terms altogether. Upon doing so, then substituting the result into Eq. (128), one obtains a closed equation (nonlocal in time) for $\langle\psi\rangle$. Such equations are frequently called *master equations.*

The least confusing expression of this procedure occurs for passive problems. It is conventional to neglect the initial-condition transient [the first term on the right-hand side of Eq. (131)]. Upon retaining only the second term on the right-hand side, one finds the *Bourret approximation* (Bourret, 1962)

$$(\partial_t + \mathrm{i}\widehat{\mathcal{L}})\langle\psi\rangle - \widehat{M}\langle\chi\rangle\langle\psi\rangle = \int_0^t \mathrm{d}\overline{\tau}\,\widehat{M}R_0(\overline{\tau})\langle\delta\chi(t)\delta\chi(t-\overline{\tau})\rangle\widehat{M}\langle\psi\rangle(t-\overline{\tau}) \tag{133}$$

(written here for the special case in which the second $\widehat{M}$ does not act on $\delta\chi$, as in a solely time-dependent problem). It involves the two-time correlation function $\Upsilon(t,t') \doteq \langle\delta\chi(t)\delta\chi(t')\rangle$ of the random coefficient; that is generally taken to be stationary: $\Upsilon(t,t') = \Upsilon(t-t')$. Furthermore, since one has already assumed a short autocorrelation time by neglecting the nonlinear terms, the *Markovian approximation* $\langle\psi\rangle(t-\overline{\tau}) \approx \langle\psi\rangle(t)$ is appropriate; one then obtains

$$(\partial_t + \mathrm{i}\widehat{\mathcal{L}})\langle\psi\rangle - \widehat{M}\langle\chi\rangle\langle\psi\rangle + \widehat{\eta}^{\mathrm{nl}}\,\langle\psi\rangle = 0, \quad \widehat{\eta}^{\mathrm{nl}} \doteq -\int_0^\infty \mathrm{d}\overline{\tau}\,\widehat{M}R_0(\overline{\tau})\Upsilon(\overline{\tau})\widehat{M}. \tag{134a,b}$$

This is, in somewhat abstract notation, the so-called *quasilinear*[106] *approximation*; $\widehat{\eta}^{\mathrm{nl}}$ represents an unrenormalized (constructed from R_0) fluctuation-induced damping (nl stands for nonlinear) and is a nontrivial prediction of the statistical closure. In problems with variables additional to the time, the presence of R_0 in Eq. (134b) has the effect of converting the Eulerian correlation function $\Upsilon(\tau)$ to a Lagrangian one taken along the zeroth-order orbits. As an illustration, consider the passive Vlasov model (the so-called *stochastic acceleration* problem) in which the underlined term in Eq. (132) is ignored. Equations (134) become

$$(\partial_t + \underbrace{\boldsymbol{v}\cdot\boldsymbol{\nabla}}_{\mathrm{i}\widehat{\mathcal{L}}} + \underbrace{\langle\boldsymbol{E}\rangle\cdot\boldsymbol{\partial}}_{-\widehat{M}\langle\chi\rangle})f + \underbrace{[-\boldsymbol{\partial}\cdot\mathsf{D}(\boldsymbol{v})\cdot\boldsymbol{\partial}]}_{\widehat{\eta}^{\mathrm{nl}}}f = 0, \quad \mathsf{D}(\boldsymbol{v}) \doteq \int_0^\infty \mathrm{d}\tau\,\langle\delta\boldsymbol{E}(\boldsymbol{x},t)\delta\boldsymbol{E}(\boldsymbol{x}-\boldsymbol{v}\tau)\rangle. \tag{135a,b}$$

For further and more explicit discussion of plasma quasilinear theory, see Sec. 4.1 (p. 90); compare Eqs. (135) with the Markovian version of Eq. (171).

It is useful to illustrate these general considerations with the stochastic oscillator model introduced in Sec. 3.3 (p. 52). Because of the random initial condition that was assumed, the mean field itself vanishes identically. Nevertheless, the mean response function does not vanish, so one may take Eq. (76) as the primitive equation analogous to Eq. (125a); alternatively, one may study the mean field conditional on $\psi(0)$. One identifies $\widehat{\mathcal{L}} \equiv 0$, $\widehat{M} = -\mathrm{i}$, $\chi = \widetilde{\omega}$, and $R_0(\tau) = H(\tau)$. Then the quasilinear approximation for R is

$$(\partial_\tau + \eta^{\mathrm{nl}})R(\tau) = \delta(\tau), \quad \eta^{\mathrm{nl}} = \int_0^\infty \mathrm{d}\overline{\tau}\,\Upsilon(\overline{\tau}) = \beta^2\tau_{\mathrm{ac}}^{\mathrm{lin}}. \tag{136a,b}$$

(I shall drop the caret on $\widehat{\eta}^{\mathrm{nl}}$ when it is merely an ordinary number, not an operator.) The solution $R(\tau) = H(\tau)\exp(-\beta^2\tau_{\mathrm{ac}}^{\mathrm{lin}}\tau)$ precisely reproduces the long-time ($\tau > \tau_{\mathrm{ac}}^{\mathrm{lin}}$) behavior of the exact solution (81) in the limit $\mathcal{K} \ll 1$.

The analogous procedure applied to self-consistent problems contains some subtleties. Because of the 0 in the second term of Eq. (131) (linear self-consistent response is included in R_0), if one wishes to work with R_0 one cannot simultaneously neglect both the initial-condition and nonlinear terms. Iteration on the nonlinear terms is one route to the self-consistent DIA (Sec. 5, p. 126). A quasilinear approximation for self-consistent problems, similar in form to the passive theory, can be obtained by iterating on g_0 rather than R_0; that is how Vlasov QLT is conventionally developed (Sec. 4.1, p. 90). For the Klimontovich equation, lowest-order classical discreteness effects arise by retaining only the initial-condition term of Eq. (131). In that case the initial conditions are singular, and their effects do not entirely phase-mix away even at infinite times.

[106] "Linear" because the fluctuations are treated linearly; "quasi" because a nonlinear correction is retained in the equation for the mean field.

3.9.3 *Exact solutions of model problems*

Although it is clearly not hard to obtain approximations suitable for weakly nonlinear cases with short autocorrelation times, in general one must deal with strongly nonlinear cases with $\mathcal{K} \geq 1$. There are two general ways of proceeding:

(1) Find approximate statistics of the exact equation;

(2) Find the exact statistics of approximate model equations.

Although the bulk of this article will focus on procedure 1, there are important instances of procedure 2. For example, Brissaud and Frisch (1974) have for passive problems discussed a class of statistical models, involving stepwise-constant random variables, that can be solved exactly. Given an exact equation with arbitrary χ, one may, for example, match the covariance of the model χ with that of the exact χ, then proceed with the exact solution of the solvable model.

Kraichnan has considered an alternate version of procedure 2 in which instead of modeling the statistics of χ one randomizes the mode-coupling coefficient $\widehat{M}$ in a particular way. The addition of extra randomness to the exact equation of motion leads to dramatic simplifications in the statistical analysis. This *random-coupling model* will be described in some detail in Sec. 5.2 (p. 131).

3.9.4 *Cumulant discard*

I now turn to a discussion of techniques that implement procedure 1, which has historically been the main focus of statistical closure theory. One procedure alternative to direct truncation of the integral equation (131) proposes to generate a sequence of ever-better closures by successively enlarging the space of variables to retain more and more cumulants. That is, exact equations for the ith cumulant are written for $i = 1, 2, \ldots, n$, and cumulant $n+1$ is set to zero. In classical kinetic theory this corresponds to truncating the BBGKY hierarchy at successively higher and higher order. Orszag and Kruskal (1968) discussed the analogous hierarchy for the NSE.

Following Kraichnan (1961), I illustrate cumulant-discard approximations for the $\mathcal{K} = \infty$ limit of the stochastic oscillator, for which the exact solution is Eq. (81). One may take $\beta = 1$. Let us begin again with Eq. (76). Define the (mixed) nth-order cumulants

$$C_n \doteq \langle\langle \delta\widetilde{\omega}^{n-1}\, \widetilde{R} \rangle\rangle = \begin{cases} 0 & (n=1) \\ \langle \delta\widetilde{\omega}\, \widetilde{R} \rangle & (n=2) \\ \langle \delta\widetilde{\omega}^2\, \widetilde{R} \rangle - \langle \delta\widetilde{\omega}^2 \rangle R & (n=3) \\ \langle \delta\widetilde{\omega}^3\, \widetilde{R} \rangle - 3\langle \delta\widetilde{\omega}^2 \rangle \langle \delta\widetilde{\omega} \widetilde{R} \rangle & (n=4) \end{cases} \tag{137a,b}$$

($\langle \delta\widetilde{\omega}^2 \rangle = \beta^2 = 1$). One has rigorously

$$\dot{R} + \mathrm{i}C_2 = \delta(t - t'), \quad \dot{C}_2 + \mathrm{i}R + \mathrm{i}C_3 = 0, \quad \dot{C}_3 + 2\mathrm{i}C_2 + \mathrm{i}C_4 = 0. \tag{138b,c}$$

Upon successively ignoring C_{n+1} for $n = 1, 2, \ldots$, one obtains

$$R(\tau) \approx H(\tau) \begin{cases} 1 & (n=1) \\ \cos(\tau) & (n=2) \\ \frac{2}{3} + \frac{1}{3}\cos(\sqrt{3}\tau) & (n=3). \end{cases} \tag{139}$$

These approximations are purely oscillatory,[107] so none captures the long-time decay of the exact

[107] The oscillatory behavior of such cumulant-discard approximations for $\mathcal{K} = \infty$ was rediscovered by

solution. This is not surprising: For $\mathcal{K} \gg 1$ the statistics are intrinsically non-Gaussian. More specifically, the dynamics are non-Markovian, so no coarse-graining in units of a small autocorrelation time is possible and thus no argument based on the central limit theorem can be invoked.

Mathematically, this sequence of cumulant discards produces better and better approximations to the *short-time* behavior of the true solution (80): $R(\tau) = H(\tau)(1 - \frac{1}{2}\tau^2 + \frac{1}{8}\tau^4 + \cdots)$. Thus approximation n in Eq. (139) matches through $O\big(\tau^{2(n-1)}\big)$. The procedure is unsuitable for describing the long-time dynamics, which evidently involve cumulants of all orders. What is needed is a way of introducing the long timescale in a nonsecular way. Various of the techniques to be described in later sections accomplish this.

Such cumulant-discard approximations have a long history of applications to the Navier–Stokes and similar equations. Millionshtchikov (1941a,b) advanced[108] the hypothesis that the four-point velocity correlation function might factor in the Gaussian way (the so-called *quasinormal approximation*). However, theoretical and numerical study of that approximation (Kraichnan 1957, 1962a,b; Tatsumi, 1957; Ogura, 1963, and references therein; Orszag, 1970a) for homogeneous, isotropic NS turbulence showed that the quasinormal approximation is ill founded; indeed, for large Reynolds numbers it allows the energy spectrum to develop catastrophically negative regions in a time characteristic of an eddy turnover time of an excited mode. Physically, the zero-fourth-cumulant approximation does not capture the irreversible decay of the correlation function due to nonlinear advective scrambling, just as the above cumulant-discard example fails to capture the exponential decay of the true solution (80).[109] A good discussion was given by Orszag (1970a).

It must be emphasized that the regime $\mathcal{K} = \infty$ is a particularly difficult one, being intrinsically nonlinear (the linear autocorrelation time is infinite). The presence of mean fields can modify the situation considerably by introducing a finite $\tau_{\rm ac}^{\rm lin}$. Some general discussion was given by Herring (1969). For example, in drift-wave problems a background gradient introduces a linear, dispersive[110] mode frequency proportional to the diamagnetic frequency ω_*. Nevertheless, in practice it frequently happens that the system saturates such that the effective $\mathcal{K} = O(1)$; i.e., it sits at the boundary between weak and strong turbulence. For such situations it is best that the closure is capable of properly dealing with strong turbulence; it can then be specialized to weaker fluctuations on a mode-by-mode basis as warranted. Accordingly, I concentrate in the next sections on methods capable of handling fully developed turbulence.

3.9.5 Regular perturbation theory

Given the previous example, one will not be surprised that a regular perturbation procedure based on formal expansion in a parameter λ (assumed to multiply the nonlinear term) that is really large will not succeed in the strong-turbulence limit if truncated at finite order. Nevertheless, this approach is particularly instructive as it suggests a necessary generalization. [For definiteness, I now

Hammett et al. (1992) in their considerations of the Landau-fluid closure problem (Appendix C.2.2, p. 278).

[108] For a historical note, see Yaglom (1994).

[109] In their study of the statistical dynamics of the guiding-center model (31), Taylor and Thompson (1973) were led to an oscillatory correlation function $C_{\boldsymbol{k}}(\tau) \propto \cos(\Omega_{\boldsymbol{k}}\tau)$ on the basis of what they called the *random-phase approximation*. They recognized that the approximation was valid only for short times and that $C_{\boldsymbol{k}}(\tau)$ should be damped for long times. See also the discussions by Vahala et al. (1974) and Taylor and Thompson (1974).

[110] It is important that the linear frequency is dispersive; otherwise, it can be transformed away.

focus exclusively on passive problems; a fully self-consistent formalism will be developed in Sec. 6 (p. 146).] A formal iteration of Eq. (131) can be performed by treating the last, nonlinear term as a small perturbation; one finds [111]

$$\delta\psi = R_0\delta\psi(0) + R_0 \star \widehat{M}\delta\chi\langle\psi\rangle + R_0 \star \widehat{M}\delta\chi R_0 \star \widehat{M}\delta\chi\langle\psi\rangle + \cdots = \widetilde{R}\,\delta\psi(0), \tag{140}$$

where $\star$ denotes time convolution and the term $\langle\delta\chi\,\delta\psi\rangle$ in Eq. (131) has not been written for simplicity as it does not contribute to the ultimate equation for $\langle\psi\rangle$. It is useful to represent this expansion diagrammatically, analogous to Feynman's diagrams for quantum electrodynamics. [Further remarks, historical background, and references on Feynman's approach are given in Sec. 6 (p. 146).] Associate a small dot with the *bare vertex* $\widehat{M}$, a light solid line with the zeroth-order Green's function R_0, and a light dashed line with the Gaussian random coefficient $\delta\chi$. The expansion (140) is then pictured in Fig. 5.

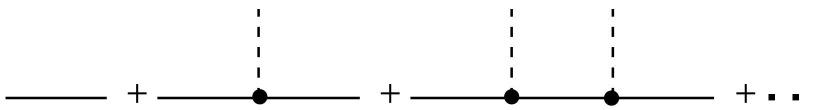

Fig. 5. Diagrammatic expansion of the passive random response function $\widetilde{R}$. Light solid line, R_0; light dashed line, $\delta\chi$; small dot, $\widehat{M}$.

When Eq. (140) is inserted into Eq. (128), the Gaussian averages produce a variety of terms, as shown in Fig. 6. [112]

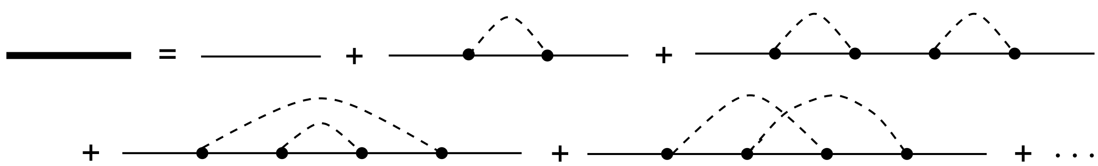

Fig. 6. Diagrammatic expansion of the passive mean response function R (heavy solid line) in terms of the zeroth-order response function R_0 (light solid line) and correlation function Υ of the Gaussian random coefficient (dashed line).

3.9.6 Failure of regular perturbation theory

Now consider truncating such series at any finite order. Whether this is a good idea depends once again on the size of the Kubo number. For $\mathcal{K} \ll 1$ the multiple time convolutions introduce higher and higher powers of $\tau_{\rm ac}$, hence $\mathcal{K}$, suggesting that the higher-order terms are small. However, for $\mathcal{K} > 1$ one expects that terms from all orders contribute.

To gain an intuitive understanding of the difficulties, consider the formal expansion $(1-x)^{-1} = 1 + x + x^2 + \cdots$. For $|x| \gg 1$ the left-hand side is small whereas the right-hand side is large at any order of truncation. In this particular case regular perturbation expansion around the origin generates an infinite series that converges for $|x| < 1$. If one were given the infinite series, one could sum it for

111 It is assumed that $\langle\psi(0)\rangle = 0$. Then the last equality in Eq. (140) follows from Eq. (127) because $\langle\psi\rangle = R\langle\psi(0)\rangle = 0$ [assuming χ and $\psi(0)$ are statistically independent] and thus $\psi = \delta\psi$.

112 For some discussion of this expansion, especially as it relates to Dupree's resonance-broadening theory (Sec. 4.3, p. 108), see Thomson and Benford (1973b).

$|x| < 1$, then use analytic continuation to deduce that the solution in the entire complex x plane is $(1-x)^{-1}$. In this context the resummation procedure is called *renormalization.* An introduction to renormalization procedures is given in the next two sections; a deeper discussion is given in Sec. 6 (p. 146).

To slightly elaborate the previous example and to place it in a more physical context, consider the function $R(\tau) \doteq H(\tau)\langle e^{-ikv\tau}\rangle = H(\tau)\exp(-\frac{1}{2}k^2v_t^2\tau^2)$, where the average is taken over a Maxwellian PDF in v. $R(\tau)$ is nothing but the response function for the stochastic oscillator (Sec. 3.3, p. 52) at infinite Kubo number with $\widetilde{\omega} = kv$. Note that $0 \leq R(\tau) \leq 1$. Now many practical renormalization procedures are conducted in the frequency domain. Therefore consider $R(\omega) = \langle[-i(\omega - kv + i\epsilon)]^{-1}\rangle$. Upon formally expanding in small kv/ω, one gets

$$R(\omega) = (-i\omega)^{-1}\langle 1 + (kv/\omega) + (kv/\omega)^2 + \cdots\rangle = (-i\omega)^{-1}[1 + (kv_t/\omega)^2 + \cdots]. \qquad \text{(141a,b)}$$

Clearly, finite truncations of this series badly misrepresent the low-frequency response.[113]

There are deep and troubling issues connected with formal perturbation expansions in statistical problems. Kraichnan (1966a) has argued that the perturbative expansions of statistical quantities such as the correlation function or mean response function of the NSE with Gaussian initial conditions in powers of the Reynolds number $\mathcal{R}$ have zero radius of convergence even though an amplitude expansion of each realization is convergent; he illustrated the point with an exactly solvable model. The logic is that (i) the radius of convergence is limited by the distance to the nearest singularity in the complex $\mathcal{R}$ plane; (ii) that distance scales inversely with the initial amplitude; (iii) in a Gaussian distribution (or any similar one without a high-amplitude cutoff), indefinitely high amplitudes are represented with nonzero weight. That leads to terms of the form $\exp(-1/\epsilon)$ in the statistical quantities, where $\epsilon \propto \mathcal{R}^{-1}$. Such functions, having an essential singularity at $\epsilon = 0$, are well known (Bender and Orszag, 1978) to have an asymptotic expansion whose coefficients all vanish.[114]

3.9.7 Propagator renormalization

I now return to the general problem posed by Eq. (140). It should be clear that in the strong-turbulence limit $\mathcal{K} > 1$ it is necessary to sum the entire perturbation series. In view of the remarks in the last paragraph, it is not entirely clear what this means, since the series may not completely represent the exact solution. For now, let us ignore this difficulty, anticipating more powerful techniques (Sec. 6, p. 146) that bypass the order-by-order expansion. Then the formal summation can be done in two steps. First, *if* one temporarily ignores the crossed lines in Fig. 6 (p. 82), it can easily be recognized that the resulting diagrams can be summed to the result shown in Fig. 7 (p. 84). This can be seen to define an integral equation for R, namely,[115] $R = R_0 - R_0\Sigma^{\text{nl}}R$, or

$$R^{-1} = R_0^{-1} + \Sigma^{\text{nl}}, \quad \Sigma^{\text{nl}}(\tau) \approx -\widehat{M}R(\tau)\Upsilon(\tau)\widehat{M}. \qquad \text{(142a,b)}$$

[113] In terms of the *plasma dispersion function* (Fried and Conte, 1966) $Z(z) \doteq \pi^{-1/2}\int_{-\infty}^{\infty} dt\,(t-z)^{-1}e^{-t^2}$ ($\operatorname{Im} z > 0$), which satisfies $Z(0) = i\sqrt{\pi}$, one has $R(\omega) = (\sqrt{2}ikv_t)^{-1}Z(\omega/\sqrt{2}kv_t)$ and $R(0) = \sqrt{\pi/2}(kv_t)^{-1}$.
[114] The convergence properties of renormalized turbulence theory have confused many people; see, for example, Thomson and Benford (1973a) and the subsequent comments by Orszag (1975).
[115] I now drop the $\star$'s and use the standard notation that operator products imply convolutions in time and the other independent variables.

At this stage, one has accomplished *propagator renormalization* (sometimes called *line* or *mass renormalization*). The resulting approximation is called the *direct-interaction approximation* (DIA) for reasons that will be explained in the next section.

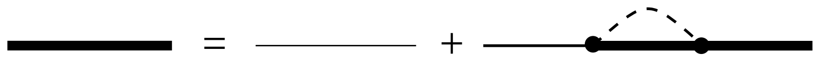

Fig. 7. The direct-interaction approximation for passive advection, obtained by ignoring vertex corrections [the crossed lines in Fig. 6 (p. 82)].

The renormalized form (142b), which contains the true response function R, should be compared with the quasilinear form (134b), which contains R_0. Even in the absence of the vertex corrections, the renormalized operator introduces a qualitatively important improvement to the description: it not only contains the linear autocorrelation time (through Υ), it also contains the nonlinear time scale τ_n, since R must be computed self-consistently. The multiplicative way in which $R(\tau)$ and $\Upsilon(\tau)$ enter shows that the time-convolution integrals will properly see[116] the true autocorrelation time $\tau_{\mathrm{ac}} = \min(\tau_{\mathrm{ac}}^{\mathrm{lin}}, \tau_n)$.

For the response function of the stochastic oscillator, the DIA is (Kraichnan, 1961)

$$\partial_\tau R + \int_0^\tau \mathrm{d}\overline{\tau}\, \Sigma^{\mathrm{nl}}(\overline{\tau}) R(\tau - \overline{\tau}) = \delta(\tau), \quad \Sigma^{\mathrm{nl}}(\tau) \doteq R(\tau)\Upsilon(\tau). \tag{143a,b}$$

It is clear that as $\mathcal{K} \to 0$, Σ^{nl} falls to zero very rapidly and Eq. (143b) reduces to the correct quasilinear description; compare Eqs. (143) with Eqs. (134). For the other extreme $\mathcal{K} = \infty$, where $\Upsilon(\tau) = \beta^2$, the solution of Eqs. (143) can readily be found[117] by Fourier transformation (Kraichnan, 1961):

$$R(\tau) = H(\tau) J_1(2\beta\tau)/(\beta\tau). \tag{144}$$

This is compared with the exact solution (and with some other approximations to be discussed later) in Fig. 8 (p. 85). The most important qualitative feature is that the DIA solution decays to zero on the proper, nonlinear timescale (although the decay is algebraic, not exponential).

Various authors have drawn opposite conclusions from the comparison in Fig. 8. Frisch and Bourret (1970) and Brissaud and Frisch (1974) labeled it a failure; the present author, following Kraichnan, considers it to be a success. Of course, the assessment depends on one's criterion. A zeroth-order measure is the area under $R(\tau)$ [i.e., $R(\omega = 0)$]; this R-based autocorrelation time τ_{ac} is relevant for the calculation of transport coefficients. For the $\mathcal{K} = \infty$ stochastic oscillator, the true $\tau_{\mathrm{ac}} = \sqrt{\pi/2}$ is approximated to within 20% by the DIA, which predicts $\tau_{\mathrm{ac}} = 1$.

[116] If R were exponential with decay constant τ_n, one would find

$$\tau_{\mathrm{ac}}^{-1} = \tau_n^{-1} + (\tau_{\mathrm{ac}}^{\mathrm{lin}})^{-1}. \tag{f-5}$$

In general, the solution of the nonlinear closure equation for R is not an exponential although Eq. (f-5) still captures in a qualitatively reasonable way the competition between the linear and nonlinear decorrelation mechanisms.

[117] For the more general finite-$\mathcal{K}$ form $\Upsilon(\tau) = \beta^2 \mathrm{e}^{-|\tau|/\tau_{\mathrm{ac}}}$, Frisch and Bourret (1970) succeeded in finding an analytical solution of Eq. (143a). Nevertheless, in practice it is easier to directly solve Eq. (143a) by numerical integration than to numerically evaluate the complicated analytical formula.

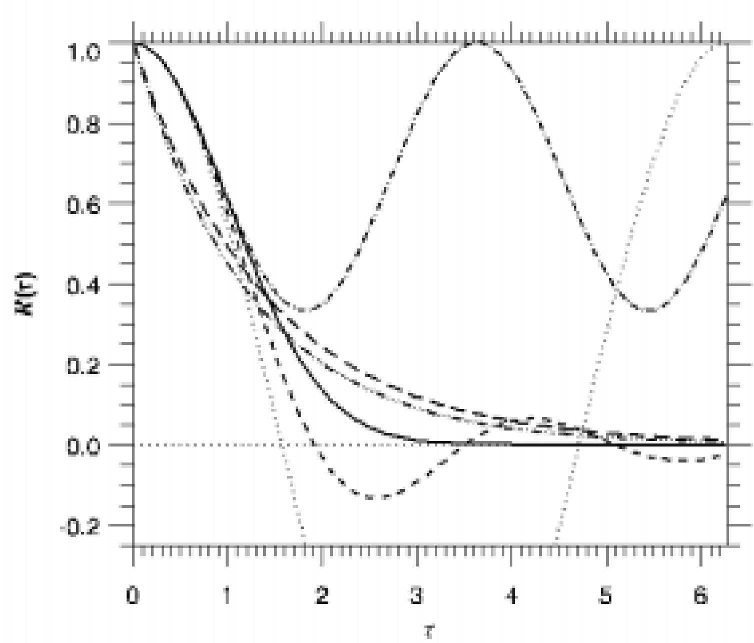

Fig. 8. Comparisons of approximations to the stochastic-oscillator response function with the exact solution. Solid line, exact solution [Eq. (81)]; short dashed line, DIA [Eq. (144)]; dotted line, $n = 2$ cumulant discard [Eq. (139)]; chain-dotted line, $n = 3$ cumulant discard [Eq. (139)]; triple chain-dotted line, Markovian approximation $\exp(-\tau/\tau_{\rm ac})$ for exact $\tau_{\rm ac} = \sqrt{\pi/2}$; long dashed line, $\exp(-\tau/\tau_{\rm ac})$ for $\tau_{\rm ac} = 1/\sqrt{2}$ [see Sec. 7.2.1 (p. 183)].

3.9.8 Vertex renormalization

Now consider the crossed lines in Fig. 6 (p. 82) in more detail. Those are called *vertex corrections*, as can be seen by redrawing the last term of Fig. 6 as Fig. 9; the boxed terms play the same role as does the right-hand bare vertex in the first diagram of Fig. 6. Brief reflection then shows that all possible terms contributing to Fig. 6 can be summed to the general *Dyson equation* shown in Fig. 10 (p. 86), where the large dot contains all possible vertex corrections and is called the *renormalized vertex* Γ. That equation can be rearranged into Eq. (142a), with $\Sigma^{\rm nl} \sim -\widehat{M}R\Upsilon\Gamma$ (in general, Γ depends on at least two time arguments, which are not shown explicitly). Unfortunately, the equation for Γ closes only in a functional, not algebraic sense [for more details, see Sec. 6.2 (p. 153)]. It is left as an exercise to show that a partial summation of the vertex corrections leads to the *first vertex renormalization* shown in Fig. 11 (p. 86). That result will be systematically rederived in Sec. 6.2.3 (p. 159).

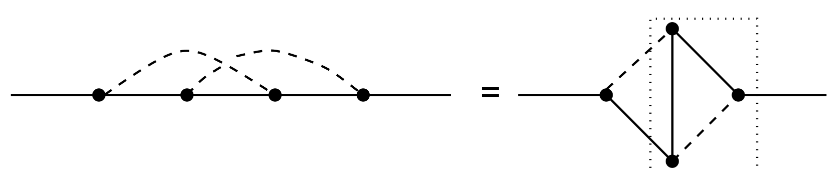

Fig. 9. An equivalent topology for the term with crossed lines. The subdiagram within the dotted box is a contribution to the renormalized vertex Γ.

One can now understand the nomenclature *direct-interaction approximation*. That closure, which involves complete neglect of vertex renormalization, is derived by considering only the most direct

Fig. 10. The complete Dyson equation for R includes both propagator and vertex renormalization. $R^{-1} = R_0^{-1} + \Sigma^{\rm nl}$, with $\Sigma^{\rm nl} = -\widehat{M} R \Upsilon \Gamma$.

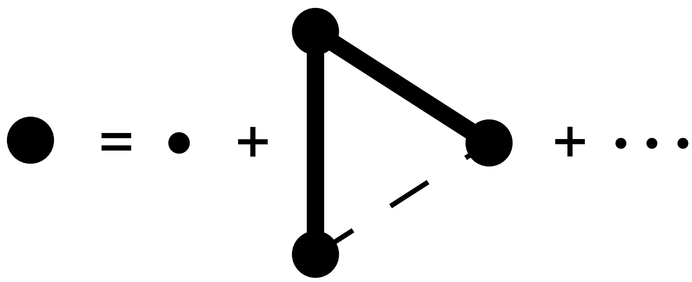

Fig. 11. Renormalized perturbation expansion for the vertex function: $\Gamma = \widehat{M} + \Gamma R \Gamma \Upsilon R \Gamma + \cdots$.

path (solid lines) through the diagrams contributing to $\Sigma^{\rm nl}$, i.e., the one that involves the least number of bare vertices. Further interpretation will be given in Sec. 5 (p. 126).

The first vertex renormalization can be completely worked out for the stochastic oscillator. As an example, I follow Kraichnan (1961) and consider the $\mathcal{K} = \infty$ limit. With time normalized to β^{-1}, one is led to the pair of equations[118]

$$-\mathrm{i}\omega R_\omega + \Sigma_\omega R_\omega = 1, \quad \Gamma_\omega = -\mathrm{i} + R_\omega^2 \Gamma_\omega^3, \tag{145a,b}$$

with $\Sigma_\omega = \mathrm{i} R_\omega \Gamma_\omega$. It can be seen that Eq. (145b) involves only the quantities $\overline{\Gamma}_\omega \doteq R_\omega \Gamma_\omega$ and $\overline{\gamma}_\omega \doteq -\mathrm{i} R_\omega$; this is a special case of a general, essentially dimensional principle (Martin et al., 1973). Upon rewriting Eq. (145b) as $\overline{\gamma}_\omega = \overline{\Gamma}_\omega - \overline{\Gamma}_\omega^3$, one sees that the vertex renormalization can be interpreted as an expansion of the bare vertex in powers of the renormalized one (Martin et al., 1973). It is a matter of straightforward algebra to eliminate $\overline{\Gamma}_\omega$, thereby obtaining the equation, first found by Kraichnan (1961) with the aid of tedious graphical analysis, $R_\omega^4 - \mathrm{i}\omega(1-\omega^2)R_\omega^3 + (3\omega^2 - 1)R_\omega^2 - 3\mathrm{i}\omega R_\omega - 1 = 0$. This fourth-degree polynomial can be analyzed completely. Its solution is compared with the exact solution and the DIA in Fig. 13 of Kraichnan (1961); the agreement is excellent.[119] As a special case, one can easily verify that $R_{\omega=0} = [\frac{1}{2}(1+\sqrt{5})]^{1/2} \approx 1.27$, to be compared with the exact value $(\frac{1}{2}\pi)^{1/2} \approx 1.25$ and the DIA value 1. One also finds that $|\overline{\Gamma}_{\omega=0}| = [\frac{1}{2}(\sqrt{5}-1)]^{1/2} \approx 0.786$, to be compared with the exact value $(2/\pi)^{1/2} \approx 0.798$ and the DIA value 1. Although $\overline{\Gamma}$ is not very small, it is at least less than one, suggesting that the vertex expansion (145b) may converge at least asymptotically.

Unfortunately, it is not true that mere polynomial extensions of the vertex renormalization, say to $O(\overline{\Gamma}^5)$, are well behaved; for further discussion, see Sec. 6.2.3 (p. 159) and especially footnote 201 (p. 160). Nevertheless, the present example shows the power and importance of vertex renormalization. Remarkably, although the renormalizations diagrammed above were derived for passive problems, it is shown in Sec. 6.2.2 (p. 155) that fully self-consistent renormalizations obey a formally identical matrix generalization (involving three independent vertex functions).

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[118] For stationary problems $\Gamma(t, t', t'')$ depends in general on two time differences; however, at $\mathcal{K} = \infty$ one time dependence goes away.

[119] For further discussion of the first vertex renormalization, see Kraichnan (1964e).

3.9.9 Markovian approximation

It is important to stress that for $\mathcal{K} = \infty$ the nonlinear time scale β^{-1} can be entirely removed from the problem by introducing the dimensionless time $\overline{\tau} \doteq \beta\tau$. Since causality is never violated, one always has $R(\tau) = H(\tau)G(\overline{\tau})$ for some function $G(\overline{\tau})$. The entire output of a closure calculation for R is the determination of an approximate form for G. I shall later discuss a variety of reasons why the DIA is in some sense a preferred approximation. Nevertheless, depending on one's goals, simpler approximations may be useful. For example, a *Markovian* approximation may be found by postulating the *form* of Eq. (136a), then demanding that $1/\eta^{\mathrm{nl}}$ agree with the true (nonlinear) autocorrelation time determined from the exact solution (assuming that is somehow known, which it is generally not). That is, $1/\eta^{\mathrm{nl}} = \int_0^\infty \mathrm{d}\tau\, R_{\mathrm{exact}}(\tau)$. This procedure is guaranteed to give the exact τ_{ac} although it does not ensure that the shape of the response function will be correct. For example, for $\mathcal{K} = \infty$ $R_{\mathrm{exact}}(\tau)$ is given by Eq. (80) whereas the Markovian approximation gives $R(\tau) = H(\tau)\exp(-\eta^{\mathrm{nl}}\tau)$ for all $\mathcal{K}$, badly misrepresenting the shape (Fig. 8, p. 85). The reason is clear: Upon recalling the exact solution and discussion of the SO in Sec. 3.3.1 (p. 54), one notes that the Markovian approximation is *justifiable* only for small $\mathcal{K}$. Nevertheless, such an approximation may be able to capture the proper timescale even for large $\mathcal{K}$.

In general, it is not fair to obtain η^{nl} from the exact solution because for realistic models that solution is not available. Markovian closures (Sec. 7.2, p. 182) *predict* η^{nl}. For example, the eddy-damped quasinormal Markovian approximation discussed in Sec. 7.2.1 (p. 183) predicts $\eta^{\mathrm{nl}} = 1/\sqrt{2}$ for the SO at $\mathcal{K} = \infty$, which differs from the exact η^{nl} by about 11%. Although this prediction appears to be not unreasonable even for strong turbulence, quantitative inaccuracies in a simple model such as the SO can hint of *qualitative* failings in more realistic nonlinear dynamical models. One expects that if the system is only weakly turbulent (here $\mathcal{K} < 1$), the approximations may be better justified. One practical example is furnished by the Landau-fluid closures of plasma physics. Those closures determine the analog of η^{nl} (or, in more complicated cases, an array of coefficients) by demanding that the *linear* dispersion relation, which can be calculated in detail from kinetic theory, be well fitted by the fluid closure. It seems clear that this method should become progressively more inadequate as the turbulence becomes stronger, but this has not been quantified. See Appendix C.2.2 (p. 278) for further discussion.

3.9.10 Padé approximants

We have seen that it is not difficult to form formal perturbation expansions for various statistical quantities such as the mean response function. As an alternative to the formal renormalization procedures described in Sec. 3.9.7 (p. 83), Kraichnan (1968a) considered the use of Padé approximants (Baker, 1965) to approximately resum the perturbation expansion. For any function f expandable in a Taylor series in a parameter λ as $f = \sum_{n=0}^{\infty} a_n\lambda^n$, the *Padé approximant* of order (r, s) is defined to be $f_{r,s} = (\sum_{m=0}^{s} b_m\lambda^m)/(\sum_{n=0}^{r} c_n\lambda^n)$ with $c_0 = 1$, where a definite procedure exists for determining the coefficients b_m and c_n from the Taylor coefficients a_n. There is an intimate relationship between Padé approximants and continued fractions.

By using the stochastic oscillator as an example, Kraichnan showed that the most straightforward expansion of the solution in powers of λ did not lead to approximants useful in the limit of large Kubo number (strong turbulence). The cure was to temporarily treat the mean response function R (itself a function of λ) as given while forming the perturbation expansion. That procedure is a kind of renormalization; the coefficients of the resulting Padé approximants then contain R.

Kraichnan's results were encouraging, and he argued for further research [cf. Kraichnan (1970a)]. But he also stressed that the convergence properties of sequences of approximants are largely unknown, and there remains the fundamental issue that the perturbation series may not adequately represent the true solution.

In plasma physics, simple examples of Padé approximants have been used by Hammett and co-workers in implementing Landau-fluid closures of the gyrokinetic equations. See Appendix C.2.2 (p. 278) for further discussion.

3.9.11 Projection operators

Ensemble averaging can be thought of as a projection operation. A projection operator P is one that is linear and obeys $\mathsf{P}^2 = \mathsf{P}$. The ensemble-averaging operation $\langle \ldots \rangle$ qualifies on both grounds. Because linear operator algebra is often formally quite concise, various authors [notably Weinstock in plasma physics; see Sec. 4.3 (p. 108)] have tried to exploit projection-operator-based manipulations to derive closed equations for statistical quantities. The technique (Zwanzig, 1961; Mori, 1965b) is useful for certain kinds of transport problems in many-particle kinetic theory (Bixon and Zwanzig, 1971; Haken, 1975). Nevertheless, it is fraught with difficulties for the general turbulence problem, as I attempt to explain in the following brief discussion.[120]

Define the projection operator P such that $\mathsf{P}\psi \equiv \langle\psi\rangle$, and define the orthogonal projector $\mathsf{Q} \doteq \mathsf{I} - \mathsf{P}$. Thus Q applied to ψ generates the fluctuating component: $\mathsf{Q}\psi = \delta\psi$. It is convenient to adopt a Dirac notation. Introduce a general time-independent probability measure $P(\varpi)$ such that $\langle\psi\rangle = \int d\varpi\, \psi[\varpi] P(\varpi)$. For example, $P(\varpi)$ could be a Gaussian distribution of initial conditions. Now define $|\,\psi\,\rangle \doteq \psi P$ and $\langle\,\psi\,| \doteq \psi$ so that for an arbitrary linear operator $\mathcal{L}$ one has $\langle a \mid \mathcal{L} \mid b\rangle = \int d\varpi\, a[\varpi]\, \mathcal{L}\, b[\varpi] P(\varpi)$. P is then formally realized by $\mathsf{P} = |\,1\,\rangle\langle\,1\,|$.

Assume that ψ obeys the *dynamically linear* equation $\partial_t|\,\psi\,\rangle + i\widetilde{\mathcal{L}}|\,\psi\,\rangle = 0$, where $\widetilde{\mathcal{L}}$ may be random and time-dependent. Problems of passive advection have this linear form, as do the equations for infinitesimal perturbations of a steady state. One may now project onto the mean and fluctuating subspaces. By inserting the identity operator $\mathsf{I} = \mathsf{P} + \mathsf{Q}$ after the $\widetilde{\mathcal{L}}$, one finds

$$\partial_t \mathsf{P}|\,\psi\,\rangle + \mathsf{P}i\widetilde{\mathcal{L}}\mathsf{P}|\,\psi\,\rangle + \mathsf{P}i\widetilde{\mathcal{L}}\mathsf{Q}|\,\psi\,\rangle = 0. \tag{146a}$$

Similarly, by interchanging P and Q, one finds

$$\partial_t \mathsf{Q}|\,\psi\,\rangle + \mathsf{Q}i\widetilde{\mathcal{L}}\mathsf{Q}|\,\psi\,\rangle = -\mathsf{Q}\widetilde{\mathcal{L}}\mathsf{P}|\,\psi\,\rangle. \tag{146b}$$

One now proceeds to formally eliminate $\mathsf{Q}|\,\psi\,\rangle$. Because all operators on the left-hand side of Eq. (146b) are linear, the elimination can be accomplished by introducing the Green's function

$$\widetilde{G}(t;t') \doteq H(t-t') \exp_+\left(-i\int_{t'}^{t} d\bar{t}\, \mathsf{Q}\widetilde{\mathcal{L}}(\bar{t})\mathsf{Q}\right), \tag{147}$$

where the + subscript denotes time ordering.[121] Upon ignoring the initial contribution from $\mathsf{Q}|\,\psi\,\rangle$,

[120] Additional remarks about the projection-operator technique are given in Chap. XIV of van Kampen (1981).

[121] Time ordering of products or a function of a time-dependent operator $\mathsf{A}(t)$, noncommutative at different times, is denoted by a + subscript and is defined by arranging the operator from left to right

which can usually be argued to phase-mix away, one finds

$$0 = \partial_t \mathsf{P}|\,\psi\,\rangle + [\mathsf{P}\widetilde{\mathcal{L}}(t)\mathsf{P}]\mathsf{P}|\,\psi\,\rangle + \int_0^t \mathrm{d}t'\,[\mathsf{P}\mathcal{L}(t)\mathsf{Q}\widetilde{G}(t;t')\mathsf{Q}\mathcal{L}(t')\mathsf{P}]\mathsf{P}|\,\psi\,\rangle(t'). \tag{148}$$

This equation can be slightly simplified by splitting $\widetilde{\mathcal{L}}$ into its mean and fluctuating parts, $\widetilde{\mathcal{L}} = \widehat{\mathcal{L}} + \delta\mathcal{L}$, and noting that $\mathsf{PQ} = \mathsf{QP} = 0$. Thus

$$[\partial_t + \mathrm{i}\widehat{\mathcal{L}}(t)]\mathsf{P}|\,\psi\,\rangle + \int_0^t \mathrm{d}t'\,\Sigma^{\mathrm{nl}}(t;t')\mathsf{P}|\,\psi\,\rangle(t') = 0, \quad \Sigma^{\mathrm{nl}}(t;t') \doteq \mathsf{P}\delta\mathcal{L}(t)\mathsf{Q}\widetilde{G}(t;t')\mathsf{Q}\delta\mathcal{L}(t')\mathsf{P}. \tag{149a,b}$$

The structure of Eq. (149a) is formally identical to that of Eq. (133). Nevertheless, whereas the kernel Σ^{nl} of the Bourret approximation is known explicitly, being derived from perturbation theory, the general representation (149b) is highly formal and intractable because of the presence of the Q operator in Eq. (147). Nonperturbative approximations to formula (149b) are very difficult to obtain, and little has been done. In the resonance-broadening theory to be described in Sec. 4.3 (p. 108), the $\widetilde{G}$ in Eq. (149b) is approximated (Weinstock, 1969) by the mean response function G, which is Green's function for the left-hand side of Eq. (149a). That procedure recovers the structure (142b) of a passive problem (recall that we assumed linear dynamics), but leaves little hint about how to properly treat self-consistency or successively improve the approximation in a convergent way.[122]

3.9.12 Approximants based on orthogonal polynomials

The cumulant description forms the basis for the most systematic formal approach to the closure problem that has yet been invented; see Sec. 6 (p. 146). Nevertheless, that truncated cumulant expansions may correspond to ill-behaved PDF's is very troubling and led Kraichnan (1985) to discuss alternate representations based on the orthonormal polynomials $p_n(x)$ that obey $\int \mathrm{d}x\, w(x) p_n(x) p_{n'}(x) = \delta_{n,n'}$, where $w(x)$ is a positive definite weight (e.g., a Gaussian) with $\int \mathrm{d}x\, w(x) = 1$. Then the PDF $\rho(x)$ has the approximants $\rho_N(x) = w(x)\sum_{n=0}^N b_n p_n(x)$, where $b_n = \langle p_n(x)\rangle$ and is constructed from the moments of order less than or equal to n. Under appropriate conditions discussed in detail by Kraichnan, the ρ_N converge in mean square as $N \to \infty$. They always exist and yield uniformly convergent approximants to the characteristic function. The theory of such approximants is central to Kraichnan's program of statistical decimation (Sec. 7.5, p. 197).

3.9.13 Summary of formal closure techniques

We have now been introduced to a variety of formal closure and statistical approximation techniques, including the Bourret approximation and quasilinear theory (Sec. 3.9.2, p. 78); exact solution of model problems, including random-coupling models (Sec. 3.9.3, p. 80); cumulant discard (Sec. 3.9.4, p. 80); propagator and vertex renormalizations [Secs. 3.9.7 (p. 83) and 3.9.8 (p. 85)] *via* summation of perturbation theory (Sec. 3.9.5, p. 81); Markovian approximations (Sec. 3.9.9, p. 87); Padé approximants (Sec. 3.9.10, p. 87); projection operator methods (Sec. 3.9.11, p. 88); and approximants based on orthogonal polynomials (Sec. 3.9.12, p. 89). Some of those can be unified with

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in order of descending time. Thus $[\mathsf{A}(t)\mathsf{A}(t')]_+ = H(t-t')\mathsf{A}(t)\mathsf{A}(t') + H(t'-t)\mathsf{A}(t')\mathsf{A}(t)]$. The solution of $\dot{\boldsymbol{x}}(t) + \mathsf{A}(t)\cdot\boldsymbol{x}(t) = \mathbf{0}$ is $\exp_+\!\big(-\int_0^t \mathrm{d}\overline{t}\,\mathsf{A}(\overline{t})\big)\cdot\boldsymbol{x}(0)$ for $t \geq 0$.

[122] For situations involving short autocorrelation times, continued-fraction representations may be useful (Mori, 1965a).

the aid of the direct-interaction approximation, which is treated in depth in Sec. 5 (p. 126). In Sec. 6 (p. 146) I show how techniques of quantum field theory can be used to elegantly derive propagator and vertex renormalizations appropriately generalized for self-consistent problems and including the DIA as a special case. First, however, I describe in the next section some of the historical development of statistical closures in plasma physics. That is interesting in its own right, and it also provides useful motivations for the more formal derivations to follow in subsequent sections.

4 HISTORICAL DEVELOPMENT OF STATISTICAL THEORIES FOR PLASMA PHYSICS

Modern work on statistical theories of plasma turbulence has focused on the direct-interaction approximation (Sec. 5, p. 126) and its Markovian relatives (Sec. 7.2, p. 182). Historically, however, the field evolved quite differently. Although the DIA for Vlasov plasma was proposed by Orszag and Kraichnan (1967) shortly after the pioneering paper by Dupree (1966) on resonance-broadening theory, it and many other developments in neutral-fluid turbulence theory were ignored for about a decade in favor of more physically based, less mathematically systematic descriptions. It is instructive to trace the development of various early statistical approximations in plasma physics. Some of those, such as quasilinear theory (QLT) and weak-turbulence theory (WTT), are well grounded in regular perturbation theory and are essentially proper subsets of the DIA valid in certain limited regimes of validity; others, such as resonance-broadening theory (RBT) and the clump algorithm, are more physically motivated and difficult to classify. After discussions of those four approaches in the present section, I shall then devote in Sec. 5 considerable space to the DIA, a central theme of this article.

4.1 Quasilinear theory

"It will be shown that the development in the non-linear regime for certain types of unstable modes can be followed in considerable detail for long times. This is illustrated for unstable electron-plasma oscillations. The result is that these waves, which are initially unstable, grow in a short time to an equilibrium spectrum . . . . The limiting of these waves . . . is a result of a diffusion in the velocity distribution due to non-linear effects . . . ." — ***Drummond and Pines (1962).***

The basic quasilinear approximation for passive advection has already been described in Sec. 3.9.2 (p. 78). However, that brief and formal discussion does not adequately capture the rich physical processes underlying the quasilinear approximation for plasmas, nor does it convey the enormous influence that quasilinear arguments had on the historical development of the field of plasma turbulence.

4.1.1 The basic equations of "strict" Vlasov quasilinear theory

As described in Sec. 2.2.2 (p. 27), the starting point for a theory of Vlasov turbulence is the Klimontovich equation (23) in the limit $\epsilon_p \to 0$. The average of that equation leads to Eq. (25), which is always retained exactly in formal statistical theories. The fluctuations obey exactly

$$D\delta N/Dt + \delta\boldsymbol{E}\cdot\boldsymbol{\partial}f = -\boldsymbol{\partial}\cdot(\delta\boldsymbol{E}\,\delta N - \langle\delta\boldsymbol{E}\,\delta N\rangle); \tag{150}$$

statistical closures are defined by the treatment of the nonlinear terms on the right-hand side. By definition, in "strict" QLT the right-hand side of Eq. (150) is neglected altogether. Orszag and Kraichnan (1967) noted that because the left-hand side of Eq. (150) is a linear operator on δN, an amplitude equation for the random variable δN is solved in each realization, so the fluctuation power spectrum must be realizable. They also emphasized that this argument does not guarantee the positive-semidefiniteness of f; see the last paragraph of Sec. 3.5.3 (p. 64). Even more importantly, they noted that strict QLT is time reversible, which they found to be unacceptable. Irreversibility can be restored by retaining and appropriately approximating the right-hand side of Eq. (150), as will be described in great detail in Sec. 5 (p. 126) in the context of the DIA. In the next several sections, however, I shall approach the problem more heuristically. In Sec. 4.1.2 I review the content of the theory in the passive limit in which the electric fields are specified with frozen intensities. It will be seen that the implicit presence of the nonlinear fluctuation terms is crucial for a sensible theory. Then in Sec. 4.1.3 (p. 95) I describe the additional complications that emerge in the fully self-consistent problem.

4.1.2 Passive quasilinear theory

Consider first the passive *stochastic acceleration* problem of a test particle moving (in 1D, for simplicity) with velocity v in a specified, random electric field $\widetilde{E}(x,t)$, which produces a random acceleration $\widetilde{a} \doteq q\widetilde{E}/m$. If this field is assumed to be rapidly varying in time on a Lagrangian timescale τ_{ac}, then the characteristic acceleration over a time τ_{ac} will be $\langle \delta a^2 \rangle^{1/2}$. One then knows from elementary short-time Langevin dynamics and the arguments of Sec. 3.2 (p. 48) that for times greater than τ_{ac} v-space diffusion will ensue with diffusion coefficient

$$D_v = \langle \delta a^2 \rangle \tau_{\mathrm{ac}}. \tag{151}$$

The quasilinear theory applies when a linearly derived autocorrelation time $\tau_{\mathrm{ac}}^{\mathrm{lin}}$ can be used.

The appropriate estimate or formal calculation of $\tau_{\mathrm{ac}}^{\mathrm{lin}}$ depends on the ultimate origin of $\widetilde{E}$. If it is merely specified as a random Gaussian time series, $\widetilde{E} = \widetilde{E}(t)$, then $\tau_{\mathrm{ac}}^{\mathrm{lin}}$ is just the given autocorrelation time as in Sec. 3.3 (p. 52). Of more interest to subsequent discussions of self-consistency is the case in which $\widetilde{E}$ is generated by a plasma, which is a dielectric medium with linear dielectric function $\mathcal{D}^{\mathrm{lin}}(k,\omega)$. Then $\widetilde{E}$ can be assumed to consist of a sum of randomly phased propagating waves with linear dispersion relation $\omega = \Omega_{\boldsymbol{k}}$, phase velocity $v_{\mathrm{ph}} \doteq \Omega_{\boldsymbol{k}}/k$, and group velocity $v_{\mathrm{gr}} \doteq \partial\Omega_{\boldsymbol{k}}/\partial k$. A physical argument (to be supported by formal mathematics below) that leads to $\tau_{\mathrm{ac}}^{\mathrm{lin}}$ is then as follows. The wave field can be considered to be a *wave packet* moving at speed v_{gr}. A test particle initially comoving with the wave packet will feel its effects only for a time $(|v - v_{\mathrm{gr}}|\Delta k)^{-1}$, where Δk is the width of the wave packet. Furthermore, in order that the test particle feel a *secular* kick[123] it must be in resonance with the waves of the wave packet: $v = v_{\mathrm{ph}}(\overline{k})$ for, say, the central wave number $\overline{k}$ of the packet. Thus for a wave field one has

$$\tau_{\mathrm{ac}}^{\mathrm{lin}} \sim [|v_{\mathrm{ph}}(\overline{k}) - v_{\mathrm{gr}}(\overline{k})|\Delta k]^{-1}. \tag{152}$$

[123] A useful discussion of the roles of resonant and nonresonant particles in quasilinear and resonance-broadening theory was given by Tetreault (1976).

To be more quantitative, one may begin with the formula

$$D_v = \int_0^\infty d\tau \, \langle \delta a\big(x(t), t\big) \delta a\big(x(t-\tau), t-\tau\big) \rangle. \tag{153}$$

Because such Taylor-like formulas arise repeatedly and are surprisingly subtle, I shall attempt to be relatively systematic in reducing Eq. (153) to the usual quasilinear formula. Consider a homogeneous ensemble of particles moving in a large box of length L. It is natural to introduce the discrete Fourier transform (Appendix A, p. 262) with wave numbers $k = n\,\delta k$, $\delta k \doteq 2\pi/L$ being the mode spacing and fundamental wave number. Thus

$$D_v = \int_0^\infty d\tau \sum_{p,q} \langle \mathrm{e}^{\mathrm{i}(p+q)x - \mathrm{i}q\Delta x(-\tau)} \delta a_p(0) \delta a_q(-\tau) \rangle, \tag{154}$$

where $x \equiv x(t)$ and $\Delta x(-\tau \mid x, t) \doteq x - x(t-\tau)$.[124] In general, evaluation of the ensemble average in Eq. (154) is extremely difficult because of hidden statistical dependences. Namely, Δx depends on (i) the Fourier amplitudes δE_k, which are random variables in a general turbulence problem; and (ii) the final condition $x(t)$, which is also random. That is, one does not know the joint PDF $f[\delta E, \Delta x, x]$. However, under the assumption of statistical homogeneity the joint PDF *conditional* on x is independent of x: $f[\delta E, \Delta x, x] = f[\delta E, \Delta x \mid x] L^{-1}$. The average over x may now be performed, yielding a factor of $L\delta_{p+q}$. Thus

$$D_v = \int_0^\infty d\tau \sum_q \langle \mathrm{e}^{-\mathrm{i}q\Delta x(-\tau)} \delta a_q^*(0) \delta a_q(-\tau) \rangle, \tag{155}$$

where the average is now interpreted in the sense of the conditional distribution.

As I have remarked, in a general turbulence problem calculation of the average required in Eq. (155) is exceedingly difficult because Δx depends on the δa_q's. It is tempting, although not justifiable for strong turbulence, to factor the average according to

$$\langle \mathrm{e}^{-\mathrm{i}q\Delta x(-\tau)} \delta a_q^*(0) \delta a_q(-\tau) \rangle \approx \langle \mathrm{e}^{-\mathrm{i}q\Delta x(-\tau)} \rangle \langle \delta a_q^*(0) \delta a_q(-\tau) \rangle. \tag{156}$$

This *independence hypothesis* was discussed by Weinstock (1976) in the fluid context. However, for the special case of test-particle motion in a wave field whose Fourier amplitudes are *completely specified*, the independence hypothesis is exact since the δa_q's can be taken to be *statistically sharp* in both amplitude and phase. The resulting stochastic acceleration problem is still interesting and makes sense from the point of nonlinear dynamics, since test-particle diffusion can ensue if certain *stochasticity criteria* are satisfied [see Appendix D (p. 279)]. In this case one has

$$D_v = \int_0^\infty d\tau \sum_q \langle \mathrm{e}^{-\mathrm{i}q\Delta x(-\tau)} \rangle \delta a_q^*(0) \delta a_q(-\tau). \tag{157}$$

[124] Although *a priori* it might appear more natural to express the random orbit $x(t)$ in terms of the initial condition $x(0)$, I have chosen instead to express it in terms of the final condition $x(t)$ in order to achieve a closer correspondence with more formal developments to be given later (Appendix E, p. 281).

In the passive quasilinear theory of test-particle diffusion, two further assumptions are made: (i) The particle position is assumed to be a Gaussian random variable[125] whose dispersion obeys the short-time ($\nu = 0$) limit of the Langevin equations (3.2). (ii) The Fourier amplitudes describe oscillatory waves: $\delta E_q(t) = \delta E_q \exp(-\mathrm{i}\Omega_q t)$. Because of assumption (i), cumulant expansion truncates exactly at second order [cf. Eq. (93b)], leading to[126]

$$\langle \mathrm{e}^{-\mathrm{i}q\Delta x(-\tau)} \rangle = \exp[-\mathrm{i}q\langle \Delta x \rangle(-\tau) - \tfrac{1}{2}q^2\langle (\delta\Delta x)^2(-\tau) \rangle] = \exp(-\mathrm{i}qv\tau - \tfrac{1}{3}q^2 D_v \tau^3). \qquad (158\text{a,b})$$

This result manifests the effects of *orbit diffusion* by introducing the diffusion time[127]

$$\tau_{d,q} \doteq (q^2 D_v)^{-1/3}. \qquad (159)$$

Thus

$$D_v = \int_0^\infty d\tau \sum_q |\delta a_q|^2 \mathrm{e}^{-\mathrm{i}(qv-\Omega_q)\tau} \mathrm{e}^{-\frac{1}{3}(\tau/\tau_{d,q})^3}. \qquad (160)$$

The most naive evaluation of formula (160) now proceeds as follows. First, the orbit-diffusion effect [contained in the nonlinear terms on the right-hand side of Eq. (150)] is ignored. (In strict QLT the particles are assumed to move on linear trajectories.) The time integral is then performed, leading to

$$D_v(v) = \pi \sum_q |\delta a_q|^2 \delta(qv - \Omega_q). \qquad (161)$$

One often sees the quasilinear diffusion coefficient written in this form. However, formula (161) is ill defined, as it mixes a Dirac delta function (which expects a continuously varying argument) with a quantized wave-number spectrum. Thus as a function of velocity $D_v(v)$ is predicted to be either 0 or ∞. Usually this difficulty is "cured" without discussion by letting $L \to \infty$ and introducing continuous Fourier transforms:

$$D_v = \pi \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{2\pi} \mathcal{A}_q \delta(qv - \Omega_q) = 2\frac{\pi(\mathcal{A}_q/2\pi)}{|v - \partial\Omega_q/\partial q|_{q=q_0}}, \qquad (162\text{a,b})$$

where the multiplier 2 accounts for the two positive and negative solutions for the resonant wave number $q_0(v)$ satisfying $v = \Omega_{q_0}/q_0$. One can see here the comforting appearance of the relative velocity between wave packet and particle discussed previously. Upon multiplying Eq. (162b) by the total width Δk of the spectrum, defined such that $\Delta k\, \mathcal{A}_{q_0} = \mathcal{A}$, one recovers precisely Eq. (151).

Thus several dubious manipulations (neglect of orbit diffusion, and passage to the continuum limit) have in combination led to a reasonable result. In fact, a much more satisfying picture is achieved

[125] This is the assumption that underlies Dupree's resonance-broadening theory; see Sec. 4.3 (p. 108) and Benford and Thomson (1972). Subtleties involving the Gaussian assumption were discussed by Pesme (1994).
[126] Several authors have suggested that this calculation of the nonlinear orbit correction is in error, and that the left-hand side of Eq. (158b) should be replaced by a form depending on t as well as τ; see, for example, Salat (1988) and Ishihara et al. (1992). In fact, Eq. (158b) is correct; see vanden Eijnden (1997) and Appendix E.1.2 (p. 284) for further discussion.
[127] A wave-number-independent diffusion time τ_d can be defined by replacing q by a characteristic wave number $\overline{k}$: $\tau_d \doteq (\overline{k}^2 D_v)^{-1/3}$.

by retaining the orbit diffusion. It can be demonstrated that the size of the orbit diffusion is always sufficient to smooth the integrand of Eq. (160) such that wave-number summation may be replaced by integration, even for a quantized spectrum,[128] whenever the Chirikov criterion for stochasticity is satisfied. That criterion is derived in Appendix D (p. 279), where the detailed justification of the continuum limit is also given. The situation is illustrated in Fig. 12. At short times the Lagrangian acceleration correlation function decays on the time scale $\tau_{\mathrm{ac}}^{\mathrm{lin}}$. In a discrete spectrum the linearly computed $C(\tau)$ would exhibit a quasirecurrence on the timescale $\tau_r \sim N\tau_{\mathrm{ac}}^{\mathrm{lin}}$. However, stochasticity induces a nonlinear envelope with timescale τ_d that is always sufficient to eliminate the recurrence, and also smooths the wave-number summations enough to justify the continuum approximation. The quasilinear regime is defined by $\tau_{\mathrm{ac}}^{\mathrm{lin}} < \tau_d$, so τ_d does not enter the final result for the diffusion coefficient.

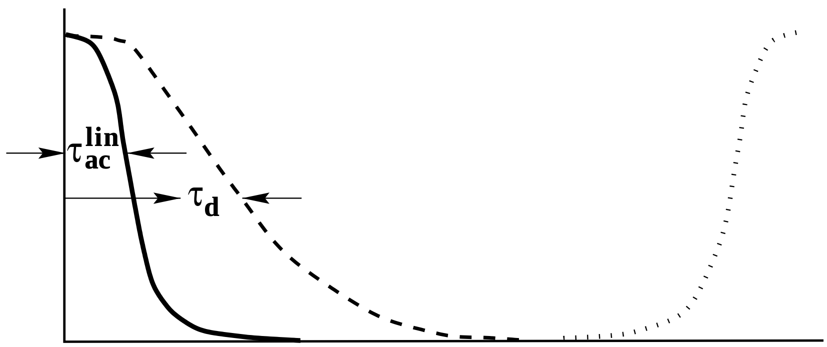

Fig. 12. Linear response (solid and dotted lines) and nonlinear envelope (dashed line, with width τ_d) in the quasilinear regime. The solid curve is the actual response, with width $\tau_{\mathrm{ac}}^{\mathrm{lin}}$.

In the previous discussion the introduction of τ_d was done somewhat heuristically. A more detailed and systematic treatment is afforded by the DIA, discussed in Sec. 5 (p. 126). One of the principle virtues of that approximation is that it permits a smooth transition between the quasilinear and strong-turbulence regimes. One should also note the work of Dewar and Kentwell (1985), who attempted to provide a precise definition and theory of the nonlinear envelope of the correlation function.

For cross-field diffusion in a strong magnetic field, particles move with the $\boldsymbol{E} \times \boldsymbol{B}$ velocity and should undergo a *spatial* random walk $\langle \delta x^2 \rangle = 2D_\perp t$. The expression analogous to Eq. (161) is then

$$D_\perp = \int_0^\infty d\tau \sum_{\boldsymbol{q}} \langle \delta V_E^2 \rangle_{\boldsymbol{q}} \mathrm{e}^{-\mathrm{i}(q_\parallel v_\parallel - \Omega_{\boldsymbol{q}})\tau} \mathrm{e}^{-\tau/\tau_{d\perp,q}}, \tag{163}$$

where the perpendicular diffusion time is $\tau_{d\perp,q} \doteq (q_\perp^2 D_\perp)^{-1}$. One can estimate

$$D_\perp \sim \langle \delta V_E^2 \rangle \tau_{\mathrm{ac}}, \tag{164}$$

where τ_{ac} is the shorter of a characteristic parallel time like Eq. (152) and the characteristic perpendicular diffusion time $\tau_{d\perp} \doteq \tau_{d\perp,\overline{k}_\perp}$. Such diffusion coefficients arose in research on the 3D guiding-center plasma; a good discussion with earlier references was given by Vahala (1974).

In systems with complicated geometries, the appropriate space in which diffusion occurs is best addressed by inquiring about the adiabatic invariants that are destroyed by resonant interactions.

[128] A brief qualitative remark that nonlinearity is necessary for smoothing was made by Kaufman (1972a).

Kaufman (1972a) formulated the appropriate quasilinear theory for the important practical case of an axisymmetric torus. A generalization of Kaufman's formalism was used by Mynick (1988) in his discussion of a Balescu–Lenard-like operator for turbulence.

4.1.3 Self-consistent quasilinear theory

Now consider the self-consistent Vlasov equation

$$\partial_t f + \boldsymbol{v} \cdot \boldsymbol{\nabla} f + \boldsymbol{E} \cdot \boldsymbol{\partial} f = 0. \tag{165}$$

The electric field is obtained from Poisson's equation. Particles are no longer test particles; they influence the field. The physical picture is that the waves (weakly damped collective oscillations) are supported by the nonresonant particles. The wave–particle resonance then transfers momentum and energy between the waves and the resonant particles. That momentum and energy are not only electromagnetic; they also comprise the mechanical momentum and energy of the nonresonant particles that participate in the wave motions. Because all of those motions are described by the same distribution function f, one must ensure that any approximate kinetic equation for f preserves the proper momentum and energy balances. This problem does not arise in the test-particle case, in which the test particles can absorb an unlimited amount of momentum and energy from the fixed bath of turbulence (provided, of course, that the particles do not diffuse from the resonant region).

Various procedures have been used to describe the self-consistent problem. Considerable confusion and controversy arose in the early days [the literature was nicely reviewed by Burns and Knorr (1972)] because of difficulties with the proper treatment of growing or damped waves. Mathematically, the problem boils down to the proper way of treating and interpreting the familiar resonance function $\delta(\omega - \boldsymbol{k} \cdot \boldsymbol{v})$, which has already appeared in the theory of steady-state fluctuations. Note that the steady-state quasilinear diffusion coefficient D_v satisfies

$$D_v \propto \pi\delta(\Omega_{\boldsymbol{k}} - \boldsymbol{k} \cdot \boldsymbol{v}) = \mathrm{Re}[-\mathrm{i}(\Omega_{\boldsymbol{k}} - \boldsymbol{k} \cdot \boldsymbol{v} + \mathrm{i}\epsilon)]^{-1}, \tag{166}$$

where $\Omega_{\boldsymbol{k}}$ is real and $\epsilon > 0$.

Now suppose one asserts that for weakly stable or unstable waves it is valid to replace ϵ by the linear growth rate $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$. Since $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$ is finite, not infinitesimal, one would then obtain $D_v \propto \mathrm{Re}\big([-\mathrm{i}(\Omega_{\boldsymbol{k}} - \boldsymbol{k} \cdot \boldsymbol{v} + \mathrm{i}\gamma_{\boldsymbol{k}}^{\mathrm{lin}})]^{-1}\big) = \gamma_{\boldsymbol{k}}^{\mathrm{lin}}/[(\Omega_{\boldsymbol{k}} - \boldsymbol{k} \cdot \boldsymbol{v})^2 + (\gamma_{\boldsymbol{k}}^{\mathrm{lin}})^2]$. Unfortunately, this is *negative* for damped waves. Since it is difficult to interpret a negative diffusion coefficient, people were somewhat confounded; it was not uncommon to assert that quasilinear theory did not apply to damped waves (Vahala and Montgomery, 1970). The paradox is that the estimates of Appendix D (p. 279) for the stochasticity criterion make no reference to the sign of the growth rate; they merely assume that the wave field is quasistationary.

In retrospect the resolution of this paradox is easy. It is simply not correct to replace ϵ by $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$. The presence of $\mathrm{i}\epsilon$ in Eq. (166) reflects causality of the particle response, so any generalization of ϵ must always be positive. However, formula (166) is valid only in steady state. Otherwise, in computing the particle dispersion one cannot reduce the two time integrals over t and t' to a single one over τ by using the assumption of statistical stationarity. The calculation is mathematically more involved.

For QLT the clearest exposition of the proper procedure was first given by Kaufman (1972b), who performed a multiple-timescale analysis of Eq. (165). See also Fukai and Harris (1972) for a related discussion based partly on a quantum-mechanical derivation. For the generalization of the procedure to include nonlinear effects in weak-turbulence theory, see Appendix G (p. 288).

In Kaufman's derivation one allows for a slow time dependence in the Fourier amplitudes due to linear growth or damping: $\mathrm{d}\ln|\delta E_k|/\mathrm{d}t \doteq \gamma_{\boldsymbol{k}}^{\mathrm{lin}}$, with $|\gamma_{\boldsymbol{k}}^{\mathrm{lin}}/\Omega_{\boldsymbol{k}}| \ll 1$. Here $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$ may be either positive or negative. One assumes and is generally able to justify that the mean distribution f evolves on a timescale slower than both $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$ and $\Omega_{\boldsymbol{k}}$. As usual, Eq. (165) is split into its mean and fluctuating parts. Upon assuming homogeneous statistics, one obtains

$$\partial_t\langle f\rangle = (q/m)\langle\delta\boldsymbol{E}\cdot\partial_{\boldsymbol{v}}\delta f\rangle, \quad (\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})\delta f = -(q/m)\delta\boldsymbol{E}\cdot\partial_{\boldsymbol{v}}\langle f\rangle + O(\delta\boldsymbol{E}\,\delta f). \qquad \text{(167a,b)}$$

In strict QLT the nonlinear terms on the right-hand side of Eq. (167b) are neglected.[129] One can solve for δf by a Green's-function technique. The transient term is neglected by a phase-mixing argument.[130] The solution involves the electric field at a retarded time τ, where τ will turn out to be $\lesssim \tau_{\mathrm{ac}}^{\mathrm{lin}}$, the timescale for the Lagrangian quasilinear correlation function. The key to the method is to be careful about that time dependence: $\delta\boldsymbol{E}_{\boldsymbol{k}}(t-\tau) \approx \delta\boldsymbol{E}_{\boldsymbol{k}}(t) - \tau\,\partial_t\delta\boldsymbol{E}_{\boldsymbol{k}} = (1-\gamma_{\boldsymbol{k}}^{\mathrm{lin}}\tau)\delta\boldsymbol{E}_{\boldsymbol{k}}(t)$. When the solution for δf is integrated over velocity and inserted into Poisson's equation, one is led to the self-consistency condition

$$\mathcal{D}^{\mathrm{lin}}(\boldsymbol{k},\Omega_{\boldsymbol{k}}) = -\mathrm{i}\gamma_{\boldsymbol{k}}^{\mathrm{lin}}\,\partial\mathcal{D}^{\mathrm{lin}}(\boldsymbol{k},\Omega_{\boldsymbol{k}})/\partial\Omega_{\boldsymbol{k}}, \qquad (168)$$

where the linear Vlasov dielectric function is

$$\mathcal{D}^{\mathrm{lin}}(\boldsymbol{k},\omega) \doteq 1 - \sum_s \left(\frac{\omega_{ps}^2}{k^2}\right)\int_0^\infty d\tau \int \mathrm{d}\boldsymbol{v}\,\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v} - \omega - \mathrm{i}\epsilon)\tau]\mathrm{i}\boldsymbol{k}\cdot\frac{\partial f}{\partial \boldsymbol{v}}. \qquad (169)$$

[Formula (169) reduces to Eq. (33).] Under the quasilinear assumption $|\gamma/\Omega| \ll 1$, Eq. (168) reduces to the results

$$0 = \operatorname{Re}\mathcal{D}^{\mathrm{lin}}(\boldsymbol{k},\Omega_{\boldsymbol{k}}), \quad \gamma_{\boldsymbol{k}}^{\mathrm{lin}} \approx -\frac{\operatorname{Im}\mathcal{D}^{\mathrm{lin}}(\boldsymbol{k},\Omega_{\boldsymbol{k}})}{\partial\operatorname{Re}\mathcal{D}^{\mathrm{lin}}/\partial\Omega_{\boldsymbol{k}}}, \qquad \text{(170a,b)}$$

which are familiar from linear wave theory. Equation (170a) determines the real frequency, and Eq. (170b) determines the growth rate of the $\boldsymbol{k}$th mode. Because $(\omega + \mathrm{i}\epsilon)^{-1} = \mathrm{P}(\omega^{-1}) - \mathrm{i}\pi\delta(\omega)$, reference to Eq. (33) shows the well-known result that the waves (mode frequency) are supported by the nonresonant particles whereas the growth is driven by the resonant particles.

Upon inserting the solution for δf into Eq. (167a), one obtains

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\int_0^{t-t_0}\mathrm{d}\tau\,\mathsf{C}(\boldsymbol{x},t;\boldsymbol{x}-\boldsymbol{v}\tau,t-\tau)\right)\cdot\frac{\partial}{\partial \boldsymbol{v}}f(\boldsymbol{v},t-\tau), \qquad (171)$$

where $\mathsf{C}(\boldsymbol{x},t;\boldsymbol{x}',t') \doteq \langle\delta\boldsymbol{a}(\boldsymbol{x},t)\delta\boldsymbol{a}(\boldsymbol{x}',t')\rangle$. The correlation function appearing in Eq. (171) is thus the Lagrangian function taken along the linear trajectory, in agreement with the calculations presented in

[129] Those terms include the diffusive effects that lead to resonance broadening. From the discussion in Appendix D.2 (p. 280), the orbit diffusion is important to justify smoothing the wave-number integrations. Here we shall pass directly to the continuum limit.

[130] The contribution to the transient *potential*, which arises from a velocity integration, can easily be shown to phase mix on the timescale $(kv_t)^{-1}$. Ballistic contributions to δf itself are more problematical and are considered in some more involved theories (Kadomtsev and Pogutse, 1971).

Sec. 4.1.2 (p. 91); Eq. (171) is the Bourret approximation (133) for this problem. By the arguments of that section, $\mathsf{C}(\tau)$ will decay on the $\tau_{\text{ac}}^{\text{lin}}$ timescale. An appropriate statistical evolution equation for f should therefore be coarse-grained in units of time greater than τ_{ac}. Therefore one can replace $f(t-\tau)$ by $f(t)$ to lowest order in τ/t; this is the Markovian assumption. Now introduce a Fourier analysis in space. From the Eulerian amplitudes arises the contribution $\langle\delta\boldsymbol{a}_{\boldsymbol{k}}(t)\delta\boldsymbol{a}_{-\boldsymbol{k}}(t-\tau)\rangle \approx \mathsf{C}_{\boldsymbol{k}}(t) - \frac{1}{2}\tau\,\partial_t\mathsf{C}_{\boldsymbol{k}}$. The first term contributes to Eq. (171) $\partial_{\boldsymbol{v}}\cdot\mathsf{D}(\boldsymbol{v})\cdot\partial_{\boldsymbol{v}}f$, where $\mathsf{D}(\boldsymbol{v}) \doteq \pi\sum_{\boldsymbol{k}}\mathsf{C}_{\boldsymbol{k}}(t)\delta(\Omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot\boldsymbol{v})$ is the same quasilinear diffusion coefficient that was computed in Sec. 4.1.2 (p. 91) on the basis of heuristic random-walk arguments, except here one is allowing for a slow temporal change in the field intensity. The $\partial_t\mathsf{C}_{\boldsymbol{k}}$ term contributes

$$\frac{\partial}{\partial\boldsymbol{v}}\cdot\left[-\frac{1}{2}\sum_{\boldsymbol{k}}\frac{\mathrm{d}\mathsf{C}_{\boldsymbol{k}}}{\mathrm{d}t}\frac{\partial}{\partial\Omega_{\boldsymbol{k}}}\mathrm{P}\left(\frac{1}{\Omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot\boldsymbol{v}}\right)\cdot\frac{\partial f}{\partial\boldsymbol{v}}\right]. \tag{172}$$

Notice that this term involves nonresonant particles and changes sign under time reversal whereas D involves resonant particles and is invariant under time reversal. One is thus motivated to write f as a zeroth-order part F, which includes both the equilibrium and the changes in the resonant particles, plus a nonresonant correction f_{nr}: $f = F + f_{\text{nr}}$, where

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial\boldsymbol{v}}\cdot\mathsf{D}(\boldsymbol{v})\cdot\frac{\partial F}{\partial\boldsymbol{v}}, \qquad \frac{\partial f_{\text{nr}}}{\partial t} = \frac{\partial}{\partial\boldsymbol{v}}\cdot\left[\dots\frac{\mathrm{d}\mathsf{C}_{\boldsymbol{k}}}{\mathrm{d}t}\dots\right]\cdot\frac{\partial F}{\partial\boldsymbol{v}}. \tag{173a,b}$$

If one considers F to be $O(1)$, one can deduce that f_{nr} is $O(\delta E^2)$. Thus it is irrelevant whether one uses f or F on the right-hand side of Eq. (173b). The correction f_{nr} describes nonresonant distortion of Kolmogorov–Arnold–Moser (KAM) surfaces. If one chooses that distortion to vanish at $t = t_0$, one can integrate Eq. (173b) explicitly:

$$f_{\text{nr}}(\boldsymbol{v},t) = -\frac{1}{2}\frac{\partial}{\partial\boldsymbol{v}}\cdot\left[\sum_{\boldsymbol{k}}\widehat{\boldsymbol{k}}\,\mathcal{E}_{\boldsymbol{k}}(t)\,\widehat{\boldsymbol{k}}\frac{\partial}{\partial\Omega_{\boldsymbol{k}}}\mathrm{P}\left(\frac{1}{\Omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot\boldsymbol{v}}\right)\cdot\frac{\partial F}{\partial\boldsymbol{v}}\right], \tag{174}$$

where $\mathcal{E}_{\boldsymbol{k}} \doteq \mathrm{Tr}\,\mathsf{C}_{\boldsymbol{k}}$.

One can show that the nonresonant correction just computed contains the mechanical "sloshing" momentum and energy of the particles in the waves. I demonstrate with the momentum:

$$\mathcal{P}_{\text{nr}} = \sum_s\int\mathrm{d}\boldsymbol{v}\,m\overline{n}\boldsymbol{v}f_{\text{nr}} = \frac{1}{2}\sum_s\omega_p^2\sum_{\boldsymbol{k}}\left(\frac{\boldsymbol{k}}{k^2}\right)\mathcal{E}_{\boldsymbol{k}}\frac{\partial}{\partial\Omega_{\boldsymbol{k}}}\mathrm{P}\int\mathrm{d}\boldsymbol{v}\left(\frac{1}{\Omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot\boldsymbol{v}}\right)\boldsymbol{k}\cdot\frac{\partial f}{\partial\boldsymbol{v}} \tag{175a,b}$$

(after integrating by parts in velocity space). Upon noting the form of the linear dielectric function, one can write this as

$$\mathcal{P}_{\text{nr}} = \frac{1}{8\pi}\sum_{\boldsymbol{k}}\mathcal{E}_{\boldsymbol{k}}\boldsymbol{k}\frac{\partial}{\partial\Omega_{\boldsymbol{k}}}(\mathrm{Re}\,\chi^{\text{lin}}) = \sum_{\boldsymbol{k}}\boldsymbol{k}\,\mathcal{N}_{\boldsymbol{k}}, \tag{176a,b}$$

where $\mathcal{N}_{\boldsymbol{k}} \doteq (\partial\,\mathrm{Re}\,\mathcal{D}^{\text{lin}}/\partial\Omega_{\boldsymbol{k}})(\mathcal{E}_{\boldsymbol{k}}/8\pi)$ is the *wave action*. Expression (176b) is the total wave momentum; recall that the electrostatic field itself carries no momentum. In a similar way, one can show that

$$\sum_s\int\mathrm{d}\boldsymbol{v}\,\frac{1}{2}m\overline{n}v^2 f_{\text{nr}} = \sum_{\boldsymbol{k}}\Omega_{\boldsymbol{k}}\mathcal{N}_{\boldsymbol{k}} - \frac{1}{8\pi}\mathcal{E}_{\boldsymbol{k}} \tag{177a}$$

$$= \text{total wave energy} - \text{electric field energy} \tag{177b}$$
$$= \text{mechanical or sloshing part of wave energy.} \tag{177c}$$

It is now easy to show that the quasilinear theory conserves momentum and energy. One wants to prove that

$$\sum_s \int \mathrm{d}\boldsymbol{v} \begin{pmatrix} m\overline{n}\boldsymbol{v} \\ \frac{1}{2}m\overline{n}v^2 \end{pmatrix} (F + f_{\mathrm{nr}}) + \sum_{\boldsymbol{k}} \frac{\mathcal{E}_{\boldsymbol{k}}}{8\pi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{178}$$

is constant. That is, one must show that, for example,

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_s \int \mathrm{d}\boldsymbol{v}\, (m\overline{n})_s \boldsymbol{v} F_s = -2 \sum_{\boldsymbol{k}} \gamma^{\mathrm{lin}}_{\boldsymbol{k}} \boldsymbol{k} \mathcal{N}_{\boldsymbol{k}}. \tag{179}$$

This follows, after a straightforward integration in velocity space, from the evolution equation for F upon using the definitions of D and γ^{lin}.

Dewar (1973) provided a compelling interpretation of the decomposition $f = F + f_{\mathrm{nr}}$ by introducing the concept of the *oscillation center*. Heuristically, the oscillation center represents the average trajectory of the real particles, the nonresonant sloshing being subtracted out. More formally, one defines "a canonical transformation such that only the resonant part of the wave–particle interaction is left in the new interaction Hamiltonian." Some discussion of such averaging transformations is given in Appendix C.1.5 (p. 273). The quasilinear diffusion equation emerges in the new coordinates whereas the sloshing momentum and energy are contained in the transformation between the oscillation-center and particle coordinates.[131] Elegant generalizations and applications of this technique were made by Johnston (1976), Cary and Kaufman (1977, 1981), and others.

A compelling and elegant experimental verification of the quasilinear theory was performed by Roberson et al. (1971) [for more details, see Roberson and Gentle (1971)]. However, carefully designed computer experiments by Adam et al. (1979) predicted discrete wave-number spectra that were not the smoothly varying functions of $\boldsymbol{k}$ that simple QLT would suggest. Those observations launched a difficult literature questioning the foundations of self-consistent QLT; for further discussion, see Sec. 6.5.6 (p. 180).

4.2 Weak-turbulence theory

"[W]e assert that in the simple situation of weakly interacting dispersive waves a sequence of closures can be obtained in a systematic and consistent manner." — *Benney and Newell (1969).*

The weak-turbulence theory goes one step beyond quasilinear theory in that it (perturbatively) incorporates nonlinear effects on the fluctuations. The result is usually written as the *wave kinetic*

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[131] The procedure is precisely defined only for the unrenormalized quasilinear limit in which resonant and nonresonant effects can be cleanly separated. Dewar (1976) made an ambitious attempt to extend the techniques to turbulence theory. Later Dewar and Kentwell (1985) used the oscillation-center apparatus to discuss the determination of the nonlinear envelope τ_d introduced in Sec. 4.1.2 (p. 93).

equation (WKE) that advances the *wave action density*. Fundamental references in the context of neutral fluids include the works of Hasselmann (1966) and Benney and Newell (1969). A very incomplete list of early references on weak plasma turbulence theory includes Rogister and Oberman (1968, 1969), Sagdeev and Galeev (1969), Davidson (1972), Tsytovich (1977), and Galeev and Sagdeev (1979). The review by Porkolab and Chang (1978) of nonlinear plasma wave effects covers WTT, including experimental verifications. A recent treatise on the general weak-turbulence problem is by Zakharov et al. (1992). The derivation of the WKE (and more general spectral balance equations) in the presence of weak variations in space and/or time is discussed in Appendix F (p. 286).

Many applications of WTT have been studied, but those are largely beyond the scope of this article, which is focused on fundamental principles. Nevertheless, the general structure of the WKE is very instructive, as it demonstrates important symmetries and suggests interpretations that are preserved in more complete renormalizations. Algorithmically, the lowest-order WKE can be simply derived from a Gaussian Ansatz applied to the four-point correlations of the wave amplitudes. When and why such an Ansatz is justified is a more difficult issue. Therefore before introducing the general weak-turbulence apparatus, I digress in the next subsection to discuss the onset of stochasticity for an ensemble of interacting waves.

4.2.1 Preamble: Random three-wave interactions

As I discussed in Sec. 1.2 (p. 10), a principal difference between neutral fluids and plasmas is the plethora of linear waves supported by the latter. In weakly turbulent plasmas the fundamental entities are waves rather than the eddies of strongly turbulent fluids.

Relatively systematic formulations can be given of the weak interaction of a collection of waves. A Hamiltonian action–angle formalism is convenient. For three weakly interacting waves with phases $\{\theta_0, \theta_1, \theta_1\} \equiv \boldsymbol{\theta}$ and actions $\boldsymbol{J}$ obeying the resonance $\Omega_0 \approx \Omega_1 + \Omega_2$, one can construct a canonical transformation $(\boldsymbol{J}, \boldsymbol{\theta}) \to (\boldsymbol{\mathcal{J}}, \boldsymbol{\Theta})$, where $\Theta_0 = \theta_0 - \theta_1 - \theta_2$, $\Theta_{1,2} = \theta_{1,2}$, $\mathcal{J}_0 = J_0$, and $\mathcal{J}_{1,2} = J_{1,2} + J_0$. One is led to the Hamiltonian $K(\mathcal{J}, \Theta) \doteq \Delta\Omega\, \mathcal{J} + \mathcal{L}(\mathcal{J}) \cos\Theta$, where Θ is the phase difference between the three waves, $\Delta\Omega \doteq \Omega_0 - \Omega_1 - \Omega_2$ is the associated frequency mismatch, $\mathcal{J} = J_0$, and $\mathcal{L}(\mathcal{J})$ is a known coupling coefficient. Being time independent, K is conserved; it describes an integrable system with one degree of freedom. This integrability of the resonant three-wave interaction strongly contrasts with the generic result for three coupled fields in the presence of dissipation; see, for example, the Lorenz system of equations (Lorenz, 1963), which supports chaos.

Because K does not depend on θ_1 or θ_2, the new actions $\mathcal{J}_{1,2}$ are conserved. This is the cleanest statement of the *Manley–Rowe relations* $\dot{J}_1 = -\dot{J}_0$ and $\dot{J}_2 = -\dot{J}_0$, which state that during a three-wave interaction in which wave 0 decays into two others, for each quantum of action lost by wave 0 one quantum of action appears in *each* of the other two waves.

Given the three conserved quantities $\{K, \mathcal{J}_1, \mathcal{J}_2\}$, the Hamiltonian equations that follow from $K(\mathcal{J}, \Theta)$ can be integrated explicitly in terms of elliptic functions; a detailed discussion was given by Sagdeev and Galeev (1969) [see also Davidson (1972)] (those authors did not use the Hamiltonian formalism). For present purposes the most important qualitative result is that when wave 0 has the highest initial excitation, a strong transfer of energy occurs between the waves on a characteristic time τ_θ that can be calculated.

When more than three waves are present, integrability is generically destroyed according to standard results of nonlinear dynamics [although see Meiss (1979)]. It is possible to develop a Chirikov-like criterion for the onset of wave stochasticity (Zaslavskiĭ and Sagdeev, 1967; Zakharov, 1984), but the details are somewhat tedious and are not of principal concern in this general discussion. Instead, I shall simply use the result that an ensemble of weakly interacting waves can be stochastic[132] in order to justify a statistical description, namely, the WKE that will be derived in the next section.

4.2.2 The random-phase approximation

I now revert to a dynamical equation of the standard form (125b). One assumes the existence of a spectrum of waves for which the stochasticity threshold has been exceeded. The goal is to develop an evolution equation for the energylike quantity $C(t) \doteq \langle \delta\psi^2(t) \rangle$. To accomplish that, one must discuss the properties of the random variable ψ. (A $\boldsymbol{k}$ index is temporarily suppressed.)

Once the wave stochasticity criterion is (moderately) exceeded, one may think of the dynamical amplitudes as complex numbers, $\delta\psi \equiv \psi_r + \mathrm{i}\psi_i = a\mathrm{e}^{\mathrm{i}\theta}$, where the amplitudes are slowly varying ($a \sim \sqrt{\mathcal{J}}$) but the phases are rapidly varying and, in particular, distributed uniformly over the interval $[0, 2\pi)$. This assumption is called the *random-phase approximation* (RPA). It is approximately true over a microscopic time interval τ_θ, the phase stochasticization time introduced above.

The RPA is similar to a Gaussian approximation. It is *not* identical, however, since realizations of ψ_r, say, take on values between $[-a, a]$ whereas realizations of Gaussian variables take on values on the entire range $(-\infty, \infty)$.[133] In the formal derivation of the WKE, to be discussed in the next section, one uses the true Gaussian assumption rather than the RPA, so such difficulties disappear. The justification for this is the same as the one given in Sec. 3.2.1 (p. 49) for Gaussian statistics of the Langevin acceleration. Namely, one coarse-grains the time axis in units Δt, where $\Delta t \gg \tau_\theta$ but scales with τ_θ. Then the central limit theorem can be used to argue for the Gaussian assumption.

4.2.3 The generic wave kinetic equation

One may now proceed to derive the evolution equation for $C(t)$. At any moment t one may assume that the complex amplitude of each wave is a Gaussian random variable. The procedure is to integrate forward for a time increment Δt that obeys $\Delta t \gg \tau_\theta$. (A crude estimate is $\tau_\theta \sim \Delta\Omega^{-1}$, where $\Delta\Omega$ is a

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[132] Numerical illustrations of the stochasticity of three interacting drift waves were given by Terry and Horton (1982).

[133] A more formal way of showing the inequivalence is to consider the joint PDF for the real and imaginary components of ψ. It is easy to prove that the characteristic function of ψ_r and ψ_i is $J_0(ka)$, where $k^2 \doteq k_r^2 + k_i^2$. On the other hand, if ψ_r and ψ_i were independent Gaussian variables with the same variance, the characteristic function would be $\exp(-\frac{1}{4}k^2a^2)$.

In the RPA and with $1 \equiv \boldsymbol{k}_1$, one has the important property $C_1 \doteq \langle \psi_1 \psi_2^* \rangle = a^2 \langle \mathrm{e}^{\mathrm{i}(\theta_1 - \theta_2)} \rangle = a^2 \delta_{1,2} = |\psi|^2 \delta_{1,2}$. More generally, odd-order correlations of randomly phased variables vanish whereas even-order correlations *factor in the Gaussian way if* the labels are at most "equal" (actually, the negatives of each other) in pairs. Thus for example,

$$\langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle = (C_1 \delta_{1,-2})(C_3 \delta_{3,-4}) + (C_1 \delta_{1,-3})(C_2 \delta_{2,-4}) + (C_1 \delta_{1,-4})(C_2 \delta_{2,-3}) \tag{f-6}$$

if no more than two of the indices are simultaneously equal. This factorization is the same as the Fourier transform of the fourth-order correlation of Gaussian variables. However, suppose that $1 = 2 = -3 = -4$. Then the left-hand side of Eq. (f-6) would be (assuming $a = 1$ for simplicity) $\langle |\psi|^4 \rangle = 1$ whereas the right-hand side of Eq. (f-6) would be $2C_1^2 = 2$. Thus one must use the RPA with some care.

characteristic frequency mismatch[134]: $\Delta\Omega \doteq \Omega_{\boldsymbol{k}} + \Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}}$, where $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$.) If one assumes that the wave amplitudes are not too large, one may use perturbation theory to compute the averaged intensity at time $t + \Delta t$, which is taken to be slowly varying. Upon subtracting off the initial intensity and dividing by Δt, one then obtains a coarse-grained approximation to the time derivative:

$$\lim_{\Delta t \to \text{“}0\text{”}} \frac{\langle |\delta\psi_{\boldsymbol{k}}|^2\rangle(t+\Delta t) - \langle |\delta\psi_{\boldsymbol{k}}|^2\rangle(t)}{\Delta t} \approx \frac{\mathrm{d}C_{\boldsymbol{k}}}{\mathrm{d}t}. \tag{180}$$

Here “0” means greater than τ_θ.

Because second-order contributions to $C(t)$ stem merely from the first-order (linear) dynamics of ψ, it is clear that one must calculate $C(t)$ beyond second order in the fluctuations in order to find interesting nonlinear behavior. Third-order contributions to C vanish because of the centered-Gaussian assumption. Contributions to the fourth-order spectrum arise from both second- and third-order contributions to ψ:

$$C^{(4)}_{\boldsymbol{k}}(t) = \langle \delta\psi^{(1)}_{\boldsymbol{k}}(t)\delta\psi^{(1)}_{\boldsymbol{k}}(t)^*\rangle + \langle \delta\psi^{(2)}_{\boldsymbol{k}}(t)\delta\psi^{(2)}_{\boldsymbol{k}}(t)^*\rangle + \langle \delta\psi^{(3)}_{\boldsymbol{k}}(t)\delta\psi^{(1)}_{\boldsymbol{k}}(t)^*\rangle + \langle \delta\psi^{(1)}_{\boldsymbol{k}}(t)\delta\psi^{(3)}_{\boldsymbol{k}}(t)^*\rangle. \tag{181}$$

The necessary calculations are straightforward, if a bit tedious. The final result is the *wave kinetic equation*

$$\partial_t C_{\boldsymbol{k}} - 2\gamma^{\mathrm{lin}}_{\boldsymbol{k}} C_{\boldsymbol{k}} + 2\,\mathrm{Re}\,\eta^{\mathrm{nl}}_{\boldsymbol{k}} C_{\boldsymbol{k}} = 2F^{\mathrm{nl}}_{\boldsymbol{k}}, \tag{182}$$

where[135]

$$\eta^{\mathrm{nl}}_{\boldsymbol{k}}(t) \approx -\sum_{\Delta} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^*_{\boldsymbol{pqk}}(t) C_{\boldsymbol{q}}(t), \quad F^{\mathrm{nl}}_{\boldsymbol{k}}(t) \approx \frac{1}{2}\sum_{\Delta} |M_{\boldsymbol{kpq}}|^2\,\mathrm{Re}[\theta_{\boldsymbol{kpq}}(t)] C_{\boldsymbol{p}}(t) C_{\boldsymbol{q}}(t), \tag{183a,b}$$

$$\theta_{\boldsymbol{kpq}} = \pi\delta(\Delta\Omega) \quad (\mathrm{WTT}). \tag{183c}$$

[134] Note that if $\Delta\Omega \equiv 0$ (nondispersive waves), the linear frequency can entirely be transformed away, so the turbulence is intrinsically strong and WTT fails.

[135] In Eqs. (183) the delta functions of the frequency mismatch arise from integrals over products of the unperturbed Green's function $G_{0,\boldsymbol{k}}(t;t') = H(\tau)\exp(-\mathrm{i}\Omega_{\boldsymbol{k}}\tau)$. For example, from the (2)–(2) term of Eq. (181) arises the integral

$$\mathrm{Re}\int_0^{\Delta t} \mathrm{d}t' \int_0^{\Delta t} \mathrm{d}t''\, \exp[\mathrm{i}\Delta\Omega(t'-t'')] = [\sin(\tfrac{1}{2}\Delta\Omega\Delta t)/\tfrac{1}{2}\Delta\Omega]^2. \tag{f-7}$$

Now the function $\sin(a\Delta t)/a$ does not have a classical limit as $\Delta t \to \infty$. To understand its significance, recall that $\lim_{\Delta t\to\infty}[\sin(a\Delta t)/a] = \pi\delta(a)$. The square of this function is even worse:

$$[\sin(\tfrac{1}{2}\Delta\Omega\Delta t)/\tfrac{1}{2}\Delta\Omega]^2 \to \pi\delta(\tfrac{1}{2}\Delta\Omega)[\sin(\tfrac{1}{2}\Delta\Omega\Delta t)/\tfrac{1}{2}\Delta\Omega] = 2\pi\Delta t\,\delta(\Delta\Omega). \tag{f-8}$$

The result depends on the integration time. That is fortunate, because the result must be divided by Δt in order to properly define the coarse-grained time derivative (180). The result is mathematically equivalent to the golden rule for quantum-mechanical scattering.

Equation (182) is called a *Markovian closure* because all quantities are local in time. $\theta_{\boldsymbol{kpq}}$ is called the *triad interaction time.*[136] The right-hand side of Eq. (182) describes the forward three-wave decay $\Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}} \to \Omega_{-\boldsymbol{k}}$; the $\eta^{\text{nl}}_{\boldsymbol{k}}$ term describes the inverse process $\Omega_{-\boldsymbol{k}} \to \Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}}$.

The constraint $\Delta\Omega_{\boldsymbol{kpq}} = 0$, enforced by the delta function in Eq. (183c), restricts three-wave interactions to modes with particular qualitative dispersion characteristics (Sagdeev and Galeev, 1969). For example, dispersion relations that as functions of wave-number magnitude pass through the origin and are concave up permit three-wave ("decay") interactions. If those are not allowed, one must turn to n-wave interactions with $n > 3$. Those emerge by continuing the iteration through higher order, thereby obtaining corrections to Eqs. (183). I shall not pursue the details here.

Just as in the analogous discussion in Sec. 4.1.2 (p. 91) of the $\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})$ quasilinear diffusion coefficient in formulas such as (161), the $\delta(\Delta\Omega)$ in Eq. (183c) makes sense only in the limit of a continuous Fourier spectrum. That need not be the case. For example, Terry and Horton (1982) derived the equations for the interaction of just three drift waves; for this case the delta function is nonsensical. Terry and Horton replaced $\pi\delta(\Delta\Omega)$ by the broadened resonance function $\nu/[(\Delta\Omega)^2 + \nu^2]$, where ν was a nonlinear decorrelation rate. However, ν is not determined within the framework of WTT; a theory of strong turbulence is required. Krommes (1982) reconsidered the three-wave problem in the direct-interaction approximation (Sec. 5, p. 126), which effectively makes a self-consistent prediction for ν, and found good agreement with numerical solutions. See Sec. 5.10.3 (p. 144) for further discussion.

4.2.4 Interpretation of the wave kinetic equation: Coherent and incoherent response

One of the most important conclusions of the renormalized theory to be described in later sections is that the general form (182) transcends its derivation from perturbation theory. The term in $\gamma^{\text{lin}}_{\boldsymbol{k}}$, of course, describes the intrinsic stirring and/or dissipation due to the linear instabilities and damping mechanisms. The right-hand side of Eq. (182), which describes the coupling of spectral intensities at two wave vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ different from $\boldsymbol{k}$, is called (the variance of) *incoherent noise.* [The justification of this nomenclature will not be fully apparent until the later discussions in Secs. 5.3 (p. 132) and 8.2.2 (p. 201) of Langevin representations of turbulence.[137]] That term is manifestly positive definite [provided that $\operatorname{Re}\theta_{\boldsymbol{kpq}} > 0$, as is required if it is to represent an interaction time; see more discussion in Secs. 8.2.1 (p. 201) and 8.2.2 (p. 201)]. If the nonlinear terms are to conserve energy or other positive definite invariants, the $\eta^{\text{nl}}_{\boldsymbol{k}}$ term must therefore be typically (for most $\boldsymbol{k}$'s) positive as well. The effects embodied in $\eta^{\text{nl}}_{\boldsymbol{k}}$ are called *coherent response*[138]; they provide a $\boldsymbol{k}$-dependent generalization of the nonlinear damping we have already seen in the stochastic-oscillator model.

To explicitly demonstrate conservation of a nonlinear invariant $\mathcal{I}$ defined by $\mathcal{I} \doteq \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}$, where $\sigma_{\boldsymbol{k}}$ is a specified weighting factor, multiply Eq. (182) by $\sigma_{\boldsymbol{k}}$ and sum over $\boldsymbol{k}$. After symmetrizing the $\eta^{\text{nl}}_{\boldsymbol{k}}$ term in $\boldsymbol{p}$ and $\boldsymbol{q}$, one finds that the nonlinear terms vanish provided that

$$\sigma_{\boldsymbol{k}} M_{\boldsymbol{kpq}} + \sigma_{\boldsymbol{p}} M_{\boldsymbol{pqk}} + \sigma_{\boldsymbol{q}} M_{\boldsymbol{qkp}} = 0. \tag{184}$$

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[136] Because of the weak-turbulence form (183c), the complex conjugate, order of the indices, and time argument of θ are unnecessary in Eqs. (183). The forms as shown permit later generalizations; see, for example, Eqs. (401).

[137] For some discussion of a Langevin equation for weak turbulence, see Elsässer and Gräff (1971).

[138] The phrase *coherent response* is originally due to Dupree (1972b). For more discussion, see Sec. 6.5.3 (p. 173).

[For some M's there may be more than one $\sigma_{\boldsymbol{k}}$ that satisfies Eq. (184).] Then $\partial_t \mathcal{I} = 2\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} \gamma_{\boldsymbol{k}}^{\text{lin}} C_{\boldsymbol{k}}$. This is the same balance between forcing and dissipation displayed by the exact equation; see Eq. (117). Conservation of the nonlinear invariant by the closure is seen to be a consequence of symmetry between the coherent and incoherent response; both must be included on equal footing.

As we will see, these results can be generalized to a more complete strong-turbulence theory. For Markovian closures the principal change will be that the delta function of Eq. (183c) is broadened, so precise frequency matching is not necessary. According to the arguments in the last paragraph, such broadening preserves the conservation properties because the symmetries are maintained. A further non-Markovian generalization is provided by the DIA (Sec. 5, p. 126).

4.2.5 Validity of weak-turbulence theory

For WTT to be valid, one must first satisfy a stochasticity condition for the waves. Additionally, however, the neglected nonlinear terms must not be too large. In order to develop a quantitative criterion, one may anticipate the renormalized expression for $\theta_{\boldsymbol{kpq}}$ discussed in Sec. 7.2.1 (p. 184). A simple generalization of Eq. (183c) is

$$\theta_{\boldsymbol{kpq}} = (-\mathrm{i}\Delta\Omega + \Delta\eta^r)^{-1}, \tag{185}$$

where $\eta^r \doteq \eta^{\text{nl}} - \gamma^{\text{lin}}$, for simplicity of presentation I am assuming that η^{nl} is real (so there is no nonlinear frequency shift), and $\Delta\eta^r \doteq \eta^r_{\boldsymbol{k}} + \eta^r_{\boldsymbol{p}} + \eta^r_{\boldsymbol{q}}$. Equation (185) plausibly states that the delta function of the frequency mismatch is broadened by the nonlinear damping. Therefore one condition for WTT is

$$|\Delta\eta^r/\Delta\Omega| \ll 1. \tag{186}$$

Because one has already assumed $|\gamma_{\boldsymbol{k}}/\Omega_{\boldsymbol{k}}| \ll 1$ in order that the waves be well developed, Eq. (186) is essentially equivalent to $|\Delta\eta^{\text{nl}}/\Delta\Omega| \ll 1$; i.e., the linear timescale must be short relative to the nonlinear one.

Frequently a validity criterion of the form $|\overline{\gamma}/\Delta\Omega| \ll 1$ is quoted, where $\overline{\gamma}$ is a typical growth rate. That would be correct if the incoherent noise could be ignored on the right-hand side of Eq. (182) so that the steady-state balance would be $\operatorname{Re}\eta^{\text{nl}}_{\boldsymbol{k}} = \gamma^{\text{lin}}_{\boldsymbol{k}}$. However, it cannot be correct in spectral regions for which $\gamma^{\text{lin}}_{\boldsymbol{k}} \approx 0$, such as an inertial range. There the steady-state balance is between $\eta^{\text{nl}}_{\boldsymbol{k}}$ and $F_{\boldsymbol{k}}$, and only the nonlinear criterion (186) makes sense. For some related discussion, see Ottaviani and Krommes (1992).

A detailed analysis of validity criteria for WTT in the context of Langmuir turbulence was given by Payne et al. (1989). The possibility of weak-turbulence Kolmogorov spectra was discussed at length by Zakharov (1984) and Zakharov et al. (1992), where more references can be found.

4.2.6 Vlasov weak-turbulence theory

A weak-turbulence analysis can also be carried out for Vlasov or other kinetic equations. The principal new qualitative feature not present in the implicitly fluid description just given is the possibility of wave–wave–particle resonances in addition to the wave–wave–wave resonance of fluid theory. That is, resonant denominators of the form $(\omega_{\boldsymbol{p}} - \boldsymbol{p}\cdot\boldsymbol{v} + \mathrm{i}\epsilon)^{-1}$ occur, where $\boldsymbol{p} \doteq -(\boldsymbol{k}+\boldsymbol{q})$ and modes $\boldsymbol{k}$ and $\boldsymbol{q}$ are assumed to label nonresonant normal modes. The reactive contributions from such resonances are called *induced scattering*; the dissipative contributions are usually called *nonlinear Landau damping*, although this phrase is sometimes taken to subsume induced scattering as well.

To deduce the general form of the two-time Vlasov spectral balance equation to lowest nontrivial order (quadratic in the intensity) in the weak-turbulence expansion, define $k \doteq \{\boldsymbol{k}, \omega_{\boldsymbol{k}}\}$ and expand the fluctuating potential deduced from the fluctuating Vlasov equation as

$$\mathcal{D}^{\text{lin}}(k)\delta\varphi_k + \frac{1}{2}\sum_{p,q}\delta_{k+p+q}\epsilon^{(2)}(k \mid p,q)(\delta\varphi_p^*\delta\varphi_q^* - \langle\ldots\rangle) \\ + \frac{1}{2}\sum_{p,q}\delta_{k+p+q}\sum_{p',q'}\delta_{p,p'+q'}\epsilon^{(3)}(k \mid q,p',q')[\delta\varphi_q^*(\delta\varphi_{p'}^*\delta\varphi_{q'}^* - \langle\ldots\rangle) - \langle\ldots\rangle] + \cdots, \tag{187}$$

where the coupling coefficients $\epsilon^{(2)}$ and $\epsilon^{(3)}$ are symmetrical in their last two arguments. It is readily shown that [139]

$$\epsilon^{(2)}(k \mid p,q) \doteq -\widehat{\Phi}_k g_{0,k}[\boldsymbol{p}\cdot\boldsymbol{\partial}\, g_{0,q}^*\boldsymbol{q}\cdot\boldsymbol{\partial} f_0 + (p \leftrightarrow q)], \tag{188a}$$

$$\epsilon^{(3)}(k \mid q,p',q') \doteq -\mathrm{i}\widehat{\Phi}_k g_{0,k}\boldsymbol{q}\cdot\boldsymbol{\partial}\, g_{0,p}^*[\boldsymbol{p}'\cdot\boldsymbol{\partial}\, g_{0,q'}^*\boldsymbol{q}'\cdot\boldsymbol{\partial} f_0 + (p' \leftrightarrow q')]. \tag{188b}$$

Expand Eq. (187) through third order and form the equation for $I_k \approx \langle|\delta\varphi_k^{(1)}|^2\rangle + \langle\delta\varphi_k^{(3)}\delta\varphi_k^{(1)*}\rangle + \langle\delta\varphi_k^{(1)}\delta\varphi_k^{(3)*}\rangle + \langle|\delta\varphi_k^{(2)}|^2\rangle$, assuming Gaussian statistics for $\delta\varphi^{(1)}$ and steady state. One obtains

$$\mathcal{D}(k)I_k = \langle\delta\widetilde{\varphi}^2\rangle_k/(\mathcal{D}^{\text{lin}})^*(k), \tag{189}$$

where $\mathcal{D} = \mathcal{D}^{\text{lin}} + \mathcal{D}^{\text{nl}}$ and (Sagdeev and Galeev, 1969)

$$\mathcal{D}_k^{\text{nl}} \doteq \sum_{p,q}\delta_{k+p+q}[\epsilon^{(3)}(k \mid q,-q,-k) - \epsilon^{(2)}(k \mid p,q)(\mathcal{D}_p^*)^{-1}\epsilon^{(2)*}(p \mid q,k)]I_q, \tag{190a}$$

$$\langle\delta\widetilde{\varphi}^2\rangle_k \doteq \frac{1}{2}\sum_{p,q}\delta_{k+p+q}|\epsilon^{(2)}(k \mid p,q)|^2 I_p I_q. \tag{190b}$$

Note that Eq. (189) is asymmetrical, involving $\mathcal{D}$ (containing second-order corrections) on the left but merely $\mathcal{D}^{\text{lin}}$ on the right. That is an artifact of the second-order perturbation theory; the renormalized theory described in Sec. 6.5 (p. 170) and Appendix G (p. 288) shows that in a more complete description $\mathcal{D}^{\text{lin}}$ should be replaced by $\mathcal{D}$, as one would expect. [140]

Reduction of the frequency- or two-time-dependent balance equation (189) to a wave kinetic equation is somewhat tedious because resonant and nonresonant effects are mixed together; it requires patience and foresight to properly reduce the results to sensible fluid equations. Important insights and guidance follow by developing analogies to discrete, quantum-mechanical balance equations; see, for example, Tsytovich (1970, 1972, 1977) and Motz (1973). Modern analyses exploit the concept of

[139] These definitions differ by a factor of 2 from those of Sagdeev and Galeev (1969) because of the different convention adopted in Eq. (187).

[140] If that is not done, certain ambiguities of sign arise because $\mathcal{D}^{\text{lin}}$ may have zeros in the upper half of the ω plane, but $\mathcal{D}$ must be stable in steady state. The same confusion surrounds the validity of QLT for damped modes. See Appendix G (p. 288) for further discussion.

oscillation centers (Johnston, 1976) and Lie transforms (Johnston and Kaufman, 1978). In any event, further reduction of Eq. (190a) leads (Appendix G, p. 288) to the WKE of WTT in the form

$$\partial_T \mathcal{N}_{\boldsymbol{k}}(T) - 2\gamma_{\boldsymbol{k}}^{\text{lin}} \mathcal{N}_{\boldsymbol{k}} - 2(\gamma_{\boldsymbol{k}}^{\text{ind}} + \gamma_{\boldsymbol{k}}^{\text{mc}}) \mathcal{N}_{\boldsymbol{k}} = \dot{\mathcal{N}}_{\boldsymbol{k}}^{\text{mc}}, \tag{191}$$

where the action density $\mathcal{N}_{\boldsymbol{k}}$ and the other terms are defined in Appendix G (p. 288). The nonlinear terms have been divided into contributions from induced scattering (involving the driven beat resonance $\omega_{\boldsymbol{p}} - \boldsymbol{p} \cdot \boldsymbol{v} = 0$) and fluid mode-coupling contributions involving the three-wave interaction $\Omega_{\boldsymbol{k}} + \Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}} = 0$. The $\epsilon^{(3)}$ term in Eq. (190a) describes scattering from bare particles [*Compton scattering*; Fig. 13(a)]; scattering from the shielding clouds [sometimes called *nonlinear scattering*; Fig. 13(b)] arises as a nonresonant contribution from the $\epsilon^{(2)}\epsilon^{(2)*}$ term of Eq. (190a). The effect involves a three-wave interaction, but with the fluctuation at $\boldsymbol{p}$ being *virtual* (driven nonresonantly), not a normal mode. The forward three-wave decay process $\Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}} \to \Omega_{-\boldsymbol{k}}$ arises from the incoherent noise $\dot{\mathcal{N}}_{\boldsymbol{k}}^{\text{mc}}$; the inverse process is described by $\gamma_{\boldsymbol{k}}^{\text{mc}}$, which arises as a resonant contribution from the $\epsilon^{(2)}\epsilon^{(2)*}$ term.

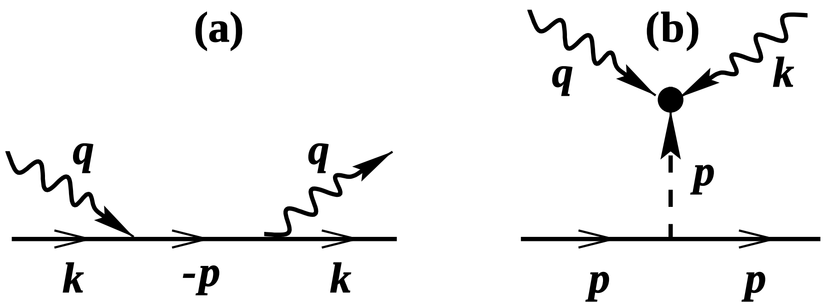

Fig. 13. Induced scattering processes in WTT. Straight line: particle propagation; wiggly line: normal mode (wave); dashed line: virtual mode. (a) Compton scattering (from bare particles); (b) nonlinear scattering (from shielding clouds).

For strong turbulence the perturbative expansion (187) fails. In order to properly calculate the nonlinear contribution to the growth rate, one needs a precise, *nonperturbative* definition of the dielectric function. That is derived in Sec. 6.5 (p. 170), where I shall demonstrate the reduction of general renormalized Vlasov theory to both kinetic WTT as well as the RBT discussed in Sec. 4.3 (p. 108).

4.2.7 Application: Ion acoustic turbulence and anomalous resistivity

A detailed application of Vlasov weak-turbulence theory was given by Horton and Choi (1979) in the context of ion acoustic turbulence, one of the few situations for which the weak-turbulence ordering can apparently be cleanly justified. Because this topic illustrates a number of key plasma-turbulence concepts (the appearance of an anomalous transport coefficient, the differing roles of electrons and ions, resonant and nonresonant response, direction of energy flow, *etc.*), I shall briefly review the essential physical points, closely following the discussion of Sagdeev (1979); see also Sagdeev (1974).

Ion acoustic turbulence is easy to excite through a current-driven instability. One then expects that the resulting fluctuations will act in such a way as to hinder the driving current. In other words, one guesses that an *anomalous resistivity* [141] will appear. Because usually the current is primarily

[141] A general formulation of the theory of anomalous resistivity in terms of correlation functions was given

carried by the electrons, one develops a picture in which the electrons emit fluctuations (just like Cerenkov emission in a near-equilibrium plasma). In order to achieve a nonlinear steady state, either those fluctuations must completely turn off the driving current (by quasilinear relaxation) or the ions must absorb the emitted fluctuations. Usually, since the system is being driven externally (e.g., by Ohmic heating), the current cannot be completely destroyed and the ion damping mechanism wins.

The anomalous resistivity is defined by considering the momentum equation for electrons: $(mn)_e \mathrm{d}\boldsymbol{u}_e/\mathrm{d}t = -(e/m_e)\boldsymbol{E} - (mn)_e \nu_{\text{eff}} \boldsymbol{u}_e$. The last term describes the loss of electron momentum due to the emission of unstable ion acoustic waves. Sagdeev found the result $\nu_{\text{eff}} \sim 10^{-2}\omega_{pi}(T_e/T_i)(u/c_{\text{s}})$, the interpretation of which will now be described.

The dielectric function for ion acoustic waves is easily obtained. One searches for waves in the frequency range $kv_{ti} < \omega < kv_{te}$. This means that to lowest order the ions are hydrodynamic, contributing a susceptibility $-\omega_{pi}^2/\omega^2$, and the electrons are adiabatic, contributing a susceptibility $1/(k\lambda_{De})^2$. Thus $\mathcal{D}^{\text{lin}}(k,\omega) \approx 1 - \omega_{pi}^2/\omega^2 + 1/(k\lambda_{De})^2 + \mathrm{i}\mathcal{D}_i^{\text{lin}}(k,\omega)$. It is useful to recall that λ_{De} can be written in the alternate ways $\lambda_{De} = v_{te}/\omega_{pe} = c_{\text{s}}/\omega_{pi}$. Assume that the electron PDF is a shifted Maxwellian with mean velocity $\boldsymbol{u}_e$. It is then easy to show that the dispersion relation for the ion acoustic waves is $\Omega_{\boldsymbol{k}}^2 = k^2 c_{\text{s}}^2/[1 + (kc_{\text{s}}/\omega_{pi})^2]$ and the growth rate is

$$\frac{\gamma_{\boldsymbol{k}}^{\text{lin}}}{\Omega_{\boldsymbol{k}}} = \left(\frac{\Omega_{\boldsymbol{k}}}{kc_{\text{s}}}\right)^2 \left[\left(\frac{\pi}{8}\right)\left(\frac{m}{M}\right)\right]^{1/2} \left(\frac{\widehat{\boldsymbol{k}}\cdot\boldsymbol{u}}{c_{\text{s}}} - \frac{\Omega_{\boldsymbol{k}}}{kc_{\text{s}}}\right) \sim \left(\frac{m}{M}\right)^{1/2} \frac{\widehat{\boldsymbol{k}}\cdot\boldsymbol{u}}{c_{\text{s}}} \quad (\Omega_{\boldsymbol{k}} \ll kc_{\text{s}},\ \widehat{\boldsymbol{k}}\cdot\boldsymbol{u} > c_{\text{s}}). \tag{192a,b}$$

In the long-wavelength limit, there is an instability when the fluid velocity in the direction of the wave vector is greater than the sound speed. Also note that the dispersion relation is of the nondecay type, so three-wave interactions can be neglected if the criterion (186) is satisfied.

In order to find a general expression for ν_{eff}, recall the quasilinear momentum conservation law: $\mathrm{d}(\boldsymbol{\mathcal{P}}_{\text{res}} + \boldsymbol{\mathcal{P}}_{\text{nr}})/\mathrm{d}t = 0$. For acoustic waves the electron interaction is primarily resonant; most of the nonresonant energy and momentum is in the ions. Thus one is entitled to write $-(mn)_e\nu_{\text{eff}}\boldsymbol{u}_e = \mathrm{d}\boldsymbol{\mathcal{P}}_{\text{res}}/\mathrm{d}t = -\mathrm{d}\boldsymbol{\mathcal{P}}_{\text{nr}}/\mathrm{d}t$. This states that the electrons emit the acoustic waves and the (nonresonant motions of the) ions absorb them. Since $\boldsymbol{\mathcal{P}}_{\text{nr}} = \sum_{\boldsymbol{k}} \boldsymbol{k}\mathcal{N}_{\boldsymbol{k}}$, one has $\mathrm{d}\boldsymbol{\mathcal{P}}_{\text{nr}}/\mathrm{d}t = \sum_{\boldsymbol{k}} \boldsymbol{k}(2\gamma_{\boldsymbol{k}}^{\text{lin}})\mathcal{N}_{\boldsymbol{k}} = 2\sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}}^{\text{lin}}(\boldsymbol{k}/\Omega_{\boldsymbol{k}})\mathcal{E}_{\boldsymbol{k}}$, where one changed from the action to the total wave energy $\mathcal{E}_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}}\mathcal{N}_{\boldsymbol{k}}$ as the fundamental unknown. Thus one arrives at the formula

$$\nu_{\text{eff}} = 2(mnu)_e^{-1} \sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}}^{\text{lin}}(\boldsymbol{k}\cdot\widehat{\boldsymbol{u}}/\Omega_{\boldsymbol{k}})\mathcal{E}_{\boldsymbol{k}}. \tag{193}$$

From now on, let us drop numerical factors. Then upon using the previous expression for γ^{lin} to estimate Eq. (193), one obtains

$$\frac{\nu_{\text{eff}}}{\omega_{pi}} \sim \left(\frac{M}{m}\right)^{1/2} \sum_{\boldsymbol{k}} (k\lambda_{De}) \left(\frac{\mathcal{E}_{\boldsymbol{k}}}{nT_e}\right). \tag{194}$$

In order to proceed, one must find the saturated level of the wave energy $\mathcal{E}_{\boldsymbol{k}}$.

Because the ion-acoustic dispersion relation is of the nondecay type, the lowest-order nonlinear process is induced scattering. [There is also a quasilinear distortion of the electron distribution function; see Rudakov and Korablev (1966) and Kovrizhnykh (1966).] The general form of the WKE

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by Tange and Ichimaru (1974).

is schematically $\partial_t \mathcal{E}_{\boldsymbol{k}} = 2\gamma_{\boldsymbol{k}}^{\text{lin}} \mathcal{E}_{\boldsymbol{k}} - \mathcal{M}_{\boldsymbol{k}} \mathcal{E}_{\boldsymbol{k}}^2$. The last term here is written in a highly schematic form; it should really involve a convolution over wave numbers. Nevertheless, very crudely one can estimate the steady-state saturation level due to induced scattering to be $\mathcal{E}_{\boldsymbol{k}} \sim \gamma_{\boldsymbol{k}}^{\text{lin}}/\mathcal{M}_{\boldsymbol{k}}$.

One must ask which species dominates in the induced scattering process. Because $\omega \ll kv_{te}$, one has $\Omega_{\boldsymbol{k}_1} + \Omega_{\boldsymbol{k}_2} \ll (\boldsymbol{k}_1 + \boldsymbol{k}_2) \cdot \boldsymbol{v}_e$. Therefore this beat resonance falls on the flat part of the electron distribution function, where there is very little net Landau interaction. However, the beat resonance $\Omega_{\boldsymbol{k}_1} - \Omega_{\boldsymbol{k}_2} = (\boldsymbol{k}_1 - \boldsymbol{k}_2) \cdot \boldsymbol{v}_i$ can interact with the heart of the ion distribution. Therefore the ions dominate the induced scattering. [The detailed proof of this remark is actually rather complicated; see Horton and Choi (1979).]

One can now estimate the size of the induced-scattering mode-coupling coefficient. On dimensional grounds one argues that $\mathcal{M}_{\boldsymbol{k}} \sim (nT_e)^{-1}\Omega_{\boldsymbol{k}}(T_i/T_e)$. The first factor, involving the inverse of the electron thermal energy density, is the natural normalization for $\mathcal{E}_{\boldsymbol{k}}$. (The waves exist even for cold ions, so it would not be appropriate to use T_i here.) The frequency factor arises because one is describing rates. One uses a frequency rather than a growth rate because the primary waves are assumed to be nonresonant. The temperature ratio appears because the beat-wave interaction is resonant with the ions, so an ion thermal spread is required in order to give an effect. Thus one estimates that $\mathcal{E}_{\boldsymbol{k}}/nT_e \sim (\gamma_{\boldsymbol{k}}^{\text{lin}}/\Omega_{\boldsymbol{k}})(T_e/T_i)$, or upon summing over all excited wave numbers to get the total fluctuation level, $(\mathcal{W}/nT_e) \sim (m/M)^{1/2}(T_e/T_i)(u/c_{\text{s}})$. If one puts this expression into expression (194) and ignore the dimensionless factor $k\lambda_{De}$, one estimates $\nu_{\text{eff}}/\omega_{pi} \sim (T_e/T_i)(u/c_{\text{s}})$, which is just Sagdeev's result. Of course, one cannot obtain the numerical coefficient 10^{-2} without performing a detailed study of the mode coupling buried inside $\mathcal{M}_{\boldsymbol{k}}$.

Because the induced scattering is nonresonant, action is conserved by the waves.[142] This differs from the resonant three-wave interaction, in which the Manley–Rowe relations show that action is not conserved for that process.

The natural flow of energy in this problem is from high frequencies to low frequencies. This can be remembered very heuristically as follows. The electron current is driving the fluctuations. Assume that one initially populates the wave spectrum with waves of characteristic frequency ω and energy $\mathcal{E}$. As other fluctuations at other frequencies are driven up, one expects $\mathcal{E}$ to decrease as energy is transferred to those other frequencies. However, because the induced-scattering process is nonresonant, action is conserved, as has already been remarked. Since most of the wave energy is in the ions (in the form of nonresonant sloshing), the wave action derives from the derivative of the ion susceptibility: $\mathcal{N} \sim (\omega_{pi}^2/\omega^3)\mathcal{E}$. Consider a process $\omega \to \omega + \delta\omega$ and $\mathcal{E} \to \mathcal{E} + \delta\mathcal{E}$ (during which $\delta\mathcal{N} = 0$). Then $\delta\mathcal{N} = 0 \sim \delta(\mathcal{E}/\omega^3)$, which leads to $\delta\mathcal{E}/\mathcal{E} = 3\delta\omega/\omega$. Since it was argued that $\delta\mathcal{E} < 0$, one finds that $\delta\omega < 0$; i.e., the energy flows to lower frequencies.

Since energy is always being pumped into the nonresonant ion waves by the nonlinear induced scattering, a further mechanism is needed in order to saturate the spectrum (assuming that the electron current and the shapes of the distribution functions are maintained). Conventionally, collisional dissipation is assumed to absorb the energy. Another possibility is that the fluctuations grow so large that they begin to interact resonantly with the ion distribution. This mechanism is related to Dupree's resonance-broadening theory, to be discussed in Sec. 4.3 (p. 108).

This concludes the discussion of the ion-acoustic application. Since the equations of WTT follow

[142] In some sense this is a tautology, since wave action can be defined as the number of plasmons. What is needed is a proper statistical definition of action in terms of fluctuation intensity. That the appropriately defined $\mathcal{N}_{\boldsymbol{k}}$ is conserved follows from the detailed algebra (Sagdeev and Galeev, 1969).

from a straightforward algorithm, it is not surprising that such theories have been derived and studied for a variety of physical problems (including ones for which the fundamental assumptions are at best only marginally satisfied); a representative work is by Hahm and Tang (1991). Ultimately the fidelity of such calculations will only become clear when the results are compared with more comprehensive strong-turbulence theories. I turn now to the development of such theories.

4.3 Resonance-broadening theory

"The nonlinear mechanism ... is the broadening of the Landau resonance due to particle 'trapping' which results from the perturbation of the particle orbits in a turbulent plasma." — *Dupree (1966).*

Both QLT and WTT are examples of systematic approaches to the problem of turbulence. They have well-defined if limited regimes of validity. Modern systematic approaches appropriate for strong turbulence, such as the DIA, are (in some sense) natural generalizations of those theories.

Dupree pioneered several approaches to strong turbulence: the *resonance-broadening theory* (RBT; Dupree, 1966, 1967), discussed in the present section; and what I shall call the *clump algorithm* (CA; Dupree, 1972b), discussed in Sec. 4.4 (p. 119). These approximations are ultimately more intuitive in nature although the original paper on RBT (Dupree, 1966) and related works of other authors (Weinstock, 1969, 1970) invoked considerable mathematical apparatus. Orszag and Kraichnan (1967) made a seminal critique of the RBT, but their excellent work was unfortunately largely ignored. Historically, it therefore took considerable time to elucidate the deficiencies in the formal trappings and the relations of Dupree's work to the DIA and associated approximations.

4.3.1 Perturbed orbits and resonance broadening

A heuristic introduction to some of the basic phenomenology of plasma and fluid turbulence was given by Dupree (1969); that is a good place to start if one wants to appreciate Dupree's motivations. The fundamental intuition behind RBT stems from the classical Langevin equations described in Sec. 3.2 (p. 48). As did Dupree (1966), I consider first the unmagnetized, collisionless plasma. If the microturbulence is idealized to be Gaussian white noise, it leads (Sec. 3.2.2, p. 49) to $\boldsymbol{v}$-space diffusion ($\langle \delta v^2 \rangle = 2D_v t$) of a test particle and an associated $\boldsymbol{x}$-space dispersion ($\langle \delta x^2 \rangle = \frac{2}{3} D_v t^3$) around the free-streaming motion; the secular development of this latter probability cone is sometimes called *orbit diffusion.*[143]

One has already seen the appearance of orbit diffusion in Eq. (158b). An alternate interpretation of that effect follows by noting that the uncertainty in position broadens the Landau resonance between the test particle and the waves. To see this quantitatively, consider the time integral in Eq. (160), which is an approximation to a renormalized single-particle propagator $g_{k\omega}$. It is rigorously an Airy function, but the principal effect can be seen by defining[144] $\nu_d \doteq \tau_d^{-1}$ and considering the simple

[143] In the unmagnetized case it should more properly be called *orbit dispersion.* In strongly magnetized situations a truly diffusive effect does appear in $\boldsymbol{x}$ space; see Sec. 4.3.3 (p. 110).
[144] Recall that the actual diffusion time depends on k, but a k-independent diffusion time τ_d can be defined in terms of a reference wave number $\overline{k}$: $\tau_d \doteq \tau_{d,\overline{k}}$.

exponential approximation $g_{k\omega} \approx \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}(\omega - kv + \mathrm{i}\nu_d)\tau}$, namely,

$$g_{k\omega} = [-\mathrm{i}(\omega - kv + \mathrm{i}\nu_d)]^{-1}. \tag{195}$$

It is the real part of g that contributes to D_v:

$$\mathrm{Re}\, g_{k\omega} = \frac{\nu_d}{(\omega - kv)^2 + \nu_d^2} \overset{\nu_d \to 0}{\to} \pi\delta(\omega - kv). \tag{196a,b}$$

The origin of the term *resonance-broadening theory* is clear from the Lorentzian form of Eq. (196a). The effect can be viewed as arising from random Doppler shifts (Dum and Dupree, 1970); it is the same phase-mixing mechanism that is responsible for the decay of the mean response function of the stochastic oscillator (Sec. 3.3, p. 52). Some explicit discussion about the relationship between RBT and the Brownian-motion problem was also given by Dum and Dupree (1970) and Benford and Thomson (1972).

4.3.2 The strong-turbulence diffusion coefficient

The role of resonance broadening in justifying the continuum wave-number representation has already been discussed in Sec. 4.1 (p. 90) and Appendix D (p. 279). There the point was made that in the quasilinear regime $\tau_{\mathrm{ac}}^{\mathrm{lin}} < \tau_d$ the size of the resonance broadening does not appear in the final expression for D_v [recall Fig. 12 (p. 94)]. However, one can admit the possibility of a *strong-turbulence regime* $\tau_d < \tau_{\mathrm{ac}}^{\mathrm{lin}}$. In general, the true autocorrelation time to be used in the random-walk formula (151) should be the smaller of τ_d and $\tau_{\mathrm{ac}}^{\mathrm{lin}}$:

$$\tau_{\mathrm{ac}} = \begin{cases} \tau_{\mathrm{ac}}^{\mathrm{lin}} & \text{(quasilinear regime)} \\ \tau_d & \text{(strong-turbulence regime).} \end{cases} \tag{197}$$

Since τ_d depends on D_v, the strong-turbulence expression (151) for D_v is actually a self-consistent formula to be solved for D_v: $D_v^{4/3} = (q/m)^2 \mathcal{E} \overline{k}^{-2/3}$. This result can be interpreted by introducing, by analogy to the expression (D4b) for the trapping frequency in a single harmonic, a macroscopic trapping frequency $\Omega_{\mathrm{tr}} \doteq [(q\overline{k}/m)^2 \mathcal{E}]^{1/4}$. This formula is built from the effective intensity that would result from strongly overlapping but randomly phased islands, a concept already discussed by Chirikov (1969). The diffusion time is simply the inverse of this frequency: $\tau_d \sim (\overline{k}^2 D_v)^{-1/3} = \Omega_{\mathrm{tr}}^{-1}$. If one also defines a macroscopic trapping velocity according to $V_{\mathrm{tr}} \doteq \Omega_{\mathrm{tr}}/\overline{k}$, then

$$D_v = V_{\mathrm{tr}}^2 \Omega_{\mathrm{tr}}, \tag{198}$$

which is the proper formula for a random-walk process [145] with characteristic velocity step V_{tr} and autocorrelation time $\Omega_{\mathrm{tr}}^{-1} = \tau_d$. These results were known to Dupree (1966).

[145] There is a troubling aspect of the interpretation of Eq. (198) that to the author's knowledge has never been satisfactorily discussed. Given that there are a finite number of unstable waves, the width of the resonant region in velocity has a finite extent. In the strongly overlapped limit, V_{tr} represents the entire width of the resonant region. It does not make sense to talk of a random-walk process with step size $\Delta v \sim V_{\mathrm{tr}}$ if a single step would scatter the particle from that region.

To show that Eq. (198) is smoothly connected to the quasilinear result, one can use Eq. (197) to write Eq. (151) in the form

$$D_v \sim \begin{cases} V_{\mathrm{tr}}^2 \Omega_{\mathrm{tr}} (\tau_{\mathrm{ac}}^{\mathrm{lin}} \Omega_{\mathrm{tr}}) & (\tau_{\mathrm{ac}}^{\mathrm{lin}} < \tau_d) \\ V_{\mathrm{tr}}^2 \Omega_{\mathrm{tr}} & (\tau_d < \tau_{\mathrm{ac}}^{\mathrm{lin}}). \end{cases} \tag{199}$$

At the point where $\tau_{\mathrm{ac}}^{\mathrm{lin}} = \tau_d$ one has $\tau_d \sim \Omega_{\mathrm{tr}}^{-1}$, so the formula smoothly interpolates between the quasilinear and strong-turbulence regimes. (If it had not done so, there would have been an extra parameter regime yet to be discovered.)

4.3.3 Saturation due to resonance broadening

Since linearly unstable waves grow due to inverse Landau damping, resonance broadening can lead to a reduction in the growth as the resonant particles (substantially perturbed from their free-streaming trajectories) sample a nonvanishing region of the background PDF around the phase velocity of the waves. More dramatically, resonance broadening can bring nonresonant particles into resonance, possibly causing onset of strong Landau damping and immediate saturation of the turbulence. Such effects related to the wave–particle resonance should be describable by nonlinear modifications to the linear dielectric function. Let us postulate, without deep understanding or justification at this point, that the generalization $(\omega - kv + \mathrm{i}\epsilon)^{-1} \to (\omega - kv + \mathrm{i}\nu_d)^{-1}$ that appeared in the discussions of the diffusion coefficient should hold also for the dielectric function. Because in this Ansatz $\mathrm{i}\nu_d$ always appears in conjunction with ω, one is led to a simple recipe for the nonlinear dielectric:

$$\mathcal{D}(\boldsymbol{k}, \omega) = \mathcal{D}^{\mathrm{lin}}(\boldsymbol{k}, \omega + \mathrm{i}\nu_d). \tag{200}$$

If a nonlinear dispersion relation $\omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} + \mathrm{i}\gamma_{\boldsymbol{k}}$ is constructed from $\mathcal{D}(\boldsymbol{k}, \omega) = 0$, the known properties of $\mathcal{D}^{\mathrm{lin}}$ can be taken over immediately; one is led to

$$\Omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}}^{\mathrm{lin}}, \quad \gamma_{\boldsymbol{k}} = \gamma_{\boldsymbol{k}}^{\mathrm{lin}} - \nu_d. \tag{201a,b}$$

If one defines the saturation criterion by $\gamma = 0$, where $\gamma \doteq \max \gamma_{\boldsymbol{k}}$, one finds that saturation occurs when

$$\nu_d \sim \gamma^{\mathrm{lin}}. \tag{202}$$

There are several noteworthy things about formula (202). First, since ν_d depends algebraically on D_v, *Eq. (202) directly determines the value of the diffusion coefficient!* Unlike the usual predictive approach in which one first obtains the fluctuation level, then uses the appropriate formula [such as Eq. (46b)] to find D_v, here D_v is given immediately in terms of the linear growth rate. If this algorithm is to be believed, it is extraordinarily convenient, since often the diffusion coefficient is the most important thing one wants to know. Later, if one desires, one can use the relation between D_v and the fluctuation level to determine the latter. (In that calculation one must ascertain whether the system saturated in the quasilinear or strong-turbulence regimes.)

Second, consider the estimate (202) from the point of view of the limits of validity of quasilinear theory. The most fundamental criterion was $\tau_{\mathrm{ac}}^{\mathrm{lin}} < \tau_d$. With Eq. (202) this can be written as $\gamma^{\mathrm{lin}} \tau_{\mathrm{ac}}^{\mathrm{lin}} < 1$, which was another of the validity criteria. This demonstrates a nice consistency.

In practice the saturation algorithm (202) is usually applied to strongly magnetized situations (Dupree, 1967). Recall the discussion of Eq. (163). If the perpendicular diffusion time $\tau_{d\perp,\boldsymbol{k}} \doteq (k_\perp^2 D_\perp)^{-1}$ dominates, then according to Eq. (202) (and temporarily ignoring the distinction between γ and $\gamma_{\boldsymbol{k}}$) the waves should saturate when $k_\perp^2 D_\perp \sim \gamma_{\boldsymbol{k}}^{\text{lin}}$, or

$$D_\perp \sim \gamma_{\boldsymbol{k}}^{\text{lin}}/k_\perp^2. \tag{203}$$

Formula (203) is one of the most frequently quoted formulas in applications-oriented plasma turbulence theory although in the literature there is relatively little demonstrated understanding of its limits of validity. Certainly one should not assume that anything that has been said up to this point can be systematically derived or justified. One obvious difficulty with formula (203) is that although $D_\perp$ is independent of $\boldsymbol{k}$ (being summed over all $\boldsymbol{k}$'s), the right-hand side of Eq. (203) will depend on $\boldsymbol{k}$ for typical growth rates. The only resolution can be that the $\boldsymbol{k}$'s on the right-hand side of Eq. (203) must be interpreted as *typical* $\boldsymbol{k}$'s; the wave-number dependence of Eq. (203) must not be taken seriously. This point will be further discussed below.

One must be strongly warned that the RBT recipe (200) for the nonlinear dielectric function is simply incorrect in detail, particularly in the strong-turbulence limit. The proper theory is subtle, as discussed in Sec. 6.5 (p. 170). Fortunately, dimensional consequences of Eq. (203) are more robust than is this initial "derivation" based on the kinetic resonance-broadening approximation to $\mathcal{D}$. In the subsequent discussions I shall frequently return to Eq. (203) in attempts to understand it in detail.

In order for random-walk phenomenology to be a valid description, there must be stochasticity. Some aspects of the stochasticity criterion for $\boldsymbol{E} \times \boldsymbol{B}$ motion are described in Appendix D (p. 279). Those considerations lead to the estimate that $\boldsymbol{x}$-space stochasticity ensues when the rms $\boldsymbol{E} \times \boldsymbol{B}$ velocity exceeds the perpendicular phase velocity. [Dupree (1967) called this a "trapping condition," but that phrase is misleading since coherent islands are destroyed in a spectrum of waves.] Now consider the physically important scenario in which the normal modes are drift waves (with frequencies $\omega \sim \omega_*$). The stochasticity criterion then reduces to the condition that $V_E \gtrsim V_*$, where V_* is the diamagnetic velocity. This same criterion was found in the crude estimates of Sec. 1.3.3 (p. 16); the stochasticity criterion offers a partial justification.

Consider the consequences of stochasticity onset for a drift-wave problem with $\omega \sim \omega_* = k_y V_*$ and $k_\parallel v_{ti} \ll \omega \ll k_\parallel v_{te}$. Random $\boldsymbol{E} \times \boldsymbol{B}$ motions cause Doppler broadening $\delta\omega = k_\perp \delta V_E = O(\omega)$, the ordering following since $V_E = O(V_*)$. Because $\omega \ll k_\parallel v_{te}$, such broadening has little effect on the electron resonance; the waves are driven unstable by the linear electron growth rate. However, $\delta\omega \sim \omega$ broadens the ion resonance into the heart of the ion distribution. In Dupree's scenario this gives rise to strongly stabilizing ion Landau damping. Temporarily ignore the possibility of a $\tau_{\text{ac}}^{\text{lin}}$ induced by parallel motion, and define $\overline{V} \doteq \langle \delta V_E^2 \rangle^{1/2}$. Then a saturation scenario is that (i) fluctuations grow according to γ_e^{lin}; (ii) when $\overline{V}/V_*$ grows to be $O(1)$, stochasticity ensues and perpendicular diffusion turns on abruptly; finally (iii) the fluctuations are stabilized due to ion dissipation. The amount of that dissipation adjusts such that $D_\perp \sim \gamma_e^{\text{lin}}/k_\perp^2$; the details of the ion distribution do not enter into this formula. Thus any form of ion dissipation, including collisions (Krommes and Hu, 1994), can provide the stabilizing sink. Indeed, ion Landau damping vanishes as $T_i \to 0$, so cannot be effective in that limit.

A crude model[146] for the onset of such diffusion is

$$D_\perp \sim H(\overline{V}^2 - V_*^2)(\overline{V}^2 - V_*^2)(\overline{k}_\perp^2 D_\perp)^{-1}. \tag{204}$$

[This form incorporates the postulate that $D_\perp$ vanishes for $\overline{V} < V_*$, and it reduces to formula (163) for $\overline{V} \gg V_*$ when $\tau_{d\perp}$ dominates.] From the solution $D \propto (\overline{V}^2 - V_*)^{1/2}$, one finds that $D(\overline{V})$ has infinite slope at the transition. This behavior is an example of a *supercritical bifurcation* of the turbulence intensity; for more discussion of such bifurcations, see Sec. 9.3 (p. 212). If the $\boldsymbol{E} \times \boldsymbol{B}$ velocity were passive, the solution of Eq. (204) would be the solid curve of Fig. 14, with $D_\perp$ asymptoting to the strong-turbulence limit $D \to \overline{V}/\overline{k}_\perp$ for $\overline{V} \gg V_*$. For self-consistent fluctuations, however, the turbulence is argued to saturate with $\overline{k}_\perp^2 D_\perp \sim \gamma_e$.

In the presence of parallel motion, diffusion need not turn on so abruptly since $v_\parallel$ should induce a $\tau_{\rm ac}^{\rm lin}$ that has the quasilinear form (152); once again, the true $\tau_{\rm ac}$ to be used in Eq. (164) should be $\min(\tau_{\rm ac}^{\rm lin}, \tau_d)$. The actual $D_\perp$ is then shown as the dashed line in Fig. 14.

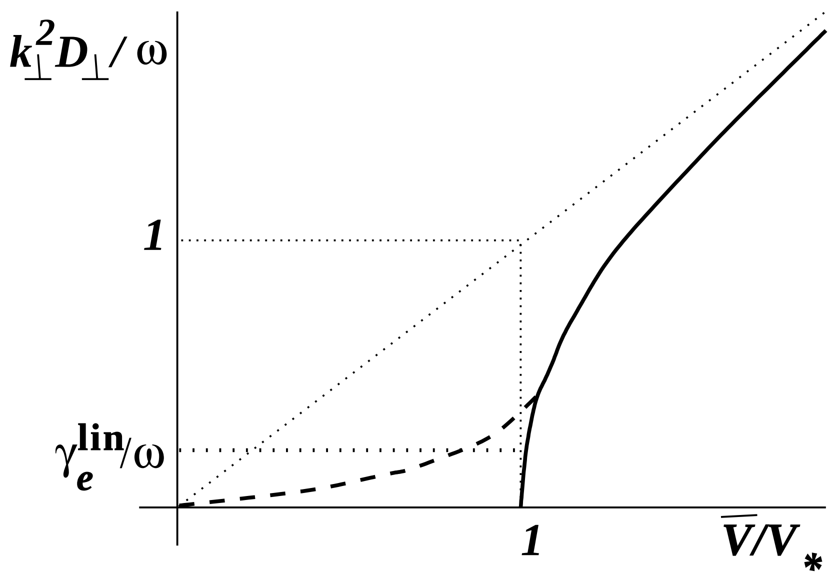

Fig. 14. Onset and saturation of cross-field diffusion due to $\boldsymbol{E} \times \boldsymbol{B}$ motions. $\overline{V} \doteq \langle \delta V_E^2 \rangle^{1/2}$; $\overline{V} \sim V_*$ is the onset of spatial stochasticity in Dupree's resonance-broadening scenario. The solid curve is the supercritical bifurcation that results when only the nonlinear correlation time τ_d is admitted; the dashed curve incorporates the possibility of a $\tau_{\rm ac}^{\rm lin}$. For self-consistent problems fluctuations are expected to saturate at $\overline{k}_\perp^2 D_\perp \sim \gamma_e^{\rm lin}$.

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[146] If formula (163) is evaluated for $k_\parallel = 0$ and $\Omega_{\boldsymbol{q}} \approx \omega_*(\boldsymbol{q})$, one obtains

$$D_\perp = \mathrm{Re} \sum_{\boldsymbol{q}} \frac{\langle \delta V_E^2 \rangle_{\boldsymbol{q}}}{-\mathrm{i}[\omega_*(\boldsymbol{q}) + \mathrm{i} q_\perp^2 D_\perp]} \approx \left(\frac{\overline{k}_\perp^2 D_\perp}{\overline{\omega}_*^2 + (\overline{k}_\perp^2 D_\perp)^2} \right) \overline{V}^2, \tag{f-9a,b}$$

where $\overline{V}^2 \doteq \sum_{\boldsymbol{q}} \langle \delta V_E^2 \rangle_{\boldsymbol{q}}$ and $\overline{k}_\perp$ is a typical perpendicular wave number. The real solution for $D_\perp$ of $1 = \overline{k}_\perp^2 \overline{V}^2 / [\overline{\omega}_*^2 + (\overline{k}_\perp^2 D_\perp)^2]$ exists only when $\overline{V}^2 > V_*^2$; then $D_\perp = (\overline{V}^2 - V_*^2)^{1/2} / \overline{k}_\perp$, which is equivalent to Eq. (204).

The saturation mechanism just described is reminiscent of the weak-turbulence ion-acoustic saturation described in Sec. 4.2.7 (p. 105); in both cases nonlinearities couple energy from the destabilizing electrons to the stabilizing ions. Clearly the RBT is intended to be some sort of strong-turbulence limit of WTT, but the details of the mode coupling underlying RBT are unclear at this point in the discussion.

4.3.4 Propagator renormalization and resonance-broadening theory

The best way of understanding the mathematical systematology, or lack thereof, underlying the RBT is to first develop a more general theory such as the DIA (Sec. 5, p. 126) or the MSR formalism (Sec. 6, p. 146), then to demonstrate the approximations necessary to recover RBT. That is done in detail from a kinetic renormalization in Sec. 6.5.5 (p. 178). Nevertheless, some introductory observations and references are appropriate here.

Fundamentally, Dupree renormalized the zeroth-order particle propagator by adding a turbulent diffusion term. A good interpretation of the resulting equations in terms of Langevin-related concepts (Sec. 3.2, p. 48) was given by Benford and Thomson (1972),[147] whose work is an important early reference on the basic formalism; some details are given in Appendix E (p. 281). Although Dupree did not clearly spell it out, the calculation is *passive*; backreaction of the particles on the fields was not considered.[148]

In more detail, consider the passive advection problem $g_0^{-1} f + \boldsymbol{V} \cdot \boldsymbol{\nabla} f = 0$, where $\boldsymbol{V} = \widehat{\boldsymbol{b}} \times \boldsymbol{\nabla}\varphi$. In Fourier space with $k \equiv (\boldsymbol{k}, \omega)$, this can be written as

$$g_{0,k}^{-1} f_k = \sum_{\Delta} M^U_{\boldsymbol{kpq}} \varphi_p^* f_q^*, \tag{205}$$

where the unsymmetrized mode-coupling coefficient is $M^U_{\boldsymbol{kpq}} = \widehat{\boldsymbol{b}} \cdot \boldsymbol{p} \times \boldsymbol{q} \equiv m_{\boldsymbol{k}}$. A standard second-order iterative renormalization following the procedures of Sec. 3.9.7 (p. 83)[149] leads to a renormalized passive propagator g that obeys $g_k^{-1} = g_{0,k}^{-1} + \Sigma_k^{(\mathrm{d})}$, where

$$\Sigma_k^{(\mathrm{d})} \doteq - \sum_{k+p+q=0} M^U_{\boldsymbol{kqp}} M^U_{\boldsymbol{pkq}} g_p^* I_q = \sum_{\Delta} \int \frac{\mathrm{d}\omega_{\boldsymbol{p}}\, \mathrm{d}\omega_{\boldsymbol{q}}}{2\pi}\, \delta(\omega + \omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}}) m_{\boldsymbol{k}}^2 g_p^* q^2 I_q \tag{206a,b}$$

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[147] An analogous discussion for magnetized plasma was given by Thomson and Benford (1978).

[148] The extra term in Eq. (102) is missing. Dupree (1966), in describing his formal theory of test waves, stated, "The method we employ for solving the Vlasov–Maxwell equation consists of two distinct pieces. First, we assume knowledge of the electric field $\boldsymbol{E}$ .... As a second step, we must ... require that the f so determined does ... produce the assumed $\boldsymbol{E}$ [via Poisson's equation]." Later he asserted, "The fact that the initial phases of the background waves in the subsidiary [test wave] problem are uncorrelated ... does not prevent the [Fourier coefficients] so calculated from being used to describe an actual system in which all the initial phases have some precise relation to each other and to f." However, freezing $\boldsymbol{E}$ in step 1 is the definition of a passive problem. Statistical correlations are lost at that point and cannot be recovered with the basic test-wave theory.

[149] An alternate technique sometimes employed in early research is to add an unknown term $\Sigma^{(\mathrm{d})}$ to both the left- and right-hand sides of the kinetic equation, then to choose $\Sigma^{(\mathrm{d})}$ to cancel undesired terms in a perturbative treatment (Rudakov and Tsytovich, 1971). That procedure works satisfactorily for second-order passive problems, but is difficult to generalize.

[note the conventions of Appendix A (p. 262)]. For definiteness, let us make the normal-mode approximation $I_q \approx 2\pi\delta(\omega_{\boldsymbol{q}} - \Omega_{\boldsymbol{q}})I_{\boldsymbol{q}}$. Then $\Sigma^{(\mathrm{d})}_{\boldsymbol{k},\omega} \approx k_\perp^2 D_{\boldsymbol{k},\omega}$ (I write D instead of $D_\perp$ to avoid clutter), where

$$D_{\boldsymbol{k},\omega} \doteq k^{-2} \sum_{\Delta} \sin^2(\boldsymbol{p},\boldsymbol{q}) g^*_{\boldsymbol{p}} q^2 I_{\boldsymbol{q}} = \sum_{\boldsymbol{q}} \sin^2(\boldsymbol{k},\boldsymbol{q}) g_{\boldsymbol{k}+\boldsymbol{q},\omega+\Omega_{\boldsymbol{q}}} q^2 I_{\boldsymbol{q}}. \qquad (207\text{a,b})$$

For isotropic spectra the first form is more convenient for manipulations at general k, but for considerations of the Markovian limit $\boldsymbol{k},\omega \to 0$ the second form is easier. Thus RBT asserts that an adequate approximation is $D_{\boldsymbol{k},\omega} \approx D \doteq \lim_{\boldsymbol{k},\omega\to 0} D_{\boldsymbol{k},\omega}$, or

$$D = \sum_{\boldsymbol{q}} \sin^2(\boldsymbol{k},\boldsymbol{q}) g_{\boldsymbol{q},\Omega_{\boldsymbol{q}}} (q^2 I_{\boldsymbol{q}}) \qquad (208)$$

[see Eq. (48) of Dupree (1968)]. Formula (208) is a natural generalization of the quasilinear expression (163).

That straightforward renormalization leads to a non-Markovian, $\boldsymbol{k}$- and ω-dependent $\Sigma^{(\mathrm{d})}_{\boldsymbol{k},\omega}$ can be the cause of significant confusion, as resonant and nonresonant effects are mixed together. This was discussed at length by Tetreault (1976).

The recipe (200) relating the linear and nonlinear dielectrics is a significant Ansatz. Dupree (1968) noticed that it is not correct in the presence of classical collisions, and he proposed modified formulas. That work was the first hint of considerable difficulties in the ultimate systematic justification of the resonance-broadening theory for self-consistent problems. [A systematic, formally exact theory of the nonlinear dielectric function is given in Sec. 6.5 (p. 170).] In related research Catto (1978) argued that the resonance-broadening approximation should be used for only the nonadiabatic part of the response. Some further discussion of that work was given by Krommes (1981).

Considerable literature on Dupree's techniques was written in the early days [see, for example, Gratzl (1970), Cook and Sanderson (1974), Peyraud and Coste (1974), Rolland (1974), and Vaclavik (1975)]. Some of their more formal features, which rely on manipulations involving a random particle propagator, are discussed in Appendix E (p. 281). The principle result, Eq. (E.7), formalizes Eq. (160) as, somewhat symbolically, $D = \int_0^\infty \mathrm{d}\tau\, \langle \delta a(t) U(\tau) \delta a(t-\tau) \rangle$, in terms of an averaged particle propagator U. A series of papers by Weinstock (1969, 1970), who used the projection-operator formalism (Sec. 3.9.11, p. 88), found this same result and clarified some of its foundations, including (Weinstock, 1968) the role of cumulant expansions (Sec. 3.5.2, p. 59). Significant attacks by Misguich (1974, 1975) and Misguich and Balescu (1975) should be noted. Additional physical and mathematical insights were given by Tetreault (1976). As will become clear from the general renormalization approaches of Secs. 5 (p. 126) and 6 (p. 146), the principle difficulty with Dupree-style renormalizations is that for self-consistent problems U differs from the response function R in a way that is difficult and unnatural to calculate; for more discussion, see Appendix E.2 (p. 285).

4.3.5 The relation of resonance-broadening theory to coherent response, incoherent response, and transfer

The interpretation of Dupree's saturation criterion $\gamma^{\mathrm{lin}} = \nu_d$ is best given by reference to the general form of the wave kinetic or spectral balance equation (182) that has already emerged in WTT but also holds more generally. The wave-number-dependent damping coefficient $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ generalizes the resonance-broadening frequency ν_d. In RBT two central approximations are made: (i) The incoherent noise on the right-hand side of the WKE is neglected. (ii) $\eta^{\rm nl}_{\boldsymbol{k}}$ is approximated (for the $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity) by $k_\perp^2 D_\perp$, where $D_\perp$ is a constant. Both of these are problematical.

If the right-hand side of Eq. (182) were negligible, saturation would occur when $\gamma^{\rm lin}_{\boldsymbol{k}} = {\rm Re}\,\eta^{\rm nl}_{\boldsymbol{k}}$, the generalization of Eq. (202). Such steady states are possible in principle because $\eta^{\rm nl}_{\boldsymbol{k}}$ is a functional of the fluctuation spectrum, which may be able to adjust in order to satisfy the balance. Nevertheless, this *coherent approximation* (Krommes and Kleva, 1979) seems difficult to justify in general because it neglects nonlinear effects of the same order as those that are retained. Such omission leads, for example, to gross violation of the quadratic conservation properties of the nonlinear terms. That error is compounded in the usual further approximation in which $\eta^{\rm nl}_{\boldsymbol{k}}$ is estimated by the diffusive operator $k_\perp^2 D_\perp$, which is at best appropriate for very small k. It seems clear (see also further discussion in the remainder of this section) that detailed wave-number dependences deduced from any such approximation should not be taken seriously. Nevertheless, if all $\boldsymbol{k}$'s are merely replaced by some typical $\overline{\boldsymbol{k}}$, the essentially dimensional balance between linear and nonlinear terms that is at the core of the coherent approximation may provide a crude estimate of the saturation level of the turbulence (note that both the coherent and incoherent effects stem from the same primitive nonlinearity, so cannot be distinguished on dimensional grounds). Yoshizawa et al. (2001) have reviewed some of the practical applications of (a matrix generalization [150] of) the coherent and diffusive approximation; for more details, see Itoh et al. (1999).

Neglect of the incoherent noise *may* be permissible when n-wave coupling effects are negligible and the kinetic wave–wave–particle interactions dominate; see the discussion of ion acoustic turbulence in Sec. 4.2.7 (p. 105). To see where those effects are buried in the resonance-broadening formalism, I follow the outlines of the seminal discussion by Rudakov and Tsytovich (1971). Consider a drift-wave problem, for which $g_{0,k} = [-{\rm i}(\omega - k_\parallel v_\parallel + {\rm i}\epsilon)]^{-1}$, and assume that the renormalized g will be used to estimate the nonlinear dielectric. Now $\omega \sim \Omega_{\boldsymbol{k}}$, so the Markovian approximation ($\omega \to 0$) is

[150] First consider a single field ψ. Suppose that the dynamics are represented in the Langevin form $(\partial_t + {\rm i}\mathcal{L}_{\boldsymbol{k}})\delta\psi_{\boldsymbol{k}}(t) = \delta\widetilde{f}_{\boldsymbol{k}}(t)$, where $\mathcal{L}_{\boldsymbol{k}}$ includes any coherent renormalizations and $\delta\widetilde{f}_{\boldsymbol{k}}(t)$ represents the incoherent noise. The steady-state spectrum obeys $\langle|\delta\psi_{\boldsymbol{k},\omega}|^2\rangle = F^{\rm nl}_{\boldsymbol{k},\omega}/|\omega - \mathcal{L}_{\boldsymbol{k}}|^2$, where $F^{\rm nl}$ is the covariance of the noise. The steady-state intensity is then

$$\langle|\delta\psi_k|^2\rangle = \int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\,\frac{F^{\rm nl}_{\boldsymbol{k},\omega}}{(\omega - \Omega_{\boldsymbol{k}} + {\rm i}\eta_{\boldsymbol{k}})(\omega - \Omega_{\boldsymbol{k}} - {\rm i}\eta_{\boldsymbol{k}})}, \tag{f-10}$$

where $\Omega_{\boldsymbol{k}} \doteq {\rm Re}\,\mathcal{L}_{\boldsymbol{k}}$ and $\eta_{\boldsymbol{k}} \doteq -\,{\rm Im}\,\mathcal{L}_{\boldsymbol{k}}$. If the integral in Eq. (f-10) is to remain nonzero as $F^{\rm nl} \to 0$, the imaginary parts of the poles stemming from the denominator must vanish: $\eta_{\boldsymbol{k}} \to 0$. This is the coherent approximation. The generalization to multiple coupled fields, represented by a coherent matrix L, is to first diagonalize L. Then the criterion for nonvanishing intensity is that the imaginary part of at least one eigenvalue vanishes. Let L be written as the sum of a Hermitian (symmetric) part L^s and an anti-Hermitian part L^a. If L is purely anti-Hermitian (dissipative), then the condition is equivalent to $\det \mathsf{L}^a = 0$. This equation is sometimes called a *nonlinear dispersion relation*, but the nomenclature is misleading. Consider again the scalar case. Then $\mathcal{L}^a$ vanishes identically, and in the frequency domain the coherent approximation is $(\omega - \Omega_{\boldsymbol{k}})\psi_{\boldsymbol{k},\omega} = 0$ or $\psi_{\boldsymbol{k},\omega} = 2\pi\psi_{\boldsymbol{k}}\delta(\omega - \Omega_{\boldsymbol{k}})$. The true nonlinear dispersion relation, $\omega = \Omega_{\boldsymbol{k}}$, does not involve the dissipative part. This analysis also demonstrates another inconsistency of the coherent approximation: it predicts a linelike frequency spectrum (Dupree and Tetreault, 1978) whereas truly turbulent states are well known to have broad spectra. That broadening is due to the incoherent noise, which clearly cannot be neglected.

inappropriate. Furthermore, if the fluctuations are sufficiently small, the waves live in the nonresonant region $k_\parallel v_{ti} \ll \omega \ll k_\parallel v_{te}$. Therefore ω is *large* with respect to the ions (the fluid limit). For them one may thus expand according to $g \approx g_0 - g_0\Sigma^{(\mathrm{d})}g_0$, thereby transferring the nonlinearity to the numerator. It will be shown in Sec. 6.5.4 (p. 176) that the result provides *half* (an unsymmetrized piece) of the induced scattering from the *bare* particles. The plausibility of this result can be seen from the presence of the propagator g_p in Eq. (207a). The long-wavelength limit describes ion diffusion, but the asymmetry means that the action conservation laws are violated.

For the electrons the resonance broadening is a small correction even for $V_E \sim V_*$; the approximation (200) then predicts a diffusive contribution, in accord with the simple random-walk theories.

It is important to note that for $T_i \to 0$ the induced-scattering contributions vanish since they require ion thermal motion. Therefore the Dupree-style passive renormalization of the *kinetic* response fails to recover any nonlinear effects related to the ion polarization-drift fluid nonlinearity $\boldsymbol{V}_E \cdot \boldsymbol{\nabla}(-\nabla_\perp^2\varphi)$ introduced in Sec. 2.4.3 (p. 34). This important conclusion is verified in more detail in Sec. 6.5.4 (p. 176).

Fluid rather than kinetic theory is of considerable practical importance in view of the model equations of Sec. 2.4 (p. 33). If one is given a robust fluid equation, it should not be necessary to engage in kinetic renormalizations at all; renormalization of the *fluid* nonlinearity should be adequate. Such renormalizations were considered by Weinstock and Williams (1971). If one renormalizes a $\boldsymbol{V}_E \cdot \boldsymbol{\nabla} n$ nonlinearity, passive diffusive renormalization straightforwardly leads to the estimate $\eta_{\boldsymbol{k}}^{\mathrm{nl}} \sim k_\perp^2 D_{\boldsymbol{k}}$. The polarization-drift nonlinearity can be similarly renormalized, leading to the result (209) discussed below.

Since simple kinetic RBT does not lead to polarization-drift effects but simple fluid RBT does, the distinction between kinetic and fluid renormalizations is evidently quite subtle. The reader is advised to return to these remarks after studying Sec. 6.5.5 (p. 178).

Usually $\boldsymbol{V}_E$ is *self-consistently related to* the vorticity $\nabla^2\varphi$; then passive renormalization *a la* RBT is not appropriate. The DIA (Sec. 5, p. 126) properly includes the effects of self-consistency and makes a prediction for $\eta_{\boldsymbol{k}}^{\mathrm{nl}}$ different from the passive one. The principle effects are already evident in the Markovian form (183a), which for the polarization-drift nonlinearity reads

$$\eta_{\boldsymbol{k}}^{\mathrm{nl}} = -\sum_{\Delta} m_{\boldsymbol{k}}^2 \left(\frac{q^2 - p^2}{1 + k^2}\right)\left(\frac{k^2 - q^2}{1 + p^2}\right)\theta_{\boldsymbol{pqk}}^* I_{\boldsymbol{q}}. \tag{209}$$

One can verify that the passive approximation leads to only the p^2k^2 term in the numerator, which is clearly positive definite for positive definite θ. The fully self-consistent form, however, is not positive definite. Let $I_{\boldsymbol{k}}$ be concentrated near a characteristic wave number $\overline{k}$—for example, $I_{\boldsymbol{k}} \approx 2\pi\overline{k}^{-1}\delta(k-\overline{k})$. Perform the summation over $\boldsymbol{p}$ to replace $\boldsymbol{p}$ by $-(\boldsymbol{k}+\boldsymbol{q})$, and integrate over $\boldsymbol{q}$ as $k \to 0$ for fixed $\overline{k}$ and isotropic statistics. One finds $\lim_{k\to 0}\eta_{\boldsymbol{k}}^{\mathrm{nl}} \sim -k^4\overline{k}^2 D < 0$. (The passive contribution has this form with a plus sign.) Crossover between negative and positive $\eta_{\boldsymbol{k}}^{\mathrm{nl}}$ occurs for $k \sim \overline{k}$. The minus sign is related to the possibility of an inverse energy cascade in 2D (Sec. 3.8.3, p. 74). I shall revisit this point in the discussion of eddy viscosity in Sec. 7.3 (p. 189).

For the self-consistent problem it is inconsistent to precisely localize $I_{\boldsymbol{k}}$ at $\overline{\boldsymbol{k}}$ because the damping $\eta_{\boldsymbol{k}}^{\mathrm{nl}} I_{\boldsymbol{k}} \sim (k^2 - q^2)I_{\boldsymbol{q}}I_{\boldsymbol{k}}$ then vanishes for $\boldsymbol{k} = \overline{\boldsymbol{k}}$. (Passive renormalization does not encounter this difficulty.) Therefore the spectrum must actually be spread over a range of $\boldsymbol{k}$'s. The transfer equation (117) provides the natural averaging. From the WKE (182), $\mathcal{T}(\Delta_{\boldsymbol{k}}) = \sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}}(\eta_{\boldsymbol{k}}^{\mathrm{nl}} I_{\boldsymbol{k}} - F_{\boldsymbol{k}}^{\mathrm{nl}})$, or

$$\mathcal{T}(\Delta_{\boldsymbol{k}}) = -\frac{1}{2}\sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}}\sum_{\Delta}\sigma_{\boldsymbol{k}}M_{\boldsymbol{kpq}}\theta_{\boldsymbol{kpq}}(\underbrace{M_{\boldsymbol{pqk}}I_{\boldsymbol{q}}I_{\boldsymbol{k}} + M_{\boldsymbol{qkp}}I_{\boldsymbol{k}}I_{\boldsymbol{p}}}_{-\eta^{\mathrm{nl}}_{\boldsymbol{k}}I_{\boldsymbol{k}}} + \underbrace{M_{\boldsymbol{kpq}}I_{\boldsymbol{p}}I_{\boldsymbol{q}}}_{F^{\mathrm{nl}}_{\boldsymbol{k}}}). \tag{210}$$

Resonance-broadening theory neglects the incoherent response (the $F^{\mathrm{nl}}_{\boldsymbol{k}}$ term), but that term is clearly required in order to provide the symmetry that guarantees conservation of the quadratic invariants. Because $\mathcal{T}(\infty) = 0$ (there is no net transfer due to the nonlinearities), $F^{\mathrm{nl}}_{\boldsymbol{k}}$ is of the same order as the $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ terms. In the presence of $F^{\mathrm{nl}}_{\boldsymbol{k}}$, the RBT balance $\gamma^{\mathrm{lin}}_{\boldsymbol{k}} = \eta^{\mathrm{nl}}_{\boldsymbol{k}}$ clearly requires further discussion; it cannot be literally true.

Assume homogeneous turbulence for simplicity and let the spectrum extend over a broad range of k's centered at $\overline{k}$ and of width Δk. In order to ensure a steady state, let there be a positive linear growth rate γ_{in} and a positive linear damping rate γ_{out} that are concentrated over regions $\Delta k_{\mathrm{in,out}}$ near $k_{\mathrm{in,out}}$—for example, $\gamma_{\mathrm{in}} \sim 2\pi\Delta k_{\mathrm{in}}\delta(k - k_{\mathrm{in}})\overline{\gamma}_{\mathrm{in}}$. Temporarily, assume that the latter widths are smaller than Δk. The situation $k_{\mathrm{out}} < \overline{k} < k_{\mathrm{in}}$ is sketched in Fig. 15. The steady-state balance for transfer of an invariant from the modes between 0 and k is then

$$-[\gamma_{\mathrm{out}}I_{\mathrm{out}}] + [\gamma_{\mathrm{in}}I_{\mathrm{in}}] = \mathcal{T}(k), \tag{211}$$

where a bracketed term contributes only if k is to the right of the appropriate source or sink region. Note that according to Eq. (210) incoherent noise always contributes negative transfer whereas $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ contributes positive or negative transfer depending on its sign. For example, consider $k_{\mathrm{out}} < k < \overline{k}$. Then the balance equation is $-\gamma_{\mathrm{out}}I_{\mathrm{out}} = \mathcal{T}(k)$ and the transfer is necessarily negative. For the polarization-drift nonlinearity $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ is negative in that region, so both coherent and incoherent response contribute to transfer with the same (negative) sign. Because the $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ and $F^{\mathrm{nl}}_{\boldsymbol{k}}$ terms both arise from the same basic nonlinearity, the nonlinear scaling of $\mathcal{T}(k)$ can be estimated from either one. The k integration [$\sum_{\boldsymbol{k}\in\Delta_{\boldsymbol{k}}}$ in Eq. (210)] basically averages $\eta^{\mathrm{nl}}_{\boldsymbol{k}}I_{\boldsymbol{k}}$ over the excited spectrum to the left of k, giving rise to the balance $\gamma_{\mathrm{out}}I_{\mathrm{out}} \sim |\eta^{\mathrm{nl}}|I$, where η^{nl} and I are typical values in the left-hand region. If k_{out} is in the middle of the excited region, then simply $\gamma_{\mathrm{out}} \sim |\eta^{\mathrm{nl}}_{\mathrm{out}}|$, which provides an estimate for I_{out} given γ_{out}. I_{in} can be estimated from the balance $\gamma_{\mathrm{in}}I_{\mathrm{in}} - \gamma_{\mathrm{out}}I_{\mathrm{out}} = 0$, which follows by integrating to $k > k_{\mathrm{in}}$.

Note that the spectral averaging is crucial. If k is taken to lie in a region where $\gamma_{\boldsymbol{k}} = 0$, then the steady-state balance is $\eta^{\mathrm{nl}}_{\boldsymbol{k}}I_{\boldsymbol{k}} = F^{\mathrm{nl}}_{\boldsymbol{k}}$, quite different from RBT. For related discussion of situations with $\gamma_{\boldsymbol{k}} = 0$, see Ottaviani and Krommes (1992).

If the roles of g_{in} and γ_{out} are reversed in the above scenario, steady state would require transfer to the right. To the extent that $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ is negative, such transfer is impossible. Even if $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ were positive, its effect would be reduced by the negative definite incoherent transfer. The situation is obviously tricky because of the possibility of dual cascade.

For forcing and dissipation distributed over a spectral range with just one characteristic wave number $\overline{k}$, the above estimates simplify with $I_{\mathrm{in}} \sim I_{\mathrm{out}} \sim I \sim \overline{I}$ and $k_{\mathrm{out}} \sim \overline{k} \sim k_{\mathrm{in}}$. Then $\gamma_{\mathrm{out}} = \gamma_{\mathrm{in}}$ and the balance $\gamma_{\mathrm{out}} \sim |\eta^{\mathrm{nl}}|_{\mathrm{out}}$ reduces to $\overline{\gamma} \sim |\eta^{\mathrm{nl}}_{\overline{k}}|$. The basic size of $|\eta^{\mathrm{nl}}_{\overline{k}}|$ might adequately be estimated from its (positive) passive contribution. For the polarization drift, that is in dimensional units $\eta^{\mathrm{nl}}_{\overline{k}} \sim (\overline{k}\rho_{\mathrm{s}})^4\overline{k}^2D$. This result obviously differs from the estimate $\overline{k}^2D$ obtained by evaluating the diffusive resonance-broadening prediction k^2D at a typical wave number $\overline{k}$.

It should now be clear that the RBT cannot be quantitatively correct; it is pointless to begin with

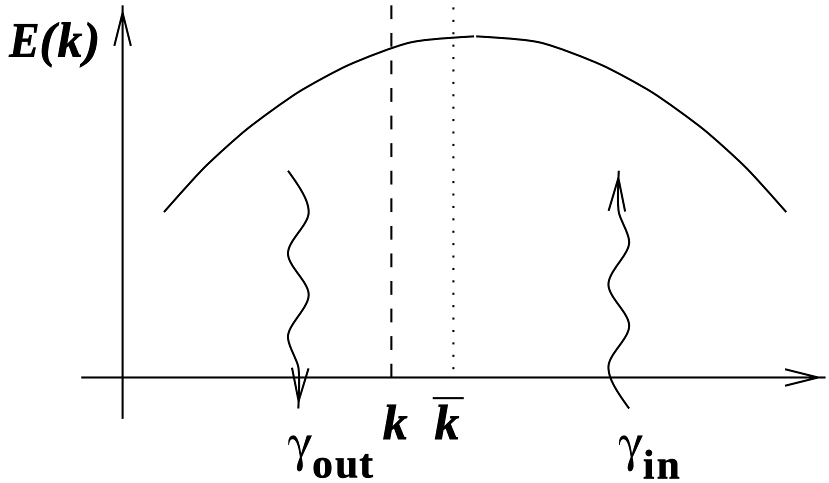

Fig. 15. Hypothetical scenario that illustrates steady-state spectral transfer.

its various approximations, then proceed to deduce values for transport coefficients correct to several decimal places, as has been done all too frequently in the literature. The importance of symmetries and conservation properties was discussed for drift-wave problems by Dupree and Tetreault (1978), whose work was a seminal contribution to the line of research that grew out of RBT. A more definitive and general discussion was given by Boutros-Ghali and Dupree (1981); see also Similon (1981). Essentially, those authors were rediscovering the symmetries of both systematically calculated WTT and robust renormalizations like the DIA. [The latter had already been emphasized in the work of Orszag and Kraichnan (1967); further illuminating discussion was given in the beautiful paper of Kraichnan (1976b) on eddy viscosity, to be discussed in Sec. 7.3 (p. 189)]. I shall defer further discussion of these topics until I develop the general apparatus of renormalized turbulence theory; see Sec. 6.5 (p. 170).

It must be emphasized that the characteristic growth rate $\overline{\gamma}$ that enters into the simplest integrated spectral balance is merely a particular integral property characteristic of the entire energy-containing range, as is $\overline{k}_{\perp}$. The rough balances given above are simply incapable of capturing details of wave-number dependence, spectral shape, *etc.* For such information one must turn to more elaborate closures such as the DIA (Sec. 5, p. 126) or its Markovian relatives (Sec. 7.2, p. 182). In Sec. 8 (p. 199) it is shown that such closures can make quantitatively accurate predictions for transport.

Finally, the estimate $D_{\perp} \sim \overline{\gamma}/\overline{k}_{\perp}^{2}$ is simply incorrect in general if $D_{\perp}$ is taken to be the fluid diffusion coefficient. For example, in HM dynamics the natural transport involves diffusion of vorticity, which, as we have seen, introduces various powers of $k_{\perp}^{2}\rho_{\mathrm{s}}^{2}$ into the calculation. Although those are $O(1)$ in the gyrokinetic ordering, they may be quantitatively significant. Indeed, since there are intimate relations between transport and dissipation (thus between transport and spectral transfer), proper estimates of spectral quantities are relevant to experimentally observable fluxes. For more discussion, see Sec. 12.2 (p. 238).

4.3.6 Summary: Approximations underlying resonance-broadening theory

In summary, a variety of conceptual approximations underly the RBT. (i) Most fundamentally, *the incoherent response is neglected.* That leads to violation of energy conservation and precludes turbulent steady states of the kind envisaged in most fluid-turbulence theories. (ii) *The RBT is at best appropriate for passive*, not self-consistent, *advection.* The passive assumption leads to the neglect of a variety of terms that reflect backreaction of the particles on the waves. (iii) The formalism is in essence a crude theory of the nonlinear dielectric function. However, the form of that function is not derived systematically, and indeed the recipe (200) will be shown below to be incorrect—importantly so for strong turbulence. (iv) *Unjustifiable Markovian approximations* (in both space and time) *are made*. Such approximations are appropriate when a separation of scales exists. Although there may be a scale disparity between microscopic fluctuations and macroscopic transport, the details of microscopic events are intrinsically non-Markovian. Analysis of spectral mode coupling and transfer requires that one study the interaction of energy-containing fluctuations that are all of the same order in k and ω.

Because of the previous points, the RBT cannot provide a quantitatively accurate description of steady-state turbulence. Nevertheless, in a very coarse-grained and dimensional sense, it can in some circumstances be used to motivate formulas such as Eq. (203) for a perpendicular transport coefficient. Furthermore, simple RBT can make qualitatively and sometimes even quantitatively successful predictions for situations in which the passive approximation is appropriate. One example is the experiment of Hershcovitch and Politzer (1979), in which an instability was suppressed by the introduction of external turbulence.

Although various approximations are made, the RBT represents a serious early attempt on a very difficult problem. Its major contributions were to focus attention on important physical processes of strong plasma turbulence (including kinetic physics) and to provide simple ways of estimating their significance. As we consider more elaborate closures [e.g., the DIA in Sec. 5 (p. 126)], it is important to keep the intuition behind the RBT firmly in mind. The prescience of the insights of Dupree, Weinstock, and the other early workers into the *physics* of strongly turbulent plasma processes, coming as they did before the results of nonlinear dynamics and stochasticity theory were widely known, is remarkable.

4.4 Clumps

"The physics picture [behind the fluid clump algorithm] is, in its emphasis on the small scales, in disagreement with well-established facts about the small-scale behavior of turbulent fluids."
— *Krommes (1986a).*

As described in the last section, RBT ignores incoherent response. Dupree (1970) recognized this difficulty relatively early. Motivated by the structure of classical Langevin equations (Sec. 3.2, p. 48) and plasma kinetic theory, in which the fluctuation effects are described by both a velocity-space-diffusion term and a polarization-drag term [cf. the Balescu–Lenard operator (32)], he proposed that a description of kinetic plasma turbulence more complete than RBT should involve a *turbulent Fokker–Planck equation*. This insight was a definite advance; the topic will be revisited in Sec. 6.5.6 (p. 180). A related paper that also introduced a turbulent Balescu–Lenard type of operator was by Kadomtsev and Pogutse (1970a); see also the more detailed calculations by Kadomtsev and Pogutse (1971).

4.4.1 Dupree's original arguments

Dupree (1972a,b) clarified his motivations by pointing out the physical importance of *phase-space granulations*, i.e., extreme distortions of phase-space fluid elements arising from nonlinear processes. His thinking was guided by the existence of *BGK modes* (Bernstein et al., 1957), which are exact nonlinear solutions of the Vlasov equation that are obviously not well described by simple diffusion theories, and by insights gained from studies of coherent trapping (O'Neil, 1965), which show how initial phase-space perturbations are distorted by nonlinearity. Of course, phase-space fluid elements are sheared at an exponentially rapid rate even in stochastic regimes, in which no trapping occurs. It is interesting to note that the early work of Dupree preceded general awareness within the plasma-physics community of the modern advances in stochasticity, chaos, and nonlinear dynamics, which occurred in the middle 1970s [see, for example, Smith and Kaufman (1975)].

Dupree's attempts at an analytical description that incorporated phase-space granulation, however, introduced a fundamental confusion that has persisted, in one form or another, to the present day. He recognized (correctly) that phase-space granulation in its various guises was not described by RBT (which contains no hint of the stochastic instability of two adjacent orbits, for example), and argued (correctly) that incoherent noise was essential. He also noted (at least implicitly) the structure of the WTT, in which the form and role of the coherent and incoherent parts of the nonlinearity are well defined; in particular, he recognized that parts of n-wave mode coupling are contained in the incoherent noise. He thus wrote

$$\delta f = \delta f^{\mathrm{coh}} + \delta f^{\mathrm{inc}}, \quad \text{where} \quad \delta f^{\mathrm{inc}} = \delta f^{\mathrm{mc}} + \delta f^{\mathrm{clumps}}. \tag{212a,b}$$

The notation mc stands for *mode coupling*. The clump contribution was intended to take account of small-scale granulations in phase space, which Dupree [echoing the remarks of Kadomtsev and Pogutse (1970a, 1971) [151]] suggested could behave as sources of fluctuations in the same sense as do the discrete particles of classical kinetic theory. Dupree stated, "δf^{mc} denotes all other effects[,] which we shall ignore."

One should note here a subtle change in the description of the fluctuations. Whereas in Sec. 4.2.4 (p. 102) coherent and incoherent terms were defined in the (ensemble-averaged) wave kinetic equation, Eq. (212a) purports to divide the *random variable* δf into coherent and incoherent parts. It is not immediately clear that such a decomposition is permissible. Dupree defined the coherent response as that part that is "phase-coherent" with the electric field. Fortunately, that concept can be generalized to statistical theory (Krommes, 1978) with the aid of the theory of Langevin representations to be described in Secs. 5.3 (p. 132) and 8.2.2 (p. 201), which shows that Eq. (212a) does indeed make sense. That coherent and incoherent response can be defined precisely in renormalized statistical dynamics is one of the major triumphs of the modern formalism (Sec. 6.5.3, p. 173).

Dupree did not attempt any systematic classification of the two pieces δf^{mc} and $\delta f^{\mathrm{clumps}}$ of the incoherent kinetic noise. Intuitively, he seems to have intended that δf^{mc} was the fluctuation calculable from WTT, while $\delta f^{\mathrm{clumps}}$ was everything else. For the kinetic problem, [152] which includes both fine-scaled $\boldsymbol{v}$-space dynamics as well as velocity-integrated potentials, that is not unreasonable.

[151] In the context of beam–plasma interactions, Kadomtsev and Pogutse explicitly calculated the clump correlation function as a ballistic remnant of initial conditions. However, they incisively remarked (Kadomtsev and Pogutse, 1971), "But of course [macroparticles] can be generated by the turbulent plasma itself so that the problem arises of considering the generation and destruction of [clumps] in the turbulent plasma and of clarifying their role in turbulent processes."

[152] Space limitations preclude a thorough treatment of the explicitly kinetic theory of clumps; for more discussion, see Hui and Dupree (1975) and Dupree (1978).

4.4.2 The clump lifetime

To describe the dynamics of the clumps, Dupree argued in part as follows. [These arguments, not all of which are correct, will be critiqued below. A concise but clear summary of Dupree's ideas was given by Liang and Diamond (1993b); see also Terry and Diamond (1984).] (i) Clumps essentially behave like point particles with a finite lifetime τ_{cl}. (ii) τ_{cl} can be determined, by a consideration of the *relative diffusion* of two adjacent trajectories in phase space,[153] to be the time for two trajectories separated by a small distance δ to separate a distance of the order of the correlation length L_c of the turbulence, i.e., a scale characteristic of the energy-containing range. One finds

$$\tau_{\text{cl}} \sim \tau_d \ln(L_c/\delta) \gg \tau_d, \tag{213}$$

where τ_c is a characteristic energy-containing time. (iii) The steady-state spectral level $\mathcal{S}$ is determined by a balance between the production term $\mathcal{P}$ [cf. the Navier–Stokes paradigm of Sec. 2.1.1 (p. 23)] and the decay $\mathcal{S}/\tau_{\text{cl}}$ due to the finite clump lifetime:

$$\mathcal{S} \sim \tau_{\text{cl}} \mathcal{P}. \tag{214}$$

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[153] Linearize Newton's laws to find

$$\Delta\dot{x} = \Delta v, \quad \Delta\dot{v} = (q/m)E'(x(t), t)\Delta x, \tag{f-11a,b}$$

where $E'(x,t) \doteq \partial_x E(x,t)$. The second moments of Eqs. (f-11) exactly obey

$$\langle \Delta x^2 \rangle^{\cdot} = 2\langle \Delta x\, \Delta v \rangle, \quad \langle \Delta x\, \Delta v \rangle^{\cdot} = \langle \Delta v^2 \rangle + (q/m)\langle E'\, \Delta x^2 \rangle, \quad \langle \Delta v^2 \rangle^{\cdot} = 2(q/m)\langle E'\, \Delta x\, \Delta v \rangle. \tag{f-12a,b,c}$$

Eliminate Δv in Eq. (f-12c) by integrating Eq. (f-11b) and making a quasinormal Markovian approximation, appropriate for short autocorrelation time τ_{ac}. [These arguments were presented in the context of stochastic magnetic field lines by Krommes et al. (1983).] Finally, drop the last term of Eq. (f-12b), which can be shown to be small in τ_{ac}. One then finds

$$\langle \Delta x^2 \rangle^{\cdot} = 2\langle \Delta x\, \Delta v \rangle, \quad \langle \Delta x\, \Delta v \rangle^{\cdot} = \langle \Delta v^2 \rangle, \quad \langle \Delta v^2 \rangle^{\cdot} = 2\overline{k}^2 D'' \langle \Delta x^2 \rangle, \tag{f-13a,b,c}$$

where $D'' \doteq \overline{k}^{-2}(q/m)^2 \int_0^\infty \mathrm{d}\tau\, \langle E'(\tau)E'(0) \rangle$ and $\overline{k}$ is a characteristic wave number. Equations (f-13) can be combined to yield $\langle \Delta x^2 \rangle^{\cdots} - (2/\tau_K)^3 \langle \Delta x^2 \rangle = 0$ with $\tau_K = (\overline{k}D''/2)^{-1/3}$, an equation first given by Dupree (1972b). Exact solution is straightforward but uninteresting in detail. The long-time solution is $\langle \Delta x^2 \rangle \sim \frac{1}{3}\Delta y^2(0)\mathrm{e}^{2t/\tau_K}$, where $\Delta y^2 \doteq \Delta x^2 + 2\Delta x\, \Delta v\, \tau_K + 2\Delta v^2 \tau_K^2$. Dupree (1972b) defined τ_{cl} to be the time to separate one wavelength, $\overline{k}^2 \langle \Delta x^2 \rangle \sim 1$; thus

$$\tau_{\text{cl}} \approx \tfrac{1}{2}\tau_K \ln\{[3\overline{k}^2 \Delta y^2(0)]^{-1}\} \qquad (\tau_{\text{cl}} > \tau_K); \tag{f-14}$$

τ_{cl} is taken to vanish for $\tau_{\text{cl}} < \tau_K$. This estimate is clearly very rough because the original linearization is valid only for $\overline{k}^2 \langle \Delta x^2 \rangle \ll 1$.

Further discussion of relative diffusion was given by Misguich and Balescu (1982). For a more refined calculation of the Liapunov time τ_K, see Rechester et al. (1979). In order of magnitude, τ_K is of the order of the diffusion time τ_d; see Eq. (159).

(***Warning:*** I show in the next section that this result is generally incorrect.) (iv) Because $\tau_{\rm cl} \gg \tau_d$, the formalism predicts an enhancement of the steady-state fluctuation level over the level one would obtain from a resonance-broadening type of theory based solely on the coherent fluctuations (which Dupree and others identified with conventional mode coupling).

4.4.3 Critiques of the clump formalism

It is clear from the early papers of Dupree (1970, 1972b) that he was initially concerned with and motivated by properties of the *kinetic* PDF. It is not difficult to argue that fine-scaled phase-space granulations are mistreated in the usual perturbative approaches, and to argue for the general importance of incoherent noise. [154] It is shown in Sec. 6.5.3 (p. 173) how a particular approximation to the kinetic incoherent noise is related to the relative diffusion calculation sketched in footnote 153 (p. 121).

Although one can give a formula for the kinetic $\tau_{\rm cl}$ [Eq. (f-14), p. 121], the general arguments of Sec. 4.4.2 (p. 121) use no specific properties of kinetic theory; they appear to apply to fluid problems as well. Indeed, although the formalism was first presented for the x–v phase space of a Vlasov plasma, Dupree (1974) soon developed a version for 2D turbulence in $\boldsymbol{x}$ space. Because of the importance of the 2D $\boldsymbol{E} \times \boldsymbol{B}$ motion for turbulence in strongly magnetized plasmas, it is the $\boldsymbol{x}$-space form of the theory that has been mostly used by subsequent authors [see, for example, Terry and Diamond (1985) or Lee and Diamond (1986)], who have treated the $\boldsymbol{x}$-space version of the "clump algorithm" as synonymous with a theory of incoherent noise. However, one must be very cautious. The particular properties of *phase*-space granulations cannot be relevant to such a formalism. If one replaces δf by a fluid fluctuation $\delta\psi$, the procedure of dividing one well-defined nonlinear effect ($\delta\psi^{\rm inc}$) into two ill-defined pieces ($\delta\psi^{\rm mc}$ and $\delta\psi^{\rm clumps}$) invites confusion. The difficulty is that "mode" coupling, in its most general manifestation, is not synonymous with n-wave processes; although those are well defined and classifiable in regular perturbation theory, they lose their identity in regimes of strong turbulence, in which well-formed linear eigenmodes do not dominate and terms of all orders in perturbation theory affect the evolution of even second-order quantities. Clean distinctions between wave–wave and other $\boldsymbol{k}$-space mode-coupling processes need not exist. Indeed, there are no waves at all in the NSE.

With reference to the $\boldsymbol{x}$-space algorithm, arguments that the philosophy of the clump formalism is flawed have been presented by Krommes (1986a), Krommes and Kim (1988), and Krommes (1997a).

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[154] As a matter of nomenclature, theories of the incoherent noise are often referred to in the plasma-physics literature as *two-point theories* because the clump lifetime depends on the motion of two adjacent trajectories; resonance-broadening calculations are called *one-point theories* because they fundamentally calculate the turbulent diffusion of a single trajectory. I do not use the terms "n-point theory" ($n = 1$ or 2) in this article since their use is precisely backwards (Krommes, 1986b) from the point of view of a general formalism based on correlation and response functions [Secs. 3.5.4 (p. 64) and 6 (p. 146)]. Namely, turbulent diffusion coefficients follow from the behavior of the response function $R(\boldsymbol{x}, t; \boldsymbol{x}', t')$, which clearly depends on *two* points in both space and time. On the other hand, the spectral level (which contains both coherent and incoherent contributions) is $\mathcal{S} = C(\boldsymbol{x}, t; \boldsymbol{x}', t')|_{\boldsymbol{x}'=\boldsymbol{x}, t'=t}$, which depends on just *one* point in both space and time (and furthermore is entirely independent of $\boldsymbol{x}$ for homogeneous turbulence). Unfortunately, a student of the plasma-physics literature needs to know that a "two-point" calculation has usually involved some sort of relative diffusion calculation whose signature is the logarithmic term $\ln(L_c/\delta)$ seen in formulas such as Eq. (213), even though that procedure is now understood to be incorrect (Krommes, 1997a).

As originally discussed by Krommes (1986a),[155] the fundamental feature that is difficult to grasp philosophically is the notion, formally expressed by Eq. (214), that the total spectral level should be determined by properties of the very small scales (at least when those scales contain negligible energy, which is almost always true). That violates well-established experimental properties of turbulence (Frisch, 1995, Chap. 5), standard Kolmogorov-like ideas about cascade (Sec. 3.8.2, p. 73), and common sense.[156] Krommes pointed out that some of the central mathematical manipulations of the CA were identical to the much earlier calculations of Batchelor (1953) for the dynamics and spectra of the very small (far-inertial-range) scales; it was clear that Batchelor did not believe that his methods or results applied to energy-containing scales. A more mathematically precise discussion was presented by Krommes and Kim (1988), who addressed the decompositions (212). They argued that (i) calculation of the incoherent noise is *not* equivalent to a theory of the small scales, and (ii) conventional wave–wave coupling is *not* synonymous with coherent response. As one has already seen from the structure of the WTT, physical processes (renormalized mode–mode coupling) whose balances determine the spectral levels in the energy-containing range (and therefore the total spectral level) appear symmetrically in both the coherent *and the incoherent parts of the fluctuations as well.* They used this observation to argue both qualitatively and quantitatively against the results of Terry et al. (1986), who had used a clump analysis to conclude that magnetic fluctuations cannot contribute to self-consistent transport to any order.[157]

4.4.4 Two-point structure function and the clump approximation

The cleanest way of understanding the mathematical mistake in the $\boldsymbol{x}$-space clump formalism is to consider an exactly solvable model. In several important works Kraichnan (1968b, 1994) developed the theory of a randomly advected passive scalar for which the velocity field $\boldsymbol{u}$ changes very rapidly in time. One considers a scalar field $T(\boldsymbol{x}, t)$ that obeys

$$\partial_t T + \boldsymbol{u} \cdot \boldsymbol{\nabla} T - \kappa \nabla^2 T = f^{\text{ext}}(\boldsymbol{x}, t), \tag{215}$$

where $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ and a Gaussian white-noise forcing f^{ext} (uncorrelated with $\boldsymbol{u}$) is used to model the (long-wavelength) production of turbulent fluctuations. I assume isotropic statistics, so one can write

$$\langle f^{\text{ext}}(\boldsymbol{x} + \boldsymbol{\rho}, t + \tau) f^{\text{ext}}(\boldsymbol{x}, t) \rangle = 2F^{\text{ext}}(\rho)\delta(\tau/\tau_{\text{ac}}^{(f)}), \tag{216}$$

where $\tau_{\text{ac}}^{(f)}$ plays the role of a microscopic autocorrelation time. ($\tau_{\text{ac}}^{(f)}$ is inserted for dimensional purposes in order that $[F] = [f^2]$; its actual value turns out to be irrelevant.) The strength of the forcing is given by $\overline{F} \doteq F^{\text{ext}}(0) = \frac{1}{2}\sigma/\tau_{\text{ac}}^{(f)}$, where σ is the production rate of scalar variance; the Fourier transform of $F^{\text{ext}}(\rho)$ is assumed to be concentrated at small (energy-containing) k's. The significant simplifying features of this model are (i) in the rapid-change limit for $\boldsymbol{u}(\boldsymbol{x}, t)$, the effects of the nonlinearity can be evaluated *exactly* (Kraichnan, 1994); and (ii) the use of random forcing precludes the need for an elaborate calculation of the actual production (which would otherwise be described by two-point cross correlations). Even so, the study of the statistical properties of

[155] A formal Reply to Krommes (1986a) was made by Terry and Diamond (1986). A reply to that Reply was given by Krommes (1986b).
[156] That is, one is concerned that the (spectral) tail wags the dog (total fluctuation level).
[157] Additional, somewhat orthogonal discussion of the work of Terry et al. (1986) was given by Thoul et al. (1987).

Eq. (215) for moments of order higher than two is difficult and the focus of much current interest [158] (HydrotConf, 2000). Fortunately, Kraichnan (1994) showed that the equation for two-space-point, equal-time correlations rigorously closes. He presented his result in terms of the second-order structure function $S_2(\rho,t) \equiv S(\rho,t)$ defined by Eq. (103), and that approach was followed by Krommes (1997a). Nevertheless, one can just as well write the equation for $C(\boldsymbol{\rho},t) \doteq \langle \delta T(\boldsymbol{x}+\boldsymbol{\rho},t)\delta T(\boldsymbol{x},t)\rangle$. For isotropic statistics one finds [159]

$$\partial_t C(\rho,t) - 2\rho^{-(d-1)}\partial_\rho[\rho^{d-1}\eta_-(\rho)\partial_\rho C] = 2F^{\rm ext}(\rho)\tau_{\rm ac}^{(f)}, \tag{217}$$

where $\eta_-(\rho) \doteq \eta_-^{\rm nl}(\rho) + \kappa$ and

$$\eta_-^{\rm nl}(\rho) \doteq \frac{1}{2}\int_0^\infty {\rm d}\tau\, \langle \Delta u_\parallel(\boldsymbol{\rho},t)\Delta u_\parallel(\boldsymbol{\rho},t-\tau)\rangle, \tag{218}$$

with $\Delta u_\parallel \doteq [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\rho},t) - \boldsymbol{u}(\boldsymbol{x},t)]\cdot\widehat{\boldsymbol{\rho}}$, is the two-particle eddy diffusivity. The $\eta_-^{\rm nl}(\rho)$ term describes the statistical effect of the advective nonlinearity in Eq. (215) *exactly* in the rapid-change limit. Therefore with reference to the general formalism developed in Sec. 6 (p. 146), especially Sec. 6.5.3 (p. 173), it can be seen that $\eta_-^{\rm nl}(\rho)$ includes contributions from both coherent and incoherent response (because $\eta_-^{\rm nl}$ is the sole vestige of the nonlinearity), in agreement with the earlier approximate calculations of Dupree (1974).

The limits

$$\eta_-^{\rm nl}(0) = 0, \quad \lim_{\rho\to\infty}\eta_-^{\rm nl}(\rho) = D, \tag{219a,b}$$

where D is the single-particle turbulent diffusivity, have long been recognized in neutral-fluid turbulence. In plasma physics they were emphasized by Dupree (1972b) and others (Misguich and Balescu, 1982), and figure prominently in the clump algorithm. Equation (219a) states that two fluid elements coincident at $t = 0$ remain so forever; Eq. (219b) states that such elements separated by more than a correlation length L_c diffuse independently. For $\rho \to 0$ it can be shown [see Krommes (1997a) for all details of the following arguments] that

$$\eta_-^{\rm nl}(\rho) \approx [\rho/\lambda_T^{(u)}]^2 D, \tag{220}$$

[158] Considerable progress in the analytical description of the 2D problem was made by Chertkov et al. (1996).

[159] The proof relies on the result that for infinitely rapid variations of $\boldsymbol{u}$ and f, first-order perturbation theory is exact [see Appendix H (p. 293)]. One has exactly

$$\partial_t C(\boldsymbol{\rho},t) + \boldsymbol{\nabla}\cdot\langle\delta\boldsymbol{u}(\boldsymbol{x},t)\delta T(\boldsymbol{x},t)\delta T(\boldsymbol{x}',t)\rangle + (\boldsymbol{x}\leftrightarrow\boldsymbol{x}') = \langle\delta f(\boldsymbol{x},t)\delta T(\boldsymbol{x}',t)\rangle + (\boldsymbol{x}\leftrightarrow\boldsymbol{x}'). \tag{f-15}$$

Expand $\langle\delta\boldsymbol{u}\,\delta T\,\delta T'\rangle = [\langle\delta\boldsymbol{u}\,\delta T^{(0)}\delta T'^{(1)}\rangle + \langle\delta\boldsymbol{u}\,\delta T^{(1)}\delta T'^{(0)}\rangle]$, where, for example, $\delta T^{(1)}(\boldsymbol{x}',t) = \int_{-\infty}^t {\rm d}\overline{t}\,[\delta f(\boldsymbol{x}',\overline{t}) - (\delta\boldsymbol{u}\cdot\boldsymbol{\nabla}\delta T^{(0)})(\boldsymbol{x}',\overline{t})]$. One ultimately finds

$$\langle\delta\boldsymbol{u}\,\delta T\,\delta T'\rangle = \int_{-\infty}^t {\rm d}\overline{t}\,[\mathsf{U}(0,\overline{t}) - \mathsf{U}(\boldsymbol{\rho},\overline{t})]\cdot\boldsymbol{\nabla}C(\boldsymbol{\rho},t), \tag{f-16}$$

where $\mathsf{U}(\boldsymbol{\rho},\tau)$ is the covariance of $\boldsymbol{u}$. The final form of Eq. (217) follows by a straightforward symmetrization and simplification of the operator $\boldsymbol{\nabla}\cdot(\ldots)\boldsymbol{\nabla}$ for homogeneous, isotropic statistics.

where $\lambda_T^{(u)}$ is the Taylor microscale (Sec. 3.6.3, p. 66) of the velocity field. This ρ^2 dependence underlies the appearance of τ_{cl} in the clump calculations. By calculations analogous to those in footnote 153 (p. 121) (or more rigorously), one can show that such dependence leads in the present case to a mean-squared divergence of adjacent trajectories (initially separated by ρ') that obeys $\langle \rho^2 \rangle (t \mid \rho') = \mathrm{e}^{2t/\tau_K} \rho'^2$, where $\tau_K \doteq [2(d+2)k_T^2 D]^{-1}$. This result leads to a "clump lifetime" $\tau_{\mathrm{cl}}(\rho') = \tau_K \ln(L_c/\rho')$.

Nevertheless, the predictions of the CA—specifically, the estimate (214), which includes the logarithmic enhancement due to trajectory divergence—do not follow in general. Equation (217) is simple enough that its steady state can be calculated unambiguously and the result compared with the approximation that would be made by the CA. This was done by Krommes (1997a), who discussed the detailed Green's-function solution of Eq. (217). In accord with the more general earlier arguments of Krommes (1986a), the steady-state Green's function contains no hint of a clump lifetime. The exponentially rapid stretching processes described by τ_{cl} do contribute to the time-*dependent* Green's function that describes transient relaxation toward the steady state, but by definition the steady state is achieved at times much longer than the times for transients to relax. In almost all cases of interest, the steady-state fluctuation level, which from Eq. (217) is rigorously

$$\mathcal{S} = \int_0^\infty \frac{\mathrm{d}\bar{\rho}}{\bar{\rho}^{d-1}\eta_-(\bar{\rho})} \int_0^{\bar{\rho}} \mathrm{d}\bar{\rho}'\, \bar{\rho}'^{d-1} F^{\mathrm{ext}}(\bar{\rho}') \tau_{\mathrm{ac}}^{(f)}, \tag{221}$$

can be estimated to be

$$\mathcal{S} \sim \tau_D \mathcal{P} = \tau_D \sigma, \tag{222}$$

where $\tau_D \doteq L_c^2/D$ is the macroscopic diffusion time and σ is again the production (forcing) rate. Thus Eq. (222) does not contain the logarithmic enhancement. The key to this estimate is the observation that formula (221) is *convergent* at $\bar{\rho} = 0$ [since $\eta_-(0) = \kappa$], but would diverge at $\bar{\rho} = \infty$ were it not that the support of $F^{\mathrm{ext}}(\rho)$ is essentially localized to $\rho \lesssim L_c$. For an explicit example of the integral (221), see Eq. (120) of Krommes (1997a).

That the correct result is Eq. (222) rather than the clump prediction (214) can be traced to the facts that (i) the Taylor microscale λ_T, which according to Eq. (220) enters into the behavior of $\eta_-^{\mathrm{nl}}(\rho)$ at very small scales, is in general quite distinct from the macroscopic autocorrelation length; and (ii) the ρ^2 behavior in Eq. (220) holds only for scales smaller than a *dissipation* scale (which is smaller than λ_T). If a well-developed inertial range exists, scale similarity predicts that $\eta_-^{\mathrm{nl}}(\rho) \propto (\rho/L_c)^\zeta$ for some ζ, but *it is not necessary that* $\zeta = 2$. It can be shown that for $\zeta < 2$ the inertial-range scalar variance is finite and does not sensibly contribute to the total fluctuation level. For $\zeta > 2$ the total variance diverges algebraically and there is no clean separation between an energy-containing range and an inertial range.

Only for the marginal case $\zeta = 2$ does the prediction (214) make sense (and then only for passive advection). This case corresponds to logarithmically divergent inertial-range variance and to purely ρ^2 scaling for $\eta_-^{\mathrm{nl}}(\rho)$. The logarithmic factor introduced by τ_{cl} (evaluated for initial separations of the order of a dissipation scale r_d) then describes the energy content of a Batchelor k^{-1} spectrum extending from r_d to L_c. See further discussion by Krommes (1997a).

With regard to $\boldsymbol{x}$-space formalisms, the final conclusion of such analysis is that incoherent noise, although essential for proper spectral balance as well as small-scale behavior, does *not* in general predict spectral enhancements of the kind envisioned by Dupree. When treated properly, calculations in either $\boldsymbol{x}$ space or $\boldsymbol{k}$ space lead to the same answers for the spectra of the small scales [cf. Batchelor

(1953)]. Energetics are dominated by the production scales; the spectral tail does not wag the spectral dog. If a spectrum is purely energy containing, so there is no clean separation between energy-containing scales and dissipative scales (often the case in plasma-physics applications), the concept of infinitesimal trajectory divergence is irrelevant. To calculate the wave-number spectrum and fluctuation level, one must consider the mode coupling of scales of comparable size. Sensible and robust formalisms for doing so include the DIA, to be discussed in the next section, and the Markovian closures discussed in Sec. 7.2 (p. 182). A particular example that can be worked out in detail is the random coupling of three modes in the eddy-damped quasinormal Markovian approximation (Appendix J, p. 297).

This leaves open the role of *kinetic* granulations in the overall spectral balance. Velocity-space clumps were observed in the simulations of Dupree et al. (1975). For drift-wave problems and to the extent that the velocity-space nonlinearity can be neglected (as conventional wisdom suggests), nonlinearly driven kinetic granulations must be absent; however, see Dupree (1978). In any event, the existence of such driven small-scale fluctuations is not in dispute; the issue is whether they materially affect the total fluctuation level. It remains a challenge to rigorously explore the nonlinear properties of physical systems with strong kinetic effects.

5 THE DIRECT-INTERACTION APPROXIMATION (DIA)

"The weak dependence principle leads to a perturbation treatment of the dynamical couplings among sets of individual Fourier amplitudes which differs importantly from conventional perturbation theory based on expansion in powers of the Reynolds number. . . . [The lowest-order] procedure, which we term the direct-interaction approximation, has a simple dynamical significance and can be shown to lead to equations which are self-consistent in the sense that they yield rigorously realizable second-order moments. [It] includes terms of all orders in an expansion in powers of the Reynolds number." — *Kraichnan (1959b).*

I now turn to a detailed discussion of Kraichnan's direct-interaction approximation (DIA). From several points of view, this closure is unique. Most importantly, it can be shown to describe the exact second-order statistics of several varieties of stochastic amplitude equations, so those statistics are realizable in the sense of Sec. 3.5.3 (p. 63). It is also the "natural" second-order renormalization [in a sense to be clarified in Sec. 6.2.2 (p. 155)], taking full account of self-consistency effects and propagator renormalization. Therefore it is a robust starting point for discussing the status of less systematic approximations such as RBT.

The discussion in this section covers mostly the period from the inception of the DIA (in the late 1950s) to about 1984, the latter date being chosen as an approximate breakpoint between early and modern DIA-related plasma research. A time line of key articles during this period can be found in Fig. 35 (p. 261).

I shall mostly consider the DIA only for quadratically nonlinear equations [although I shall mention a cubic DIA in Sec. 6.2 (p. 153)]. For much of the discussion it is adequate to consider the self-consistent dynamics

$$\partial_t \psi_{\boldsymbol{k}} + \mathrm{i}\mathcal{L}_{\boldsymbol{k}} \psi_{\boldsymbol{k}} = \frac{1}{2} \sum_{\Delta} M_{\boldsymbol{kpq}} \psi^*_{\boldsymbol{p}} \psi^*_{\boldsymbol{q}} + \widehat{\eta}_{\boldsymbol{k}}(t), \tag{223}$$

where $M_{\boldsymbol{kpq}}$ is symmetrical in its last two indices and $\widehat{\eta}_{\boldsymbol{k}}(t)$ is a statistically sharp source. For the time being, one may think of $\psi_{\boldsymbol{k}}$ as the Fourier transform of a scalar fluid variable such as the electrostatic potential $\varphi_{\boldsymbol{k}}$; cf. the Hasegawa–Mima equation (Sec. 2.4.3, p. 34). Ultimately, however, the index $\boldsymbol{k}$ may include field labels[160] s or velocity variables $\boldsymbol{v}$ as well, and one can formulate a kinetic DIA for the PDF $f_s(\boldsymbol{x}, \boldsymbol{v}, t)$; the most general development will be given in Sec. 6 (p. 146). I shall first state the DIA, then discuss various derivations. Useful discussions and many details about Kraichnan's earlier works on the DIA were given by Leslie (1973b); a more recent and accessible account is by McComb (1990).

The DIA closure consists of the exact equation for the mean field $\langle\psi\rangle$ plus two coupled equations for the two-point correlation function $C_{\boldsymbol{k}}(t, t') \doteq \langle\delta\psi_{\boldsymbol{k}}(t)\delta\psi^*_{\boldsymbol{k}}(t')\rangle$ and mean response function $R_{\boldsymbol{k}}(t; t') \doteq \langle\delta\psi_{\boldsymbol{k}}(t)/\delta\widehat{\eta}_{\boldsymbol{k}}(t')\rangle|_{\widehat{\eta}=0}$, written here in the absence of mean fields[161] and external forcing:

$$(\partial_t + \mathrm{i}\mathcal{L}_{\boldsymbol{k}})R_{\boldsymbol{k}}(t; t') + \int_{t'}^{t} \mathrm{d}\bar{t}\, \Sigma^{\mathrm{nl}}_{\boldsymbol{k}}(t; \bar{t})R_{\boldsymbol{k}}(\bar{t}; t') = \delta(t - t'), \tag{224a}$$

$$(\partial_t + \mathrm{i}\mathcal{L}_{\boldsymbol{k}})C_{\boldsymbol{k}}(t, t') + \int_{0}^{t} \mathrm{d}\bar{t}\, \Sigma^{\mathrm{nl}}_{\boldsymbol{k}}(t; \bar{t})C_{\boldsymbol{k}}(\bar{t}, t') = \int_{0}^{t'} \mathrm{d}\bar{t}\, F^{\mathrm{nl}}_{\boldsymbol{k}}(t, \bar{t})R^*_{\boldsymbol{k}}(t'; \bar{t}). \tag{224b}$$

Here $\Sigma^{\mathrm{nl}}_{\boldsymbol{k}}$ is a nonlocal "turbulent collision operator" (causal in time). It describes the tendency of the turbulence to scramble (damp out) a perturbation, and is the turbulent generalization of the time-local ν term of classical Langevin theory (Sec. 3.2, p. 48). The $F^{\mathrm{nl}}_{\boldsymbol{k}}$ term on the right-hand side of Eq. (224b) is the mean square of an internally produced incoherent noise. (If external forcing is present, its covariance $F^{\mathrm{ext}}_{\boldsymbol{k}}$ should be added to $F^{\mathrm{nl}}_{\boldsymbol{k}}$.) All of these assertions will be discussed further, ultimately with the aid of an underlying Langevin representation of the DIA statistics (Sec. 5.3, p. 132).

Equations (224) are called the *Dyson equations* of turbulence. Under certain reasonable assumptions [see Sec. 6 (p. 146)], they are formally exact. A closure provides specific forms for the nonlinear terms $\Sigma^{\mathrm{nl}}_{\boldsymbol{k}}$ and $F^{\mathrm{nl}}_{\boldsymbol{k}}$. In the DIA those are

$$\Sigma^{\mathrm{nl}}_{\boldsymbol{k}}(t; t') \approx -\sum_{\Delta} M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}R^*_{\boldsymbol{p}}(t; t')C^*_{\boldsymbol{q}}(t, t'), \;\; F^{\mathrm{nl}}_{\boldsymbol{k}}(t, t') \approx \frac{1}{2}\sum_{\Delta} |M_{\boldsymbol{kpq}}|^2 C^*_{\boldsymbol{p}}(t, t')C^*_{\boldsymbol{q}}(t, t'). \tag{225a,b}$$

The Dyson equations are obviously two-time, nonlocal generalizations of the wave kinetic equation (182) of weak-turbulence theory. A spectral balance equation for the fluctuation

[160] A Cartesian component index for vector fields (e.g., $\psi \to \boldsymbol{u} \equiv u_i$) is a special instance of the general case of arbitrarily coupled fields $\psi^{(s)}$ such as n, φ, or T.

[161] The effect of mean fields is to add extra contributions to the zeroth-order operator $R_0^{-1} \doteq \partial_t + \mathrm{i}\mathcal{L}_{\boldsymbol{k}}$ on the left-hand side of Eqs. (224). If the nonlinearity is schematically $\frac{1}{2}\widehat{M}\psi\psi$, then with $\Delta\psi$ standing for R or C one must add the the first-order variation $-\widehat{M}\langle\psi\rangle\Delta\psi$. [For a formal proof of this statement, see Sec. 6.2.2 (p. 155), especially Eq. (283a).] Since $\widehat{M}$ is a symmetrized operator, this leads to two extra terms. For example, if a self-consistent velocity $\boldsymbol{V}$ is linearly related to ψ *via* $\boldsymbol{V}[\psi] = \widehat{\boldsymbol{V}}\psi$, then a $\boldsymbol{V}[\psi]\cdot\boldsymbol{\nabla}\psi$ term leads to $\langle\boldsymbol{V}\rangle\cdot\boldsymbol{\nabla}\Delta\psi + (\widehat{\boldsymbol{V}}\Delta\psi)\cdot\boldsymbol{\nabla}\langle\psi\rangle$. The last term describes the interaction of the fluctuations with background profile gradients. If it is assumed that those gradients are constant (for example, $\boldsymbol{\nabla}\langle n\rangle = -L_n^{-1}\langle n\rangle\widehat{\boldsymbol{x}}$), then a statistically homogeneous theory results in which the effects of the gradients can be incorporated into $\mathcal{L}_{\boldsymbol{k}}$ and the equations can be reinterpreted as describing the fluctuations $\delta\psi$.

intensity $C_{\boldsymbol{k}}(t) \equiv C_{\boldsymbol{k}}(t,t)$ follows from Eq. (224b) by noting that $\partial_t C_{\boldsymbol{k}}(t,t) = 2\,\mathrm{Re}\,\partial_t C_{\boldsymbol{k}}(t,t')|_{t'=t}$. Then

$$\partial_t C_{\boldsymbol{k}}(t) = 2\gamma^{\mathrm{lin}}_{\boldsymbol{k}} C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\int_0^t \mathrm{d}\overline{t}\,[F^{\mathrm{nl}}_{\boldsymbol{k}}(t,\overline{t})R^*_{\boldsymbol{k}}(t;\overline{t}) - \Sigma^{\mathrm{nl}}_{\boldsymbol{k}}(t;\overline{t})C^*_{\boldsymbol{k}}(t,\overline{t})]. \tag{226}$$

The forms (225) guarantee that quadratic conservation properties of the primitive equation (223) are preserved; specifically, the nonlinear terms are readily shown to conserve $\mathcal{I}(t) \doteq \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}(t)$ provided that Eq. (184) is satisfied.

Equations (224) are causal. From a specified initial condition $C_{\boldsymbol{k}}(0)$, they can be integrated in time by using Eq. (226) to advance from $C_{\boldsymbol{k}}(t)$ to $C_{\boldsymbol{k}}(t+\Delta t)$, then using Eqs. (224) to construct the time-lagged functions for $\tau \doteq t - t' = \Delta t,\ 2\Delta t,\ \ldots,\ t$.[162] Brief remarks on a practical numerical implementation used by the author and his colleagues are given in Appendix I (p. 295).

The DIA properly reduces to lowest-order[163] WTT. Explicit reduction to Eqs. (182) and (183) is carried out in Sec. 7.2.1 (p. 183) for fluid problems. Reduction of the Vlasov DIA is described in Sec. 6.5.4 (p. 176).

5.1 Kraichnan's original derivation of the DIA

Although a variety of algorithms that lead to the DIA are now known, the original derivation of Kraichnan (1959b) remains one of the most heuristically compelling. The difficulty of a theory of strong turbulence is that no small parameter is apparent. The regular perturbation theory of WTT, which retains only a few primitive terms, is clearly inappropriate. Instead, Kraichnan argued that one should assess the importance of any particular elementary interaction by *removing* it from the sea of all fully developed interactions. At least for a continuum of wave numbers, the effect of a single such interaction should be infinitesimally small, suggesting a perturbation treatment. Kraichnan argued that the *direct interactions* of wave vectors $\boldsymbol{k}$, $\boldsymbol{p}$, and $\boldsymbol{q}$ (where $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$) should dominate. Since the properties of the fully turbulent system are unknown *a priori*, the method leads to coupled integral equations for the two-point correlation and response functions.

As plausible as this argument may appear to be, there is an important subtlety. Although in a wave-number continuum a single indirect interaction may be subdominant to a direct one, there are infinitely many more indirect interactions than direct ones. The possibility therefore remains that the net effect of all indirect interactions may be comparable to that of the direct ones. That is, in fact, the case. For strong turbulence the DIA is at best an order-unity approximation for at least some frequencies or wave numbers [the precise meaning of this statement will be clarified below and in Sec. 6 (p. 146)]; as a consequence, it possesses at least one qualitative deficiency, its lack of random Galilean invariance (Sec. 5.6.3, p. 138). Nevertheless, the DIA is remarkably robust and effective in practice, and reasons for this will be explained.

In addition to Kraichnan's own work, the mathematics of the original derivation was given by Montgomery (1977) and Krommes (1984a). Here I shall give a version appropriate for systems of

[162] The Hermitian symmetry $C_{\boldsymbol{k}}(t,t') = C^*_{\boldsymbol{k}}(t',t)$ permits one to consider just $t \geq 0$ and $\tau \geq 0$.

[163] Here "lowest-order" means at the level of three-wave interactions quadratic in the intensity of the turbulence. Perturbation expansion of the DIA contains higher-order interactions as well, through all orders, but their description is not complete; vertex corrections (Sec. 6.2, p. 153) absent in the DIA contribute to WTT at orders higher than quadratic (Thompson and Krommes, 1977).

multiple coupled fields, for which the correlation functions need not be diagonal with respect to the field indices. Accordingly, let us consider the dynamical equation

$$R_0^{-1}u_\alpha(t) \equiv \partial_t u_\alpha(t) + \sum_\beta \mathrm{i}\mathcal{L}_{\alpha\beta}u_\beta = \frac{1}{2}\sum_{\beta,\gamma} M_{\alpha\beta\gamma}u_\beta^*(t)u_\gamma^*(t) + f_\alpha^{\mathrm{ext}}(t) + \widehat{\eta}_\alpha(t), \tag{227a,b}$$

where u is taken to have zero mean for simplicity. The covariance matrix is defined by $C_{\alpha\alpha'}(t,t') \doteq \langle\delta u_\alpha(t)\,\delta u_{\alpha'}^*(t')\rangle$, and the random response function is defined by $\widetilde{R}_{\alpha\alpha'}(t;t') \doteq \delta u_\alpha(t)/\delta\widehat{\eta}_{\alpha'}(t')|_{\widehat{\eta}=0}$. These functions rigorously obey

$$R_0^{-1}C_{\alpha\alpha'}(t,t') = \frac{1}{2}\sum_{\beta,\gamma} M_{\alpha\beta\gamma}\langle\delta u_\beta^*(t)\delta u_\gamma^*(t)\delta u_{\alpha'}^*(t')\rangle + \langle\delta f_\alpha^{\mathrm{ext}}(t)\delta u_{\alpha'}(t')\rangle, \tag{228a}$$

$$R_0^{-1}\widetilde{R}_{\alpha\alpha'}(t;t') - \sum_{\beta,\gamma} M_{\alpha\beta\gamma}u_\beta^*(t)\widetilde{R}_{\gamma\alpha'}(t;t') = \delta_{\alpha,\alpha'}\delta(t-t'). \tag{228b}$$

Consider the specific triad of modes $\{\alpha,\beta,\gamma\}$ as well as its primed counterpart.[164] Diagrammatic perturbation theory such as described in Secs. 3.9.5 (p. 81) and 3.9.7 (p. 83) may be used to classify the various possible interactions into direct ones (propagator renormalizations) and indirect ones (vertex corrections). The direct interactions correspond to the shortest route through the bare-vertex space. To assess the effect of a particular direct interaction, define $\Delta u_{\alpha|\beta\gamma} \equiv \Delta u_\alpha$ to be the difference between the exact solution u_α and the value $\overline{u}_\alpha$ that the solution would take if the specific triads under consideration were deleted from the right-hand side of Eq. (227b); i.e.,

$$R_0^{-1}\overline{u}_\alpha = \frac{1}{2}\sum_{\rho,\sigma} M_{\alpha\rho\sigma}\overline{u}_\rho^*\overline{u}_\sigma^* - \frac{1}{2}\sum_{\beta,\gamma} M_{\alpha\beta\gamma}\overline{u}_\beta^*\overline{u}_\gamma^*, \tag{229}$$

where the last sum is only over the specific (unprimed and primed) deleted triads. It is asserted—the so-called *weak-dependence principle*; see extensive discussion by Kraichnan (1958b, 1959b)—that Δu_α should be small (but see the further remarks at the end of this section). Thus to lowest order it obeys

$$R_0^{-1}\Delta u_\alpha - \sum_{\rho,\sigma} M_{\alpha\rho\sigma}u_\rho^*\Delta u_\sigma^* = \frac{1}{2}\sum_{\beta,\gamma} M_{\alpha\beta\gamma}(u_\beta^*u_\gamma^* - \langle\ldots\rangle) + f_\alpha^{\mathrm{ext}}. \tag{230}$$

[To this order it is immaterial whether one writes u or $\overline{u}$ on the right-hand side of Eq. (230).] Upon comparing Eq. (230) to Eq. (228b), one can see that Green's function for Δu_α is the exact (random) infinitesimal response function $\widetilde{R}$; thus the solution of Eq. (230) is

$$\Delta u_\alpha(t) = \frac{1}{2}\int_{-\infty}^{t} \mathrm{d}\overline{t} \sum_{\overline{\alpha},\overline{\beta},\overline{\gamma}} \widetilde{R}_{\alpha\overline{\alpha}}(t;\overline{t})\{M_{\overline{\alpha}\overline{\beta}\overline{\gamma}}[u_{\overline{\beta}}^*(\overline{t})u_{\overline{\gamma}}^*(\overline{t}) - \langle\ldots\rangle] + f_{\overline{\alpha}}^{\mathrm{ext}}(\overline{t})\}. \tag{231}$$

This result may be used to evaluate the correlation functions needed on the right-hand side of Eq. (228a). For example, one has without approximation

$$\langle\delta u_\beta\,\delta u_\gamma\,\delta u_{\alpha'}\rangle = \langle\delta\overline{u}_\beta\,\delta\overline{u}_\gamma\,\delta\overline{u}_{\alpha'}\rangle + (\langle\Delta u_\beta\,\delta\overline{u}_\gamma\,\delta\overline{u}_{\alpha'}\rangle + 2\text{ terms}) + O(\Delta u^2). \tag{232}$$

[164] The primed triad must be included because one is considering general, inhomogeneous statistics.

Because $\overline{u}_\alpha$ is missing the effects of the direct interaction, it is argued that the first term on the right-hand side of Eq. (232) is small, and that the dominant contribution comes from the second term. Upon inserting the result (231), one is left with the evaluation of terms like

$$\tfrac{1}{2}\langle \widetilde{R}^*_{\beta\overline{\beta}}(t;\overline{t})M^*_{\overline{\beta}\overline{\gamma}\overline{\alpha}}\delta u^*_\gamma(t)\delta u^*_{\alpha'}(t)[\delta u_{\overline{\gamma}}(\overline{t})\delta u_{\overline{\alpha}}(t') - \langle\ldots\rangle]\rangle \approx R^*_{\beta\overline{\beta}}(t;\overline{t})C^*_{\gamma\overline{\gamma}}(t,\overline{t})C_{\overline{\alpha}\alpha'}(\overline{t},t'), \tag{233}$$

where the symmetry of M was used and higher-order correlations were neglected according to the weak-dependence principle. The final result is

$$R_0^{-1}C_{\alpha\alpha'}(t,t') + \int_{-\infty}^{t} \mathrm{d}\overline{t}\, \sum_{\overline{\alpha}} \Sigma^{\mathrm{nl}}_{\alpha\overline{\alpha}}(t;\overline{t})C_{\overline{\alpha}\alpha'}(\overline{t},t') = \int_{-\infty}^{t'} \mathrm{d}\overline{t}\, \sum_{\overline{\alpha}} (F^{\mathrm{nl}} + F^{\mathrm{ext}})_{\alpha\overline{\alpha}}(t,\overline{t})R^*_{\alpha'\overline{\alpha}}(t';\overline{t}), \tag{234}$$

$$\Sigma^{\mathrm{nl}}_{\alpha\overline{\alpha}}(t;\overline{t}) \doteq -\sum_{\beta,\gamma,\overline{\beta},\overline{\gamma}} M_{\alpha\beta\gamma}M^*_{\overline{\beta}\overline{\gamma}\overline{\alpha}}R^*_{\beta\overline{\beta}}(t;\overline{t})C^*_{\gamma\overline{\gamma}}(t,\overline{t}), \tag{235a}$$

$$F^{\mathrm{nl}}_{\alpha\overline{\alpha}}(t,\overline{t}) \doteq \frac{1}{2}\sum_{\beta,\gamma,\overline{\beta},\overline{\gamma}} M_{\alpha\beta\gamma}M^*_{\overline{\alpha}\overline{\beta}\overline{\gamma}}C^*_{\beta\overline{\beta}}(t,\overline{t})C^*_{\gamma\overline{\gamma}}(t,\overline{t}), \tag{235b}$$

and $F^{\mathrm{ext}}_{\alpha\overline{\alpha}}(t,\overline{t}) \doteq \langle \delta f^{\mathrm{ext}}_\alpha(t)\delta f^{\mathrm{ext}*}_{\overline{\alpha}}(\overline{t})\rangle$.

Equations (234) and (235) contain the as yet unknown mean response function R. A similar technique can be used to find its equation in the DIA. The result is that R obeys

$$R_0^{-1}R_{\alpha\alpha'}(t;t') + \int_{t'}^{t} \mathrm{d}\overline{t}\, \sum_{\overline{\alpha}} \Sigma^{\mathrm{nl}}_{\alpha\overline{\alpha}}(t;\overline{t})R_{\overline{\alpha}\alpha'}(\overline{t},t') = \delta_{\alpha,\alpha'}\delta(t-t'). \tag{236}$$

Since the lower limit t' can be replaced by $-\infty$ because $R(t;t')$ is causal, the operator acting on R is the same as the one on the left-hand side of Eq. (234).

Equations (234) and (236) are the multifield DIA. For the special case of a single-field model ($\alpha \to \boldsymbol{k}$) with spatially homogeneous statistics, so that $C_{\alpha\alpha'}(t,t') \to C_{\boldsymbol{k}}(t,t')\delta_{\boldsymbol{k},\boldsymbol{k}'}$, they reduce to the results (224) and (225) quoted at the beginning of this section. A straightforward generalization[165] is to multiple coupled fields, giving rise to a theory nondiagonal in the field indices but diagonal in wave number (homogeneous in space). If α is further permitted to contain a velocity variable, a

[165] Another kind of generalization is to the theory of predictability initiated by Kraichnan (1970b). He posited two statistically identical ensembles with velocity fluctuations $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ correlated only through either (i) the initial value $\Delta(\boldsymbol{x},t;\boldsymbol{x}',t') \doteq \frac{1}{2}\langle[\boldsymbol{u}^{(1)}(\boldsymbol{x},t) - \boldsymbol{u}^{(2)}(\boldsymbol{x},t)][\boldsymbol{u}^{(1)}(\boldsymbol{x}',t') - \boldsymbol{u}^{(2)}(\boldsymbol{x}',t')]\rangle = \mathsf{C}(\boldsymbol{x},t;\boldsymbol{x}',t') - \mathsf{W}(\boldsymbol{x},t;\boldsymbol{x}',t')$, where $\mathsf{W}(\boldsymbol{x},t;\boldsymbol{x}',t') \doteq \langle \boldsymbol{u}^{(1)}(\boldsymbol{x},t)\boldsymbol{u}^{(2)}(\boldsymbol{x}',t')\rangle$; or (ii) correlated forcing $F^{\mathrm{ext}}_{12}(\boldsymbol{x},t;\boldsymbol{x}',t') \doteq \langle f^{(1)\mathrm{ext}}(\boldsymbol{x},t)f^{(2)\mathrm{ext}}(\boldsymbol{x}',t')\rangle$. The evolution of Δ is taken as a measure of uncertainties of measurement or of instability in the flow. By taking $\alpha = 1$ and $\alpha' = 2$ in Eq. (234), employing the statistical symmetries, and noting that $R_{\alpha\beta} = 0$ for $\beta \neq \alpha$, one is readily led to

$$(R_0^{-1} + \Sigma^{\mathrm{nl}}_{\boldsymbol{k}}) \star W_{\boldsymbol{k}}(t,t') = (F^{\mathrm{nl}}_{12} + F^{\mathrm{ext}}_{12})_{\boldsymbol{k}} \star R^*_{\boldsymbol{k}}(t';t), \tag{f-17}$$

where $F^{\mathrm{nl}}_{12}(t,\overline{t}) \doteq \frac{1}{2}\sum_\Delta |M_{\boldsymbol{kpq}}|^2 W^*_{\boldsymbol{p}}(t,\overline{t})W^*_{\boldsymbol{q}}(t,\overline{t})$ and $\Sigma^{\mathrm{nl}}_{\boldsymbol{k}}$ is the conventional one-field result (225a). These results reproduce Kraichnan's Eqs. (3.9)–(3.13) (Dubin, 1984a). The predictability equations were compared with DNS by Herring et al. (1973), who also gave additional references.

kinetic DIA results [see, for example, DuBois and Espedal (1978)]. Discussion of the kinetic DIA is given in Sec. 6.5 (p. 170).

However plausible the weak-dependence principle may be, it has *not* been demonstrated that its lowest-order application, the DIA, is dominant in any sense. I have already suggested that the net effects of all of the indirect interactions omitted from the DIA may, depending on the question asked, be comparable to those of the direct ones; see Sec. 5.6 (p. 137) below. Kraichnan (1958a) suggested that the procedure can be extended to higher order by systematically deriving a set of more and more complicated closures of which the DIA is the simplest. With the techniques of the present section, the algebra of even the next approximation (first vertex correction) becomes decidedly tedious; it is remarkable that Kraichnan was able to formulate that theory (Kraichnan, 1961) and deduce nontrivial consequences (Kraichnan, 1964e). In Sec. 6 (p. 146) I shall describe a more elegant and compact procedure that leads to such closures with a minimum of tedium. Nevertheless, Kraichnan's original calculations are unsurpassed for their physical insights.

5.2 Random-coupling models

The original justification offered for the DIA, the weak-dependence principle, made explicit reference to a continuum of wave numbers. The resulting DIA equations, however, can be applied to systems with a finite (possibly small) number of coupled amplitudes. This observation makes it clear that in general one must look elsewhere for a justification of the perturbation procedures that define the DIA algorithm.

In seminal work Kraichnan (1961) showed that the DIA provides the *exact description of second-order statistics* for a certain *random-coupling model* (RCM). Several varieties of such models are now known (Herring and Kraichnan, 1972; Kraichnan, 1991); their mere existence answers several important questions.

Equation (223) together with a PDF of random initial conditions defines an ensemble of realizations. Consider now a superensemble (ensemble of ensembles) consisting of N identical and independent copies of the original ensemble; let Roman letters identify the particular copy. In an RCM a new dynamics is constructed by inducing statistical dependence among the copies. In one version of the procedure (Kraichnan, 1991), this is done by modifying the original dynamics

$$R_0^{-1} u_\alpha^{(n)}(t) = \frac{1}{2} \sum_{\beta,\gamma} M_{\alpha\beta\gamma} u_\beta^{(n)*} u_\gamma^{(n)*} \tag{237}$$

(I omit f^{ext} for simplicity), in which each system evolves independently of any other, to the form

$$R_0^{-1} u_\alpha^{(n)}(t) = \frac{1}{2} N^{-1} \sum_{r,s,\beta,\gamma} \phi_{nrs} M_{\alpha\beta\gamma} u_\beta^{(r)*} u_\gamma^{(s)*}, \tag{238}$$

in which the systems are coupled. The dimensionless coupling coefficient ϕ_λ, where $\lambda \equiv \{n, r, s\}$, is randomly assigned the value ± 1 as a function of λ (while preserving symmetry under arbitrary permutations of the elements of λ). The factor of N^{-1} preserves the variances of the nonlinear terms at $t = 0$. It can then be shown (Kraichnan, 1958c, 1961) that as $N \to \infty$ the DIA for $u_\alpha^{(n)}$ becomes *exact*. That is, as $N \to \infty$ the vertex corrections vanish because of the randomly phased couplings induced by the random ϕ_λ's. A related derivation of the DIA as an infinite-N limit was given by Mou and Weichman (1993).

Thus the DIA has been demonstrated to have a *primitive amplitude representation*. The consequences are profound. The very existence of such a representation means that statistical moments formed from the solution of Eq. (238) are *realizable* in the sense of Sec. 3.5.3 (p. 63). In particular, the DIA covariance matrix is guaranteed to remain positive definite as time evolves. This property is extremely difficult to prove directly from the DIA equations themselves, and is not shared by many other superficially plausible closures (Kraichnan, 1961).

An alternate derivation of the RCM of Kraichnan (1961) was given by Frisch and Bourret (1970), who were able to prove a variety of theorems and asymptotic results.

Use of the RCM is not restricted to deriving the standard DIA equations for second-order statistics; because a primitive amplitude equation is written explicitly, higher-order statistics can be predicted as well. For further discussion, see Sec. 10.2 (p. 221).

The random-coupling representation of the DIA highlights a principal deficiency of the approximation, namely, its failure to properly represent *coherent structures*. By definition, a coherent structure is represented by well-specified, statistically sharp phase relations between Fourier amplitudes. When mode-coupling coefficients are randomized, as in the RCM, those phase relations are destroyed; the retained statistical information is insufficient to reconstruct the coherent structure. [Nevertheless, a sea of interacting coherent structures may adequately be described by the DIA. For more discussion, see Sec. 10.5 (p. 228).]

5.3 Langevin representation of the DIA

Although the RCM is very important, it is rather abstract; it is perhaps difficult to intuitively relate the meaning of random couplings in a superensemble to the approximate statistical behavior of the original equation. It is therefore useful to know that the DIA has a Langevin representation (Leith, 1971; Kraichnan, 1970a). Consider the primitive amplitude equation

$$\partial_t \psi_{\boldsymbol{k}} + \mathrm{i}\mathcal{L}_{\boldsymbol{k}}\psi_{\boldsymbol{k}} + \Sigma^{\mathrm{nl}}_{\boldsymbol{k}} \star \psi_{\boldsymbol{k}} = \widetilde{f}^{\mathrm{nl}}_{\boldsymbol{k}}(t), \tag{239}$$

where $\Sigma^{\mathrm{nl}}_{\boldsymbol{k}}$ has the DIA form (225a) and

$$\widetilde{f}^{\mathrm{nl}}_{\boldsymbol{k}}(t) = \frac{1}{\sqrt{2}} \sum_{\Delta} M_{\boldsymbol{kpq}} \widetilde{\xi}^{*}_{\boldsymbol{p}}(t) \widetilde{\xi}^{*}_{\boldsymbol{q}}(t), \tag{240}$$

$\widetilde{\xi}_{\boldsymbol{k}}$ being a random variable whose covariance is constrained to be that of $\psi_{\boldsymbol{k}}$ itself. Green's function for the left-hand side of Eq. (239) clearly obeys Eq. (224a) for the response function $R_{\boldsymbol{k}}$ of the DIA.[166] To verify that Eq. (224b) is obeyed, it is necessary to show that $\langle \widetilde{f}^{\mathrm{nl}}_{\boldsymbol{k}}(t)\psi^{*}_{\boldsymbol{k}}(t')\rangle$ is equal to the right-hand side of Eq. (224b). This follows upon solving Eq. (239) for $\psi_{\boldsymbol{k}}$ using the Green's function $R_{\boldsymbol{k}}$, then performing the required average over the assumed statistics of ξ.

This Langevin representation provides a route, alternative to that of the random-coupling model, to the proof that the second-order statistics described by the DIA equations are realizable. It also has a pleasing physical interpretation that parallels that of the original Langevin equations discussed in

[166] Perhaps it is not so clear, since $\Sigma^{\mathrm{nl}}_{\boldsymbol{k}}$ depends functionally on ψ through the covariance. However, in a continuously distributed ensemble of realizations, the contribution of a particular $\psi_{\boldsymbol{k}}$ to that covariance is infinitesimal.

Sec. 3.2 (p. 48). The statistical effects of the original nonlinearity [Eq. (223)] are seen to be broken into two pieces: *incoherent noise* (internally created random stirring) $\widetilde{f}^{\rm nl}_{\boldsymbol{k}}$; and mean *turbulent damping* $\Sigma^{\rm nl}_{\boldsymbol{k}}$ (typically positive). The specific form of $\widetilde{f}^{\rm nl}_{\boldsymbol{k}}$ is just such that energy is conserved by the nonlinear terms; the relation between $\Sigma^{\rm nl}_{\boldsymbol{k}}$ and the covariance of $\widetilde{f}^{\rm nl}_{\boldsymbol{k}}$ is a generalization of Einstein's relation relating the ν and D_v of Langevin's original theory (Sec. 3.2, p. 48).

It is a Langevin representation like that for the DIA that is missing from the more heuristic plasma closures such as RBT. Thus while Dupree sometimes speaks of incoherent noise, he cannot [167] refer to a *realizable* representation such as Eq. (240) that unambiguously captures the second-order statistics of that noise. [168] From this point of view, the elegance of the DIA is compelling.

It should be emphasized that the Langevin equation (239) is appropriate for deriving only *second-order* statistics. At higher order, statistics predicted from Eq. (239) *do not agree* with those predicted from the RCM (Krommes, 1996). Further discussion of this point is given in Sec. 10.2 (p. 221).

5.4 The spectral balance equation

The balance between incoherent noise and coherent damping can be demonstrated directly at the covariance level. Upon noting that Green's function for the left-hand side of Eq. (224b) is precisely $R_{\boldsymbol{k}}$, one can write the formal solution of Eq. (224b) (now allowing for external forcing) as

$$C_{\boldsymbol{k}}(t,t') = \int_0^t \mathrm{d}\overline{t} \int_0^{t'} \mathrm{d}\overline{t}'\, R_{\boldsymbol{k}}(t;\overline{t}) F_{\boldsymbol{k}}(\overline{t},\overline{t}') R^*_{\boldsymbol{k}}(t';\overline{t}'), \tag{241a}$$

where $F \doteq F^{\rm nl} + F^{\rm ext}$. It is conventional to introduce $\tau \doteq t - t'$ and $T \doteq \frac{1}{2}(t+t')$, then write [169] $C_{\boldsymbol{k}}(t,t') \equiv C_{\boldsymbol{k}}(\tau \mid T)$. Clearly both the intensity $C_{\boldsymbol{k}}(0 \mid T) \equiv C_{\boldsymbol{k}}(0)$ and the two-time shape must be found. Although those are coupled, it is useful to think of the former as determined by the T dynamics, the latter by the τ dynamics. [Some related discussion was given by Boutros-Ghali and Dupree (1981).] Because two-time correlations are expected to decay, it is natural to Fourier-transform with respect to τ: $C_{\boldsymbol{k},\omega}(T) \doteq \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} C_{\boldsymbol{k}}(\tau \mid T)$. T dependence disappears in steady state ($T \to \infty$), for which Eq. (241a) transforms to the spectral balance equation

$$C_{\boldsymbol{k},\omega} = |R_{\boldsymbol{k},\omega}|^2 F_{\boldsymbol{k},\omega}. \tag{241b}$$

Both the τ and the ω versions of the spectral balance determine the ultimate steady-state fluctuation level as a balance between nonlinear forcing and damping. They generalize a familiar result of classical Langevin theory; see the discussion of Eqs. (70) and (72) in Sec. 3.2.2 (p. 49). For the DIA Kraichnan (1964d) has given a thorough discussion of the interpretation of Eqs. (241) in terms of impedance and

[167] It is not a matter of ignorance. The practical equations of RBT involve asymmetric approximations to the effects of the mode coupling (e.g., long-wavelength, low-frequency limits) that preclude a demonstration of realizability. That is to be contrasted with the Markovian closures discussed in Sec. 7.2 (p. 182), which in appropriate cases possess Langevin equations similar to that of the DIA and can be shown to be realizable.
[168] Nevertheless, Dupree understood that the *coherent response* (the $\Sigma^{\rm nl}_{\boldsymbol{k}}$ term) was the portion of the nonlinear effects *phase-coherent* with the turbulent field (i.e., ψ is evaluated at $\boldsymbol{k}$ in $\Sigma^{\rm nl}_{\boldsymbol{k}} \star \psi_{\boldsymbol{k}}$), and the *incoherent response* was the remaining portion involving coupling between modes $\boldsymbol{p}$ and $\boldsymbol{q}$ not equal to $\boldsymbol{k}$.
[169] For a systematic treatment of the dependence on the slow variable T, as well as the generalization to weak spatial inhomogeneity, see Appendix F (p. 286).

related concepts. He also related them to earlier work by Edwards (1964), who employed a Fokker–Planck description. For some discussion of Edwards's approach, see McComb (1990, Chap. 6.2).

As shown in Sec. 6.2.2 (p. 155), the forms (241) transcend the DIA; they are a general statement of the balance between forcing and dissipation that determines the overall fluctuation level. In conjunction with specific expressions for Σ^{nl} and F^{nl}, they also determine the details of the "microscopic" turbulent noise. As I discussed in Sec. 3.2.2 (p. 49), that possibility does not exist in the classical Langevin model, which compresses those details into unspecified dynamics with vanishing autocorrelation time. General turbulence theory "opens up" those details.

A number of subtleties surround the balance equations (241). First consider the question of ordering. Let $\epsilon \ll 1$ denote an appropriately normalized fluctuation intensity $C(0)$ such as $\langle(\delta n/\overline{n})^2\rangle$. According to Eq. (225b), $F = O(\epsilon^2)$ whereas the left-hand sides of Eqs. (241) are $O(\epsilon)$. Thus it is not always correct to take R to be of order unity, as is sometimes asserted.[170]

One can use Eq. (241b) to demonstrate that the DIA is compatible with the Gibbsian equilibrium spectra found in Sec. 3.7.2 (p. 68); the single-field version of the argument[171] is given here for simplicity. One begins with the FDT (110). One can decompose $C(\tau)$ into one-sided pieces according to $C(\tau) = C_+(\tau) + C_-(\tau)$; it is a consequence of time stationarity that (for scalar fields) $C_-(\boldsymbol{k},\omega) = C_+^*(\boldsymbol{k},\omega)$. Thus $C_{\boldsymbol{k},\omega} = (R_{\boldsymbol{k},\omega} + R^*_{\boldsymbol{k},\omega})C_{\boldsymbol{k}}(0)$, or

$$C_{\boldsymbol{k},\omega} = 2\,\mathrm{Re}\,R_{\boldsymbol{k},\omega}C_{\boldsymbol{k}}(0). \tag{242}$$

One has $R_{\boldsymbol{k},\omega} = [-\mathrm{i}(\omega - \mathcal{L}_{\boldsymbol{k}} + \mathrm{i}\Sigma^{\mathrm{nl}}_{\boldsymbol{k},\omega})]^{-1}$, with $\mathrm{Im}\,\mathcal{L} \equiv \gamma^{\mathrm{lin}} = 0$ as a consequence of thermal equilibrium; F^{ext} must also be taken to vanish. Then $\mathrm{Re}\,R = \mathrm{Re}\,\Sigma^{\mathrm{nl}}/|\omega - \mathrm{Re}\,\mathcal{L} + \mathrm{i}\Sigma^{\mathrm{nl}}|^2 = |R|^2\,\mathrm{Re}\,\Sigma^{\mathrm{nl}}$, so Eq. (241b) reduces to

$$2\,\mathrm{Re}\,\Sigma^{\mathrm{nl}}_{\boldsymbol{k},\omega}C_{\boldsymbol{k}}(0) = F^{\mathrm{nl}}_{\boldsymbol{k},\omega}. \tag{243}$$

Insert the forms (225) into Eq. (243) and again use Eq. (242). After appropriate symmetrization, one finds that Eq. (243) is satisfied provided that

$$M_{\boldsymbol{kpq}}/C_{\boldsymbol{k}}(0) + \mathrm{c.p.} = 0. \tag{244}$$

With $M_{\boldsymbol{k}} \equiv M_{\boldsymbol{kpq}}$, the triple $\boldsymbol{M} \doteq (M_{\boldsymbol{k}}, M_{\boldsymbol{p}}, M_{\boldsymbol{q}})$ can be interpreted as proportional to the set of direction cosines that determine a line perpendicular to a "constraint plane"; Eq. (244) is thus an orthogonality condition that requires the vector $\big(1/C_{\boldsymbol{k}}(0), 1/C_{\boldsymbol{p}}(0), 1/C_{\boldsymbol{q}}(0)\big)$ to lie in that plane. It is assumed that the M's obey $\sigma^{(i)}_{\boldsymbol{k}}M_{\boldsymbol{k}} + \mathrm{c.p.} = 0$ for one or more multipliers $\sigma^{(i)}_{\boldsymbol{k}}$. Thus the vectors $\boldsymbol{\sigma}^{(i)} \doteq (\sigma_{\boldsymbol{k}}, \sigma_{\boldsymbol{p}}, \sigma_{\boldsymbol{q}})^{(i)}$ also lie in the plane. (There can be at most two linearly independent such $\boldsymbol{\sigma}$'s.) Clearly the linear superposition $1/C_{\boldsymbol{k}}(0) = \sum_i \alpha_i\sigma^{(i)}_{\boldsymbol{k}}$ also lies in the plane, so Eq. (244) is satisfied; see Fig. 16 (p. 135). This result is just the Gibbsian spectrum.

Equation (241b) is quite reminiscent of the Test Particle Superposition Principle discussed in Sec. 2.3.2 (p. 30). There are subtleties, however. As will become clearer in Sec. 6.3 (p. 165), the derivation of the DIA in the form presented so far holds only for Gaussian initial conditions. Unfortunately, as Rose (1979) has stressed, the statistical dynamics of *discrete* particles are

[170] The other possibility, that the solution of Eqs. (241) is $C = O(1)$, can be dismissed on physical grounds, in general; observable drift-wave fluctuations are, in fact, small.
[171] See related discussion by Ottaviani (1990).

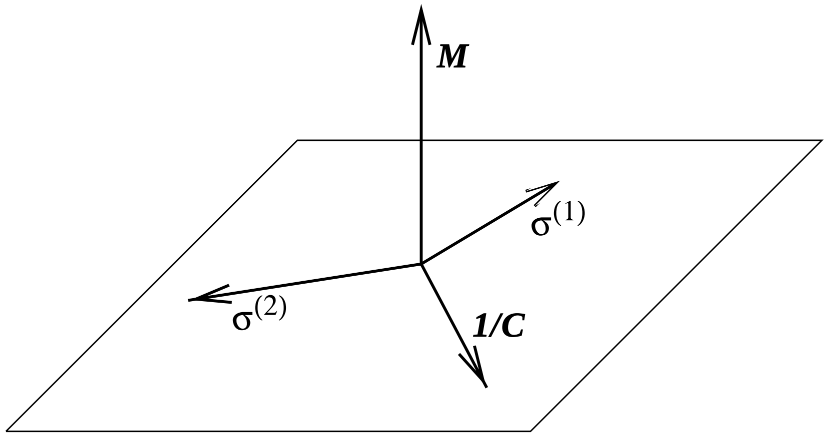

Fig. 16. Geometrical interpretation (Johnston, 1989) of the result that the DIA is compatible with the Gibbsian thermal-equilibrium solutions.

intrinsically non-Gaussian. (This can easily be seen in thermal equilibrium, in which the Gibbs ensemble holds; potential-energy contributions to that ensemble are non-Gaussian.) The construction of a proper renormalized theory that handles particle discreteness on equal footing with continuum dynamics is quite difficult, but was done elegantly by Rose (1979), who developed a general formalism and proposed a *Particle Direct-Interaction Approximation* (PDIA).

Further discussion of the spectral balance (241) is given in Sec. 6.5 (p. 170).

5.5 The DIA for passive advection

To this point the derivations of the DIA that I have presented have been for self-consistent problems, in which both of the ψ terms on the right-hand side of Eq. (223) are treated on equal footing and therefore the mode-coupling coefficient $M_{\boldsymbol{kpq}}$ can be taken to be symmetrical in its last two indices. However, problems of passive advection are also of interest. To accomodate those, consider instead of Eq. (223) the passive dynamics

$$\partial_t \psi_{\boldsymbol{k}} + \mathrm{i}\mathcal{L}_{\boldsymbol{k}}\psi_{\boldsymbol{k}} = \sum_{\Delta} M^U_{\boldsymbol{kpq}} \psi^*_{\boldsymbol{p}} \chi^*_{\boldsymbol{q}} + f^{\mathrm{ext}}_{\boldsymbol{k}}(t), \tag{245}$$

where χ is a specified random variable and M^U has no particular symmetries. I allow the possibility that χ is correlated with the external forcing f^{ext}. In addition to the response function R defined in the usual way, the independent two-point correlation functions for this problem are the six independent entries of the matrix $\langle\langle \Phi(t)\Phi^T(t')\rangle\rangle$, where $\Phi \doteq (\psi,\, \chi,\, f^{\mathrm{ext}})^T$, namely,

$$C_{\boldsymbol{k}}(t,t') \doteq \langle \delta\psi_{\boldsymbol{k}}(t)\delta\psi^*_{\boldsymbol{k}}(t')\rangle, \quad V_{\boldsymbol{k}}(t,t') \doteq \langle \delta\psi_{\boldsymbol{k}}(t)\delta\chi^*_{\boldsymbol{k}}(t')\rangle, \quad W_{\boldsymbol{k}}(t,t') \doteq \langle \delta\psi_{\boldsymbol{k}}(t)\delta f^*_{\boldsymbol{k}}(t')\rangle, \tag{246a,b,c}$$

$$X_{\boldsymbol{k}}(t,t') \doteq \langle \delta\chi_{\boldsymbol{k}}(t)\delta f^*_{\boldsymbol{k}}(t')\rangle, \quad S_{\boldsymbol{k}}(t,t') \doteq \langle \delta\chi_{\boldsymbol{k}}(t)\delta\chi^*_{\boldsymbol{k}}(t')\rangle, \quad F^{\mathrm{ext}}_{\boldsymbol{k}}(t,t') \doteq \langle \delta f_{\boldsymbol{k}}(t)\delta f^*_{\boldsymbol{k}}(t')\rangle \tag{246d,e,f}$$

($\delta f \equiv \delta f^{\mathrm{ext}}$); of these, C, V, and W must be solved for whereas X, S, and F^{ext} are specified functions. The usual perturbative algorithm leads to the passive DIA in the form

$$R^{-1}_{0,\boldsymbol{k}} R_{\boldsymbol{k}}(t;t') + \int_{t'}^{t} \mathrm{d}\bar{t}\, \Sigma_{\boldsymbol{k}}(t;\bar{t}) R_{\boldsymbol{k}}(\bar{t};t') = \delta(t-t'), \tag{247a}$$

$$R_{0,\boldsymbol{k}}^{-1}C_{\boldsymbol{k}}(t,t') + \int_0^t \mathrm{d}\overline{t}\,[\Sigma_{\boldsymbol{k}}(t;\overline{t})C_{\boldsymbol{k}}(\overline{t},t') + \Sigma'_{\boldsymbol{k}}(t;\overline{t})V^*_{\boldsymbol{k}}(t',\overline{t}]$$

$$= \int_0^{t'} \mathrm{d}\overline{t}\,F^{\mathrm{nl}}_{\boldsymbol{k}}(t,\overline{t})R^*_{\boldsymbol{k}}(t';\overline{t}) + W^*_{\boldsymbol{k}}(t',t), \tag{247b}$$

$$R_{0,\boldsymbol{k}}^{-1}V_{\boldsymbol{k}}(t,t') + \int_0^t \mathrm{d}\overline{t}\,[\Sigma_{\boldsymbol{k}}(t;\overline{t})V_{\boldsymbol{k}}(\overline{t},t') + \Sigma'_{\boldsymbol{k}}(t;\overline{t})S^*_{\boldsymbol{k}}(t',\overline{t})] = X^*_{\boldsymbol{k}}(t',t), \tag{247c}$$

$$R_{0,\boldsymbol{k}}^{-1}W_{\boldsymbol{k}}(t,t') + \int_0^t \mathrm{d}\overline{t}\,[\Sigma_{\boldsymbol{k}}(t;\overline{t})W_{\boldsymbol{k}}(\overline{t},t') + \Sigma'_{\boldsymbol{k}}(t;\overline{t})X^*_{\boldsymbol{k}}(t',\overline{t})] = F^{\mathrm{ext}*}_{\boldsymbol{k}}(t',t). \tag{247d}$$

Here

$$\Sigma_{\boldsymbol{k}}(t;\overline{t}) \doteq -\sum_\Delta M^U_{\boldsymbol{kpq}}M^{U*}_{\boldsymbol{pkq}}R^*_{\boldsymbol{p}}(t;\overline{t})S^*_{\boldsymbol{q}}(t,\overline{t}), \quad \Sigma'_{\boldsymbol{k}}(t,\overline{t}) \doteq -\sum_\Delta M^U_{\boldsymbol{kpq}}M^{U*}_{\boldsymbol{pqk}}R^*_{\boldsymbol{p}}(t;\overline{t})V_{\boldsymbol{q}}(\overline{t},t), \tag{248a,b}$$

$$F^{\mathrm{nl}}_{\boldsymbol{k}}(t,\overline{t}) \doteq \sum_\Delta |M^U_{\boldsymbol{kpq}}|^2 C^*_{\boldsymbol{p}}(t,\overline{t})S^*_{\boldsymbol{q}}(t,\overline{t}) + \sum_\Delta M^U_{\boldsymbol{kpq}}M^{U*}_{\boldsymbol{kqp}}V^*_{\boldsymbol{p}}(t,\overline{t})V_{\boldsymbol{q}}(\overline{t},t). \tag{248c}$$

Note that V is driven by the externally specified cross correlation X. If X and V are set to zero, then Σ' vanishes, one finds $W_{\boldsymbol{k}}(t,t') = \int_0^t \mathrm{d}\overline{t}\,R_{\boldsymbol{k}}(t;\overline{t})F^{\mathrm{ext}*}_{\boldsymbol{k}}(t',\overline{t})$, and the right-hand side of Eq. (247b) becomes $\int_0^{t'} \mathrm{d}\overline{t}\,[F^{\mathrm{nl}}_{\boldsymbol{k}}(t,\overline{t}) + F^{\mathrm{ext}}_{\boldsymbol{k}}(t,\overline{t})]R^*_{\boldsymbol{k}}(t';\overline{t})$, showing that F^{ext} adds to the internal noise F^{nl} as expected. Equation (247a) with formula (248a) reproduces the passive propagator renormalization derived in Sec. 3.9.7 (p. 83). Some features of these equations in the presence of nonzero cross correlation X were discussed by Krommes (2000b); for further discussion, see Sec. 12.7 (p. 248).

Although the structure of Eqs. (247) may appear to be more complicated than that of the self-consistent DIA, the passive equations can actually be derived from the self-consistent ones if one is careful. Thus imagine reworking the self-consistent calculation for a multispecies nonlinear coupling $\sum M^{U,\alpha\beta\gamma}_{\boldsymbol{kpq}}\psi^{\beta*}_{\boldsymbol{p}}\psi^{\gamma*}_{\boldsymbol{q}}$, with $\psi^\beta = \psi$ and $\psi^\gamma = \chi$. That is, the first and second indices of M^U always refer to ψ (an active variable that responds under perturbations), while the third always refers to χ (a passive function). The statistics are determined by the symmetrized mode-coupling coefficient $M_{\boldsymbol{kpq}} = M^U_{\boldsymbol{kpq}} + M^U_{\boldsymbol{kqp}}$. The strategy is to first retain all terms of the self-consistent calculation, then to discard ones that do not enter in a passive problem. For example, with $\widehat{C} \equiv \widehat{C}^{\alpha\beta}$ the internal noise term for the self-consistent DIA is, from Eq. (225b),

$$F^{\mathrm{nl}}_{\boldsymbol{k}}(t,\overline{t}) = \frac{1}{2}\sum_\Delta |M_{\boldsymbol{kpq}}|^2 \widehat{C}^*_{\boldsymbol{p}}(t,\overline{t})\widehat{C}^*_{\boldsymbol{q}}(t,\overline{t}) \tag{249a}$$

$$= \sum_\Delta (|M^U_{\boldsymbol{kpq}}|^2 + M^U_{\boldsymbol{kpq}}M^{U*}_{\boldsymbol{kqp}})\widehat{C}^*_{\boldsymbol{p}}(t,\overline{t})\widehat{C}^*_{\boldsymbol{q}}(t,\overline{t}) \tag{249b}$$

$$= \sum_\Delta [|M^U_{\boldsymbol{kpq}}|^2 C^*_{\boldsymbol{p}}(t,\overline{t})S^*_{\boldsymbol{q}}(t,\overline{t}) + M^U_{\boldsymbol{kpq}}M^{U*}_{\boldsymbol{kqp}}V^*_{\boldsymbol{p}}(t,\overline{t})V_{\boldsymbol{q}}(\overline{t},t)]. \tag{249c}$$

This result is identical to Eq. (248c). Similarly, the self-consistent mass operator operating on some unspecified function $Z_{\boldsymbol{k}}$ is, from Eq. (225a),

$$\Sigma^{\mathrm{nl}}_{\boldsymbol{k}}(t;\overline{t})Z_{\boldsymbol{k}}(\overline{t},t') = -\sum_\Delta M_{\boldsymbol{kpq}}M^*_{\boldsymbol{pqk}}R^*_{\boldsymbol{p}}(t;\overline{t})\widehat{C}^*_{\boldsymbol{q}}(t,\overline{t})Z_{\boldsymbol{k}}(\overline{t},t') \tag{250a}$$

$$= -\sum_\Delta (M^U_{\boldsymbol{kpq}}M^{U*}_{\boldsymbol{pqk}} + M^U_{\boldsymbol{kpq}}M^{U*}_{\boldsymbol{pkq}} + M^U_{\boldsymbol{kqp}}M^{U*}_{\boldsymbol{pqk}} + M^U_{\boldsymbol{kqp}}M^{U*}_{\boldsymbol{pkq}})$$

$$\times R^*_{\boldsymbol{p}}(t;\overline{t})\widehat{C}^*_{\boldsymbol{q}}(t,\overline{t})Z_{\boldsymbol{k}}(\overline{t},t') \tag{250b}$$

$$= -\sum_{\Delta} M^U_{\boldsymbol{kpq}} M^{U*}_{\boldsymbol{pqk}} R^*_{\boldsymbol{p}}(t;\bar{t}) V_{\boldsymbol{q}}(\bar{t},t) Z_{\boldsymbol{k}}(\bar{t},t')$$
$$- \sum_{\Delta} M^U_{\boldsymbol{kpq}} M^{U*}_{\boldsymbol{pkq}} R^*_{\boldsymbol{p}}(t;\bar{t}) S^*_{\boldsymbol{q}}(t;\bar{t}) Z_{\boldsymbol{k}}(\bar{t},t') + 0 + 0. \quad (250c)$$

The third and fourth contributions vanish because the $\boldsymbol{p}$ index of $R_{\boldsymbol{p}}$ is the third one of $M^U_{\boldsymbol{kqp}}$; however, that index refers to χ, which does not respond under perturbations in a passive problem. The first term is absent from the $R_{\boldsymbol{k}}$ equation ($Z_{\boldsymbol{k}} = R_{\boldsymbol{k}}$) because the third index of $M^U_{\boldsymbol{pqk}}$ is passive; however, it remains for $Z_{\boldsymbol{k}} = C_{\boldsymbol{k}}$ because $\psi_{\boldsymbol{k}}$ responds under perturbations. It is readily seen that this result reproduces the $\Sigma_{\boldsymbol{k}}$ contribution to Eq. (247a) and the $\Sigma_{\boldsymbol{k}}$ and $\Sigma'_{\boldsymbol{k}}$ contributions to Eq. (247b).

Finally, a Langevin representation for the passive DIA can be given by simply modifying formula (240) such that $\widetilde{\xi}_{\boldsymbol{q}}$ is replaced by the externally specified $\chi_{\boldsymbol{q}}$. A random-coupling model can also be given. Thus the passive DIA is realizable.

5.6 Early successes and failures of the DIA

The DIA is one of the most extensively tested and thoroughly researched approximations in modern nonlinear physics. It has enjoyed considerable success, but possesses some (well-understood) flaws as well.

5.6.1 Application to stochastic-oscillator models

The application of the DIA to the pedagogically useful stochastic oscillator model (74) has already been briefly described in Sec. 3.9.7 (p. 83). In the difficult limit of strong turbulence (Kubo number $\mathcal{K} \to \infty$), the DIA succeeds in predicting irreversible decay of the response function (on the proper time scale), unlike naive approximations such as cumulant-discard approximations or regular perturbation theory. Various measures of success may be formulated; for example, the area under the response function (an autocorrelation time) is approximated by the DIA to within about 20%.

This general irreversibility of the DIA is an inherent property of the approximation, as can readily be appreciated from its derivation from the random-coupling model. Unfortunately, it is not always appropriate. Consider, for example, the modified oscillator model described by Eq. (84), whose most important property is that its response function does not decay as $\tau \to \infty$. (This imitates the behavior of various integrable systems.) The solution of the DIA for this model is compared with the exact solution in Fig. 17 (p. 138). Although the DIA succeeds in capturing the general period of the nonlinear oscillations, it superimposes an irreversible envelope such that the mean response asymptotes to 0 as $\tau \to \infty$.

The lesson to be learned from this simple example is that in a certain sense the DIA is "too irreversible." The randomly phased coupling coefficients of the RCM wipe out delicate phase correlations between cumulants of high orders. Such correlations are crucial for properly representing various kinds of interesting physics such as coherent (possibly integrable) structures embedded in the flow or intermittent statistics. One important consequence of the neglect of such correlations is that the DIA does not properly describe the interactions of scales of very disparate sizes; see the discussion of random Galilean invariance in Sec. 5.6.3 (p. 138).

Another deficiency of the DIA shows up in the context of passive advection with nonzero mean fields. It is a rigorous consequence of Eq. (74) that the mean field conditional on unit amplitude at $t = t'$ should be identical to the mean response function $R(t;t')$. However, that is not true in

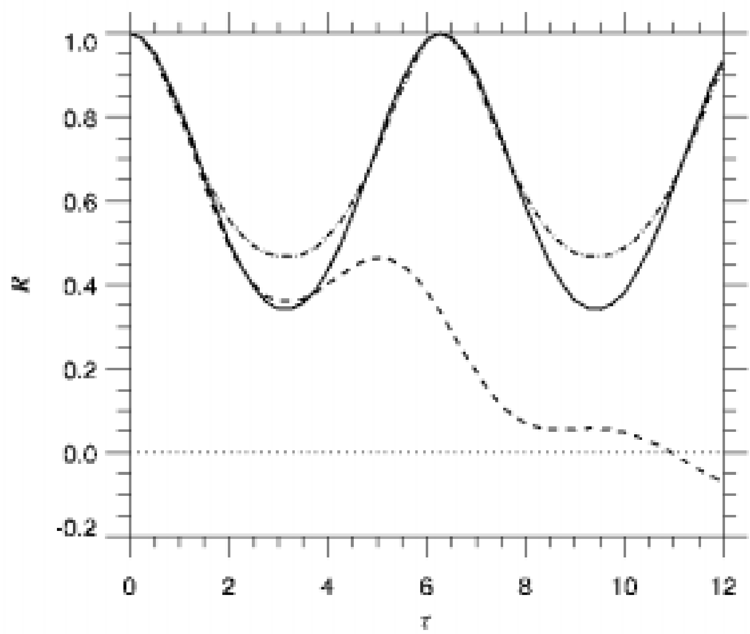

Fig. 17. The response function for the variant of the stochastic oscillator model described by Eq. (84) with $a = 0.9$. Solid line, exact solution; dashed line, DIA; chain-dotted line, exact solution for a Gaussian distribution of a with unit variance $[R(\tau) = \exp[-f(\tau)]I_0(f(\tau))$, with $f(\tau) \doteq \sin^2(\frac{1}{2}\tau)]$.

the DIA (Orszag and Kraichnan, 1967), which predicts spurious long-time oscillations in the mean field (Rose, 1985).

Finally, generalized stochastic-oscillator models have been used by Kraichnan (1976a) to illustrate a deep failing of the DIA when applied to the kinematic-dynamo problem; see Secs. 5.8 (p. 141) and 10.3 (p. 223).

5.6.2 Turbulence at moderate Reynolds numbers

In spite of those difficulties, the RCM preserves the basic dimensional and scaling properties of the original dynamics as well as the essence of the quadratic nonlinearity as a convolution in the mode or wave-number labels, and the second-order DIA statistics are realizable. One can infer that the energetics of turbulence may be well represented by the DIA. This hypothesis was tested by Kraichnan (1964b), who compared detailed numerical solutions of the DIA for homogeneous, isotropic turbulence at moderate Reynolds numbers with experimental measurements of grid turbulence. Satisfactory quantitative agreement was found. In later work Herring (1969) concluded that the DIA may also provide a satisfactory description for certain problems of thermal convection.

5.6.3 Random Galilean invariance

In his original work Kraichnan (1959b) considered the predictions of the DIA for the inertial-range spectrum of fully developed Navier–Stokes turbulence. He found $E(k) \sim k^{-3/2}$ rather than Kolmogorov's K41 prediction $k^{-5/3}$. At the time experimental data were insufficiently precise to distinguish the exponents 1.50 and 1.67, and Kraichnan briefly admitted the possibility that the $k^{-3/2}$ result was correct; however, as measurements have been refined over the years and computing power has improved dramatically, the $k^{-3/2}$ prediction appears to have been definitively ruled out.

A lucid and quantitative explanation was given by Kraichnan (1964e), although he was clearly already aware of the issue in his original papers on the DIA a half-decade earlier. He traced the incorrect $k^{-3/2}$ spectrum to a spurious interaction between the long-wavelength, energy-containing scales and the short-wavelength inertial scales. Kraichnan argued that although the primitive dynamics were invariant under a Galilean transformation and the equal-time correlation functions (energy spectra) should be similarly invariant in an ensemble of *random* such transformations, the DIA was not so invariant. Whereas a small-scale eddy advected by a very-long-wavelength flow should in reality be affected only by the *shear* in that flow, in the DIA the small eddy is distorted at a rate proportional to the *energy* in the long wavelengths. Thus transfer through the inertial range is represented incorrectly, resulting in an incorrect spectrum.

To understand the origins of the $-\frac{3}{2}$ exponent, recall that we have already encountered this exponent in Eq. (123), the form of $E(k)$ in the presence of a particular independent autocorrelation time. Now the DIA predicts an Eulerian τ_{ac} that is the characteristic time to advect a small-scale eddy of size k^{-1} through a distance of the order of its own size by a *macroscopic* flow of rms velocity $\overline{u} \doteq \langle \delta u^2 \rangle$:

$$\tau_{\mathrm{ac}} = (k\overline{u})^{-1}; \tag{251}$$

This is the same assumption used in the derivation of Eq. (123), which reproduces Kraichnan's original result.[172] In the hydrodynamic DIA, that contains a spurious dependence on the size of the long-wavelength fluctuations. If $\overline{u}$ is replaced by the local velocity difference Δu across an eddy—$\overline{u} \to [kE(k)]^{1/2}$, so that $\tau_{\mathrm{ac}} = \tau_{\mathrm{eddy}}$—one recovers the Kolmogorov result.

To illustrate the technical difficulty with eliminating such spurious nonlocal effects, Kraichnan considered the random advection problem $\partial_t \psi + \widetilde{\boldsymbol{V}} \cdot \boldsymbol{\nabla} \psi = 0$, where $\widetilde{\boldsymbol{V}}$ is a spatially uniform random Gaussian vector. The Fourier transform of this equation is thus $\partial_t \psi_{\boldsymbol{k}} + \mathrm{i}\boldsymbol{k} \cdot \widetilde{\boldsymbol{V}} \psi_{\boldsymbol{k}} = 0$ [just the $\mathcal{K} = \infty$ SO model (74)]. Now moment-based closures typically work with three-point correlation functions. For the present problem, consider $T(\boldsymbol{k}, t; \boldsymbol{p}, t'; \boldsymbol{q}, t'') \doteq \langle \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\widetilde{\boldsymbol{V}}t} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\widetilde{\boldsymbol{V}}t'} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\widetilde{\boldsymbol{V}}t''} \rangle$. For calculations of the equal-time energy spectrum, the triplet correlation is needed only for equal times. Then the argument of the exponential vanishes because of the triangle constraint $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}$, and one finds $T(\boldsymbol{k}, t; \boldsymbol{p}, t; \boldsymbol{q}, t) \equiv 1$ for the exact solution. Closures, however, typically approximate the many-time correlations. For example, the DIA for this problem approximates

$$T(\boldsymbol{k}, t; \boldsymbol{p}, t'; \boldsymbol{q}, t'') \approx \langle \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\widetilde{\boldsymbol{V}}t} \rangle \langle \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\widetilde{\boldsymbol{V}}t'} \rangle \langle \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\widetilde{\boldsymbol{V}}t''} \rangle. \tag{252}$$

Each of the averages separately decays because of phase mixing; cf. the discussion of the stochastic oscillator in Sec. 3.3 (p. 52). Therefore even when all of the times are taken to be equal, T decays.

The characteristic time (251) is actually correct for the mean two-time response to a small perturbation. The DIA, fundamentally a two-time theory, represents that effect qualitatively correctly in the decay of the response function. Unfortunately, the Eulerian nature of the closure mixes the one- and two-time information together in a way that is difficult to untangle.[173] Kraichnan noted that this

[172] The essence of Kraichnan's arguments was also sketched by Tennekes (1977).

[173] Technically, the difficulty is manifested as a long-wavelength divergence when inertial-range forms are substituted into the equation for the response function. The details can be found in Chap. 7.1 of McComb (1990).

kind of spurious coupling of the one- and two-time statistics would plague Eulerian-based closures through all orders. He actually calculated in detail the first vertex correction (Sec. 3.9.8, p. 85) for the random advection model; although the spurious effect was reduced, it was not eliminated.

At approximately the same time as the work of Kraichnan (1964e), Kadomtsev (1965) published his monograph on plasma turbulence. He discussed what he called the *weak-coupling approximation*, which is essentially the DIA couched in the frequency domain rather than the time domain. [DuBois and Pesme (1985) discussed errors in Kadomtsev's work, and showed that when his algorithm is implemented consistently one recovers precisely the DIA.] In qualitative terms, he also pointed to the difficulty that one now understands to be the lack of RGI; he referred to the need for a proper treatment of the "adiabatic" interaction between the long and the short scales. It is clear that the works of Kadomtsev and of Kraichnan were independent, and that although Kadomtsev's work was influential in plasma physics, Kraichnan should be credited with the more incisive, mathematically and physically precise analysis of the issue.

The problem of random Galilean invariance can be ameliorated or cured by alternate approaches to the closure problem. The physical idea that τ_{eddy} should be the appropriate time seen in a *Lagrangian* frame can be made precise by the Lagrangian schemes mentioned briefly in Sec. 7.1 (p. 181); a general "decimation" approach is described in Sec. 7.5 (p. 197). More prosaically, Kraichnan (1964e) suggested several ways of cutting off the wave-number integrations to ensure RGI. For instance, in the equations for the two-time functions one can restrict the p–q integration domain of Fig. A.1 (p. 263) to $q > \alpha^{-1}k$ and $p > \alpha^{-1}k$, where $\alpha \gtrsim 2$. Examples of calculations in which this approach has been used include the works of Sudan and Keskinen (1977) and Sudan and Pfirsch (1985).

The difficulty with random Galilean invariance has been frequently invoked as sufficient reason to dismiss the DIA (and, in some extreme cases, any approach based on statistical closures). The latter reaction is obviously logically flawed; the $k^{-5/3}$ law, possibly corrected for intermittency effects (Frisch et al., 1978), is certainly an observable statistical property of a particular nonlinear system, so must yield in principle to rigorous mathematical justification. Furthermore, even dismissal of just the specific Eulerian DIA is indefensible in important situations, especially many of interest to plasma physics. In particular, for situations with low or moderate Reynolds numbers (so that inertial ranges are not well developed) failure to preserve random Galilean invariance does not seem to be crucial. Furthermore, even a well-developed inertial range possesses little energy relative to that of the energy-containing range (by definition); it is the latter that determines macroscopic transport coefficients. One therefore expects that the DIA and similar closures should make reasonable predictions for transport. One plasma-physics example that has been studied in some detail, the Hasegawa–Wakatani system, bears out this claim, as is discussed in Sec. 8.5 (p. 208).

5.7 Eddy diffusivity

For homogeneous, isotropic fluid turbulence, the DIA is most simply couched in a Fourier wave-number representation; the mean velocity field ($\boldsymbol{k} = \mathbf{0}$ component) may usually be taken to vanish. However, for inhomogeneous, anisotropic turbulence (the usual case in practice), an $\boldsymbol{x}$-space representation is more convenient and the mean field is nontrivial. Taylor (1915) showed that "turbulent motion is capable of diffusing heat and other diffusible properties through the interior of a fluid in much the same way that molecular agitation gives rise to molecular diffusion" (Taylor, 1921). According to the discussion in Sec. 1.3.1 (p. 13), a turbulent diffusion coefficient scales as $D \sim \overline{v}\ell$, where $\overline{v}$ is a characteristic rms velocity fluctuation [174] and ℓ is a characteristic Lagrangian correlation length, frequently called a *mixing length* after Prandtl (1925). [175] Thus for passive advection the rigorous equation for the mean field, $\partial_t\langle T\rangle = -\partial_x\langle\delta V_x\,\delta T\rangle$, is assumed to reduce to $\partial_t\langle T\rangle = \partial_x D\,\partial_x\langle T\rangle$. [$D$ may itself depend on $\partial_x\langle T\rangle$, as would be typical in problems of drift-wave turbulence. The solution for the analogous equation for fluid velocity in a turbulent jet was reviewed by Kadomtsev (1965).]

Kraichnan (1964c) gave a systematic and detailed analysis of the DIA for arbitrary turbulent flows. He showed that the DIA provides a natural generalization of mixing-length concepts to the practical situation in which the scale lengths of the fluctuations and the mean fields are not cleanly separated. That paper also contains some valuable discussion of the fidelity and interpretation of the DIA. More such discussion can be found in the work of Kraichnan (1976b), where difficulties with the naive concept of eddy viscosity were analyzed in detail. That work will be discussed in Sec. 7.3 (p. 189).

5.8 Diffusion of magnetic fields by helical turbulence

A natural generalization of the concept of eddy viscosity is to the problem of the kinematic dynamo, defined in Sec. 2.4.8 (p. 43). If one assumes helical, isotropic turbulence (for which $\langle\boldsymbol{u}\rangle = \boldsymbol{0}$) and that $\langle\boldsymbol{B}\rangle$ has weak gradients, then general considerations of tensor symmetry lead to the form

$$\partial_t\langle\boldsymbol{B}\rangle = \mu_m\nabla^2\langle\boldsymbol{B}\rangle - \boldsymbol{\nabla}\times(\alpha\langle\boldsymbol{B}\rangle), \tag{253}$$

involving the two undetermined coefficients μ_m and α. The latter term is conventionally called the *α effect.* For *small* $\mathcal{R}_m$ straightforward quasilinear analysis of Eq. (62a) leads (Steenbeck and Krause, 1969) to Eq. (253) with $\mu_m = \mu_{m,\mathrm{cl}}$ [$\mu_{m,\mathrm{cl}}$ is defined after Eqs. (62)] and α being a particular moment of the helicity spectrum that is also proportional to $\mu_{m,\mathrm{cl}}$. Kraichnan (1976a) instead considered $\mathcal{R}_m = \infty$ and discussed the problem from the point of view of the DIA. He showed that when the velocity field was rapidly decorrelated on a timescale $\tau_{\mathrm{ac}}^{(u)}$, the form of Eq. (253) still held provided that $\mu_m = \overline{u}^2\tau_{\mathrm{ac}}$ and $\alpha = \frac{1}{3}\tau_{\mathrm{ac}}\mathcal{H}$, where $\overline{u}$ is the rms velocity, $\mathcal{H}$ is the helicity density, and τ_{ac} is the shorter of $\tau_{\mathrm{ac}}^{(u)}$ and the eddy turnover time. The turbulent α effect is usually said to exist only in helical flows, but see Gilbert et al. (1988) for a counterexample. Linear analysis of Eq. (253) shows that sufficiently large α can cause $\langle\boldsymbol{B}\rangle$ to grow. Such an instability is due to the vector character; there is no corresponding instability of a passive scalar.

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174 Velocity fluctuations are $\delta u \sim \ell|\partial U/\partial y|$. See Kadomtsev (1965), Chap. III, Sec. 2(a).

175 It is worthwhile to recall the warning of Tennekes and Lumley (1972): "Let us recall that mixing-length expressions can be understood as the combination of a statement about the stress ($-\langle u_x u_y\rangle \sim \overline{u}^2$) and a statement about the mean-velocity gradient ($\partial U_y/\partial x \sim \overline{u}/\ell$). These statements do not give rise to inconsistencies if there is only one characteristic velocity, but they cannot be used to obtain solutions to the equations of motion if there are two or more characteristic velocities that contribute to $\overline{u}$ in unknown ways. In other words, *mixing-length theory is **useless** because it cannot predict anything substantial*; it is often confusing because no two versions of it can be made to agree with each other. Mixing-length and eddy-viscosity models should be used only to generate analytical expressions for the Reynolds stress and the mean-velocity profile if those are desired for curve-fitting purposes in turbulent flows characterized by a single length scale and a single velocity scale. *The use of mixing-length theory in turbulent flows whose scaling laws are not known before-hand should be avoided.*" This admonition has frequently been ignored in the plasma-physics literature.

As discussed by Kraichnan (1976a), the appearance of $\tau_{\rm ac}$ (possibly computed self-consistently) in the previous formulas is the principal qualitative contribution of the DIA; it appears to be successful for nonvanishing $\mathcal{H}$. But Kraichnan also emphasized that helicity *fluctuations* in mirror-*symmetric* turbulence lead to what he called an α^2 *effect* that is *not* captured by the DIA. The reason is that the α^2 effect first enters at fourth order in perturbation theory whereas the structure of the DIA is determined algorithmically from statistical approximations made at second order. For further discussion of this difficulty and of possible resolutions, see Sec. 10.3 (p. 223).

5.9 Vlasov DIA

It was at least half a decade after Kraichnan's proposal of the DIA for strong Navier–Stokes turbulence before plasma physicists began to develop theories of strong plasma turbulence that attempted to go beyond WTT. In the U.S. the resonance-broadening theory (Sec. 4.3, p. 108) of Dupree (1966) was very influential, but see also Galeev (1967). Shortly after Dupree's original paper, Orszag and Kraichnan (1967) published a critique of the RBT (and other related approximations) in which they proposed the DIA for Vlasov turbulence. At the time, that work was largely ignored by the plasma-physics community.[176] That was very unfortunate, as the paper contained a large number of important insights. An incomplete list includes the following:

(1) Orszag and Kraichnan clearly distinguished between the passive (stochastic-acceleration) problem and the self-consistent Vlasov problem.
(2) They used the random-coupling approach to write down realizable models, including the DIA, for both passive and self-consistent dynamics.
(3) They gave an elegant and pedagogical discussion of the properties of the models, emphasizing both their strong points and their weak points.
(4) They pointed out that Dupree had not provided a complete prescription for closing the statistics of the electric field, and they offered a consistent one.
(5) They observed that Dupree's theory did not conserve energy and momentum, and they traced that to the passive approximation that was implicitly used.[177]
(6) They took issue with an assertion of Dupree that his test-wave expansion of the exact particle propagator in powers of the mean one was convergent.

The work of Martin et al. (1973) on the general field theory of classical statistical dynamics led to a revival of interest in the formal and systematic description of plasma turbulence. Krommes, independently DuBois, and co-workers of those authors showed how to reduce the formal theory to simpler approximations such as RBT. Their work will be described after the general MSR formalism is developed in Sec. 6 (p. 146).

5.10 Early plasma applications of the DIA

In this section I survey various early research on the DIA as applied to plasma physics. This discussion covers work done through about 1984 except that detailed applications to the Vlasov

[176] It was discussed a decade later by Montgomery (1977), and influenced DuBois and Espedal (1978) and Krommes (1978).
[177] Analogous observations in the plasma literature were not made until much later with the work of Dupree and Tetreault (1978).

equation (including considerable important work by DuBois and co-workers) are reserved for Sec. 6.5 (p. 170). For an earlier, much more cursory review, see Krommes (1984a). Subsequent, more quantitative research is described in Sec. 8 (p. 199).

5.10.1 Renormalized plasma collision operator and convective cells in magnetized plasma

Classical Coulomb collisions in weak magnetic fields lead to cross-field transport coefficients that scale as B^{-2} (Braginskii, 1965). $\boldsymbol{E}\times\boldsymbol{B}$ motion, however, leads to B^{-1} scaling in the absence of parallel dynamics (Taylor and McNamara, 1971). It is of interest to consider a unified formalism that embraces both of those regimes. The cleanest situation arises for the thermal-equilibrium statistical dynamics of a 2D system of charged rods. Pioneering numerical simulations of that model were performed by Dawson et al. (1971) and Okuda and Dawson (1973). They actually observed three regimes for the magnetic-field scaling of the test-particle diffusion coefficient: (i) the classical B^{-2} regime; (ii) a plateau regime in which the transport was independent of B; and (iii) a B^{-1} regime. In order to explain those results, they carefully considered the normal modes of a 2D magnetized plasma in thermal equilibrium. In addition to the well-known hybrid oscillations and Bernstein modes, they identified a new "zero-frequency" mode (damped by ion shear viscosity) that they called a *convective cell*. They employed the fluctuation–dissipation theorem to show that the convective cell carried the bulk of the energy as $B \to \infty$, and used simple strong-turbulence estimates (not necessarily identified as such) to argue that the convective cells were responsible for the B^{-1} scaling.

Although Dawson and Okuda did not recognize it, their work was closely related to the then-recent discovery of *long-time tails* in certain correlation functions arising in neutral-fluid kinetic theory. The observation, first made by Alder and Wainwright (1970) on the basis of their computer simulations, was that the Green–Kubo integrands, time integrals of which define transport coefficients, exhibited an algebraic decay $\sim \tau^{-d/2}$ (d being the dimension of space) on hydrodynamic timescales. The result was subsequently reproduced by a variety of theoretical calculations [pedagogically reviewed by Reichl (1980, Chap. 16)] and interpreted as the effect of the hydrodynamic modes (those whose frequency vanishes as k^2). Whereas in 3D the $\tau^{-3/2}$ tail is integrable, in 2D the integral of the τ^{-1} tail is logarithmically divergent, signaling a breakdown of the usual local description of transport.

Krommes (1975) and Krommes and Oberman (1976b) discussed the connection between the work of Dawson and Okuda and the theory of long-time tails. They developed a general formalism (Krommes and Oberman, 1976a), based on a two-time BBGKY hierarchy developed in earlier work by Williams (1973), that could be used to predict hydrodynamic contributions to plasma transport but also embraced the usual classical regimes.[178] They showed that a consistent treatment of the self-interactions of the hydrodynamic modes required that terms of all orders in the BBGKY hierarchy be retained. They argued for a particularly natural subset; the resulting renormalized approximation was identical in form to the *self-consistent field approximation*[179] of Herring (1965).

In fact, since Krommes and Oberman were working in thermal equilibrium, in which the FDT holds, the approximation turned out to be identical to the DIA for their problem. It led to a renormalized Balescu–Lenard-like collision operator for plasmas in strong magnetic fields that indeed

[178] The two-time hierarchy also provides the most natural way of deriving the Test-Particle Superposition Principle of Rostoker (1964b); see Krommes (1976).

[179] The self-consistent field approximation (Herring, 1973) retains the DIA equations for $C(t,t)$ and $R(t;t')$, but drops the equation for $C(t,t')$ in favor of the fluctuation–dissipation Ansatz $C(t,t') = R(t;t')C(t',t')$.

embraced all of the scaling regimes previously found by Dawson and Okuda.

The methods used by Krommes and Oberman were relatively primitive. Direct renormalization of the BBGKY hierarchy is clumsy and not to be recommended; the generating-functional methods of Martin et al. (1973), to be discussed in Sec. 6 (p. 146), are far superior. Nevertheless, a variety of useful physical and technical insights followed from the work, which was one of the first practical applications of the DIA in plasma physics (albeit not to a true turbulence problem involving fluctuations that are far from equilibrium).

5.10.2 Turbulence in the equatorial electrojet

Sudan and Keskinen (1977, 1979) formulated the DIA for a model of weakly ionized plasma turbulence driven by the equatorial electrojet. Following Kadomtsev (1965), they wrote the equations in the frequency representation. They did not faithfully solve the coupled equations for correlation and response functions, but postulated a plausible form for the spectrum, then estimated the turbulent linewidth. Reasonable agreement was found with the experimental data. Their calculations [see also Sudan et al. (1997)] provided an early plasma-physics example of how to estimate for complicated practical situations the self-consistency effects inherent in the DIA. Some of that work was reviewed by Sudan (1988) and Similon and Sudan (1990).

5.10.3 Forced and dissipative three-wave dynamics

One of the important early tests of the DIA was given by Kraichnan (1963), who compared its predictions for a system of three coupled "shear waves" with direct numerical simulations. The model equations studied had the form

$$\partial_t \psi_K = M_K \psi_P \psi_Q \tag{254}$$

plus the cyclic permutations $K \to P \to Q$; i.e., linear effects were absent. An appropriately defined energy is conserved. In fact, the system (254) is integrable (Meiss, 1979; Terry and Horton, 1982), so random behavior enters only extrinsically (from random initial conditions). Although general arguments suggest that the performance of closures such as the DIA should not depend on whether the turbulence is extrinsic or intrinsic in origin, it is useful to verify that explicitly. Therefore as a paradigm Krommes (1982) considered the generalization of Eq. (254) to include a complex linear frequency $\omega_k = \Omega_k + \mathrm{i}\gamma_k$:

$$\partial_t \psi_K + \mathrm{i}\omega_K \psi_K = M_K \psi_P^* \psi_Q^*. \tag{255}$$

Such models can possess strange attractors and exhibit chaos in certain regimes (Wersinger et al., 1980). The particular form of the M_k's was derived from the three-mode version of the Terry–Horton equation (43). Direct simulations of the new system exhibited linear growth, then nonlinear saturation. Numerical solution of the DIA showed that it was able to adequately reproduce the steady-state fluctuation level. This calculation was technically trivial, but it made the important conceptual point that the DIA provided a reasonable description of renormalized drift-wave mode coupling.

When all of the γ_k's are taken to vanish, it is not hard to show that Eq. (255) is derivable from the Hamiltonian

$$\widetilde{H} \doteq 2\,\mathrm{Im}(\psi_K\psi_P\psi_Q) + \left(\frac{\Omega_K}{M_K}|\psi_K|^2 + \mathrm{c.p.}\right) \tag{256}$$

written in the canonical coordinates $q_k = \psi_k/\sqrt{M_k}$ and $p_k = \mathrm{i}\psi_k^*/\sqrt{M_k}$. $\widetilde{H}$ is a third constant of the motion; this underlies the integrability mentioned above and uniquely determines the final covariances in terms of their initial values. The random-coupling model (Sec. 5.2, p. 131) or direct calculation can be used to prove that the DIA also conserves $\widetilde{H}$, so it predicts the final covariances exactly in this case (Bowman, 1992; Bowman et al., 1993).

5.10.4 A Markovian approximation to the DIA

Waltz (1983) used a parametrization of the two-time correlation functions to derive a Markovian approximation; he compared its predictions with direct numerical simulations of the forced HM and TH equations. The agreement was quite reasonable given the relatively low resolution. It must be stressed that, as he recognized, Waltz was *not* solving the DIA itself, which is a very specific set of nonlinear integro-differential equations with memory. Nevertheless, his work added further evidence to the belief that appropriately symmetric second-order closures could be successful in plasma-physics applications. The modern theory of Markovian closures is described in Sec. 7.2 (p. 182).

5.10.5 Self-consistency and polarization effects

In Sec. 3.5.4 (p. 64) I referred in general terms to the difference between self-consistent and passive problems; see Eq. (102). When the self-consistent DIA is written out for the Vlasov equation, the extra term in Eq. (102) leads to a plethora of terms in the closure equations. To gain physical insight into their meaning, Krommes and Kotschenreuther (1982) considered an analogy to the Balescu–Lenard operator of classical transport theory. As was described in Sec. 2.3.2 (p. 30), the two terms of Eq. (32) are well known to correspond to (i) (passive) velocity-space diffusion of test particles by the random fields, and (ii) a self-consistent polarization effect that describes the formation of a shielding cloud around the test particle. Krommes and Kotschenreuther showed how the terms from the Vlasov DIA could be put into more or less one-to-one correspondence with terms from the *linearized* Balescu–Lenard operator; the extra terms arising from self-consistency were identified with and henceforth called the *polarization effects* (Krommes and Kleva, 1979). I defer further discussion of this point until Sec. 6.5 (p. 170), where the theory of the nonlinear dielectric function is described.

By considering the derivation of the BL operator from Fokker–Planck theory (Ichimaru, 1973), Krommes and Kotschenreuther were also able to see how the extra terms were related to linearization of Fokker–Planck coefficients, an observation that was independently made by Boutros-Ghali and Dupree (1981).

5.10.6 The DIA and stochastic particle acceleration

Maasjost and Elsässer (1982) criticized the use of the DIA for the stochastic-acceleration problem on the grounds that it either agreed with much simpler Fokker–Planck theory or did not agree with simulation data. Their arguments were analyzed in detail by Dimits and Krommes (1986). The latter authors noted that Maasjost and Elsässer had used a highly non-Gaussian acceleration for which none of the standard closures, including the DIA, would be expected to be correct (Sec. 6.3, p. 165). Dimits and Krommes argued that for nearly Gaussian acceleration the DIA was qualitatively superior to the Fokker–Planck and Bourret approximations.

5.10.7 Miscellaneous references

Among a variety of additional papers on the DIA that could be cited, I shall mention just three representative ones selected for their diversity of applications.

DuBois and Rose (1981) gave an authoritative and detailed discussion of the DIA in the context of Langmuir turbulence.

Krommes et al. (1983) discussed the DIA in the context of particle transport in stochastic magnetic fields by treating the magnetic field in the drift-kinetic streaming term $v_\parallel \widehat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}F$ [see Sec. 2.3.1 (p. 28)] as a passive random variable. Although the DIA was deemed to be qualitatively successful in some regimes, in others it failed to capture some detailed consequences of the stochastic instability of magnetic field lines. Krommes *et al.* argued that such problems would be cured by appropriate vertex renormalizations (Sec. 6.2.2, p. 155), and made some initial steps in that direction.

Finally, Dubin (1984b) considered the application of the DIA to the logistic map (May, 1976) defined by Eq. (98). For the important case of $\lambda = 4$ (whose invariant measure is known analytically and covers the entire interval $0 \leq x \leq 1$), he showed that the basic DIA failed because it did not constrain x to lie in the unit interval. Instead, a periodic extension of the DIA appeared to be promising. Dubin also discussed a variety of deep issues and techniques relating to realizability constraints (Sec. 3.5.3, p. 63) that might be used to formulate superior closures. The lines of research initiated by Dubin have not been pursued to the extent that they deserve.

6 MARTIN–SIGGIA–ROSE FORMALISM

"[We] present what we believe to be the elusive generalization which is necessary for deriving a renormalized set of equations and thus to deduce the renormalized statistical theory of a classical field satisfying a nonlinear dynamical equation." — *Martin et al. (1973).*

We have by now encountered a variety of approaches to the derivation of closed equations for low-order statistical moments. Some are semiheuristic [resonance-broadening theory (Sec. 4.3, p. 108) and the clump algorithm (Sec. 4.4, p. 119)]; some are more systematically based, especially the methods based on regular perturbation theory [quasilinear theory (Sec. 4.1, p. 90) and weak-turbulence theory (Sec. 4.2, p. 98)] and diagrammatic summation of perturbation theory through all orders (Sec. 3.9.7, p. 83). However, none of them is easily extendable to justifiable, quantitatively accurate descriptions of strong turbulence.

Of the methods mentioned, diagrammatic summation has the most generality. However, diagrammatic representations are tedious to work with, especially for self-consistent problems; one can be plagued by both combinatoric and topological questions. An example of the plethora of terms that can result is provided by the work of Wyld (1961) on the diagrammatic renormalization of the NSE; see also the work of Kraichnan (1961) on the simpler stochastic oscillator model, Eq. (74). At the level of the DIA, Kraichnan called the diagrammatic resummations *line renormalization*, to be distinguished from a further *vertex renormalization* [called *higher-order interactions* by Kraichnan (1958a)]. Indeed, he derived (Kraichnan, 1961) a vertex renormalization for the $\mathcal{K} = \infty$ stochastic oscillator (Sec. 3.3, p. 52) that was very successful, and referred in several publications (Kraichnan, 1958a, 1964e) to the corresponding approximation for Navier–Stokes turbulence although he did not publish the details. A simpler example of the diagrammatic method is the work of Horton and Choi (1979) on renormalized ion acoustic turbulence.

In fact, renormalization procedures had been highly developed in quantum field theory (QFT) long before the classical attempts outlined in the last several paragraphs; a brief history is given in Sec. 6.1 (p. 147). However, the nonlinear equations of concern in the present article are classical, and taking the direct classical limit of quantum mechanics is difficult and subtle; moreover, one needs to handle dissipation, not incorporated in either standard quantum-level descriptions or the thermal-equilibrium classical descriptions frequently used in condensed-matter physics. It was therefore a major triumph when, in one of the most elegant papers of modern classical statistical physics, Martin et al. (1973) showed how to treat the *nonequilibrium statistical dynamics of classical field theories* directly by replacing the infinity of primitive diagrams by a few (functionally) closed equations: (i) the exact equation for the mean field; (ii) a 2×2 matrix Dyson equation linking the two-point correlation function C and the mean infinitesimal response function R; and (iii) a functional equation for a vertex matrix Γ (containing three distinct entries) related to three-point correlation and response functions. The method has come to be known as the *MSR formalism*; based on generating functionals and path-integral representations, it is the classical generalization of Schwinger's nonperturbative approach to quantum field theory (Schwinger, 1951a). It accomplishes renormalization in one fell stroke, banishing tedious combinatoric difficulties. The DIA emerges as the natural lowest-order approximation, and Kraichnan's higher-order approximation follows as a logical generalization (first vertex renormalization). The method also suggests a variety of nonperturbative techniques that are still incompletely explored.

Various considerations to be discussed below show that the MSR formalism is "not a panacea" (Martin, 1976). Nevertheless, it provides a compelling unification of various earlier cumbersome techniques, highlights strong and beautiful links between a variety of fields, and permits an economy of description that can be extremely useful in practice (Krommes and Kim, 2000). I therefore give a relatively thorough discussion. In Sec. 6.1 (p. 147) I provide a highly condensed historical background on the general problem of renormalization. The actual equations of MSR (correct for Gaussian initial conditions) are derived in Sec. 6.2 (p. 153). The treatment of non-Gaussian initial conditions is discussed in Sec. 6.3 (p. 165). The path-integral representation of the formalism is described in Sec. 6.4 (p. 166). In Sec. 6.5 (p. 170) the MSR techniques are used to derive a formally exact representation of the nonlinear dielectric function. The theory of the nonlinear plasma dielectric permits a unification of various superficially disparate lines of research in plasma physics, including derivations of the wave kinetic equation (Sec. 6.5.4, p. 176), resonance-broadening formalism (Sec. 6.5.5, p. 178), and self-consistent quasilinear theory (Sec. 6.5.6, p. 180).

6.1 Historical background on field-theoretic renormalization

Although the MSR formalism of classical renormalization is self-contained, it did not spring from a (field-theoretic) vacuum. In presenting the following abbreviated history, I draw heavily on the excellent works of Pais (1986; a "history and memoir" of elementary-particle physics) and

Mehra (1994; a scientific biography of Richard Feynman).[180] A historical account of Feynman's early contributions to quantum electrodynamics was given by Schweber (1986). Further historical and philosophical remarks can be found in the collection of articles on renormalization edited by Brown (1993). The reader is urged to consult those works for a much more complete perspective and many references impossible to list here. A very clear and pedagogical introduction to renormalization in the context of critical phenomena was given by Binney et al. (1992). A modern and detailed account of renormalized QFT is by Zinn-Justin (1996).

6.1.1 Mass and charge renormalization

"The elementary phenomena in which divergences occur, in consequence of virtual transitions involving particles with unlimited energy, are the polarization of the vacuum and the self-energy of the electron .... The basic result of these fluctuation interactions is to alter the constants characterizing the properties of the individual fields." — *Schwinger (1948).*

By the end of the 1930s it had become clear that the nascent analytical theory of quantum electrodynamics (QED), initiated by Dirac (1927), was in serious trouble due to various infinities. Certain primitive infinities such as infinite energy due to zero-point oscillations of the electromagnetic sea or infinite charge due to a Dirac sea filled by an infinite number of negative-energy electrons were easily eliminated by redefinitions of the zero points (technically, by "normal ordering" of the creation operator $\widehat{\psi}$ and annihilation operator ψ).[181] Nevertheless, although a small parameter (the fine-structure constant) $\alpha \doteq e^2/\hbar c \approx \frac{1}{137}$ had been identified so that perturbation theory seemed appropriate, effects thus calculated still exhibited high-energy (ultraviolet) divergences. Although such infinities became visible only at extremely short distances, where modifications to the theory could be expected, the divergences could not simply be ignored; for example, Oppenheimer (1930) predicted an infinite shift of the spectral lines of the hydrogen atom due to interaction of the electron with the radiation field. Of course, the measurable experimental value of that shift (Lamb and Retherford, 1947) is finite and very small.

It became appreciated that the remaining divergences were related to two distinct physical effects: the *self-energy of the electron*; and the *polarization of the vacuum*. A short and readable discussion can be found in the Introduction to the paper by Schwinger (1948). A very clear technical explanation of the self-energy correction was given by Feynman (1949).[182] Briefly, at first order[183] in α an electron can interact with itself by emitting a photon and later reabsorbing it, as shown in Fig. 18 (p. 149).

Unfortunately, the process diagrammed in Fig. 18 is divergent at large energies. A partial solution to this difficulty of apparently infinite electron self-energy was found by noting that standard field theories of QED begin with a Lagrangian containing an electron mass parameter m_0. The crucial

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[180] The definitive biography of Julian Schwinger by Mehra and Milton (2000) was not available to me at the time of writing, but provides a wealth of useful information and insights.

[181] A nice account of the role of creation and annihilation operators in the second-quantization route to many-particle quantum field theory, and of the relation of that formalism to solution of the many-particle Schrödinger equation, can be found in Fetter and Walecka (1971).

[182] These papers are cited for their clarity, not historical precedence.

[183] It would be more consistent with the discussion in the rest of the article to introduce $\widehat{\alpha} \doteq \sqrt{\alpha}$. Then one would speak of processes of second order in $\widehat{\alpha}$, emphasizing that the effects are related to quadratic nonlinearity.

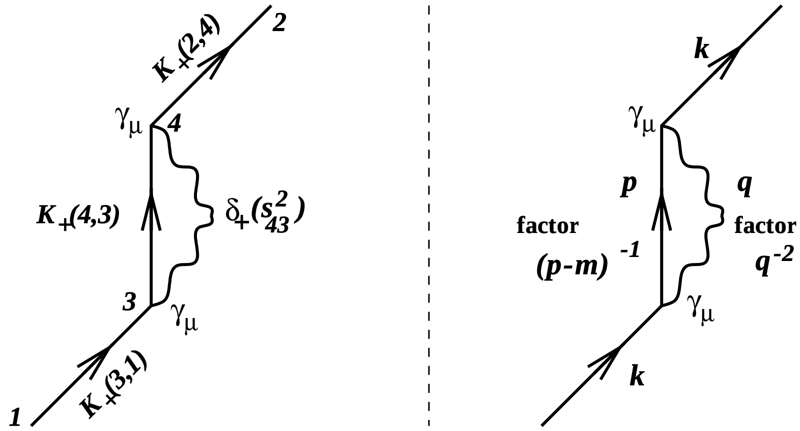

Fig. 18. First-order self-energy contribution of an electron interacting with itself by emitting, then absorbing a photon; after Figs. 2 (real space) and 3 (momentum space) of Feynman (1949) (momentum labels have been changed to conform to present notation). Straight lines represent the electron; wiggly lines represent the photon. s is the invariant space-time distance $t^2 - r^2$. Compare these figures with Fig. 7 (p. 84). In general renormalization theory (Sec. 6.2, p. 153), the electron propagator becomes the infinitesimal response function R, and the photon term becomes the two-point correlation function C.

insight was to recognize that m_0 is *not* the experimentally measurable mass m of the electron. Instead, the "bare" parameter m_0 should be *renormalized*[184] by a correction δm due to the electron–radiation interaction: $m = m_0 + \delta m$. Perturbation theory predicts an infinite value for δm. Presumably a more complete theory would provide ultra-high-energy corrections that would render δm finite (and small, proportional to α). Therefore physical phenomena observable at modest energies should be insensitive to the precise form of the high-energy cutoff. Feynman (1948a) regularized the integrals by replacing in a relativistically covariant way a Dirac delta function by a regular function. Alternatively, the theory can be reworked by using m rather than m_0 as the fundamental mass by adding to the Lagrangian appropriate *counterterms* that are chosen to cancel the infinities; a detailed discussion is given by Zinn-Justin (1996). Gratifyingly, with the theory expressed solely in terms of the experimental mass m, a finite value can indeed be obtained for the Lamb shift; the nonrelativistic calculation of Bethe (1947) was followed by the exceedingly tedious but successful relativistic calculation of Kroll and Lamb (1949).[185]

Mass renormalization is a specific example of the propagator renormalization introduced in Sec. 3.9.7 (p. 83). An analogous discussion can be given of *charge renormalization*, which is the proper solution to the problem of polarization of the vacuum (by virtual pairs of electrons and positrons). Charge renormalization is an example of the vertex renormalization introduced in Sec. 3.9.7 (p. 83). We will encounter both kinds of renormalization again in the general MSR formalism (Sec. 6.2, p. 153).

[184] Pais (1986) attributes the first use of the word *renormalization* to a paper by Serber (1936). Ideas of Kramers were also significant; see Mehra (1994, Chap. 11) for references.

[185] This single reference does not do justice to the intense activity of the time. For more discussion, see Mehra (1994, Chap. 13) and Feynman (1949, footnote 13). Coincidentally, this particular section of the present article was written on Friday the 13th (really).

6.1.2 Renormalization and intermediate asymptotics

"The concepts of self-similarity are widely used by physicists in quantum field theory and in the theory of phase transitions (where self-similarity is called 'scaling'). ... In fact, scaling is precisely what we understand today by self-similarity of the second kind. It would seem to me useful for those interested in scaling to look at how this concept works in other situations, where ... the origins of the self-similar asymptotics can be traced directly." — *Barenblatt (1979).*

Infinite bare masses, regularization of divergent integrals, and the addition of infinite counterterms may be unpalatable, confusing, or both. Although these concepts will not be used in the formal MSR procedure to be described in Sec. 6.2 (p. 153), they are discussed frequently enough in the literature that it is useful to have a simple model in mind. To illustrate some of the issues, I shall return to the stochastic oscillator (Sec. 3.3, p. 52) at infinite Kubo number ($\tau_{\rm ac} = \infty$). Thus consider

$$\partial_t \psi = -\mathrm{i}\widetilde{\omega}\psi, \tag{257}$$

where $\widetilde{\omega}$ is a Gaussian random *number* with standard deviation β. Let the goal be to find the *damping rate* η of the mean response function R, eschewing details of the actual time dependence. I shall distinguish the total rate η from the nonlinear contribution $\eta^{\rm nl}$, the difference being a possible linear damping rate ν to be added later. I shall consider the Markovian approximation (Sec. 3.9.2, p. 78) $\eta^{\rm nl} = \int_0^\infty \mathrm{d}\tau\, \Sigma^{\rm nl}(\tau)$, where $\Sigma^{\rm nl}$ is the mass operator discussed in Sec. 3.9.7 (p. 83). Both dimensional analysis and the exact solution lead to $\eta^{\rm nl}(\beta) \propto \beta$. However, let us examine the problem from the point of view of perturbation theory.

When $\beta = 0$ the oscillator does not decay, so $\eta^{\rm nl} = 0$. For $\beta \neq 0$, however, one has in lowest-order perturbation theory $\Sigma^{\rm nl} = \beta^2 R_0(\tau)$ [cf. Eqs. (134b) or (247)], so one finds $\eta^{\rm nl} = \int_0^\infty \mathrm{d}\tau\, \beta^2 = \infty$. The discontinuous lowest-order result

$$\eta_0^{\rm nl} = \begin{cases} 0 & (\beta = 0) \\ \infty & (\beta \neq 0) \end{cases} \tag{258}$$

is a characteristic signature of unusual asymptotics and is reminiscent of the infinities encountered in QFT.

To regularize the infinity, let us add a linear damping term to the left-hand side of Eq. (257):

$$\partial_t \psi + \nu\psi = -\mathrm{i}\widetilde{\omega}\psi, \tag{259}$$

where ν is arbitrary but should be thought of as small. This extra term causes the zeroth-order response function to decay exponentially with rate ν, so has the effect of terminating the τ integral that defines $\eta^{\rm nl}$ at a long-time cutoff $\Lambda \doteq \nu^{-1}$. Now one has $\Sigma_0^{\rm nl} = \beta^2/\nu$, a continuous function of β. Notice that β enters to the power 2 [quasilinear scaling; see Sec. 3.3.2 (p. 55)]. This *classical exponent* is analogous to the prediction of mean-field theory for a scaling exponent in the Ginzburg–Landau model of phase transitions (Binney et al., 1992).

With perturbation theory regularized, one can resum according to the propagator renormalization procedure of Sec. 3.9.7 (p. 83). The result is the direct-interaction approximation $\Sigma^{\rm nl}(\tau) = \beta^2 R(\tau)$. Since in the Markovian approximation one has $R(\tau) = H(\tau)\mathrm{e}^{-\eta\tau}$, with $\eta = \nu + \eta^{\rm nl}$, one finds

$\eta^{\rm nl} = \beta^2/(\nu + \eta^{\rm nl})$, a self-consistent nonlinear equation[186] to be solved for $\eta^{\rm nl}(\beta;\nu)$. But now there is no difficulty with taking the limit $\nu \to 0$ ($\Lambda \to \infty$). The solution of $\eta^{\rm nl}(\beta) = \beta^2/\eta^{\rm nl}$ is the expected result $\eta^{\rm nl}(\beta) = \beta$. Here β is raised to the power 1, a value differing by an amount of order unity from the classical power 2. This appearance of an *anomalous exponent* is analogous to the way in which critical exponents in the theory of critical phemonena differ from the predictions of mean-field theory. The self-consistently calculated $\eta^{\rm nl}$ is analogous to the renormalized mass of QFT; the infinity in Eq. (258) is analogous to the infinity in the perturbation theory for the bare mass in the Lagrangian of QFT.

Deep insights into the appearance of anomalous exponents may be had by considering the theory of scaling, self-similarity, and intermediate asymptotics as described by Barenblatt (1996) and very briefly reviewed in Appendix B (p. 264). Barenblatt (1979) had noted the essential identity between (i) the renormalization (scaling) theories employed in QFT and critical phenomena, and (ii) self-similarity of the second kind (defined in Appendix B). The analogy was developed in detail by Goldenfeld and co-workers and was reviewed by Goldenfeld (1992); see also Barenblatt (1996). For the $\mathcal{K} = \infty$ stochastic oscillator, and upon using the notation introduced in Appendix B, one has

$$\frac{\eta^{\rm nl}}{\beta} = \Phi\left(\frac{\beta}{\nu};\epsilon\right), \tag{260}$$

where ϵ is assumed to multiply the right-hand side of Eq. (257). Whereas first-order perturbation theory would give $\Phi = \epsilon^2(\beta/\nu)$, the correct result in the limit $\beta/\nu \to \infty$ is

$$\frac{\eta^{\rm nl}}{\beta} = \left(\frac{\beta}{\nu}\right)^0 \Phi_1(\epsilon), \tag{261}$$

with $\Phi_1(\epsilon) = \epsilon$.

In the present calculation the cutoff time $\Lambda \doteq \nu^{-1}$ was not present in the original model equation (257) (and, of course, is not present in the final result). That is analogous to the situation in QFT, in which no large-energy cutoff is apparent (at least in the absence of gravity).[187] The situation is typically different in classical applications. In the application to critical phenomena (Binney et al., 1992; Goldenfeld, 1992), where time integrals are replaced by d-dimensional momentum integrals, a natural large-k cutoff is provided by the inverse of the lattice spacing a: $k_{\rm max} = 2\pi/a$. For dissipative systems such as the NSE, linear dissipation $k^2\mu_{\rm cl}$ provides a physical analog to ν. To the extent that there are no finite-time singularities, one can calculate the response for all times, both below and above the natural cutoff, and for all spatial scales, both larger and smaller than the Kolmogorov dissipation scale. Thus the MSR procedure makes no reference to a cutoff. Nevertheless, various long-wavelength limits can sometimes be profitably treated by renormalization-group ideas; for more discussion, see Sec. 7.4 (p. 196).

[186] Compare the closely related Eq. (f-9b), p. 112, for the perpendicular diffusion coefficient of magnetized RBT.
[187] Although I shall not discuss the details here, one method of calculating the anomalous exponents by exploiting the independence of the solution on Λ is the *renormalization group*; see Zinn-Justin (1996).

6.1.3 Path-integral formulation of quantum mechanics

"The formulation to be presented contains as its essential idea the concept of a probability amplitude associated with a completely specified motion as a function of time." — *Feynman (1948b).*

So far I have concentrated on the necessity and intuitive content of field-theoretic renormalization. I now turn to the technical advances that are the natural antecedents of the MSR formalism, which is rooted in the two main threads of development of post-war QED, namely, the works of Schwinger (and closely related work of Tomonaga) and of Feynman. An important unification was achieved by Dyson (1949), who demonstrated the ultimate equivalence of the "radiation theories of Tomonaga, Schwinger, and Feynman"; for a short account, see Dyson (1965).

Actually, the specific work of Schwinger that underlies the MSR formalism came later and is described in Sec. 6.1.4 (p. 152). Here I concentrate on Feynman's contribution [see, for example, Schweber (1986)]. In his Ph.D. dissertation (Feynman, 1942), key parts of which were published by Feynman (1948b), he stressed the importance of action principles [thoroughly reviewed by Mehra (1994); see also Lanczos (1949)]. He then showed that the Schrödinger equation $-(\hbar^2/2m)\psi'' + V\psi = \mathrm{i}\hbar\,\partial_t\psi$ follows from the propagation law $\psi(X, t+\epsilon) = A^{-1}\int\!\mathrm{d}x\,\mathrm{e}^{\mathrm{i}\epsilon\hbar^{-1}L(X,t+\epsilon;x,t)}\psi(x,t)$, where $L \doteq \frac{1}{2}m\dot{x}^2 - V$ is the Lagrangian and $A \doteq (2\pi\hbar\epsilon/m)^{1/2}$. By compounding that law, he was then led to a *path-integral representation* of the propagator $K(x,t;x',t')$, defined such that $\psi(x,t) = \int\!\mathrm{d}x'\,K(x,t;x',t')\psi(x',t')$. Namely,

$$K(x,t;x',t') = \lim_{\epsilon\to 0}\int \dots \frac{\mathrm{d}x_i}{A}\frac{\mathrm{d}x_{i+1}}{A}\dots \exp\left(\mathrm{i}\hbar^{-1}\sum_j S(x_j, x_{j+1})\right), \tag{262}$$

where $S \doteq \int_{t'}^{t}\mathrm{d}\bar{t}\,L(\bar{t})$ is the classical action and the velocity dependence of the Lagrangian is appropriately differenced in time.

The beauty of the path-integral formulation is that it deals at once with the entire solution of the problem through all of space-time. It is therefore intrinsically nonperturbative. In some cases, that feature can be usefully exploited by integrating away the dependence of S on certain variables (such as the field coordinates in a theory of coupled particles and radiation). However, it also lends itself naturally to perturbative calculations and thus facilitates physical interpretation; the famous Feynman diagrams are nothing but graphical representations of various terms in perturbation theory, in which L is written as a zeroth-order part L_0 plus a perturbed part δL and Eq. (262) is expanded in δL. Feynman's methods were written up most formally by Feynman (1950). A path-integral representation of the classical MSR formalism will be discussed in Sec. 6.4 (p. 166).

6.1.4 The role of external sources

"The temporal development of quantized fields, in its particle aspects, is described by propagation functions, or Green's functions. The construction of these functions for coupled fields is usually considered from the point of view of perturbation theory. Although the latter may be resorted to for definite calculations, it is desirable to avoid founding the formal theory of the Green's functions on the restricted basis provided by the assumption of expandability in powers of coupling constants." — *Schwinger (1951a).*

Although I have pointed out that path-integral representations are intrinsically nonperturbative, all of the practical work on QED was initially implemented perturbatively. An alternate approach to a nonperturbative formalism was initiated in deceptively short papers by Schwinger (1951a,b) [elaborated by Schwinger (1951c)]. He considered a Lagrangian for interacting matter and radiation that included coupling to *external sources* of particles and currents. He then showed that appropriate functional variations with respect to those sources leads to an exact functional differential equation that relates various propagators. It is this approach that most immediately underlies the work of MSR; the technical details are described in the next section. For a retrospective on Schwinger's work, see Martin (1979).

6.2 Generating functionals and the equations of Martin, Siggia, and Rose

"The formal quantity which will play a central role in our discussion is an operator which serves to infinitesimally change the classical random variable at a given point in space and time. With the aid of this quantity we will be able to ask questions about the response of the system in a representation-free fashion and thus, to determine the response in a state, the details of which are only determined at the end of an exact (or approximate) self-consistent calculation." — *Martin et al. (1973).*

In the present section I review the MSR formalism as presented in the original paper of Martin et al. (1973). In Sec. 6.3 (p. 165) I discuss the generalization needed to treat non-Gaussian initial conditions. The path-integral version of the formalism is described in Sec. 6.4 (p. 166).

6.2.1 Classical generating functionals and cumulants

The MSR formalism is based on a sophisticated use of *generating functionals*. The concepts of moment and cumulant generating *functions* have already been introduced in Sec. 3.5.2 (p. 59), the moment generating function being simply the characteristic function (Fourier transform) of the PDF. For the random field $\psi(1)$, where 1 denotes the complete set of independent variables, a moment generating functional is $Z[\eta] = \langle \exp[\eta(\overline{1})\psi(\overline{1})]\rangle$. Here and subsequently the Einstein summation–integration convention is adopted for repeated indices. The statistically sharp field $\eta(1)$ plays the role of the $-\mathrm{i}k$ in Eq. (89).[188] For example, the mean field is the first functional derivative of Z: $\langle\psi(1)\rangle = \delta Z/\delta\eta(1)|_{\eta=0}$; compare this and similar equations with Eq. (90b).

The corresponding cumulant generating functional is $W[\eta] \doteq \ln Z[\eta]$; cf. Eq. (91a). The n-point, η-dependent cumulants are defined by

$$\langle\langle\psi(1)\dots\psi(n)\rangle\rangle_\eta \equiv C_n(1,\dots,n) = \frac{\delta^n W[\eta]}{\delta\eta(1)\dots\delta\eta(n)} = \frac{\delta\langle\langle\psi(1)\dots\psi(n-1)\rangle\rangle}{\delta\eta(n)}; \qquad (263\text{a,b,c})$$

cf. Eq. (91b). The usual Eulerian correlation functions are obtained by setting $\eta = 0$. For example, the two-point correlation function can be generated from the second functional derivative of W: $C(1,1') = \delta^2 W[\eta]/\delta\eta(1)\delta\eta(1')|_{\eta=0}$. Usually it is unnecessary to indicate the η dependence explicitly.

[188] One need not worry here about possible convergence difficulties in various regions of the complex η plane. The formal manipulations are independent of the value of η, and analytic continuations can be performed if necessary.

For example, one often writes $C(1,1') = \delta\langle\langle\psi(1)\rangle\rangle/\delta\eta(1')$. η can always be set to zero at the end of the calculation.[189]

A multitime BBGKY-like cumulant hierarchy readily follows by taking successive functional derivatives[190] of the equation resulting from the time derivative of $\langle\langle\psi(1)\rangle\rangle$; see analogous calculations in Sec. 6.2.2 (p. 155). However, mere generation of such a hierarchy does not in itself effect statistical closure. The difficulty is that the cumulant hierarchy as described so far contains only correlation functions. However, it seems intuitively reasonable that an efficient description of turbulence should involve not only correlation functions but also response functions. Speaking very loosely, one may say that correlation functions describe the intensity of fluctuations that have arisen in the turbulent state whereas response functions describe the time-dependent fate of fluctuations once they have appeared. The steady turbulent state represents a balance between correlations and response. The fluctuation–dissipation theorem (Martin, 1968) of thermal equilibrium is an important special case, and the DIA spectral balance equation (241) provides a key approximate realization of the general nonequilibrium balance.

In the very pedagogical introduction to the paper of Martin et al. (1973), MSR discussed at length the difficulties of statistical closure of a classical theory. They pointed out that whereas in quantum theory the existence of nontrivial commutation relations for the field operators lead naturally to both correlation functions (anticommutators) and response functions (commutators), in classical theory the commutation properties of ψ are trivial. Thus although n-point correlation functions can readily be derived from W, no means is immediately apparent for the derivation of response functions. The solution of MSR was to extend the system to include not only the original field ψ but also the operator

$$\widehat{\psi} \doteq -\delta/\delta\psi. \tag{264}$$

The need for such a (functional) differential operator arises from the fact that the state of a classical field can change by an infinitesimal amount, as stressed by MSR.

In fact, Martin *et al.* never wrote Eq. (264) explicitly; they preferred a more abstract discussion based on the commutation properties of ψ and $\widehat{\psi}$ [see Eq. (265a)]. Many details underlying the MSR paper were given by Rose (1974) and Phythian (1975, 1976). An explicit construction that leads to Eq. (264) is afforded by the path-integral representation to be described in Sec. 6.4 (p. 166). For additional discussion of the relationships between the MSR formalism and quantum field theory, see Eyink (1996).

At equal times ψ and $\widehat{\psi}$ exhibit canonical (boson) commutation relations analogous to those of the position and momentum variables q and $p = -\mathrm{i}\hbar\,\partial_q$ of quantum mechanics, which obey $[p,q] = -\mathrm{i}\hbar$. That is, upon using an underline to represent all variables except the time and with the conventional notation $[A,B] \doteq AB - BA$, one has $[\psi(\underline{1},t), \widehat{\psi}(\underline{1}',t)] = \delta(\underline{1},\underline{1}')$. If an extended field vector $\Phi(1) \doteq \big(\psi(1),\, \widehat{\psi}(1)\big)^T$ is introduced, then

$$[\Phi(\underline{1},t), \Phi(\underline{1}',t)] = \mathrm{i}\sigma\delta(\underline{1},\underline{1}'), \quad \mathrm{i}\sigma \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{265a,b}$$

[189] η must not be set to zero prematurely because derivatives with respect to η may be taken.

[190] Compare the two-time hierarchy discussed by Krommes and Oberman (1976a); (less sophisticated) generating-function techniques were used in its derivation as well. See also Dawson and Nakayama (1967).

Now consider the extended, *time-ordered* generating functional[191]

$$Z[\eta_1] = \langle e^{\psi(1)\eta(1)+\widehat{\psi}(1)\widehat{\eta}(1)}\rangle_+ = \langle e^{\Phi(1)\eta_1(1)}\rangle_+, \tag{266}$$

where[192] $\eta_1 \doteq (\eta, \widehat{\eta})^T$. It is not difficult to show that

$$\delta^2 Z/\delta\eta(1)\delta\widehat{\eta}(1') = \langle[\psi(1), \widehat{\psi}(1')]e^{\Phi(1)\eta_1(1)}\rangle_+; \tag{267}$$

the commutator is introduced by the time ordering. A key result is that the right-hand side of Eq. (267) evaluated at $\eta_1 = 0$ is precisely the mean infinitesimal response function $R(1; 1')$:

$$R(1; 1') = \langle[\psi(1), \widehat{\psi}(1')]\rangle_+ = \delta^2 Z/\delta\eta(1)\delta\widehat{\eta}(1')\big|_{\eta_1=0}\,. \tag{268}$$

For the detailed arguments, see Rose (1974), Phythian (1975), and the discussion of path integrals in Sec. 6.4 (p. 166). One can adopt the convention that the expectation of any function of ψ and $\widehat{\psi}$ that begins with $\widehat{\psi}$ on the far left vanishes; thus one finds the alternate represention

$$R(1; 1') = \langle\psi(1)\widehat{\psi}(1')\rangle \quad (t > t'). \tag{269}$$

Extended cumulants with spinor indices can now be defined by generalizing Eqs. (263):

$$G(1) = \frac{\delta W[\eta_1]}{\delta\eta_1(1)} \equiv \langle\langle\Phi(1)\rangle\rangle, \quad G(1, \ldots, n) = \frac{\delta G(1, \ldots, n-1)}{\delta\eta_1(n)}. \tag{270a,b}$$

In particular, upon noting Eq. (269) one finds that the two-point, time-ordered correlation matrix of the extended field contains both of the usual correlation and response functions:

$$G_{\eta_1}(1, 1') \doteq \frac{\delta^2 W}{\delta\eta_1(1)\delta\eta_1(1')} = \langle\langle\Phi(1)\Phi(1')\rangle\rangle_+, \quad G(1, 1')\big|_{\eta_1=0} = \begin{pmatrix} C(1, 1') & R(1; 1') \\ R(1'; 1) & 0 \end{pmatrix}. \tag{271a,b,c}$$

Diagrammatically, $G_2 \equiv G$ will be represented by a heavy solid line [Fig. 20(a), p. 161].

6.2.2 The Dyson equations

"The elimination of graphs with self-energy parts is a most important simplification of the theory." — *Dyson (1949).*

In the usual approach to renormalized field theory, one does not attempt to calculate the correlation and response functions directly from their definitions; instead, one deduces evolution equations for them. For the formal work in this section, let us take the primitive dynamical equation

[191] The + subscript denotes time ordering [discussed in footnote 121 (p. 88)]. A shorthand notation is used in which, for arbitrary functional $A[\Phi]$, $\langle A\rangle_+ \equiv \langle(A)_+\rangle$; i.e., the time ordering must be performed before the statistical averaging.

[192] The 1 subscript indicates that $\eta_1(1)$ depends on a single argument, and also distinguishes the vector η_1 from the scalar η and, later, from a two-body source $\eta_2(1, 2)$.

to have the form [193]

$$\partial_t\psi(1) = U_1(1) + U_2(1,2)\psi(2) + \tfrac{1}{2}U_3(1,2,3)\psi(2)\psi(3), \tag{272}$$

where $\partial_t \equiv \partial_{t_1}$ and the U_n coefficients are given. In general, the U_n's may be random. Random U_1 describes external forcing; cf. Eq. (7a). Random U_2 describes the usual passive advection problem (linear in dynamical variables but quadratically nonlinear in random variables); cf. the stochastic oscillator model (74) or the kinematic dynamo problem (62). The corresponding equation for $\widehat{\psi}$ can be shown to be

$$-\partial_t\widehat{\psi}(1) = U_2(2,1)\widehat{\psi}(2) + U_3(2,3,1)\widehat{\psi}(2)\psi(3). \tag{273}$$

Equations (272) and (273) can be combined (Martin et al., 1973) into the symmetrical equation

$$-\mathrm{i}\sigma\dot{\Phi}(1) = \gamma_1(1) + \gamma_2(1,2)\Phi(2) + \tfrac{1}{2}\gamma_3(1,2,3)\Phi(2)\Phi(3), \tag{274}$$

where the arguments now include the spinor indices in the 2×2 extended state space. The nonvanishing elements of the fully symmetric matrices γ_i (called *bare vertices*) have precisely one $-$ index and are defined by $\gamma_{1-}(1) = U_1(1)$, $\gamma_{2-+}(1,2) = U_2(1,2)$, and $\gamma_{3-++}(1,2,3) = U_3(1,2,3)$. For the remainder of this section I shall assume that the γ's are *not* random (i.e., are statistically sharp), so the present formalism deals with self-consistent nonlinearity. A generalization of the formalism that uses statistically sharp γ's but handles passive advection and other kinds of random coefficients can be accomplished by extending the Φ vector to include the random coefficient as its third component and using 2×3 spinors instead of 2×2 ones; see Deker and Haake (1975). A superior technique is to allow random γ's and employ a path-integral formalism, as described in Sec. 6.4 (p. 166).

Successive functional differentiations with respect to η_1 generate higher-order correlation matrices and multipoint generalizations of $R(t;t')$ such as the "two in, one out" response function $R(t;t',t'')$. One has

$$\delta W/\delta\eta_1(1) = \langle\Phi(1)\rangle_{\eta_1}/Z_{\eta_1} \equiv \langle\langle\Phi(1)\rangle\rangle \equiv G(1) \tag{275}$$

(I now drop the η_1 subscripts); the time derivative of this expression introduces both the right-hand side of Eq. (274) and, because of the time ordering, an extra forcing term η_1:

$$-\mathrm{i}\sigma\partial_t G(1) - \gamma_2(1,2)G(2) - \tfrac{1}{2}\gamma_3(1,2,3)[G(2)G(3) + G(2,3)] = \gamma_1(1) + \eta_1(1). \tag{276}$$

The appearance of η_1 on the right-hand side of this equation makes its significance as an external probe of the system apparent; the present methods stem from the seminal work of Schwinger (1951a) mentioned in Sec. 6.1 (p. 147). Note that because of the $-\mathrm{i}\sigma$ it is $\widehat{\eta}$ that perturbs ψ: $\partial_t\psi + \cdots = \widehat{\eta}$, $\partial_t\widehat{\psi} + \cdots = -\eta$. Equation (276) at $\eta_1 = 0$ reproduces the exact equation for the mean field, which is retained without approximation in the MSR formalism. [The $-$ component of Eq. (276) is the statistical average of Eq. (272).] Fluctuation-induced contributions to the mean fields arise, of course, from $G(2,3)$; those are generalized Reynolds stresses. [194]

[193] In the original development of MSR, the U's were taken to be local in time. That restriction is unnecessary, however; see Sec. 6.4 (p. 166).

[194] Note that a possible solution for any closure that includes Eq. (276) is one for which fluctuations vanish

The functional derivative of Eq. (276) with respect to $\eta_1(1')$ gives an equation for $G(1,1')$:

$$-\mathrm{i}\sigma\partial_t G(1,1') - \gamma_2(1,2)G(2,1') - \gamma_3(1,2,3)G(2)G(3,1') - \tfrac{1}{2}\gamma_3(1,2,3)G(2,3,1') = \delta(1,1'). \quad (277)$$

Additional functional derivatives could be taken, but that procedure would merely generate more members of an unclosed multipoint statistical hierarchy. Instead, the key to effecting at least a formal statistical closure of that hierarchy is to perform a *Legendre transformation* (de Dominicis, 1963; de Dominicis and Martin, 1964a,b; Krommes, 1978, 1984a) from $\eta_1(1)$ to the mean field $G(1)$, i.e., to consider [195]

$$L[G_1] \doteq W[\eta_1] - \eta_1(1)G_1(1). \quad (278)$$

Functional derivatives of L with respect to G_1 are called the *vertex functions* Γ_n:

$$\Gamma_n(1,\ldots,n) \doteq \frac{\delta^n L[G_1]}{\delta G_1(1)\ldots\delta G_1(n)}; \quad (279)$$

cf. Eq. (263b). The vertices are clearly completely symmetrical in their arguments. The first few of them are

$$\begin{aligned}
\Gamma(1) &= -\eta_1(1), && (280a)\\
\Gamma(1,2) &= -G^{-1}(1,2), && (280b)\\
\Gamma(1,2,3) &= G^{-1}(1,\overline{1})G^{-1}(2,\overline{2})G^{-1}(3,\overline{3})G(\overline{1},\overline{2},\overline{3}), && (280c)\\
\Gamma(1,2,3,4) &= G^{-1}(1,\overline{1})G^{-1}(2,\overline{2})G^{-1}(3,\overline{3})G^{-1}(4,\overline{4})G(\overline{1},\overline{2},\overline{3},\overline{4})\\
&\quad - \Gamma(1,5,2)G(5,\overline{5})\Gamma(3,\overline{5},4) - [\Gamma(1,5,3)G(5,\overline{5})\Gamma(2,\overline{5},4) + (3 \leftrightarrow 4)]. && (280d)
\end{aligned}$$

Equation (280b) follows by writing $\delta\eta_1(1)/\delta G(2) = [\delta G(2)/\delta\eta_1(1)]^{-1}$ and using the result (280a). Equations (280c) and (280d) additionally use the result that follows by varying the operator identity $G^{-1}G = 1$, namely, $\delta(G^{-1}) = -G^{-1}(\delta G)G^{-1}$. Note that Eqs. (279) and (280b) imply that

$$\Gamma(1,2,3) = -\delta G^{-1}(1,2)/\delta G(3). \quad (281)$$

A quick graphical way of deriving the relations between G_n and Γ_n is to represent G_2 by a heavy solid line and Γ_3 by a filled circle, then to note that $\delta G_2/\delta\eta_1 = G_3 = GG(\Gamma_3 G)$; i.e., *differentiation of a line with respect to* η_1 *inserts a (third-order, renormalized) vertex*. Additionally, the result $\delta\Gamma_n/\delta\eta_1 = (\delta\Gamma_n/\delta G_1)(\delta G_1/\delta\eta_1) = \Gamma_{n+1}G_2$ shows that *differentiation of a vertex raises its order by one*. The results for G_2, G_3, and G_4 are diagrammed in Fig. 19.

Note that $\Gamma_3 \equiv \Gamma$ is a matrix in the spinor indices. The symmetry of that matrix, Eq. (280c), and the vanishing of G_{---} can be used to show that there are exactly three nontrivial elements of Γ, namely, Γ_{-++}, Γ_{--+}, and Γ_{---}; Γ_{+++} vanishes identically.

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identically; i.e., the exact dynamics (with singular initial condition) satisfy the closure. Presumably, however, in turbulent regimes such solutions are unstable.

[195] Frequently an overall minus sign is included on the right-hand side of Eq. (278). A graphical explanation of the Legendre transform can be found in Box 1.2 of Binney et al. (1992).

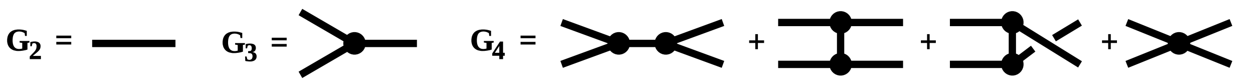

Fig. 19. Relationships between the first few cumulants and renormalized vertices. Successive diagrams follow by differentiation with respect to η_1, using the rules that (i) the derivative of a line inserts a vertex, and (ii) the derivative of a vertex raises its order by one. The first term of G_4 is the DIA; the last three terms represent all of the doubly connected contributions omitted in the DIA.

Because the three-point vertex (matrix) function is intimately related to the three-point correlation function, it is a good target for statistical closure. Indeed, Eq. (277) can be formally closed by introducing the result (280c):

$$[G_0^{-1}(1,\overline{1}) - \Sigma(1,\overline{1})]G(\overline{1},1') = \delta(1,1'), \tag{282}$$

where [Fig. 20(d), p. 161]

$$G_0^{-1}(1,\overline{1}) \doteq -\mathrm{i}\sigma\partial_t\delta(1,\overline{1}) - \gamma_2(1,\overline{1}) - \gamma_3(1,2,\overline{1})G(2), \tag{283a}$$
$$\Sigma(1,\overline{1}) \doteq \tfrac{1}{2}\gamma_3(1,2,3)G(2,\overline{2})G(3,\overline{3})\Gamma_3(\overline{1},\overline{2},\overline{3}). \tag{283b}$$

Equation (282) is called the *Dyson equation* in view of the work of Dyson (1949). In the form $G = G_0 + G_0\Sigma G$ it is diagrammed in Fig. 20(c) (p. 161), which should be compared with Fig. 10 (p. 86). Σ is analogous to the *renormalized mass* of QFT. In this classical formalism it is a 2×2 matrix equation whose $(--)$ and $(-+)$ components are

$$R^{-1}(1,\overline{1})R(\overline{1};1') = \delta(1,1'), \quad R^{-1}(1,\overline{1})C(\overline{1},1') = F^{\mathrm{nl}}(1,\overline{1})R(1';\overline{1}), \tag{284a,b}$$

where

$$R^{-1}(1,\overline{1}) = R_0^{-1}(1,\overline{1}) + \Sigma^{\mathrm{nl}}(1,\overline{1}), \tag{285a}$$
$$R_0^{-1}(1,\overline{1}) \doteq \partial_t\delta(1,\overline{1}) - U_2(1,\overline{1}) - U_3(1,2,\overline{1})G(2), \tag{285b}$$

$F^{\mathrm{nl}} \equiv \Sigma_{--}$, and [196] $\Sigma^{\mathrm{nl}} \equiv -\Sigma_{-+}$. One has already seen this general structure in the form of the DIA; see Eqs. (224). However, under certain technical restrictions Eqs. (284) are exact. [197] Σ^{nl} plays the role of a *generalized resonance broadening* of the linear response; F^{nl} describes a generalized internal stirring that can be called *incoherent noise*. [198] An appealing alternate form of Eq. (284b) is [Fig. 20(g), p. 161]

$$C(1,1') = R(1;\overline{1})F^{\mathrm{nl}}(\overline{1},\overline{1}')R(1';\overline{1}'). \tag{286}$$

We have already encountered this balance form in classical Langevin theory [Eq. (72)] and in the discussion of the DIA [Eq. (241)]; it is now seen to be a general property of renormalized theory.

[196] The minus sign that defines Σ^{nl} [so that it appears with a plus sign in Eq. (285a)] emphasizes its role as a dissipative (generalized) resonance broadening. Krommes (1984a) incorporated that sign into the definition of the Σ matrix, but that convention leads to confusion with the general field-theory literature.
[197] Specifically, the initial conditions must be *Gaussian*. The generalization to non-Gaussian initial conditions is given in Sec. 6.4 (p. 166).
[198] More precisely, F^{nl} is a positive definite form that can be viewed as the variance of an internally produced random noise.

Early discussion of the physical interpretation of this form was given by Kraichnan (1964d) in the context of the DIA (Sec. 5.4, p. 133). Additional interpretation is given in Sec. 6.5 (p. 170).

Since the Dyson equation is formally exact, the focus of statistical closure shifts to the determination of the unknown renormalized vertex matrix Γ, which is analogous to the *renormalized charge* of QFT. To derive an equation for Γ, write Eq. (282) in the form

$$G^{-1}(1,2) = G_0^{-1}(1,2) - \Sigma(1,2), \tag{287}$$

recall Eqs. (281) and (283a), and functionally differentiate Eq. (287) with respect to $G(3)$:

$$\Gamma(1,2,3) = \gamma(1,2,3) + \frac{\delta\Sigma(1,2)}{\delta G(3)}. \tag{288}$$

If the last term of Eq. (288) is ignored [see Sec. 6.2.3 (p. 159) below for more discussion], the result is the (Eulerian) DIA in an elegant and compact matrix form:

$$\Gamma_{\mathrm{DIA}} = \gamma \tag{289}$$

(i.e., the DIA omits vertex renormalization), or [Fig. 20(f), p. 161]

$$\Sigma_{\mathrm{DIA}}(1,\overline{1}) = \tfrac{1}{2}\gamma(1,2,3)G(2,\overline{2})G(3,\overline{3})\gamma(\overline{1},\overline{2},\overline{3}). \tag{290}$$

The scalar components of Eq. (290) are [Fig. 20(g), p. 161]

$$\Sigma^{\mathrm{nl}}_{\mathrm{DIA}}(1;\overline{1}) = -U(1,2,3)R(2;\overline{2})C(3,\overline{3})U(\overline{2},\overline{3},\overline{1}), \tag{291a}$$
$$F^{\mathrm{nl}}_{\mathrm{DIA}}(1,\overline{1}) = \tfrac{1}{2}U(1,2,3)C(2,\overline{2})C(3,\overline{3})U(\overline{1},\overline{2},\overline{3}). \tag{291b}$$

Compare these space- and time-dependent formulas with Eqs. (224), (234), and (236).

6.2.3 Vertex renormalizations

The remaining and most difficult issue in the formal statistical-closure problem is to identify the conditions under which vertex renormalization may be ignored—in other words, when is the Eulerian DIA an adequate approximation? The earlier discussion of random Galilean invariance in Sec. 5.6.3 (p. 138) suggests that the strictly mathematical answer is "never," because the MSR formalism is built on a physically inappropriate Eulerian framework. Nevertheless, the structure of the theory beyond DIA order is intrinsically interesting. To understand the significance of Eq. (288) more clearly, note the form (283b), which together with Eq. (288) can be used to argue that Σ depends on G_1 only implicitly[199] *via* its dependence on G_2:

$$\left.\frac{\delta\Sigma}{\delta G_1}\right|_{\eta_1} = \left.\frac{\delta\Sigma}{\delta G_2}\right|_{G_1}\left.\frac{\delta G_2}{\delta G_1}\right|_{\eta_1} = IGG\Gamma, \tag{292a,b}$$

[199] The absence of explicit dependence is correct only for Gaussian initial conditions; see Rose (1974) and Sec. 6.3 (p. 165).

where

$$I(1,2;1',2') \doteq \frac{\delta\Sigma(1,2)}{\delta G(1',2')} = \left(\gamma G\Gamma + GG\,\frac{\delta\Gamma}{\delta G}\right) GG\Gamma. \tag{293a,b}$$

Thus from Eq. (288) the exact (functional) equation for Γ is

$$\Gamma = \gamma + IGG\Gamma. \tag{294}$$

Equation (294) may be used in conjunction with Eq. (293b) to develop approximate equations for Γ. In regimes of weak turbulence or small Kubo number, it is reasonable to expand in powers of[200] γ. However, more generally it is better to develop an expansion for Γ in terms of itself[201]:

$$\Gamma = \gamma + \Gamma G\Gamma G\Gamma G + \cdots. \tag{295}$$

Note that if γ is $O(1)$, as can be assumed in a strong-turbulence theory, and if the unwritten higher-order terms of Eq. (295) are neglected, then solutions of the cubic equation (295) are themselves $O(1)$. This result holds more generally; in the absence of a small parameter, the renormalized vertex Γ must be $O(1)$. It can, however, be numerically smaller than 1.

The present development describes self-consistent turbulence (γ is fully symmetric). The formalism was extended to passive problems by Deker and Haake (1975) [for a superior approach, see the path-integral formalism in Sec. 6.4 (p. 166)]. One can thereby deduce (Krommes, 1984a) the analog of the first vertex renormalization (295) for the SO model of Sec. 3.3 (p. 52). The resulting equation was first derived and studied by Kraichnan (1961); it is very successful, as discussed in Sec. 3.9.8 (p. 85).

Nevertheless, although the MSR equations are formally exact (at least for Gaussian initial conditions), they are "not a panacea" (Martin, 1976). One problem is that they provide equations for *Eulerian* correlation and response functions whereas in some cases a Lagrangian basis would be more suitable. Thus none of the nth vertex renormalizations is invariant to random Galilean transformations. For further discussion, see Kraichnan (1964e) and Sec. 5.6.3 (p. 138). A further difficulty is that the equations beyond DIA order are largely intractable for physical problems with nontrivial mode couplings.

Although the general MSR formalism provides coupled Dyson equations for C and R, in special cases just one of those suffices. In addition to the early work by Kraichnan (1959a), general conditions under which fluctuation–dissipation relations hold were discussed by Deker and Haake (1975); see also Forster et al. (1977). Furthermore, even though for general nonequilibrium situations the time dependences of $C(\tau)$ and $R(\tau)$ are different in detail, it is frequently useful to assert a fluctuation–dissipation *Ansatz*. For more discussion, see Sec. 7.2 (p. 182).

The formalism as developed thus far is diagrammed in Fig. 20.

So far we have considered quadratic nonlinearity. However, in a number of instances cubic nonlinearity is also of importance. It forms the basis of a statistical description of Langmuir turbulence

[200] A more cumbersome calculation that led to the γ expansion $\Gamma \approx \gamma + G^3\gamma^3$ was done by Mond and Knorr (1980).

[201] Although the first vertex renormalization (295) appears to be well behaved, higher ones in the polynomial Γ expansion need not be. As an alternative, Krommes (1984a) suggested a continued-fraction development that introduced successively more complicated n-body connected functions. That has not been explored since it is presumably intractable in practice.

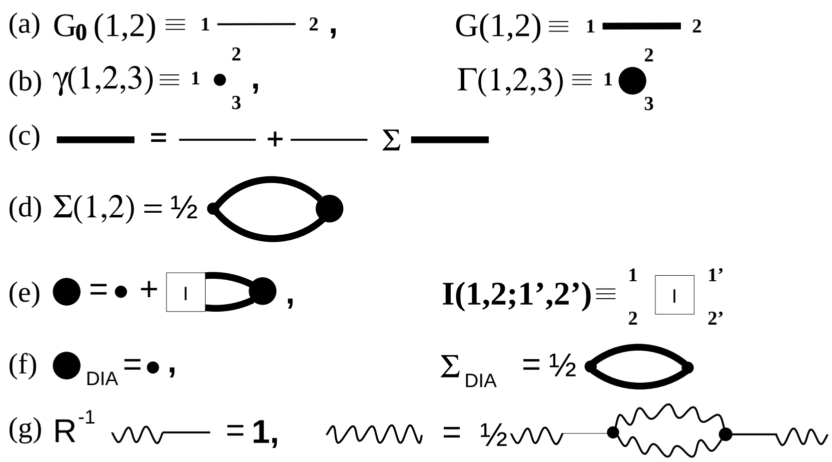

Fig. 20. Diagrammatic expression of the MSR formalism for quadratic nonlinearity in the vertex representation. (a) Bare propagator G_0 and correlation matrix G; (b) bare three-point vertex γ and renormalized vertex Γ; (c) the (matrix) Dyson equation $G = G_0 + G_0 \Sigma G$ (or $G^{-1} = G_0^{-1} - \Sigma$); (d) mass operator $\Sigma = \gamma G G \Gamma$; (e) exact equation for Γ ($I \doteq \delta\Sigma/\delta G$); (f) DIA for Γ and Σ; (g) DIA equations for response and correlation functions.

(Sun et al., 1985), arises naturally in quantum field theory (de Dominicis and Martin, 1964a), emerges in the theory of random $U(1,2)$ [as shown in Sec. 6.4 (p. 166)], and was used by Rose (1979) in his elegant treatment of particle discreteness. Without proof (de Dominicis and Martin, 1964a), I summarize the theory for the field equation

$$-\mathrm{i}\sigma\dot{\Phi}(1) = \gamma_1(1) + \gamma_2(1,2)\Phi(2) + \tfrac{1}{2}\gamma_3(1,2,3)\Phi(2)\Phi(3) + \tfrac{1}{3!}\gamma_4(1,2,3,4)\Phi(2)\Phi(3)\Phi(4). \tag{296}$$

The equation for the mean field,

$$\begin{aligned} -\mathrm{i}\sigma\partial_t G(1) = \gamma_1(1) + \gamma_2(1,2)\Gamma(2) &+ \tfrac{1}{2}\gamma_3(1,2,3)[G(2)G(3) + G(2,3)] \\ &+ \tfrac{1}{3!}\gamma_4(1,2,3,4)[G(2)G(3)G(4) + 3G(2)G(3,4) + G(2,3,4)] \end{aligned} \tag{297}$$

is retained without approximation. Differentiation of Eq. (297) with respect to γ_1 leads to the Dyson equation

$$G^{-1} = G_0^{-1} - \overline{\Sigma}, \tag{298}$$

where $G_0 \doteq -\mathrm{i}\sigma\partial_t - \gamma_2$ (i.e., it is defined *without* mean-field terms, which are now put into $\overline{\Sigma}$). One finds [Fig. 21(a), p. 162]

$$\begin{aligned} \overline{\Sigma}(1,1') = \gamma(1,2,1')G(2) &+ \tfrac{1}{2}\gamma(1,2,3)G(2,\overline{2})G(3,\overline{3})\Gamma(\overline{2},\overline{3},1') \\ &+ \tfrac{1}{3!}\gamma(1,2,3,4)\{3\delta(2,1')[G(3)G(4) + G(3,4)] + 3G(2)G(3,\overline{3})G(4,\overline{4})\Gamma(\overline{3},\overline{4},1') \\ &+ G(2,\overline{2})G(3,\overline{3})G(4,\overline{4})[3\Gamma(\overline{2},\overline{3},5)G(5,\overline{5})\Gamma(\overline{5},\overline{4},1') + \Gamma(\overline{2},\overline{3},\overline{4},1')]\}, \end{aligned} \tag{299}$$

$$\Gamma(1,2,3) = \frac{\delta\overline{\Sigma}(1,2)}{\delta G(3)} + I(1,2;4,5)G(4,\overline{4})G(5,\overline{5})\Gamma(\overline{4},\overline{5},3), \tag{300a}$$

$$\Gamma(1,2,3,4) = D(1,2;3,4) - [\Gamma(1,5,3)G(5,\overline{5})\Gamma(2,\overline{5},4) + (3 \leftrightarrow 4)]. \tag{300b}$$

The function D represents all of the *doubly connected* graphs. Specifically, $GGDGG$ represents the last three terms of Fig. 19 (p. 158), and Eq. (300b) represents just the last diagram in that figure (shorn of its legs). It will be shown in the next section that D obeys

$$\tfrac{1}{2}D = I + IK_0(\tfrac{1}{2}D), \quad I \doteq \delta\overline{\Sigma}/\delta G_2, \tag{301a,b}$$

where $K_0 \doteq (GG)_{\mathrm{s}}$ and the subscript s denotes symmetrization.

For purely cubic nonlinearity ($\gamma_3 \equiv 0$), the formulas simplify. In particular, a *cubic direct-interaction approximation* (CDIA) can be defined by the approximation $\Gamma_4 \approx \gamma_4$. Then (in the absence of mean fields) one finds [Fig. 21(b)]

$$\overline{\Sigma}_{\mathrm{CDIA}}(1,1') = \tfrac{1}{2}\gamma_4(1,1',3,4)G(3,4) + \tfrac{1}{6}\gamma_4(1,2,3,4)G(2,\overline{2})G(3,\overline{3})G(4,\overline{4})\gamma_4(\overline{2},\overline{3},\overline{4},1'). \tag{302}$$

Appropriate spinor components of this formula readily lead to the equations of Hansen and Nicholson (1981) and Sun et al. (1985).

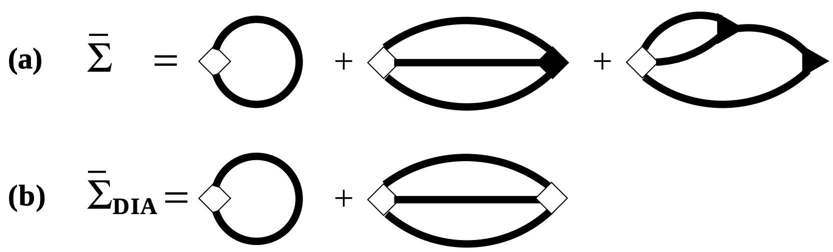

Fig. 21. The mass operator for cubic nonlinearity with vanishing mean field. An alternate representation of the vertices is used in which open n-gons represent γ_n and filled n-gons represent Γ_n. (a) Exact result [cf. Fig. 2 of Martin (1979)]; (b) DIA.

The cubic formalism can also be used to demonstrate that first-order perturbation theory is exact for a Gaussian advective nonlinearity white in time. The proof is sketched in Appendix H.3 (p. 294).

6.2.4 The Bethe–Salpeter equation

Although for quadratic nonlinearity we have succeeded in developing approximate closures entirely in terms of the three-point vertex function Γ, the appearance of I in Eq. (292b) suggests that four-point functions are central to the theory, a point made in Appendix A of Martin et al. (1973); see also Schwinger (1951b). To introduce such functions conveniently, I follow Krommes (1978, 1984a) and introduce the extended generating functional

$$W[\eta_1,\eta_2] = \ln\langle\exp[\eta_1(1)\Phi(1) + \tfrac{1}{2}\eta_2(1,2)\Phi(2)\Phi(2)]\rangle_+, \tag{303}$$

where η_2 is symmetric. Derivatives with respect to η_2 may be said to define *two-body functions*, which are to be compared with the usual *one-body functions* defined by derivatives with respect to η_1. For example, the natural generalization of the two-point, one-body function G is the four-point, two-body function

$$K(1,2;1',2') \doteq \left.\frac{\delta G(1,2)}{\delta\eta_2(1',2')}\right|_{G_1}, \tag{304}$$

sometimes called the *two-body scattering matrix.*[202] To develop the analogy further, introduce (Krommes, 1978) the *two-body Legendre transformation*

$$L[G_1, G_2] = W[\eta_1, \eta_2] - \eta_1(1)G(1) - \eta_2(1,2)G(1,2) \tag{305}$$

and define renormalized vertices according to

$$\Gamma_i(1, \dots, n) \doteq \delta^n L/\delta G_i(1) \dots \delta G_i(n), \tag{306}$$

where $G_1 \equiv F$ and $G_2 \equiv G$. For example,

$$\begin{aligned} \Gamma_1(1) &= -\eta_1(1), & \Gamma_2(1,2) &= -\eta_2(1,2), \\ \Gamma_1(1,1') &= -G^{-1}(1,1'), & \Gamma_2(1,2;1',2') &= -K^{-1}(1,2;1',2'). \end{aligned} \tag{307}$$

The symmetry between the one- and two-body functions is apparent.

The equation for K follows (Martin et al., 1973; Krommes, 1978, 1984a) by adding η_2 to the γ_2 of Eq. (287), then differentiating Eq. (287) with respect to η_2 at fixed η_1. The result, the *Bethe–Salpeter equation* (BSE), can be written in either of the forms

$$K = K_0 + K_0 I K \quad \text{or} \quad K^{-1} = K_0^{-1} - I. \tag{308a,b}$$

Equation (308a) is diagrammed in Fig. 22.

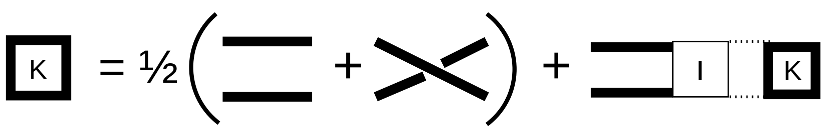

Fig. 22. The Bethe–Salpeter equation (308a); compare with the Dyson equation [Fig. 20(c), p. 161].

The interaction kernel I, which describes "intrinsic" two-body correlations, plays the same role in Eqs. (308) as Σ plays in the Dyson equation; compare Eq. (308b) with Eq. (287). Because the formal solution of Eq. (308a) is $K = (1 - K_0 I)^{-1} K_0$, Eq. (300a) can be written as $K_0 \Gamma = K \delta\overline{\Sigma}/\delta G_1 \to K\gamma$, the latter result holding only for quadratic nonlinearity. For that case, upon inserting this result into Eq. (283b) one finds the symmetrical representation[203] [Fig. 23(a), p. 164]

$$\Sigma(1, \overline{1}) = \tfrac{1}{2}\gamma(1,2,3)K(2,3;\overline{3},\overline{2})\gamma(\overline{3},\overline{2},\overline{1}). \tag{309}$$

The DIA follows by neglecting the intrinsic two-body correlations altogether:

$$I_{\text{DIA}} = 0, \quad K_{\text{DIA}} = K_0 = (GG)_{\text{s}}. \tag{310}$$

There is a rather precise analogy to the distinction between the Balescu–Lenard and Boltzmann collision operators. The DIA is analogous to the BL operator, which is derived by neglecting the

[202] A textbook that provides quantum-field-theoretical background in the spirit of the present section is by Nishijima (1969).

[203] The representation (309) can be used to show that the fully renormalized theory preserves the same nonlinear invariants as the DIA, by virtue of the detailed conservation property (309) and the symmetries of K.

Coulomb interaction between two bare particles [thereby permitting the factorization of a two-particle Green's function, as noted in Sec. 2.3.2 (p. 32)]. The interaction kernel I describes correlational effects analogous to the large-angle scattering described by the Boltzmann operator. Indeed, K figures prominently in the theory of Rose (1979), who showed how to incorporate into a renormalized description discrete (Klimontovich) and continuous particle distributions on equal footing.

The various equations of the Bethe–Salpeter representation for quadratic nonlinearity are diagrammed in Fig. 23.

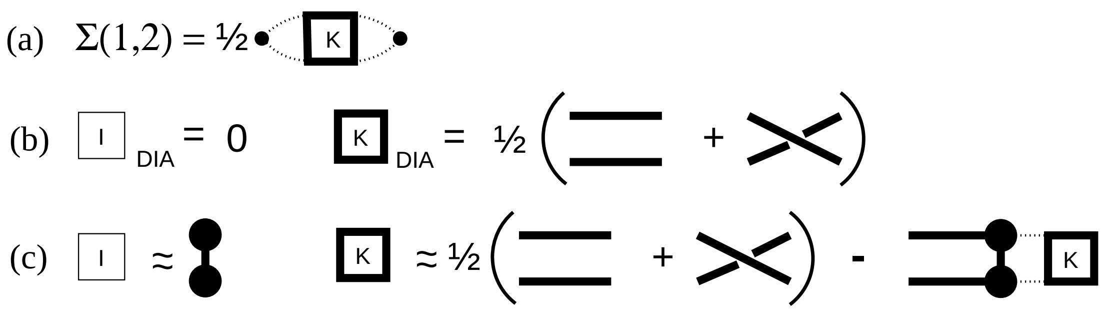

Fig. 23. Four-point representation of the MSR formalism for quadratic nonlinearity. (a) Renormalized mass operator Σ in terms of the two-body scattering matrix K; (b) DIA for the four-point functions; (c) first vertex renormalization in terms of four-point functions.

The remaining task is to relate the interaction kernel I to the doubly connected graphs D. It can be shown [204] (Martin et al., 1973) that

$$\tfrac{1}{2}D = IKK_0^{-1}. \tag{311}$$

Upon multiplying Eq. (308a) on the left with I and on the right with K_0^{-1}, one is led immediately to Eq. (301a).

The formalism of this section provides an elegant description of the statistical dynamics of nonlinear systems. It was used, for example, by DeDominicis and Martin (1979) [see also Martin and de Dominicis (1978)] to discuss the energy spectra of randomly stirred fluids. Connections to the theory of the renormalization group were discussed by Eyink (1994).

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[204] To find a simple expression for the doubly connected graphs, write (Martin et al., 1973)

$$G_4 = \frac{\delta^2 G_2}{\delta\eta_1 \delta\eta_1} = G_2 \frac{\delta}{\delta G_1} G_2 \frac{\delta G_2}{\delta G_1} = G^2(\Gamma G \Gamma)G^2 + GG\frac{\delta^2 G_2}{\delta G_1 \delta G_1}; \tag{f-18}$$

the last term is the doubly-connected part $K_0 D K_0$. We will see shortly that the identity

$$\left.\frac{\delta^2 G(1,2)}{\delta G(3)\delta G(4)}\right|_{\eta_2} = 2\left.\frac{\delta\overline{\Sigma}(3,4)}{\delta\eta_2(1,2)}\right|_{G_1} \tag{f-19}$$

holds. If that is accepted, then $\frac{1}{2}DK_0 = \delta\overline{\Sigma}/\delta\eta_2 = IK$, which reproduces the result (311). The steps in the proof of the identity are as follows: (i) Differentiate Eq. (298) with respect to η_2. (ii) Note that $G_2^{-1} = \delta\eta_1/\delta G_1|_{\eta_2}$. (iii) Interchange the derivatives to get $\delta\eta_1/\delta\eta_2|_{G_1} = -G_2^{-1}\,\delta G_1/\delta\eta_2|_{\eta_1}$. (iv) Calculate $\delta G_1/\delta\eta_2$ by explicitly differentiating Eq. (303).

In plasma physics, applications of the MSR formalism have been mostly restricted to formal derivations of the DIA and discussions of the nonlinear dielectric function (Sec. 6.5, p. 170). However, Krommes (1996) recently employed the formalism in his discussion of higher-order statistics (Sec. 10.2, p. 221), and Krommes and Kim (2000) found it to be useful in their discussion of the theory of long-wavelength growth rates nonlinearly driven by short scales (Sec. 7.3.3, p. 194). It is also convenient for formal discussions of the Landau-fluid closure problem (Appendix C.2.2, p. 278).

6.2.5 Ward identities

Ward identities are relationships between the vertex functions of different orders that are consequences of particular continuous symmetries. I shall give only a brief introduction. Suppose the Φ dynamics are invariant under the transformation $\Phi \to \overline{\Phi} = T\Phi$. Then $W[\eta_1]$ is invariant provided that $\eta_1 \to \overline{\eta}_1 = T^{-1}\eta_1$. Let the transformation be parametrized by ϵ and let $T = \exp(\epsilon\widehat{L})$ (i.e., $\widehat{L}$ is the infinitesimal generator of the transformation[205]). Then invariance of W under an infinitesimal transformation leads to

$$0 = \int \mathrm{d}1 \, \frac{\delta W}{\delta \eta_1(1)} \widehat{L}\eta_1(1) \tag{312}$$

(as usual, a sum over spinor components is implied). Upon recalling Eqs. (275) and (280a), one can rewrite Eq. (312) as the fundamental Ward identity

$$0 = \int \mathrm{d}1 \, G(1)\widehat{L}\Gamma(1). \tag{313}$$

Further differentiations with respect to G lead, upon recalling Eqs. (279), to an infinite number of further identities that can provide important constraints on the structure of the renormalized theory. An example involving rotational symmetry can be found in Sec. 7–11 of Amit (1984). Finally, the invariance of the Burgers equation (19) to the Galilean transformation $u(x,t) \to \overline{u}(x,t) \doteq u(x - Vt, t) + V$ for constant V can be used to prove that vertices in the Burgers turbulence do not renormalize in the long-wavelength limit; see Appendix B of Forster et al. (1977).

6.3 Non-Gaussian initial conditions and spurious vertices

As presented so far, the MSR formalism is valid only for Gaussian initial conditions and in the absence of random coefficients. Note that if one considers one-sided functions $\psi_+(t) \doteq H(t - t_0)\psi(t)$, initial conditions can explicitly be incorporated into the dynamics as $\partial_t\psi_+ + \cdots = \psi(t_0)\delta(t - t_0)$. In this form they are additive to U_1 and can thus be dealt with as a special case of random forcing. For the latter, Rose (1974) proved that the formalism remains valid provided that the bare vertices γ_n are replaced by $\gamma_n + \Gamma_n^{(0)}$, where the *spurious vertices* $\Gamma_n^{(0)}$ are the nth-order cumulants of U_1 when all spinor indices are $-$, and are zero otherwise. This important result can be recovered most easily from the path-integral formalism of Sec. 6.4 (p. 166), as can the generalization to random coefficients. For some related discussion, see Krommes (1996).

[205] For example, translations through a distance ϵ, $Tf(x) = f(x + \epsilon)$, are generated by $\widehat{L} = \partial_x$. For some further discussion of such transformations, see Appendix C.1.5 (p. 273).

An important instance of non-Gaussian initial conditions is the many-particle system described by the Klimontovich microdensity $\widetilde{N}$. The singular nature of $\widetilde{N}$ leads (Rose, 1974) to spurious vertices of all orders, each of $O(1)$. This feature presents significant technical difficulties; a superior procedure for analyzing such singular systems was given in the elegant work by Rose (1979).

Rose (1985) used the theory of non-Gaussian initial conditions to develop an interesting, efficiently computable closure alternative to the DIA. The DIA makes a time-dependent prediction for the triplet correlation function; computational difficulties ensue because of a time-history integral that must in principle be taken all the way from $t = 0$. Rose instead proposed dividing the calculation into time intervals and using the value of the triplet correlation at the end of each interval to determine the non-Gaussian initial condition for the next interval. The resulting *cumulant update DIA* (CUDIA) can provide a substantial computational savings. For recent discussion of the CUDIA, see Frederiksen et al. (1994).

6.4 Path-integral representation

I have already mentioned in Sec. 6.1.3 (p. 152) the influential role of Feynman's action-integral formulation of quantum mechanics. The representation of the probability amplitude of a particular event as an appropriate sum over all possible histories expedites intuitive visualizations of the elementary processes that contribute to the total amplitude. The famous Feynman diagrams are graphical representations of those processes; each diagram corresponds to a specific term in the perturbative expansion of the total amplitude in powers of a coupling constant g that is assumed to be small. In quantum electrodynamics g is the fine-structure constant α.

If one recalls that Dyson (1949) was able to demonstrate the mathematical equivalence between the superficially disparate formalisms of Schwinger and Feynman, one will not be surprised to learn that a path-integral representation underlies the classical MSR formalism as well. The most general discussion was given by Jensen (1981), who relied heavily on the work of Jouvet and Phythian (1979); those authors, in turn, cited a variety of earlier references. Useful introductions to the formalism were given by Dubin (1984b) and Frisch (1995, Sec. 9.5.2). In the context of QFT, detailed technical discussion can be found in Zinn-Justin (1996). For some related remarks on dynamic critical phenomena, see Chang et al. (1992).

The fundamental conceptual idea underlying the path-integral representation is the identity for the PDF $P(x)$ of a random variable $\widetilde{x}$: $P(x) = \langle\delta(x-\widetilde{x})\rangle$. The delta function can be Fourier-transformed, so averages of an arbitrary function $F(\widetilde{x})$, $\langle F(\widetilde{x})\rangle = \int \mathrm{d}x\, F(x)P(x)$, can be represented as

$$\langle F\rangle = \int \mathrm{d}x \int \frac{\mathrm{d}k}{2\pi} F(x)\mathrm{e}^{\mathrm{i}kx}\langle \mathrm{e}^{-\mathrm{i}k\widetilde{x}}\rangle. \tag{314}$$

Therefore a generating function for averages of polynomial functions of x is

$$Z(\eta) = \int \mathrm{d}x \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}kx}\langle \mathrm{e}^{-\mathrm{i}k\widetilde{x}}\rangle \mathrm{e}^{\eta x}; \tag{315}$$

for example, $\langle x^2\rangle = \partial^2 Z(\eta)/\partial\eta^2|_{\eta=0}$. This is, of course, nothing more than an elaborate way of stating that the characteristic function of a PDF is a moment generating function (Sec. 3.5.2, p. 59) [for imaginary η, the x integration can be performed, yielding $\delta(k-\eta)$], but the present form, integrated symmetrically in x and k, is useful for later generalization.

In the application to field theory one must consider all times and space points simultaneously and incorporate the information that the dynamical variables obey an equation of motion. To illustrate the construction with the least amount of clutter, I follow Jouvet and Phythian (1979) and consider the evolution of a function $Q(t)$ that obeys

$$\partial_t Q - A[Q] = f^{\text{ext}}(t). \tag{316}$$

The brackets indicate functional dependence of A on the values of Q in some time interval $0 \le t \le T$. The functional A and the function f^{ext} may be random[206]; one's ultimate goal is to calculate the ensemble averages of specified functionals $F[Q]$. For example, the choice $F[Q] = Q(t_1)Q(t_2)$ leads upon averaging to the two-point correlation function $C(t_1, t_2)$. Nevertheless, let us temporarily refrain from introducing statistics and first attempt to develop a convenient representation of $F[Q]$, where $Q(t)$ is constrained to obey Eq. (316) with the particular initial condition $Q(0) = Q_0$. One has formally $F[Q] = \int \mathrm{D}[q]\, F[q]\delta[q - Q]$, where a functional integral and Dirac delta functional have been introduced. As in Sec. 3.5 (p. 59), those can be defined by discretizing the time axis into N intervals of length $\Delta t = T/N$,

$$t_n = n\Delta t,\ Q_n = Q(t_n) \quad (n = 0, 1, \ldots, N \doteq T/\Delta t), \tag{317}$$

and passing to the limit $\Delta t \to 0$ ($N \to \infty$):

$$F[Q] = \lim_{\Delta t \to 0} F(Q_0, Q_1, \ldots, Q_N) \tag{318a}$$

$$= \lim_{\Delta t \to 0} \prod_{n=0}^{N} \mathrm{d}q_n F(q_0, q_1, \ldots, q_N)\delta(q_0 - Q_0)[\delta(q_1 - Q_1) \ldots \delta(q_N - Q_N)]. \tag{318b}$$

To explicitly exhibit the dynamical constraint (the equation of motion) satisfied by the Q_n's, one may write

$$\delta[q - Q] = J\, \delta(q_0 - Q_0)\delta[\dot{q} - A[Q] - f^{\text{ext}}], \tag{319}$$

where J is the Jacobian on the transformation between Q and f^{ext}: $J = \det(\delta f^{\text{ext}}/\delta Q)$. The value of J depends on the way in which the equation of motion is discretized. The simplest discretization (used, for example, by Jensen) is

$$(Q_{n+1} - Q_n)/\Delta t - A(Q_n) = f^{\text{ext}}(t_n), \tag{320}$$

written here for the special but practically important case for which A is local in time.[207] For this convention the transformation of variables is between $\{Q_1, \ldots, Q_N\}$ and $\{f_0^{\text{ext}}, \ldots, f_{N-1}^{\text{ext}}\}$, so $J = \Delta t^{-N}$. In the continuum limit J is infinite [$J \sim \exp(N \ln N)$], but that infinity will be canceled by another; physical observables will be finite.

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[206] Clearly f^{ext} could be incorporated into $A[Q]$, but it seems more pedagogical to write it explicitly here.

[207] For the Klimontovich nonlinearity $(\widetilde{\boldsymbol{E}} + c^{-1}\boldsymbol{v} \times \widetilde{\boldsymbol{B}}) \cdot \partial_{\boldsymbol{v}} \widetilde{N}$, where $\widetilde{\boldsymbol{E}}$ and $\widetilde{\boldsymbol{B}}$ are related to $\widetilde{N}$ *via* Maxwell's equations, temporal locality holds for electrostatic but not electromagnetic interactions. One of Jensen's contributions was to show that it is not necessary to assume locality; a general nonlocal dependence $A[Q]$ is permissible.

In order to deal conveniently with the delta functional in Eq. (319), one can introduce conjugate momentum variables p by writing, for arbitrary real function $G(t)$,

$$\delta[G] \approx \int \frac{\mathrm{d}p_1}{2\pi} \ldots \frac{\mathrm{d}p_N}{2\pi}\, \mathrm{e}^{\mathrm{i}p_1 G_1 + \cdots + \mathrm{i}p_N G_N} = \left(\frac{\mathrm{i}\Delta t}{2\pi}\right)^N \int \mathrm{D}[\widehat{q}]\, \exp\left(-\int_0^T \mathrm{d}t\, \widehat{q}(t) G(t)\right), \tag{321a,b}$$

where I have defined $\widehat{q} \doteq p/(\mathrm{i}\Delta t)$ and exploited Riemann's representation of the time integral. Upon combining Eqs. (318b), (319), and (321b) and recalling that J is independent of Q, one obtains

$$F[Q] = \mathcal{N} \int \mathrm{D}[q]\, \mathrm{D}[\widehat{q}]\, F[q] \mathrm{e}^{-\widetilde{\mathcal{S}}}, \tag{322}$$

where $\mathcal{N}$ is a normalization coefficient, the random *action* is $\widetilde{\mathcal{S}} \doteq \int_{0_-}^{T} \mathrm{d}t\, \widetilde{\mathcal{L}}(t)$, and the random *Lagrangian* is $\widetilde{\mathcal{L}}(t) \doteq \widehat{q}(t)\{\dot{q}(t) - A[q] - f^{\mathrm{ext}}(t) - \delta(t) Q_0\}$. Notice that the initial condition has been incorporated into the equation of motion. That could have been done at the outset by considering the equation of motion for the one-sided function $Q_+(t) \doteq H(t) Q(t)$; that was, in fact, the procedure followed by Jensen. That the terms involving A, f^{ext}, and Q_0 appear symmetrically in $\widetilde{\mathcal{L}}$ suggests that a completely unified treatment of random coefficients, forcing, and initial conditions will stem from the path-integral formalism.

To understand the significance of $\widehat{q}$, first recall the relationship between the functional derivative and ordinary partial differentiation (Beran, 1968). From the definition

$$\frac{\delta}{\delta \psi(t_n)} \int \mathrm{d}t\, A[\psi] = \left.\frac{\partial A}{\partial \psi}\right|_{t_n}, \tag{323}$$

one has in a discretization $\delta/\delta\psi(t_n) = \Delta t^{-1} \partial/\partial \psi_n$. If p_n is a Fourier variable conjugate to ψ_n, then the functional derivative transforms to $\mathrm{i}p_n/\Delta t$; thus $\widehat{q} \doteq p/(\mathrm{i}\Delta t)$ is the Fourier transform of $-\delta/\delta\psi$: $\widehat{q} \Leftrightarrow -\delta/\delta\psi \equiv \widehat{\psi}$. Because functional derivatives describe the effects of infinitesimal changes in functional form, one suspects that $\widehat{q}$ is related to infinitesimal response functions. Indeed,

$$\frac{\delta^n F[Q]}{\delta f^{\mathrm{ext}}(t_1) \ldots \delta f^{\mathrm{ext}}(t_n)} = \mathcal{N} \int \mathrm{D}[q]\, \mathrm{D}[\widehat{q}]\, F[q]\, \widehat{q}(t_1) \ldots \widehat{q}(t_n) \mathrm{e}^{-\widetilde{\mathcal{S}}}. \tag{324}$$

One is now motivated to generalize in several ways: (i) Consider arbitrary functions $F[Q, \widehat{Q}]$, where $\widehat{Q}$ obeys a suitable equation of motion adjoint to Q [see Eq. (328)]. (ii) Perform the statistical average (statistics reside only in $\widetilde{\mathcal{S}}$) by writing $\langle \mathrm{e}^{-\widetilde{\mathcal{S}}} \rangle = \mathrm{e}^{-S}$ [see Eq. (93a)], thereby defining an *effective action* S. (iii) Introduce statistically sharp source terms $\eta(t)$ and $\widehat{\eta}(t)$ such that $S \to S + \int \mathrm{d}t\, [\eta(t) q(t) + \widehat{\eta}(t) \widehat{q}(t)]$. (Those can be introduced either before or after the averaging. If they are introduced before, note that $\widehat{\eta}$ is additive to f^{ext}.) (iv) Define the *generating functional*

$$Z[\eta, \widehat{\eta}] \doteq \mathcal{N} \int \mathrm{D}[q]\, \mathrm{D}[\widehat{q}]\, \mathrm{e}^{-S} \mathrm{e}^{\int \mathrm{d}t\, [\eta(t) q(t) + \widehat{\eta}(t) \widehat{q}(t)]}. \tag{325}$$

Clearly averages of all polynomial functions of Q and $\widehat{Q}$ can be derived from functional differentiation of Z. For example, with $\boldsymbol{\eta} \doteq (\eta, \widehat{\eta})^T$ one has $\langle Q(t_1) Q(t_2) \rangle = \delta^2 Z/\delta\eta(t_1)\, \delta\eta(t_2)|_{\boldsymbol{\eta}=\mathbf{0}}$ and $R(t_1; t_2) \doteq \langle \delta Q(t_1)/\delta f^{\mathrm{ext}}(t_2) \rangle|_{f^{\mathrm{ext}}=0} = \delta^2 Z/\delta\eta(t_1)\, \delta\widehat{\eta}(t_2)|_{\boldsymbol{\eta}=\mathbf{0}}$. Cumulants can be generated from $W[\boldsymbol{\eta}] \doteq \ln Z[\boldsymbol{\eta}]$.

These results strongly suggest that the Z thus defined is identical to the generating functional of MSR, which I shall here write as

$$Z_{\rm MSR}[\eta, \widehat{\eta}] = \langle e^{\int dt\,[\eta(t)\widetilde{Q}(t)+\widehat{\eta}(t)\widetilde{\widehat{Q}}(t)]}\rangle_+, \tag{326}$$

where $\widetilde{Q}$ and $\widetilde{\widehat{Q}}$ are *random variables*. It is perhaps not immediately obvious that formulas (325) and (326) are identical: Equation (326) involves statistical averaging and time ordering whereas Eq. (325) involves neither of those (explicitly), but instead contains a functional integral weighted with the exponential of an effective action. But averaging has been performed in the definition of S, and the functional integration conceals a delta-functional constraint that reflects the evolution of the causal dynamics; indeed, the information content of Z and $Z_{\rm MSR}$ is the same. Consider, for example, $\langle\langle q(t)\rangle\rangle_{\boldsymbol{\eta}} \doteq \delta W[\boldsymbol{\eta}]/\delta\eta(t)$. Its time derivative introduces $\dot{q}(t)$, which can be replaced by $-\delta/\delta\widehat{q}(t)$ acting on the $\exp[-\int dt\,\widehat{q}(t)\dot{q}(t)]$ that resides in S. After an integration by parts, the $\widehat{q}$ derivative brings down the right-hand side of the equation of motion augmented by $\widehat{\eta}$. Thus

$$\partial_t\langle\langle q(t)\rangle\rangle_{\boldsymbol{\eta}} = \langle\langle\{A[q] + f^{\rm ext}(t) + \delta(t)Q_0\}\rangle\rangle_{\boldsymbol{\eta}}(t) + \widehat{\eta}(t), \tag{327}$$

which is the appropriate MSR average of the equation of motion [cf. Eq. (272)]. Similarly, upon noting $\int dt\,\widehat{q}(t)\dot{q}(t) = -\int dt\,\dot{\widehat{q}}(t)q(t)$, one finds

$$-\partial_t\langle\langle\widehat{q}(t)\rangle\rangle_{\boldsymbol{\eta}} = \langle\langle\widehat{q}\,\delta A[q]/\delta q\rangle\rangle_{\boldsymbol{\eta}}(t) + \eta(t), \tag{328}$$

which is the average of the adjoint equation of MSR [cf. Eq. (273)]. The matrix MSR development in terms of the vector $\Phi = (q,\,\widehat{q})^T$ [see Eq. (274)] follows immediately.[208]

An alternate way of deriving the cumulant equations is to differentiate the identity

$$0 = \int D[\Phi]\frac{\delta}{\delta\Phi}\left(e^{-S}e^{\int dt\,\eta_1(t)\Phi(t)}\right). \tag{329}$$

This procedure is convenient for dealing with the effects of random forcing and/or coefficients because the average over those quantities has been taken into account in deriving the explicit form for S. Thus if the average over the random force is performed explicitly, the resulting cumulant expansion involves only $\widehat{q}$. For example, for a centered Gaussian

$$\langle e^{\int dt\,\widehat{q}(t)f^{\rm ext}(t)}\rangle = \exp\left(\frac{1}{2}\int dt\,dt'\,\widehat{q}(t)F^{\rm ext}(t,t')\widehat{q}(t')\right), \tag{330}$$

where $F^{\rm ext}(t,t') \doteq \langle\langle f^{\rm ext}(t)f^{\rm ext}(t')\rangle\rangle$; higher-order cumulants just introduce more powers of $\widehat{q}$. This behavior can readily be seen to lead to the result of Rose (1974) that forcing can be taken into account by the spurious vertices defined in Sec. 6.3 (p. 165), because $\delta/\delta\Phi$ of Eq. (330) is nonvanishing only for its $-$ component (which couples only to $\widehat{q}$ fields). For example (generalizing now to the possible presence of independent variables other than the time), Eq. (330) can be written as $\exp[\frac{1}{2}\gamma_2'(1,2)\Phi(1)\Phi(2)]$, where $\gamma_2'(1,2) \doteq F^{\rm ext}(1,2)$ for all indices $-$, and vanishes otherwise. The effect is to add a term $\gamma_2'(1,2)\Phi(2)$ to the right-hand side of the equation of motion: $-i\sigma\partial_t\langle\langle\Phi(1)\rangle\rangle =$

[208] Causality is built in naturally. The MSR rule that cumulant averages beginning on the left with $\widehat{q}$ vanish appears here as a generalization of the result that $(2\pi)^{-1}\int_{-\infty}^{\infty}dx\,dk\,k\,e^{ikx}e^{\widehat{\eta}k}$ involves (at $\hat{\eta} = 0$) $k\,\delta(k) = 0$.

$\gamma_2'(1,2)\langle\langle\Phi(2)\rangle\rangle + \cdots$. This term does not contribute to the averaged equation of motion at $\boldsymbol{\eta} = \mathbf{0}$ because $\langle\langle\widehat{q}\rangle\rangle|_{\boldsymbol{\eta}=\mathbf{0}} = 0$. However, to the G equation derived by functional differentiation with respect to $\boldsymbol{\eta}$, it contributes $\gamma_2'(1,2)G(2,1')$, so γ_2' is additive to the mass operator Σ; in particular, its $--$ component $F^{\text{ext}}(1,\overline{1})$ adds to the internal noise $F^{\text{nl}}(1,\overline{1}) \doteq \Sigma_{--}$, as would be expected.

Multiplicative passive statistics can be dealt with in a similar fashion. An important special case is random U_2 (passive advection). For Gaussian U_2,

$$\langle e^{\widehat{q}(1)U_2(1,\overline{1})q(\overline{1})}\rangle = \exp[\tfrac{1}{2}\widehat{q}(1)\widehat{q}(2)\langle\langle U_2(1,\overline{1})U_2(2,\overline{2})\rangle\rangle q(\overline{1})q(\overline{2})] \tag{331}$$

can be written as $\exp[\frac{1}{4!}\gamma_4'(1,2,3,4)\Phi(1)\Phi(2)\Phi(3)\Phi(4)]$, where γ_4' has precisely two $-$ indices and $\gamma_{4--++}'(1,2,3,4) = \langle\langle U_2(1,3)U_2(2,4)\rangle\rangle$; thus one finds a *cubic* contribution to the equation of motion: $-\mathrm{i}\sigma\partial_t\langle\langle\Phi(1)\rangle\rangle = \frac{1}{3!}\gamma_4'(1,2,3,4)\Phi(2)\Phi(3)\Phi(4) + \cdots$. The associated functional formalism was described in Sec. 6.2 (p. 153). When U_2 is white in time, further reduction is possible; see Appendix H.3 (p. 294).

As discussed by Jensen (1981), one useful feature of the path-integral formalism is that the final renormalized equations involve only cumulants of the random coefficients, not mixed correlations between those coefficients and the dynamical variables. The formalism of Deker and Haake (1975) produces more unwieldy equations involving mixed correlations. The information content of both approaches is the same, however (Johnston and Krommes, 1990).

Dubin (1984a) has shown that the path-integral representation can also be generalized to deal efficiently with the predictability problem (footnote 165, p. 130).

Finally, although I have stressed the application of the path-integral representation to the derivation of renormalized cumulant equations, the explicit form of the generating functional raises the possibility that it can be evaluated directly. Most such work has been restricted to discrete mappings; see, for example, Rechester et al. (1981). In a related development, recently generating-functional techniques have been used successfully to deduce considerable information about the PDF for intermittent Burgers turbulence; see Sec. 10.4.3 (p. 227).

6.5 The nonlinear dielectric function

"The MSR techniques allow a more succinct and, we believe, more understandable derivation of [the Orszag–Kraichnan equations (Vlasov DIA), the Krommes–Oberman ... equation for equilibrium fluctuations, and DuBois' space-time formulation of weak-turbulence theory]." — ***Krommes (1978).***

One of the most important physical distinctions between plasmas and neutral fluids lies in the dielectric properties of the two mediums. The differences can be seen most starkly at linear order. The linear response of the usual incompressible Navier–Stokes model consists merely of weak viscous decay. However, plasmas have an exceedingly complicated linear response, particularly in a magnetic field; they support a rich spectrum of linear waves and instabilities driven by gradients in both $\boldsymbol{v}$ space and $\boldsymbol{x}$ space.

I have already mentioned in Sec. 4.3.3 (p. 110) Dupree's early work on the resonance-broadening approximation to the nonlinear plasma dielectric function $\mathcal{D}$. Much confusion has ensued by the failure of many workers to recognize that the resonance-broadening dielectric was not derived systematically. It is therefore important to have at hand a formally exact formula for the true, fully nonlinear $\mathcal{D}(\boldsymbol{k},\omega)$. The MSR formalism provides a convenient tool. Conversely, the representation of $\mathcal{D}$ provides a central interpretation of the mean infinitesimal response function R. Note that Martin *et al.* did not explicitly discuss dielectric response; the following material (DuBois and Espedal, 1978; Krommes and Kleva, 1979) represents one of the rare contributions to classical statistical dynamics that is original to plasma-physics research.

6.5.1 Definition of the dielectric function

As I shall show, a dielectric function can be defined for all random mediums, including ones having nothing to do with electrodynamics (like the NSE). Nevertheless, elementary electromagnetism provides the most familiar and compelling motivation. Recall (Jackson, 1962) that the electric displacement $\boldsymbol{D}$ obeys $\boldsymbol{\nabla}\cdot\boldsymbol{D} = 4\pi\rho^{\text{free}}$, where ρ^{free} is the free charge density. In general, $\boldsymbol{D}$ is a nonlinear functional of the (macroscopic) mean electric field $\boldsymbol{E} \equiv \langle\boldsymbol{E}\rangle$: $\boldsymbol{D} = \boldsymbol{D}[\boldsymbol{E}]$. Here "macroscopic" means an average over any microscopic fluctuations of the medium. In elementary discussions those are considered to arise from particle discreteness, but the concept applies to any source of random fluctuations, including turbulent collective motions. It is very important to distinguish the macroscopic field $\boldsymbol{E}$ from the random microscopic field $\widetilde{\boldsymbol{E}}$.

If the dependence $\boldsymbol{D}[\boldsymbol{E}]$ is analytic near $\boldsymbol{E} = \boldsymbol{0}$, then $\boldsymbol{D} = \mathsf{D}\cdot\boldsymbol{E} + O(E^2)$. The matrix D is called the *dielectric tensor*; it describes the first-order polarizability properties of the medium. Most generally, D is a causal linear operator in the space and time domains; however, for homogeneous and stationary turbulence one may Fourier transform with respect to space and time differences to obtain the function $\mathsf{D}(\boldsymbol{k},\omega)$.

For definiteness, consider the electrostatic limit, in which magnetic fluctuations are negligible. Then the electric field is the (negative) gradient of a scalar potential φ, the dielectric tensor reduces to a scalar $\mathcal{D}$, and

$$\mathcal{D}\langle\varphi\rangle^{\text{tot}} = \varphi^{\text{free}} \equiv \varphi^{\text{ext}} \quad \text{or} \quad \langle\varphi\rangle^{\text{tot}} = \mathcal{D}^{-1}\varphi^{\text{ext}}. \tag{332}$$

Here $k^2\varphi^{\text{free}} = 4\pi\rho^{\text{free}}$, tot stands for *total*, and ext stands for *external*. The tot superscript indicates the total potential in the medium, including that due to the free charge. One often refers to the free charge as external because its functional form is specified; it is not governed by the internal nonlinear equations of the particles that polarize in response to the free charge. By analogy, in turbulence theory one assumes that the fundamental nonlinear equation of motion describes the *internal* or *induced* dynamics of the medium. Therefore the first-order response to an external test potential $\Delta\varphi^{\text{ext}}$ is $\Delta\langle\varphi\rangle^{\text{tot}} = \Delta\langle\varphi\rangle^{\text{ind}} + \Delta\varphi^{\text{ext}}$. In conjunction with Eq. (332), this leads to

$$\Delta\langle\varphi\rangle^{\text{ind}} = (\mathcal{D}^{-1} - 1)\Delta\varphi^{\text{ext}}. \tag{333}$$

The fundamental algorithm for determining the nonlinear dielectric is therefore as follows: (i) Consider a *pre-existing* turbulent state. (ii) Insert an infinitesimal test charge or potential into the turbulence. (iii) Calculate the mean induced potential $\Delta\langle\varphi\rangle^{\text{ind}}$. The (inverse of the) dielectric function is then defined by Eqs. (332) or (333). When multiple potentials are involved, there is a natural generalization that defines a dielectric tensor. [209]

[209] The dielectric tensor need not be 3×3. For calculations that employed a 2×2 dielectric tensor describing the coupling between φ and $A_\parallel$ (the component of the vector potential parallel to the magnetic field), see Krommes and Kim (1988) and Krommes (1993c).

6.5.2 General form of the renormalized dielectric function

For Hamiltonian dynamics the Klimontovich equation of motion is $\partial_t \widetilde{N} = [H, \widetilde{N}]$, so additive contributions to H enter as multiplicative contributions to the equation of motion. Since one knows from the discussion of the MSR formalism in Sec. 6 (p. 146) that a general turbulence dynamics can be derived from a Hamiltonian functional, consider an equation of motion of the form

$$\partial_t \psi + \mathrm{i}\widehat{\mathcal{L}}\psi + (\widehat{\boldsymbol{\mathcal{F}}}\psi) \cdot \boldsymbol{\partial}\psi = 0, \tag{334}$$

where $\widehat{\boldsymbol{\mathcal{F}}}$ is a linear operator that determines a generalized force from ψ and $\boldsymbol{\partial}$ is the derivative with respect to the associated generalized momentum. For example, in Vlasov electrostatics $\psi = f_s(\boldsymbol{x}, \boldsymbol{v}, t)$ and $\widehat{\boldsymbol{\mathcal{F}}} = q\widehat{\boldsymbol{\mathcal{E}}}$, where the Fourier transform of $\widehat{\boldsymbol{\mathcal{E}}}$ is given by Eq. (27) and $\boldsymbol{\partial} \doteq m^{-1}\partial_{\boldsymbol{v}}$. To encompass the case of multiple generalized potentials (an example would be the set $\{\varphi, A_\parallel\}$ of weakly electromagnetic plasma turbulence, where $\boldsymbol{A}$ is the vector potential), write $\widehat{\boldsymbol{\mathcal{F}}} = \mathsf{M} \cdot \widehat{\boldsymbol{\Phi}}$, where M is a specified linear matrix operator. Note that in general ψ depends on a kinetic momentum variable whereas the dielectric function is a property of $\boldsymbol{x}$-space response alone.

Now write $\psi = \widetilde{\psi} + \Delta\psi^{\mathrm{ind}}$ and consider the response to an infinitesimal perturbation $\Delta\psi^{\mathrm{ext}}$:

$$(\partial_t + \mathrm{i}\widehat{\mathcal{L}})(\widetilde{\psi} + \Delta\psi^{\mathrm{ind}}) + [\widehat{\boldsymbol{\mathcal{F}}}(\widetilde{\psi} + \Delta\psi^{\mathrm{ind}} + \Delta\psi^{\mathrm{ext}})] \cdot \boldsymbol{\partial}(\widetilde{\psi} + \Delta\psi^{\mathrm{ind}}) = 0. \tag{335}$$

The zeroth-order equation, $\left(\partial_t + \mathrm{i}\widehat{\mathcal{L}}\right)\widetilde{\psi} + (\widehat{\boldsymbol{\mathcal{F}}}\widetilde{\psi}) \cdot \boldsymbol{\partial}\widetilde{\psi} = 0$, provides the (fully nonlinear) description of the base turbulent state $\widetilde{\psi}$ [step (i)]. The first-order response obeys [step (ii)]

$$\left(\partial_t + \mathrm{i}\widehat{\mathcal{L}}\right)\Delta\psi^{\mathrm{ind}} + (\widehat{\boldsymbol{\mathcal{F}}}\widetilde{\psi}) \cdot \boldsymbol{\partial}\Delta\psi^{\mathrm{ind}} + (\widehat{\boldsymbol{\mathcal{F}}}\Delta\psi^{\mathrm{ind}}) \cdot \boldsymbol{\partial}\widetilde{\psi} = -\boldsymbol{\partial}\widetilde{\psi} \cdot \widehat{\boldsymbol{\mathcal{F}}}\Delta\psi^{\mathrm{ext}}. \tag{336}$$

The Green's function for the left-hand side of Eq. (336) is just the random infinitesimal response function $\widetilde{R}$. Therefore

$$\Delta\psi^{\mathrm{ind}} = -\widetilde{R}(\boldsymbol{\partial}\widetilde{\psi} \cdot \widehat{\boldsymbol{\mathcal{F}}}\Delta\psi^{\mathrm{ext}}), \quad \langle\Delta\psi\rangle^{\mathrm{ind}} = -\langle\widetilde{R}\,\boldsymbol{\partial}\widetilde{\psi}\rangle \cdot \widehat{\boldsymbol{\mathcal{F}}}\Delta\psi^{\mathrm{ext}}. \tag{337a,b}$$

Finally, one accomplishes the first part of step (iii) by applying the potential operator $\widehat{\boldsymbol{\Phi}}$:

$$\langle\Delta\varphi\rangle^{\mathrm{ind}} = (-\widehat{\boldsymbol{\Phi}}\langle\widetilde{R}\,\boldsymbol{\partial}\widetilde{\psi}\rangle \cdot \mathsf{M}) \cdot \Delta\varphi^{\mathrm{ext}}. \tag{338}$$

By comparing this result to Eq. (333), one can see that the parenthesized term of Eq. (338) defines $\mathsf{D}^{-1} - \mathsf{I}$.

Unfortunately, the average in Eq. (337b) does not immediately close because it involves the product of two fluctuating quantities, $\widetilde{R}$ and $\widetilde{\psi}$. Nevertheless, the MSR result

$$\langle\widetilde{R}(1;2)\widetilde{\psi}(3)\rangle = \langle\widetilde{\psi}(1)\widehat{\psi}(2)\widetilde{\psi}(3)\rangle = R(1;2)\langle\psi\rangle(3) + G\begin{pmatrix} 1 & 2 & 3 \\ + & - & - \end{pmatrix} \tag{339a,b}$$

together with Eq. (280c), which relates[210] G_{+-+} to the vertex functions Γ_{-++} and Γ_{--+}, closes the equations in terms of quantities already introduced. The result can be written in the form

$$\mathsf{D}^{-1} = \mathsf{I} - \widehat{\boldsymbol{\Phi}}R\,\boldsymbol{\partial}\overline{f} \cdot \mathsf{M}, \tag{340}$$

[210] By systematically summing over all possible combinations of spinor indices and recalling that Γ_{+++} vanishes, one finds

where $\overline{f} \doteq \langle F\rangle + \delta\overline{f}$ and $\boldsymbol{\partial}\delta\overline{f}$ is a symbolic notation defined by

$$\boldsymbol{\partial}\delta\overline{f}(\overline{1},\underline{2}) \doteq \int \mathrm{d}\boldsymbol{p}_2 \left[\boldsymbol{\partial}_2 C(2,\overline{2})\Gamma\begin{pmatrix}\overline{1} & \overline{2} & \overline{3}\\ - & + & +\end{pmatrix} + \boldsymbol{\partial}_2 R(2;\overline{2})\Gamma\begin{pmatrix}\overline{1} & \overline{2} & \overline{3}\\ - & - & +\end{pmatrix}\right] R(\overline{3};2). \tag{341}$$

It is important to note that this procedure naturally provides an expression for the *inverse* of the dielectric tensor, a point stressed in a more specific context by Taylor (1974a). Because formula (340) is built around the full response function R, its structure cannot immediately be compared with that of the well-known linear dielectric (33). To accomplish that I follow DuBois and Espedal (1978) and introduce a new *particle response function* $g(1;2) \doteq \delta\langle f(1)\rangle/\delta\widehat{\eta}(2)|_{\varphi}$, a properly renormalized version of the single-particle propagator g_0. One finds that $(g_0^{-1} + \Sigma_g^{\mathrm{nl}})g = \mathsf{I}$, where Σ_g^{nl} is the part of Σ^{nl} that does not involve the $\widehat{\boldsymbol{\Phi}}$ operator acting to the right. This can be shown to introduce the same combination $\boldsymbol{\partial}\overline{f}$ that appeared in Eq. (340) above, and leads to an intuitively plausible alternative to the R equation:

$$g^{-1}R + \boldsymbol{\partial}\overline{f}\cdot\widehat{\boldsymbol{\mathcal{F}}}R = \mathsf{I}. \tag{342}$$

Thus one has obtained a rigorous decomposition, valid through all orders of renormalization, of the response into particle and self-consistent field contributions. By noting that Eq. (342) has a form identical to that of Eq. (35), one can see that R is the renormalized generalization of Green's function for the linearized Vlasov equation. Straightforward operator manipulations (Krommes, 1984a) then lead to the formal solution

$$R = g - g\,\boldsymbol{\partial}\overline{f}\cdot\mathsf{M}\cdot\mathsf{D}^{-1}\cdot\widehat{\boldsymbol{\Phi}}g, \tag{343a}$$

the alternate representation

$$\mathsf{D} = \mathsf{I} + \widehat{\boldsymbol{\Phi}}g\,\boldsymbol{\partial}\overline{f}\cdot\mathsf{M}, \tag{343b}$$

and the physically pleasing result

$$\widehat{\boldsymbol{\Phi}}R = \mathsf{D}^{-1}\cdot\widehat{\boldsymbol{\Phi}}g. \tag{343c}$$

When the nonlinear terms are neglected, so that $g \to g_0$ and $\overline{f} \to \langle f\rangle$, Eq. (343b) reduces to the linearized Vlasov dielectric (33).

6.5.3 Coherent and incoherent response

Equations (343) generalize the familiar results (36) known to all students of Landau's solution of the initial-value problem for the linearized Vlasov equation. They also provide the renormalized

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$$G\begin{pmatrix}1 & 2 & 3\\ + & - & +\end{pmatrix} = C(1,\overline{1})R(\overline{2};2)R(3;\overline{3})\Gamma\begin{pmatrix}\overline{1} & \overline{2} & \overline{3}\\ + & + & -\end{pmatrix} + R(1;\overline{1})R(\overline{2};2)C(3,\overline{3})\Gamma\begin{pmatrix}\overline{1} & \overline{2} & \overline{3}\\ - & + & +\end{pmatrix}$$
$$+ R(1;\overline{1})R(\overline{2};2)R(3;\overline{3})\Gamma\begin{pmatrix}\overline{1} & \overline{2} & \overline{3}\\ - & + & -\end{pmatrix}. \tag{f-20}$$

In the application to the dielectric function, the first term can be shown to vanish by causality; the common factor of $R(1;\overline{1})$ of the second and third terms is the R of Eq. (340).

description of a shielded test particle, which was discussed in perturbation theory by Rostoker (1964b). Consider a scalar (single-potential) case for simplicity, and construct (Krommes, 1978, 1984a) the potential correlation function by applying the $\widehat{\Phi}$ operator to both arguments of $C(1,1')$ written in the spectral-balance form (286). In a homogeneous steady state, for which Fourier transformation in both space and time is appropriate, one obtains

$$\langle\delta\varphi^2\rangle(\boldsymbol{k},\omega) = \frac{\langle\delta\widetilde{\varphi}^2\rangle(\boldsymbol{k},\omega)}{|\mathcal{D}(\boldsymbol{k},\omega)|^2}, \tag{344}$$

where $\langle\delta\widetilde{\varphi}^2\rangle \doteq \widehat{\Phi}g\mathcal{F}(\widehat{\Phi}g)^T$. The interpretation of the completely rigorous Eq. (344) is that the true (observable) potential spectrum arises from an incoherent noise spectrum $\langle\delta\widetilde{\varphi}^2\rangle$ appropriately shielded by the dielectric properties of the turbulent medium. Related discussion of such spectral balance equations in the context of the fluid DIA was given in Sec. 5.4 (p. 133). Dupree employed the general form (344) in his theory of clumps (Sec. 4.4, p. 119), which makes specific predictions for $\langle\delta\widetilde{\varphi}^2\rangle$ (Dupree, 1972b, 1978).

As intuitive as they may appear, Eqs. (343b) and (343c) are not necessarily useful for practical calculations because the form of g is not known; it is determined only implicitly in terms of R or g itself. The full response function R is the natural object of renormalized turbulence theory; it obeys a relatively simple causal evolution equation. Although in principle g can be considered instead, the expressions of Σ_g^{nl} and $\boldsymbol{\partial}\delta\overline{f}$ in terms of g are extremely complicated (Krommes and Kleva, 1979; Krommes, 1984a) and do not appear to be appropriate for numerical evaluation.

Nevertheless, the expressions involving g are useful both formally, in the practical reduction of the formulas to WTT, and in other considerations relating to self-consistency. Let me first comment on the formal definitions of coherent and incoherent response. Equation (344) is the turbulent generalization of a well-known result from classical many-particle kinetic theory, in which the incoherent noise arises from an uncorrelated collection of discrete test particles. [See the construction $\boldsymbol{\epsilon_k}\boldsymbol{\epsilon}^*_{\boldsymbol{k}}/|\mathcal{D}^{\text{lin}}(\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v})|^2$ that appears in the kernel of the Balescu–Lenard operator (32).] An analogous interpretation can be given to the particle-level spectral balance (286), $C = RFR^T$. Now all of the formal development so far has been given in terms of statistically averaged functions; however, it is useful heuristically to consider the random fluctuations δf themselves and to divide them(Dupree, 1972b; Krommes, 1978, 1984a; DuBois and Espedal, 1978) into[211] a *coherent part* δf_c and an *incoherent part*[212] $\delta\widetilde{f}$:

$$\delta f = \delta f_c + \delta\widetilde{f}. \tag{345}$$

By definition, the coherent fluctuations are the renormalized *particle* response to an internal fluctuation $\delta\boldsymbol{E}$,

$$\delta f_c \doteq -g\,\boldsymbol{\partial}\overline{f}\cdot\delta\boldsymbol{E}. \tag{346}$$

[Dupree (1972b) wrote the form (346) with $\langle f\rangle$ instead of $\overline{f}$ and a resonance-broadening approximation for the propagator renormalization Σ_g^{nl}.] The incoherent fluctuations $\delta\widetilde{f} \doteq \delta f - \delta f_c$ are to be thought of as "bare" phase-space fluid elements that are propagated by g (the generalization of test-particle

[211] DuBois and Pesme (1985) referred to *diagonal* and *nondiagonal parts*.

[212] Here the tilde denotes incoherent, not random. Both δf_c and $\delta\widetilde{f}$ are random.

streaming in linear Klimontovich kinetic theory) and shielded by the dielectric properties of the medium. In particular, arising from $\delta\widetilde{f}$ is an incoherent field $\delta\widetilde{\boldsymbol{E}}$ such that

$$\delta\boldsymbol{E} = \mathcal{D}^{-1}\delta\widetilde{\boldsymbol{E}}; \tag{347}$$

the mean square of Eq. (347) reproduces the spectral balance (344) for the potential. With these definitions and the result (343a), one can easily show that

$$\delta f = R\,\delta\widetilde{f}_0, \quad \text{with} \quad \delta\widetilde{f}_0 \doteq g^{-1}\delta\widetilde{f}; \tag{348a,b}$$

i.e., the plasma response to an incoherent fluctuation develops with the full response function R.[213]

In these formulas one is mixing random variables (e.g., $\delta\boldsymbol{E}$) together with statistical observables (e.g., g or R). Thus one is really working with a generalized Langevin interpretation. Now the quantity $F^{\rm nl}(1,\overline{1})$ that appears in the formal MSR development is positive definite, so it can be written as the covariance of some (non-Gaussian)[214] random variable $\delta\widetilde{f}_0$. Consistent with Eq. (348b), one may define

$$\delta\widetilde{f} = g\,\delta\widetilde{f}_0. \tag{349}$$

The mean square of Eq. (349) then gives the covariance of the incoherent fluctuations as (DuBois and Espedal, 1978; Krommes, 1978)

$$\widetilde{C} = gF^{\rm nl}g^T, \tag{350}$$

and the mean square of Eq. (348a) reproduces the balance $C = RF^{\rm nl}R^T$. The similarity of these two forms for $\widetilde{C}$ and C is appealing. In the DIA one finds from formula (291b) that

$$F^{\rm nl}(1,2) = \langle\delta\boldsymbol{E}(1)\delta\boldsymbol{E}(2)\rangle : \boldsymbol{\partial}_1\boldsymbol{\partial}_2 C(1,2) + \langle\delta\boldsymbol{E}(1)\boldsymbol{\partial}_2\delta f(2)\rangle : \langle\boldsymbol{\partial}_1\delta f(1)\delta\boldsymbol{E}(2)\rangle. \tag{351}$$

The distinction between g and R is fundamental: g describes the propagation of bare, incoherent fluctuations; R includes additionally the effects of dielectric polarization [the second term of Eq. (343a)]. Because the nonlinear terms $\Sigma^{\rm nl}_g$ and $\delta\overline{f}$ are defined in terms of R, the decomposition (343a) leads to a corresponding decomposition of those terms that is useful for later discussion of the relationship between the general formalism and simpler approximations such as RBT. I shall restrict the discussion to the DIA, for which (DuBois and Espedal, 1978)

$$\Sigma^{\rm nl}_g(1,\overline{1}) = -\boldsymbol{\partial}_1\cdot[R(1;2)\langle\delta\boldsymbol{E}\,\delta\boldsymbol{E}\rangle(2,1) + \widehat{\boldsymbol{\mathcal{E}}}R(1;2)\langle\delta\boldsymbol{E}(2)\delta f(1)\rangle]\cdot\boldsymbol{\partial}_2\delta(2-\overline{1}), \tag{352a}$$

$$\boldsymbol{\partial}\delta\overline{f}(1,\underline{2}) = \int \mathrm{d}\boldsymbol{v}_2\,[\boldsymbol{\partial}_2\langle\delta f(2)\delta\boldsymbol{E}(\underline{1})\rangle\cdot\boldsymbol{\partial}_1 R(1;2) + \boldsymbol{\partial}_2\boldsymbol{\partial}_{\overline{1}}C(2,1)\cdot\widehat{\boldsymbol{\mathcal{E}}}R(\underline{1};2)]. \tag{352b}$$

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[213] The presence of g^{-1} in formula (348a) can be understood by noting that if $\delta\widetilde{f}$ is propagated by g from some initial condition $\delta\widetilde{f}(0)$, then $g^{-1}\delta\widetilde{f}$ is just that initial condition, which is then propagated by R to give the full response.

[214] $\delta\widetilde{f}_0$ must be non-Gaussian in order that a Langevin representation of the dynamics will generate the proper higher-order statistics. For related discussion, see Sec. 10.2 (p. 221).

Into Eqs. (352) insert the decomposition (343a) for R. Let us call the contributions from the first (g) term of Eq. (343a) *diffusion terms* and the contributions from the second term *polarization terms*. Then one can write (Krommes and Kleva, 1979; Krommes, 1984a)

$$\Sigma_g^{\mathrm{nl}} = \Sigma^{(\mathrm{d})} + \Sigma^{(\mathrm{p})}, \quad \delta\overline{f} = \delta\overline{f}^{(\mathrm{d})} + \delta\overline{f}^{(\mathrm{p})}; \tag{353a,b}$$

for example, $\Sigma^{(\mathrm{d})}(1,\overline{1}) = -\boldsymbol{\partial}_1 \cdot g(1;2)\langle\delta\boldsymbol{E}\,\delta\boldsymbol{E}\rangle(2,1)\cdot\boldsymbol{\partial}_2\delta(2-\overline{1})$. Explicit formulas for the remaining quantities in the DIA were given by Krommes (1984a).

The diffusion term $\Sigma^{(\mathrm{d})}$ and the first term of Eq. (351) are the only ones that survive in a passive calculation; the remaining terms are due to self-consistency. Some additional insights follow from comparisons to the various terms of the linearized Balescu–Lenard operator (Krommes and Kotschenreuther, 1982). In a specific context the existence of $\delta\overline{f}^{(\mathrm{d})}$ was noted by Dupree and Tetreault (1978), who called it the *β term*. Of course, all of these terms were already present in the discussion by Orszag and Kraichnan (1967) of the Vlasov DIA.

The g representation is convenient for efficient reduction of the renormalized theory to weak-turbulence theory (Krommes, 1978; DuBois and Espedal, 1978; Krommes and Kleva, 1979), as summarized in the next section, and for discussion of the relation of RBT to the full DIA (Sec. 6.5.5, p. 178). It has also been used in discussions of self-consistent QLT (Sec. 6.5.6, p. 180).

6.5.4 *The wave kinetic equation of weak-turbulence theory*

The weakly nonstationary version of Eq. (344) is the starting point for the derivation of the wave kinetic equation of WTT. It is assumed that the slowly evolving system is stable against infinitesimal perturbations, so $\mathcal{D}(\boldsymbol{k},\omega)$ must have zeros $\omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}} + \mathrm{i}\gamma_{\boldsymbol{k}}$ only in the lower half of the ω plane. That is, $\mathcal{D}$ is the *fully nonlinear* dielectric; linear instability may exist, but must be overcompensated by nonlinear damping so that $\gamma_{\boldsymbol{k}} < 0$. In WTT it is assumed that $\omega_{\boldsymbol{k}}$ is located close to the real axis, with $|\gamma_{\boldsymbol{k}}/\Omega_{\boldsymbol{k}}| \ll 1$; it is then consistent that $I_{\boldsymbol{k}}(\omega) \approx 2\pi\delta(\omega-\Omega_{\boldsymbol{k}})I_{\boldsymbol{k}}$. With these assumptions it is shown in Appendix G.1 (p. 288) that the action density $\mathcal{N}_{\boldsymbol{k}}$ evolves according to

$$\frac{\partial\mathcal{N}_{\boldsymbol{k}}}{\partial t} - 2\gamma_{\boldsymbol{k}}\mathcal{N}_{\boldsymbol{k}} = \frac{\langle\delta\widetilde{\varphi}^2\rangle_{\boldsymbol{k}}(\Omega_{\boldsymbol{k}})}{|\mathcal{D}'_{\boldsymbol{k}}|}, \tag{354}$$

where as usual[215] $\gamma_{\boldsymbol{k}} \approx -\,\mathrm{Im}[\mathcal{D}(\boldsymbol{k},\Omega_{\boldsymbol{k}})]/\mathcal{D}'_{\boldsymbol{k}}$ with $\mathcal{D}'_{\boldsymbol{k}} \doteq \partial\,\mathrm{Re}\,\mathcal{D}(\boldsymbol{k},\Omega_{\boldsymbol{k}})/\partial\Omega_{\boldsymbol{k}}$.

Further reduction of Eq. (354) requires that $\gamma_{\boldsymbol{k}}$ and $\langle\delta\widetilde{\varphi}^2\rangle_{\boldsymbol{k}}$ be evaluated in the weak-turbulence limit. Now in general it is very difficult to find the relative sizes of δf_c and $\delta\widetilde{f}$; it is the very essence of renormalized theory that order is ill defined. Nevertheless, progress can be made in the order-by-order expansion appropriate for WTT, for which one can argue that $F^{\mathrm{nl}} = O(I^2)$. If one inserts $C = \langle\delta f_c\,\delta f_c\rangle + \langle\delta f_c\,\delta\widetilde{f}\rangle + \langle\delta\widetilde{f}\,\delta f_c\rangle + \langle\delta\widetilde{f}\,\delta\widetilde{f}\rangle$, compatible with both Eqs. (345) and (348a), and ignores the possibility that the velocity derivatives may change the nominal order of the terms, one can see

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[215] Note that γ is defined in terms of the renormalized $\mathcal{D}$. Although γ contains nonlinear contributions, it does *not* define the full time evolution of $\mathcal{N}_{\boldsymbol{k}}$. Unfortunately, some authors define γ from $\partial_t\mathcal{N}_{\boldsymbol{k}} = 2\gamma_{\boldsymbol{k}}\mathcal{N}_{\boldsymbol{k}}$ (Pesme and DuBois, 1985), implicitly including the mode-coupling contribution from the right-hand side of Eq. (354) in $\gamma_{\boldsymbol{k}}$.

that formula (351) is dominated by the coherent parts:

$$C \approx \langle \delta f_c \, \delta f_c \rangle = (g\boldsymbol{\partial}\overline{f}) \cdot \langle \delta \boldsymbol{E} \, \delta \boldsymbol{E} \rangle \cdot (g\boldsymbol{\partial}\overline{f})^T, \quad \langle \delta f \, \delta \boldsymbol{E} \rangle \approx \langle \delta f_c \, \delta \boldsymbol{E} \rangle = -(g\boldsymbol{\partial}\overline{f}) \cdot \langle \delta \boldsymbol{E} \, \delta \boldsymbol{E} \rangle. \quad \text{(355a,b)}$$

In a strict ordering g can be replaced by its unrenormalized form and $\delta\overline{f}$ can be replaced by $\langle f \rangle$. The resulting contribution to $\langle \delta\widetilde{\varphi}^2 \rangle$ is written in detail in Appendix G.2 (p. 290).

The weak-turbulence expansion of the dielectric function follows from an intensity expansion of formula (343b) in which it is assumed that the primary waves are *nonresonant*, $\Omega_{\boldsymbol{k}} \neq \boldsymbol{k} \cdot \boldsymbol{v}$. This inequality permits the propagator expansion $g_k \approx g_{0,k} - g_{0,k}\Sigma^{\rm nl}_{g,k} g_{0,k}$, where $k \equiv (\boldsymbol{k}, \omega_{\boldsymbol{k}})$. Through second order one finds $\mathcal{D} = \mathcal{D}^{\rm lin} + \mathcal{D}^{\rm nl}$, where (Krommes, 1984a)

$$\mathcal{D}^{\rm nl}_k \approx \widehat{\boldsymbol{\mathcal{E}}} \cdot g_{0,k}(\boldsymbol{\partial}\delta\overline{f}_k - \Sigma^{\rm nl}_{g,k} g_{0,k} \boldsymbol{\partial}\langle f \rangle). \tag{356}$$

After tedious but straightforward calculations, one finds that Eq. (356) reduces precisely to Eq. (190a); this is an important consistency check. If one keeps track of the various pieces, one finds that $\Sigma^{\rm (d)}$ and $\delta\overline{f}^{\rm (d)}$ contribute symmetrically to $\epsilon^{(3)}$, while $\Sigma^{\rm (p)}$ and $\delta\overline{f}^{\rm (p)}$ contribute symmetrically to $\epsilon^{(2)}(p \mid q, k)$ [each being proportional to $\epsilon^{(2)}(k \mid p, q)$].

Because the induced-scattering contributions proportional to $\gamma^{\rm ind}_{\boldsymbol{k}}$ involve particle motion, they must vanish in the fluid approximation $T = 0$. In the absence of $\gamma^{\rm ind}_{\boldsymbol{k}}$ Eq. (191) has the same form as the fluid WKE (182) deduced in Sec. 4.2 (p. 98), with $\gamma^{\rm mc}_{\boldsymbol{k}} = -\eta^{\rm nl}_{\boldsymbol{k}}$; the kinetic-based procedure just discussed therefore provides one route to the derivation of the fluid mode-coupling coefficients $M_{\boldsymbol{kpq}}$. This assertion is explicitly demonstrated for HM dynamics in Appendix G.3 (p. 291).

Let us focus on the fluid mode-coupling effects and consider their relation to the diffusion and polarization parts in more detail. Superficially, the (d)–(p) decomposition does not appear to be invariant to the route used to derive the final fluid WKE. If one were to begin immediately with a fluid nonlinearity, e.g., the polarization-drift nonlinearity $\boldsymbol{V}_E \cdot \boldsymbol{\nabla}(-\nabla^2\varphi)$, a diffusive contribution $\Sigma^{\rm (d)}_{\rm fluid}$ would arise by treating the $\boldsymbol{E} \times \boldsymbol{B}$ velocity as passive. When the potential is treated self-consistently, leading to symmetrized mode-coupling coefficients, three extra terms $\Sigma^{\rm (p)}_{\rm fluid}$, $\delta\overline{f}^{\rm (d)}_{\rm fluid}$, and $\delta\overline{f}^{\rm (p)}_{\rm fluid}$ arise. However, in the kinetic route all of the fluid effects [which involve $\epsilon^{(2)}$; see the discussion of Eq. (191)] arise from the polarization terms $\Sigma^{\rm (p)}_{\rm kin}$ and $\delta\overline{f}^{\rm (p)}_{\rm kin}$.

The resolution of this paradox is that the kinetic decomposition (352) that has been discussed previously (Krommes and Kleva, 1979; Krommes, 1984a) is not sufficiently detailed. In order to properly track the passage to the fluid limit, each of the kinetic terms should carry a second d or p label to define its fluid-related passive or self-consistent structure. Terms explicitly involving the spectral level are assigned a fluid d label; the other terms, involving the cross correlation $\langle \delta\boldsymbol{E} \, \delta f \rangle$, are assigned a fluid p label. Thus in a two-index notation in which the first index is kinetic, the second fluid, one writes $\Sigma^{\rm nl}_g = \Sigma^{\rm (dd)} + \Sigma^{\rm (pd)} + \Sigma^{\rm (pp)}$ and similarly for $\delta\overline{f}$. Note that there are no (dp) contributions. Then the structure of Eq. (190a) can be elaborated as follows:

$$\mathcal{D}^{\rm nl}_k = \sum_{k+p+q=0} [\underbrace{\epsilon^{(3)}(k \mid q, -q, -k)}_{\Sigma^{\rm (dd)} - \delta\overline{f}^{\rm (dd)}} - \underbrace{\epsilon^{(2)}(k \mid p, q)(\mathcal{D}^*_p)^{-1}\epsilon^{(2)*}(p \mid q, k)}_{[\Sigma^{\rm (pd)} + \Sigma^{\rm (pp)}] - [\delta\overline{f}^{\rm (pd)} + \delta\overline{f}^{\rm (pp)}]}] I_q. \tag{357}$$

For example, one finds for the ion polarization-drift nonlinearity the dependences

$$[\underbrace{\Sigma^{(\mathrm{pd})}}_{p^2k^2} + \underbrace{\Sigma^{(\mathrm{pp})}}_{-q^2k^2}] - [\underbrace{\delta\overline{f}^{(\mathrm{pd})}}_{p^2q^2} + \underbrace{\delta\overline{f}^{(\mathrm{pp})}}_{-q^2q^2}] \propto (p^2 - q^2)(k^2 - q^2), \tag{358}$$

in accord with the four terms that arise from the $M_{\boldsymbol{kpq}}M_{\boldsymbol{pqk}}$ of Eq. (183a). The passive contribution proportional to k^2p^2 arises from $\Sigma^{(\mathrm{pd})}$, a kinetic *polarization* effect.

6.5.5 Resonance-broadening theory redux

This general discussion has consequences for the interpretation of the RBT. It implies that if a standard passive resonance-broadening approximation is applied directly to a kinetic equation (so that, by definition, only $\Sigma^{(\mathrm{dd})} \equiv \Sigma^{(\mathrm{d})}_{\mathrm{kin}}$ is retained), the resulting theory does not properly describe *any* of the nonlinearities seen at the fluid level. Let us consider this assertion in the two limits of weak turbulence and strong turbulence.

For definiteness, consider kinetic ions and adiabatic electrons. For the case of $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity, $\Sigma^{(\mathrm{d})}_{\mathrm{kin}}$ was written in Eq. (206b). In WTT the Markovian approximation is inappropriate; instead, $\Sigma^{(\mathrm{dd})}$ contributes half of the induced scattering from the bare particles. (The resulting asymmetry of $\epsilon^{(3)}$ leads to violation of the action conservation laws.) Meanwhile, the contribution of $\Sigma^{(\mathrm{pd})}$ produces the passive fluid contribution to $\gamma^{(\mathrm{d})}_{\boldsymbol{k}} \equiv -\operatorname{Re}\eta^{\mathrm{nl(d)}}_{\boldsymbol{k}}$:

$$\operatorname{Re}\eta^{\mathrm{nl(d)}}_{\boldsymbol{k}} = k^2 \sum_{\Delta} \frac{|\widehat{\boldsymbol{b}}\cdot\boldsymbol{p}\times\boldsymbol{q}|^2}{(1+k^2)(1+p^2)} \operatorname{Re}\theta_{\boldsymbol{kpq}} q^2 I_{\boldsymbol{q}}, \tag{359}$$

where in WTT θ has the unrenormalized form (183c). Although this result is also proportional to k^2, the predicted fluid diffusion coefficient differs from the form (207) stemming from $\Sigma^{(\mathrm{d})}_{\mathrm{kin}}$.

Now consider the strong-turbulence limit. Perturbation theory is inappropriate, but if one takes $T_i \to 0$ then $v_\parallel$ can be ignored, $J_0 \to 1$, the GKE can trivially be integrated over $\boldsymbol{v}$, and the relevant kinetic equation reduces to the GK ion continuity equation. Importantly, the potential operator $\widehat{\Phi}$ becomes *purely multiplicative*, $\widehat{\Phi}_{\boldsymbol{k}} = (1+k^2)^{-1}$. Therefore the fundamental shielding relationship (343c) between g and R, $\widehat{\Phi}R = \mathcal{D}^{-1}\widehat{\Phi}g$, reduces to $R = \mathcal{D}^{-1}g$, or

$$\mathcal{D}(\boldsymbol{k},\omega) = g(\boldsymbol{k},\omega)R^{-1}(\boldsymbol{k},\omega). \tag{360}$$

Thus R^{-1} and $\mathcal{D}$ have identical zeros in the complex ω plane. That is, a proper nonlinear theory of the fluid dielectric function is essentially equivalent to the theory of the *full* response function R, not the renormalized particle propagator g on which RBT focuses. In particular, the nonlinear damping is η^{nl}, not $\Sigma^{(\mathrm{d})}$.

The observations of the previous paragraph were first made by Krommes and Similon (1980) in the context of the 2D, electrostatic *guiding-center model* (31), in which charged rods move cross-field with the $\boldsymbol{E}\times\boldsymbol{B}$ velocity. This problem was originally considered by Taylor (1974a). For the special case of thermal equilibrium, Taylor used the general theory of linear response (Martin, 1968) to derive the approximate expression

$$\mathcal{D}(\boldsymbol{k},\omega) \approx 1 + (k_D^2/k^2)[1 + \mathrm{i}\omega g(\boldsymbol{k},\omega)], \tag{361a}$$

$$g(\boldsymbol{k},\omega) = \frac{1}{-\mathrm{i}(\omega + \mathrm{i}\Sigma^{\mathrm{nl}}_{g,\boldsymbol{k}})}, \qquad \Sigma^{\mathrm{nl}}_{g,\boldsymbol{k}} \approx \left(\frac{k^2}{k^2 + k_D^2}\right) k^2 D. \tag{361b,c}$$

This result is to be compared with the prediction of one version of RBT (Lee and Liu, 1973), which retains the form of Eqs. (361a) and (361b) but asserts that $\Sigma^{\mathrm{nl}}_{\mathrm{g,RBT}} \approx k^2 D$. Note that Taylor's result, the form of which is correct, predicts a long-wavelength modification to the usual resonance-broadening term that renormalizes the particle propagator. Also note that the solution of $\mathcal{D}(\boldsymbol{k},\omega) = 0$ is $\omega = -\mathrm{i}k^2 D$. This result is heuristically reasonable; nevertheless, it does *not* arise as a zero of the RBT dielectric; the zero of the correct Eq. (361a) is $\omega_{\boldsymbol{k}} = -\mathrm{i}(\Sigma^{\mathrm{nl}}_{g,\boldsymbol{k}}/k^2)(k^2 + k_D^2) = -\mathrm{i}k^2 D$, not $\omega_{\boldsymbol{k}} = -\mathrm{i}\Sigma^{\mathrm{nl}}_{g,\boldsymbol{k}}$. Now (361a) can be written as

$$\mathcal{D} = 1 + \left(\frac{k_D^2}{k^2}\right)\left(\frac{\Sigma^{\mathrm{nl}}_{g,\boldsymbol{k}}}{-\mathrm{i}(\omega + \mathrm{i}\Sigma^{\mathrm{nl}}_{g,\boldsymbol{k}})}\right), \tag{362}$$

so that although $\mathrm{i}\Sigma^{\mathrm{nl}}_g$ is indeed additive to ω *in the expression (361b) for the* particle *propagator*, the original RBT recipe $\mathcal{D}^{\mathrm{nl}}(\boldsymbol{k},\omega) = \mathcal{D}^{\mathrm{lin}}(\boldsymbol{k},\omega + \mathrm{i}\Sigma^{\mathrm{nl}}_g)$ does *not* hold; indeed, $\mathcal{D}^{\mathrm{lin}} \equiv 1$ for this entirely nonlinear model. Furthermore, the proper Σ^{nl}_g, Eq. (361c), differs from $k^2 D$ in just such a way that the expected $k^2 D$ damping is recovered.

The connection of Taylor's approach to the general theory of renormalized response was subsequently discussed by Krommes and Similon (1980), who again emphasized the importance of Eq. (360). (In the guiding-center model one has $\widehat{\Phi}_{\boldsymbol{k}} = 4\pi/k^2$.) Note that since the mean field vanishes for this model, the zeroth-order dielectric has just the vacuum value 1; the nonlinear correction $\delta\overline{f}$ can in no way be ignored. By writing out the DIA formulas for $\Sigma^{(\mathrm{d})}$, $\Sigma^{(\mathrm{p})}$, $\delta\overline{f}^{(\mathrm{d})}$, and $\delta\overline{f}^{(\mathrm{p})}$, Krommes and Similon showed that (i) the numerator $\Sigma^{\mathrm{nl}}_{g,\boldsymbol{k}}$ in Eq. (362) (a nonlinear correction that is not additive to ω) arises from $\boldsymbol{\partial}\delta\overline{f}$; and (ii) for each of Σ^{nl}_g and $\delta\overline{f}$ the diffusion and polarization contributions combine to give results proportional to the factor $p^2 - q^2$, which vanishes as $k \to 0$. This cancellation, arising from the self-consistent backreaction described by the polarization terms, reduces Σ^{nl}_g by a factor of k^2 relative to its size when only the diffusion terms are retained. [This cancellation is analogous to the one in weak-turbulence theory between the Compton scattering from a bare test particle (diffusion effect) and the nonlinear scattering from the shielding cloud of that test particle.] One is ultimately led to a result having the same structure as Eqs. (361a) and (361c) except that D is replaced by a wave-number- and frequency-dependent function $D_{\boldsymbol{k},\omega}$ closely related to the nonlocal transport coefficient for convective cells analyzed earlier by Krommes and Oberman (1976b). Its form, involving integrations over all triads, was discussed in detail by Krommes and Similon (1980).

The thermal-equilibrium guiding-center model provides an ideal test bed for illustrating many of the generally complicated formulas of renormalized turbulence theory. An expanded form of the present discussion was used by Krommes and Similon (1980) to argue in favor of the general theory of linear response, the consistency of linear response theory with specific approximations such as the DIA, the importance of self-consistency in renormalized descriptions, and the superiority of the full response function R over the particle propagator g. It serves as a nicely unifying example of the general formalism.

In summary, one can now see more clearly that properly renormalized equations make a smooth and ultimately straightforward transition from the weak-turbulence limit to the strong-turbulence limit. Perturbation theory already leads to the proper form (183a) of the nonlinear damping $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ [with the weak-turbulence form (183c) of the triad interaction time θ]. For strong fluctuations the form of the nonlinear dielectric is just such that the modal damping is still described by $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$, now with a renormalized θ. Qualitatively, the variation of $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ with fluctuation level is as Dupree suggested [see Fig. 14 (p. 112)], but strict application of the resonance-broadening recipe does not lead in detail to the proper formula for $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$.

6.5.6 Kinetic self-consistency redux

In lowest-order WTT the approximation $\langle\delta f\,\delta\boldsymbol{E}\rangle \approx \langle\delta f_c\,\delta\boldsymbol{E}\rangle$ is appropriate for evaluating the nonlinear terms Σ^{nl}_g, $\delta\overline{f}$, and F^{nl}. Consider, however, the equation for the one-particle PDF f of a Klimontovich or Vlasov description. f evolves according to $\partial_t f + \cdots = -\boldsymbol{\partial}\cdot\langle\delta\boldsymbol{E}\,\delta f\rangle$. One has exactly $\langle\delta\boldsymbol{E}\,\delta f\rangle = \langle\delta\boldsymbol{E}\,\delta f_c\rangle + \langle\delta\boldsymbol{E}\,\delta\widetilde{f}\rangle$, or more explicitly,

$$\langle\delta\boldsymbol{E}\,\delta f\rangle = -g\langle\delta\boldsymbol{E}\,\delta\boldsymbol{E}\rangle\cdot\boldsymbol{\partial}\overline{f} + \mathcal{D}^{-1}\widehat{\boldsymbol{\mathcal{E}}}gFg^T. \tag{363}$$

The first term generalizes the usual quasilinear solution, and with $\overline{f}\to\langle f\rangle$ would be present for passive advection as well. The last term, however, is a consequence of self-consistency; it is the formal analog of the polarization-drag term of classical kinetic theory. Dupree (1970) assumed that the incoherent noise was localized in velocity space, and he reduced an approximate version of Eq. (363) to a Fokker–Planck equation, in close analogy to derivations of the Balescu–Lenard operator. Those calculations formed the basis for his work on clumps (Sec. 4.4, p. 119). Mynick (1988) followed a similar procedure in his derivation of his generalized Balescu–Lenard operator. The incoherent contribution is required in order to conserve kinetic momentum and energy.

The polarization effects in self-consistent problems are clearly important. Nevertheless, it is crucial to understand that Eqs. (344) and (363) are *formally exact*, highly nonlinear balance equations. An analog to particle discreteness effects must be pursued with care since, as we have seen, even the spectral balance equation of weak-turbulence theory, which describes a collection of weakly interacting, spatially extended waves rather than pointlike entities, can be written in the form (344). Lengthy discussion of this point was given by Krommes and Kim (1988).

Serious attempts to understand the consequences of self-consistent polarization have been made in the context of 1D Vlasov theory. Adam et al. (1979) argued that even in the limit of very short autocorrelation time, where QLT would be expected to apply, if the ordering $\gamma^{\mathrm{lin}} < (k^2 D_v)^{1/3} \equiv \tau_d^{-1}$ holds, then nonlinear wave–particle resonances through all orders contribute an order-unity correction to the growth rate; to maintain conservation laws, the diffusion coefficient must be enhanced as well. The possibility of corrections to the diffusion coefficient can be seen in the extra terms of Eq. (363).

Adam *et al.* advanced a *turbulent trapping model* (TTM) that made the definite prediction $\gamma = 2.2\gamma^{\mathrm{lin}}$; the formalism was a kind of weak-turbulence limit of the clump formalism of Dupree (1972b). Pesme and DuBois (1985) made a heroic attempt to exploit the ordering $\gamma^{\mathrm{lin}} < \tau_d$ in order to reduce the Vlasov DIA to a simpler form. They showed what further approximations lead to the TTM. They brought their reduced approximation to dimensionless form and found that any correction to the linear growth rate must be necessarily positive. The enhancement was explicitly seen to be due to the effects of self-consistency; they advanced the heuristic argument that self-consistent response leads to an intrinsically non-Gaussian electric field. That such a field can contribute order-unity corrections to the usual Gaussian Fokker–Planck coefficients was shown in a readable paper by Pesme (1994), whose work serves as a useful review; see also Pesme and DuBois (1982).

Theilhaber et al. (1987) made a serious numerical attempt to measure enhancements to the growth rate; although such were found, they were modest and smaller than the predictions of the TTM, and numerical uncertainties precluded definite conclusions.[216] Their paper contains a useful review of the physical ideas, which include the effect of partial trapping in barely overlapping wave packets. An experiment by Tsunoda et al. (1987) specifically designed to test the theoretical ideas did not find the expected enhancement. Liang and Diamond (1993b) revisited the theory and argued that calculations using the TTM calculated the production of fluctuations incorrectly by ignoring a momentum-conservation constraint. Their discussion of the basic ideas is a very clear and useful reference. Nevertheless, the formalism needs to be reassessed in view of the difficulties with the x-space version of the theory (Krommes, 1997a) that were discussed in Sec. 4.4 (p. 119). [Note that the work of Liang and Diamond went beyond the calculations of Krommes (1997a) in that the former authors attempted to calculate the production of fluctuations self-consistently.]

The most recent review of these topics is by Laval and Pesme (1999), who give additional references. Ultimately, it is unlikely that the theoretical aspects of this problem will be completely and convincingly understood until a numerical solution of the 1D Vlasov DIA is undertaken. [One should study the full DIA in the R representation, not the reduced version of Pesme and DuBois (1985).] Such work is feasible in principle, but is entirely nontrivial.

7 ALTERNATE THEORETICAL APPROACHES

Although the MSR formalism provides an elegant unification of traditional approaches to renormalized perturbation theory, its foundations in Eulerian correlation and response functions makes it intrinsically unsuitable for addressing various important issues such as random Galilean invariance and nuances of higher-order statistics. The general level of complexity is also extremely high. In the present section some alternate approaches are briefly described. Those include Lagrangian schemes (Sec. 7.1, p. 181), Markovian approximations (Sec. 7.2, p. 182), eddy viscosity and large-eddy simulations (Sec. 7.3, p. 189), use of the renormalization group (Sec. 7.4, p. 196), and statistical decimation (Sec. 7.5, p. 197). Applications of the Markovian schemes to plasma physics will be elaborated in Sec. 8 (p. 199).

7.1 Lagrangian schemes

"Closure approximations which involve only low-order Eulerian moments do not retain sufficient information to represent properly the energy transfer among small scales which are convected by large scales. . . . [However, a heuristic, Lagrangian-history alteration of the DIA implies] high-Reynolds-number inertial and dissipation ranges which obey Kolmogorov's laws."
— *Kraichnan (1965a).*

A key conceptual problem with the standard DIA has already been identified in Sec. 5.6.3 (p. 138) to be its lack of invariance to random Galilean transformations. As observed by Kraichnan (1965a), the fundamental technical difficulty is the use of Eulerian rather than Lagrangian correlation functions.[217] In an attempt to cure the problem, Kraichnan (1965a) introduced the Lagrangian

[216] For more recent related numerical work of very high quality, see Cary et al. (1992) and Stoltz and Cary (1994).

[217] Eulerian and Lagrangian were defined in Sec. 1.3.1 (p. 13). For a more mathematical discussion, see Lumley (1962).

History Direct Interaction Approximation (LHDIA). By considering a generalized velocity field $\boldsymbol{u}(\boldsymbol{x}, t \mid s)$, defined as the velocity measured at time s of the fluid element that passes through $\boldsymbol{x}$ at time t, and imposing a variety of constraints (including random Galilean invariance), Kraichnan was able to formulate closure equations for the mixed Eulerian–Lagrangian correlation functions of $\boldsymbol{u}$. The resulting approximation (and an abridged, easier-to-compute version thereof) reproduces the Kolmogorov $-\frac{5}{3}$ law (by construction) and has been shown to behave quantitatively reasonably in a variety of situations (Herring and Kraichnan, 1979); a comprehensive discussion of tests of various closures is given in Chap. 8 of McComb (1990). Unfortunately, the partially heuristic derivation did not ascend to a level of systematology the same as enjoyed by the Eulerian schemes so elegantly unified by MSR, and no realizable primitive amplitude representation is known. Later Kraichnan (1977) showed how to derive a variety of Lagrangian schemes, including the LHDIA, by a reversion procedure based on perturbation theory; however, that only emphasized the substantial ambiguity in the resulting approximations.

Kraichnan's Lagrangian-history closures were formulated in terms of the labeling time t. Kaneda (1981) instead proposed a Lagrangian renormalized approximation involving the measuring time s. Some of its predictions were explored by Kaneda (1986).

If this were primarily a review of statistical closures for neutral-fluid turbulence, a further extensive discussion of literature on the Lagrangian-history schemes would be warranted. However, Lagrangian closures have been very little studied in the context of plasma-turbulence theory; a practically unique exception is the work of Orszag (1969) on stochastic acceleration. The general argument in favor of Eulerian plasma closures, that inertial ranges are often not well developed, has already been given in Sec. 5.1 (p. 128). Nevertheless, the lack of Lagrangian calculations represents an unfortunate and significant gap in one's understanding of the analytical theory of plasma turbulence.

7.2 Markovian approximations

"It is possible to make a simple, though crude, modification ... that eliminates many of the deficiencies of the quasi-normal theory .... Our purpose in discussing [the resulting EDQNM closure] is not to propose it as a basic theory of turbulence but rather to illustrate the sort of effects that must be included in a satisfactory theory." — *Orszag (1977).*

A virtue of closures at the DIA level of sophistication is that they attempt to relatively faithfully describe the details of the two-time response. That is also their Achilles heel, since they are very computationally intensive. Because in one way or another they attempt to self-consistently determine an autocorrelation time, which at core requires time-history information, they exhibit a very adverse computation-time scaling with the number of time steps. [For more details, see Sec. 8.3.1 (p. 205).]

If one is interested primarily in equal-time behavior [which according to Eq. (3) is sufficient to determine transport fluxes], one may attempt to develop *Markovian approximations*, which are renormalized versions of the wave kinetic equation (Sec. 4.2, p. 98) that evolve solely equal-time spectra. In practice those are surprisingly successful. A general approach to their derivation, couched in the unifying language of classical statistical field theory, was given by Carnevale and Martin (1982).[218]

[218] For discussion of a subtle conceptual difficulty with the work of Carnevale and Martin (1982), see Appendix F (p. 286).

7.2.1 *The eddy-damped quasinormal Markovian (EDQNM) approximation*

The *eddy-damped quasinormal Markovian* (EDQNM) *approximation* is a popular and successful moment-based closure for neutral-fluid turbulence. Its original derivation was reviewed in detail by Orszag (1977). In brief: The exact equation for the triplet correlation function $T(t,t',t'')$ is written, then the effects of the fourth-order cumulants are approximated as an (as yet unknown) nonlinear *eddy damping* $\eta^{\rm nl}$ of T. The resulting equation for $T(t) \equiv T(t,t,t)$ is solved with a Green's function G; the result is schematically $T(t) \sim \int_0^t \mathrm{d}t'\, G(t;t')C^2(t')$. At this point one has the original *non*-Markovian quasinormal closure, whose properties were discussed by Orszag (1970a). One now makes the *Markovian approximation* $\int_0^t \mathrm{d}t'\, G(t;t')C^2(t') \approx [\int_0^t \mathrm{d}t'\, G(t;t')]C^2(t)$, thereby arriving at a closed equation for $C(t)$. The step $C(t') \to C(t)$ is the same one made in the passage from the Bourret approximation to quasilinear theory (Sec. 3.9.2, p. 78) except that in strong-turbulence theory one cannot take advantage of a short autocorrelation time to justify the approximation. This reduced description does not attempt to predict the detailed shapes of two-time correlation functions, positing instead an exponential decay [see Fig. 8 (p. 85) and associated discussion]. Nevertheless, it can still capture the appropriate, self-consistently determined nonlinear timescale.

There are two approaches to the determination of $\eta^{\rm nl}$. Heuristically (Leith, 1971), one can estimate $\eta^{\rm nl}_{\boldsymbol{k}}$ as the rms value of an eddy turnover rate estimated from Kolmogorov arguments. In the absence of wave physics, one has that the contribution from one wave-number band $\overline{k}$ of width $\Delta\overline{k}$ is $\Delta\langle\nu^2_{\rm eddy}\rangle_{\overline{k}} \sim \overline{k}^2(\Delta\langle\delta u^2\rangle_{\overline{k}}/\Delta\overline{k})\Delta\overline{k}$, or

$$\eta^{\rm nl}_{\boldsymbol{k}} = C\left(\int_0^k \mathrm{d}\overline{k}\,\overline{k}^2 E(\overline{k})\right)^{1/2}, \tag{364}$$

where C is an undetermined constant. (The upper limit of k follows by arguing that the effects of fluctuations of wavelength shorter than k^{-1} should average away.) The expression (364) is just an estimate for the rms vorticity in the long wavelengths.[219] It does not involve the spurious $ku_{\rm rms}$ advection frequency that contaminates the energetics of the DIA; formula (364) is random-Galilean-invariant (RGI).[220]

An alternate derivation of an EDQNM approximation proceeds directly from the DIA (Carnevale and Martin, 1982; Bowman et al., 1993). Consider the DIA in the form (224b). The balance equation for $C(t) \equiv C(t,t)$ follows as Eq. (226). Let the response function obey the Markovian equation

$$\partial_t R_{\boldsymbol{k}}(t;t') + \mathrm{i}\mathcal{L}_{\boldsymbol{k}} R_{\boldsymbol{k}} + \eta^{\rm nl}_{\boldsymbol{k}}(t) R_{\boldsymbol{k}} = \delta(t-t'). \tag{365}$$

[219] If the k integral in Eq. (364) were extended to ∞ (or, essentially, the Kolmogorov dissipation wave number k_d), formula (364) would be proportional to the total dissipation rate ε; see the discussion in Sec. 3.6.3 (p. 66).
[220] Variation of $E(k)$ at $k = 0$ does not change the value of expression (364).

Finally, make the *Fluctuation–Dissipation Ansatz* (FDA)[221]

$$C_{\boldsymbol{k}}(t;t') \approx R_{\boldsymbol{k}}(t;t')C_{\boldsymbol{k}}(t) \quad (t \geq t'). \tag{366}$$

Given these various approximations, the time integrals in the nonlinear terms of Eq. (226) can be performed. The result has the same form as the spectral evolution equation (182) of WTT, repeated here for convenience:

$$\partial_t C_{\boldsymbol{k}}(t) - 2\gamma_{\boldsymbol{k}}^{\text{lin}} C_{\boldsymbol{k}} + 2\operatorname{Re}\eta_{\boldsymbol{k}}^{\text{nl}} C_{\boldsymbol{k}} = 2F_{\boldsymbol{k}}^{\text{nl}}, \tag{367}$$

where

$$\eta_{\boldsymbol{k}}^{\text{nl}}(t) \doteq -\sum_{\Delta} M_{\boldsymbol{kpq}} M_{\boldsymbol{pqk}}^{*} \theta_{\boldsymbol{pqk}}^{*}(t) C_{\boldsymbol{q}}(t), \quad F_{\boldsymbol{k}}^{\text{nl}}(t) \doteq \frac{1}{2}\sum_{\Delta} |M_{\boldsymbol{kpq}}|^2 \operatorname{Re}(\theta_{\boldsymbol{kpq}}(t)) C_{\boldsymbol{p}}(t) C_{\boldsymbol{q}}(t). \tag{368a,b}$$

Instead of the weak-turbulence form (183c) for the triad interaction time $\theta_{\boldsymbol{kpq}}$, however, its renormalized form is defined as[222]

$$\theta_{\boldsymbol{kpq}}(t) \doteq \int_0^t \mathrm{d}t'\, R_{\boldsymbol{k}}(t;t') R_{\boldsymbol{p}}(t;t') R_{\boldsymbol{q}}(t;t'). \tag{369}$$

An evolution equation for $\theta_{\boldsymbol{kpq}}$ follows by differentiating Eq. (369) with respect to t and using Eq. (365):

$$\partial_t \theta_{\boldsymbol{kpq}}(t) + [\eta_{\boldsymbol{k}}(t) + \eta_{\boldsymbol{p}}(t) + \eta_{\boldsymbol{q}}(t)]\theta_{\boldsymbol{kpq}}(t) = 1, \quad \theta_{\boldsymbol{kpq}}(0) = 0, \tag{370a,b}$$

[221] Pragmatically, some such Ansatz is necessary in order to close the system. Although the relation (366) is exact only in thermal equilibrium (Kraichnan, 1959a), experience has shown that in general it is not unreasonable even in strongly nonequilibrium situations (LoDestro et al., 1991; Bowman and Krommes, 1997). The choice of t rather than, say, t' in $C_{\boldsymbol{k}}(t)$ defines the EDQNM. Additional discussion of the FDA is given in Sec. 8.2.3 (p. 203).

[222] Compare the appearance of this $\theta \equiv \theta_3$ (integrated symmetrically over three response functions) in the R equation with the $\Sigma^{(\mathrm{d})}$ of RBT and Kraichnan (1964d), which involves $\theta_2 \doteq \int \mathrm{d}t'\, R_{\boldsymbol{p}} R_{\boldsymbol{q}}$. To see how θ_2 might emerge, consider the schematic form of a steady-state R equation (mode-coupling sums omitted), where $R(\tau) = H(\tau)G(\tau)$ and $G(0_+) = 1$:

$$\partial_\tau G_{\boldsymbol{k}}(\tau) + \int_0^\tau \mathrm{d}\overline{\tau}\, G_{\boldsymbol{p}}^{*}(\overline{\tau}) G_{\boldsymbol{q}}^{*}(\overline{\tau}) G_{\boldsymbol{k}}(\tau - \overline{\tau}) = 0. \tag{f-21}$$

A seemingly direct way to derive an equation for the area under the curve $1/\eta_{\boldsymbol{k}}^{\text{nl}} = \int_0^\infty \mathrm{d}\tau\, G_{\boldsymbol{k}}(\tau)$ is to integrate Eq. (f-21) from 0_+ to ∞ (Leslie, 1973b); upon interchanging the order of integration in the convolution term, one finds $0 = -1 + \left[\int_0^\infty \mathrm{d}\overline{\tau}\, G_{\boldsymbol{p}}(\overline{\tau}) G_{\boldsymbol{q}}(\overline{\tau})\right] \left[\int_0^\infty \mathrm{d}\tau\, G_{\boldsymbol{k}}(\tau)\right] = \theta_{2,\boldsymbol{pq}}/\eta_{\boldsymbol{k}}^{\text{nl}}$, or $\eta_{\boldsymbol{k}}^{\text{nl}} = \theta_{2,\boldsymbol{pq}}$. But this procedure is not unique (one could have multiplied by an arbitrary weight function before integrating), and furthermore does not work when a term $\mathrm{i}\Omega_{\boldsymbol{k}} G_{\boldsymbol{k}}$ is added to the left-hand side of Eq. (f-21). To eliminate $\Omega_{\boldsymbol{k}}$ and deduce an equation for $\operatorname{Re}\eta_{\boldsymbol{k}}^{\text{nl}}$, form the equation for $\frac{1}{2}|G_{\boldsymbol{k}}|^2$ by multiplying Eq. (f-21) by $G_{\boldsymbol{k}}^{*}$. Brief manipulations assuming the form $G_{\boldsymbol{k}}(\tau) = \exp(-\mathrm{i}\Omega_{\boldsymbol{k}}\tau - \eta_{\boldsymbol{k}}^{\text{nl}}\tau)$ now lead to $\operatorname{Re}\eta_{\boldsymbol{k}}^{\text{nl}} = \operatorname{Re}\theta_3^{*}$, compatible with the result of the Markovian formalism.

The distinction between θ_2 and θ_3 is not a trivial point. Formulas based on θ_2 need not be realizable and may have spurious, nonphysical solutions (Koniges and Krommes, 1982). However, a Langevin representation of the formalism with θ_3 can be given (Sec. 8.2.2, p. 201), proving realizability.

where

$$\eta_{\boldsymbol{k}} \doteq \mathrm{i}\omega_{\boldsymbol{k}} + \eta^{\mathrm{nl}}_{\boldsymbol{k}}. \tag{371}$$

Note that this evolves to the steady state

$$\theta_{\boldsymbol{kpq}}(\infty) = [\eta_{\boldsymbol{k}}(\infty) + \eta_{\boldsymbol{p}}(\infty) + \eta_{\boldsymbol{q}}(\infty)]^{-1} \tag{372}$$

(assuming that $\Delta\eta \doteq \eta_{\boldsymbol{k}} + \eta_{\boldsymbol{p}} + \eta_{\boldsymbol{q}} > 0$ as $t \to \infty$).

Because the basic symmetries of both WTT and the DIA are preserved by this Markovian procedure, the nonlinear terms still conserve the appropriate quadratic invariants.[223] Thus Gibbs distributions based on those invariants are inviscid solutions of the closure. However, there is no guarantee that nonquadratic invariants will be conserved. One interesting case is the generalization of the three-wave model introduced in Sec. 5.10.3 (p. 144) to include real linear frequencies Ω_k. The M's can be arranged so that the nonlinear terms conserve both energy and enstrophy. As noted in Sec. 5.10.3 (p. 144), the Hamiltonian $\widetilde{H}$ [Eq. (256)] is also conserved, both exactly and by the DIA. However, that invariant is destroyed by the EDQNM (as well as the RMC to be derived below). The final Markovian covariances, obeying *two*-parameter Gibbs distributions, are therefore quantitatively in error for nonzero frequencies[224] (Bowman, 1992; Bowman et al., 1993).

The three-wave model can further be generalized to include linear growth rates γ_k. For the EDQNM the predicted steady states can be obtained analytically, as was shown by Ottaviani (1991); the calculation is presented in Appendix J (p. 297). The results illustrate that constraints on the growth rates are required in order that steady states exist [for example, $\sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}} < 0$, a constraint that emerges more generally in the theory of entropy evolution (Sec. 7.2.3, p. 188)], and provide a special case of growth-rate scalings for steady-state intensities I that can be deduced from general considerations. Specifically,

$$I \sim \begin{cases} \gamma & \text{(weak-turbulence theory)} \\ \gamma^2 & \text{(strong-turbulence theory).} \end{cases} \tag{373}$$

Further discussion of these scalings is given in Appendix J (p. 297).

The EDQNM closure is readily computable at least for small numbers of modes. Orszag (1977) proved that the EDQNM for NS turbulence is realizable. Unfortunately, that turns out to be false when linear waves are present. I shall take up this important issue in Sec. 8.2.3 (p. 203).

The forced Burgers equation (19), which arises in various physical contexts, including the theory of self-organized criticality (Sec. 12.4, p. 241), serves as a useful illustration of calculations with the EDQNM. I shall consider specifically $d = 1$. Such models were originally studied by Forster, Nelson, and Stephen (FNS; Forster et al., 1976, 1977). They considered various forcing scenarios

[223] Conservation of quadratic nonlinear invariants is true for a single scalar field. Generalization of the DIA-based EDQNM to systems of multiple coupled fields is not entirely straightforward (Bowman, 1992). Bowman (1992) and Bowman et al. (1993) discussed the possibility of a realizable multiple-field EDQNM [see discussion in Sec. 8.2.3 (p. 203)]; however, the resulting form did not conserve all nonlinear invariants in general.

[224] When all frequencies vanish, $\widetilde{H}$ vanishes identically and the two-parameter Gibbs PDF's are adequate.

defined by the spectral characteristics of f^{ext}, i.e., by $F^{\text{ext}}_{k,\omega}$, where

$$F^{\text{ext}}(\rho,\tau) \doteq \langle f^{\text{ext}}(x+\rho, t+\tau) f^{\text{ext}}(x,t)\rangle = F^{\text{ext}}(\rho)\delta(\tau). \tag{374}$$

In *Model A* the noise is conservative: $F^{\text{ext}}_{k,\omega} = 2k^2 D$. In *Model B* it is nonconservative: $F^{\text{ext}}_{k,\omega} = F^{\text{ext}}_0 =$ const. In *Model C* it is band-limited (and shown to have infrared characteristics identical to those of Model A). For the NSE, Model A can be considered to be the description of fluctuations in thermal equilibrium [the forcing being made self-consistent with the aid of the fluctuation–dissipation theorem (Sec. 3.7.1, p. 67)], and one can make contact with the theory of convective cells and long-time tails (Sec. 5.10.1, p. 143). Here I shall consider Model B. Forster *et al.* did not study that model explicitly for the Burgers equation; however, some of their results can be extrapolated to that case. Medina et al. (1989) studied the Burgers Model B implicitly through their analysis of the KPZ equation (20) with long-ranged forcing, where the power spectrum of f^{ext}_h scales at small k as $k^{-2\rho}$. Since the KPZ equation for the height fluctuation h of an interface can be transformed to the Burgers equation for velocity u by $u = -\partial_x h$, Model B corresponds to $\rho = 1$. Hwa and Kardar (HK; 1992) studied the Burgers Model B explicitly. Diamond and Hahm (DH; 1995) reiterated many of the results of HK in the context of a discussion of self-organized criticality (Sec. 12.4, p. 241).

The present EDQNM calculation was originally given by Krommes (2000a). Let us focus on the long-wavelength behavior of $R_{k,\omega}$, which describes the mean propagation of small pulses. From Eq. (19), one has $M_{kpq} = -\mathrm{i}k$. In steady state one has $R_{k,\omega} = [-\mathrm{i}(\omega + \mathrm{i}\eta_k)]^{-1}$ and can use the asymptotic result (372); then Eq. (368a) can be self-consistently solved for the nonlinear damping η^{nl}_k once the spectrum C_q is known. The steady-state spectral balance equation is

$$\operatorname{Re}\eta_k C_k = F^{\text{nl}}_k + F^{\text{ext}}_k \equiv F_k. \tag{375}$$

Detailed analysis of the k dependence of F^{nl}_k shows (Krommes, 2000a) that in Model B and for $d = 1$, the internal noise is not negligible as $k \to 0$; rather, it renormalizes F^{ext}_k to give a total forcing F_k that approaches a constant as $k \to 0$. Then Eq. (368a) becomes

$$\eta^{\text{nl}}_k = \int \frac{\mathrm{d}q}{2\pi}\, k(k+q) \left(\frac{1}{\eta^{\text{nl}}_k + \eta^{\text{nl}}_{k+q} + \eta^{\text{nl}}_q}\right)\left(\frac{F_q}{\eta^{\text{nl}}_q}\right), \tag{376}$$

where one has anticipated that the classical dissipation is negligible. That can be justified by noting that (i) because of the last factor of $1/\eta^{\text{nl}}_q$, the q integral in Eq. (376) is dominated by the small q's; and (ii) η^{nl}_k will be $O(k)$. Then upon scaling q to k, one readily finds that

$$\eta^{\text{nl}}_k = |k|\overline{V}, \quad \overline{V} \sim F_0^{1/3}[\ln(k/k_{\min}) + \tfrac{1}{2}(k_{\min}/k)]. \tag{377a,b}$$

For further discussion of this *ballistic scaling* ($\eta^{\text{nl}} \propto |k|$), see Sec. 12.4 (p. 241).

The scaling $\eta^{\text{nl}}_k \propto |k|$ agrees with general results of FNS, Medina *et al.*, and HK. Forster *et al.* discussed the form

$$R^{-1}_{k,\omega} \sim |k|^z g_R(\omega/|k|^z) \tag{378}$$

for some undetermined scaling function g_R and calculated z for the forced NSE. If their result is extrapolated to $d = 1$ and applied to the Burgers equation, one finds $z = 1$. That exponent agrees with an explicit result of HK as well as the $\rho = 1$ result that can be extracted from the work of

Medina et al. (1989). It disagrees with the Ansatz $\eta_k^{\mathrm{nl}} = k^2 D$ used by DH. Accordingly, the proper, benign dependence on $k_{\min}$ [Eq. (377b)] is much weaker than the divergence $(\int dq/q^4)^{1/3}$ found by DH. Although the k^2D Ansatz is ubiquitous in resonance-broadening theory (Sec. 4.3, p. 108), it is clearly not universally applicable.

This example illustrates the use of the spectral balance equation and the expression for the renormalized nonlinear damping in a simple context; the predictions are sensible and correct [225] for $k \to 0$. Calculations for large, inertial-range k's are more problematical because the DIA-based EDQNM is not RGI, inheriting that failing from the DIA itself. As I have previous discussed, this need not be a serious issue for many problems in plasma physics in which studies of the energy-containing part of the spectrum are paramount. A Markovian closure that is RGI is the test-field model, discussed in the next section.

An important paper on the EDQNM was by Fournier and Frisch (1978), who discussed its properties analytically continued into noninteger spatial dimension d. A key result was the existence of a critical dimension $d_c \approx 2.05$ at which the direction of the energy cascade changes sign. See the related remarks in footnote 101 (p. 74).

7.2.2 Test-field model

Although the DIA-based EDQNM is realizable at least in the absence of waves, it is an Eulerian description that possesses the same difficulty with RGI as does the DIA itself. Kraichnan (1971a) proposed the *test-field model* (TFM) to rectify this difficulty; the method was reviewed by Sulem et al. (1975). In the TFM the rate of interaction of an eddy with an advecting velocity field $\boldsymbol{u}$ is estimated from the rate at which advection couples the solenoidal (S) and compressive (C) components of a test velocity field $\boldsymbol{v}$ (in the absence of pressure); by definition, $\boldsymbol{\nabla} \cdot \boldsymbol{v}^S = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{v}^C = \boldsymbol{0}$. For 2D incompressible flow, for which $\boldsymbol{u}$ is derivable from a potential φ according to [226] $\boldsymbol{u} = \widehat{\boldsymbol{z}} \times \boldsymbol{\nabla}\varphi$, those couplings obey (in the absence of linear waves)

$$(\partial_t + \nu_{\boldsymbol{k}})\boldsymbol{v}_{\boldsymbol{k}}^S = \mathsf{P}^S(\boldsymbol{k}) \cdot \sum_{\Delta} M_{\boldsymbol{kpq}}^U \varphi_{\boldsymbol{p}}^* \boldsymbol{v}_{\boldsymbol{q}}^{C*}, \quad (\partial_t + \nu_{\boldsymbol{k}})\boldsymbol{v}_{\boldsymbol{k}}^C = \mathsf{P}^C(\boldsymbol{k}) \cdot \sum_{\Delta} M_{\boldsymbol{kpq}}^U \varphi_{\boldsymbol{p}}^* \boldsymbol{v}_{\boldsymbol{q}}^{S*}, \qquad \text{(379a,b)}$$

where

$$\mathsf{P}^S(\boldsymbol{k}) \doteq \mathsf{I} - \mathsf{\Pi}(\boldsymbol{k}), \quad \mathsf{P}^C(\boldsymbol{k}) \doteq \mathsf{\Pi}(\boldsymbol{k}), \quad \mathsf{\Pi}(\boldsymbol{k}) \doteq \widehat{\boldsymbol{k}}\,\widehat{\boldsymbol{k}}. \qquad \text{(380a,b,c)}$$

A Markovian renormalization of these equations can be done straightforwardly; the result retains the form of Eqs. (367) and (368), but involves a modified evolution equation for $\theta_{\boldsymbol{kpq}}$. In terms of the modified mode-coupling coefficients

$$M_{\boldsymbol{kpq}}^G \doteq [\mathsf{P}(\boldsymbol{k}) : \mathsf{\Pi}(\boldsymbol{q})]^{1/2} M_{\boldsymbol{kpq}}^U, \qquad \text{(381)}$$

[225] There is an important reason why for the present problem DIA-based statistical closures (i.e., with no vertex corrections) provide reasonable results for small k's. As stressed by FNS, the Burgers equation is invariant under the Galilean transformation $\overline{x} = x + u_0 t$ and $\overline{u} = u + u_0$, where u_0 is a constant. This invariance can be shown to imply that vertex corrections vanish as $k \to 0$. Formally, it is a consequence of a Ward identity (Sec. 6.2.5, p. 165).

[226] Such a velocity field is always solenoidal. Thus in 2D the method does not attempt to postulate a modified equation for φ, but directly evolves a test velocity field. Such an advected field can develop a compressive part.

one finds

$$\partial_t \theta_{\boldsymbol{kpq}} + (\eta^S_{\boldsymbol{k}} + \eta^S_{\boldsymbol{p}} + \eta^S_{\boldsymbol{q}})\theta_{\boldsymbol{kpq}} = 1, \quad \theta_{\boldsymbol{kpq}}(0) = 0, \tag{382a,b}$$

$$\eta^S_{\boldsymbol{k}} \doteq \nu_{\boldsymbol{k}} + A \sum_{\Delta} M^G_{\boldsymbol{kqp}} M^{G*}_{\boldsymbol{pqk}} \theta^{G*}_{\boldsymbol{pqk}} C_{\boldsymbol{q}}, \quad \eta^C_{\boldsymbol{k}} \doteq \nu_{\boldsymbol{k}} + \lambda A \sum_{\Delta} M^G_{\boldsymbol{kqp}} M^{G*}_{\boldsymbol{pqk}} \theta^{G*}_{\boldsymbol{pqk}} C_{\boldsymbol{q}}, \tag{382c,d}$$

$$\partial_t \theta^G_{\boldsymbol{kpq}} + (\eta^C_{\boldsymbol{k}} + \eta^S_{\boldsymbol{p}} + \eta^S_{\boldsymbol{q}})\theta^G_{\boldsymbol{kpq}} = 1, \quad \theta^G_{\boldsymbol{kpq}}(0) = 0. \tag{382e,f}$$

Here A is an undetermined constant often taken to be 1,[227] and λ is the number of solenoidal components associated with each compressive component ($\lambda = 1$ in 2D[228]).

Random Galilean invariance of these equations follows from the presence of the coefficient $\mathsf{P}(\boldsymbol{k}) : \mathsf{\Pi}(\boldsymbol{p})$ in the expressions for $\eta^S_{\boldsymbol{k}}$ and $\eta^C_{\boldsymbol{k}}$; note that $\mathsf{P}(\boldsymbol{k}) \cdot \mathsf{\Pi}(-\boldsymbol{k} - \boldsymbol{q}) \to 0$ as $\boldsymbol{q} \to 0$ because $\mathsf{P}(\boldsymbol{k})$ and $\mathsf{\Pi}(\boldsymbol{k})$ are orthogonal. As $k \to 0$ that coefficient introduces two extra powers of $p \sim q$, hence the long-wavelength contributions to η^S and η^C are proportional to the mean-square shear in those wavelengths rather than to the fluctuation energy itself. In 2D $\mathsf{P}(\boldsymbol{k}) : \mathsf{\Pi}(\boldsymbol{p}) = \sin^2(\boldsymbol{k}, \boldsymbol{p})$.

Kraichnan proved that the TFM is realizable for Hermitian linear damping by demonstrating a Langevin representation. Holloway and Hendershott (1977) extended the calculation (but not the proof of realizability!) to include linear waves (complex $\nu_{\boldsymbol{k}}$),[229] and made detailed closure calculations for Rossby waves that are relevant for HM dynamics. However, those authors ignored the contributions of $\operatorname{Im} \eta^{S,C}_{\boldsymbol{k}}$ (nonlinear frequency shifts) to the triad interaction times.

For further discussion of and references to the TFM, see Bowman and Krommes (1997). Those authors showed that the TFM is not realizable in the presence of linear waves and proposed a realizable modification; see Sec. 8.2 (p. 201).

7.2.3 Entropy and an H theorem for Markovian closures

Carnevale et al. (1981) showed that the nonlinear terms of a wide class of Markovian closures possess an *H theorem*[230] such that in the absence of forcing and dissipation a particular entropy functional S increases monotonically in time, achieving a maximum value at absolute thermal equilibrium. Specifically, considerations based on information theory [see also Jaynes (1965)] lead to $S(t) \doteq \frac{1}{2} \ln \det \mathsf{C}(t)$, where C is the correlation matrix. For homogeneous turbulence and a single field, this becomes $S(t) = \frac{1}{2} \sum_{\boldsymbol{k}} \ln C_{\boldsymbol{k}}(t)$. In the inviscid limit it is proven that $S(t) \geq S(0)$.

To demonstrate this result explicitly from a DIA-based Markovian closure, use Eqs. (367)

[227] Kraichnan (1971a) suggested that A could be calculated by comparing the TFM with the predictions of the DIA for the interactions of dynamically identical modes of comparable k. For 3D turbulence he found $A = 1.064$, a value that gave reasonable predictions for the Kolmogorov constant.

[228] Note that in 3D $\eta^C_{\boldsymbol{k}} \neq \eta^S_{\boldsymbol{k}}$, so $\theta^G_{\boldsymbol{kpq}}$ is symmetrical only in its last two arguments; however, in 2D $\eta^C_{\boldsymbol{k}} = \eta^S_{\boldsymbol{k}}$ and $\theta^G_{\boldsymbol{kpq}}$ is completely symmetrical.

[229] Those authors used $\operatorname{Re} \theta_{\boldsymbol{kpq}}$ rather than just $\theta_{\boldsymbol{kpq}}$ everywhere in their definitions of the various η^{nl}'s; thus they did not consider nonlinear frequency shifts.

[230] Boltzmann's original H function was defined by $H = \mathcal{N} \int \mathrm{d}z f \ln f$, where $\mathcal{N}$ is a positive normalization constant, $z \doteq \{\boldsymbol{q}, \boldsymbol{p}\}$, and $f(z)$ is the one-particle PDF. Boltzmann proved that for his collision operator H decreases until a Maxwellian distribution is obtained; that same property holds for the Balescu–Lenard operator. A lucid interpretation of Boltzmann's H was given by Jaynes (1965). Although entropy functionals S are the negatives of H functionals, one still refers to "H theorems" even when discussing entropies.

and (368) for symmetrical $\theta_{\boldsymbol{kpq}}$ to calculate $\dot{S}$. After straightforward manipulations, one finds

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}} + \frac{1}{6} \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}} \operatorname{Re} \theta_{\boldsymbol{kpq}} C_{\boldsymbol{k}} C_{\boldsymbol{p}} C_{\boldsymbol{q}} \left| \frac{M_{\boldsymbol{kpq}}}{C_{\boldsymbol{k}}} + \text{c.p.} \right|^2, \tag{383}$$

where $C_{\boldsymbol{k}} \equiv C_{\boldsymbol{k}}(t)$. For $\gamma_{\boldsymbol{k}} = 0$, one has[231] $\dot{S} \geq 0$. $\dot{S}$ vanishes only when Eq. (244) is satisfied; as was shown in Sec. 5.4 (p. 133), this leads to the Gibbs equilibria.

Define $\Delta\gamma \doteq \sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}}$. For $\Delta\gamma > 0$ Eq. (383) shows that no steady state is possible; this can be interpreted as the inexorable expansion of phase-space volume elements. Nonequilibrium steady states are possible only for $\Delta\gamma < 0$ (Carnevale and Holloway, 1982; Horton, 1986). An explicit example of this constraint is given by the solution of the EDQNM for three modes, discussed in Appendix J (p. 297).

If such manipulations are repeated for the DIA, the last term of Eq. (383) is replaced by

$$\begin{aligned} &\frac{1}{6} \operatorname{Re} \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}} \int_0^t \mathrm{d}\bar{t}\, R_{\boldsymbol{k}}(t;\bar{t}) R_{\boldsymbol{p}}(t;\bar{t}) R_{\boldsymbol{q}}(t;\bar{t}) C_{\boldsymbol{k}}(t) C_{\boldsymbol{p}}(t) C_{\boldsymbol{q}}(t) \\ &\qquad \times \left(\frac{M_{\boldsymbol{k}}}{C_{\boldsymbol{k}}(t)} + \text{c.p.} \right) \left(\frac{M_{\boldsymbol{k}}}{C_{\boldsymbol{k}}(t)} c_{\boldsymbol{p}}(t;\bar{t}) c_{\boldsymbol{q}}(t;\bar{t}) + \text{c.p.} \right)^*, \end{aligned} \tag{384}$$

where $c_{\boldsymbol{k}}(t;\bar{t}) \doteq C_{\boldsymbol{k}}(t;\bar{t})/[R_{\boldsymbol{k}}(t;\bar{t}) C_{\boldsymbol{k}}(t)]$. In thermal equilibrium the FDT guarantees that $c(\tau) = 1$, so one recovers a positive-semidefinite form that again vanishes for the Gibbs equilibria. Otherwise, however, the sign of Eq. (384) is indeterminate. The DIA does not possess a monotonic H theorem, a consequence of its temporal nonlocality. This same remark holds for the realizable Markovian closure discussed in Sec. 8.2.3 (p. 203).

A discussion of entropy in the context of predictability theory [defined in footnote 165 (p. 130)] was given by Carnevale and Holloway (1982). Some applications of maximum-entropy procedures to plasma-turbulence models were reviewed by Montgomery (1985).

7.3 Eddy viscosity, large-eddy simulations, and the interactions of disparate scales

"Perhaps the principal achievement of DIA-type theories is that they deduce from the equations of motion a generalized dynamical damping which embodies the idea of eddy viscosity but takes account of nonlocalness in space and time due to the absence of clean scale separations in turbulence." — *Kraichnan and Chen (1989).*

Statistical renormalization of a multiplicatively nonlinear equation leads to closed equations for correlation and response functions or PDF's that in principle apply to all excited scales of the turbulence. Physically, however, some scales behave very differently from others. For example, large-scale coherent structures may importantly influence macroscale transport properties but may be poorly represented by standard moment-based closures.

A more practical difficulty with a totally closure-based approach is that it is very difficult to deal with realistic geometry, for which boundary conditions and/or shapes may be inhospitable to

[231] It is required that $\operatorname{Re} \theta_{\boldsymbol{kpq}} > 0$. That will be the case provided that the closure is realizable; see further discussion in Sec. 8.2 (p. 201).

the usual Fourier decomposition. One can in principle proceed in $\boldsymbol{x}$ space (Kraichnan, 1964c; Martin et al., 1973), but the resulting equations are generally extremely complicated.

A long-recognized alternate approach is *large-eddy simulation* (LES), in which a particular class of modes (usually the large-scale, energy-containing ones) is simulated directly, while the effects of the unresolved modes are estimated by a statistical approximation. If the unresolved modes are at short wavelengths, it is not unreasonable to believe that their principle effect is to produce an *eddy viscosity* in the equation for the resolved modes. The subject has a long history. Smagorinsky (1963) concluded on the basis of heuristic arguments that $\mu_{\text{eddy}} = Cd^2S$, where d is the subgrid scale, $S \doteq \sqrt{\mathsf{S}\cdot\mathsf{S}}$ is the local strain rate,[232] and C is an undetermined numerical constant. This result, although often successful in practice, clearly does not take into account the details of fluctuations and transfer in k space. Further discussion of the Smagorinsky and more elaborate sub-grid-scale models can be found in Yoshizawa et al. (2001).

I have already noted in Sec. 5.7 (p. 140) that an eddy viscosity can be extracted from the DIA, given an appropriate separation of space and time scales. That does not generally exist, however, so the concept of eddy viscosity must be reexamined. Important theoretical advances were made by Kraichnan (1976b). For isotropic turbulence he defined the quantity[233] $\nu(q \mid k_m)$, where k_m is the cutoff wave number that separates the resolved modes ($q < k_m$) from the unresolved ones. If $\nu(q \mid k_m)$ is to have the usual behavior of an eddy viscosity, it should become independent of q for $q \ll k_m$, and that is consistent with the behavior of the closures. However, Kraichnan pointed out that the usual interpretation breaks down for the description of wavelengths in the vicinity of the boundary between resolved and unresolved modes, where for $q \sim k_m$ $\nu(q \mid k_m)$ exhibits a cusp because of the close competition between coherent damping and incoherent drive, better described as a diffusion in wave-number space than as an eddy damping. Explicit calculations based on the TFM (Sec. 7.2.2, p. 187) permitted a rich and detailed description of the transfer processes.

One important qualitative result is that when the analysis is applied to energy transfer for the 2D NSE, the appropriate eddy viscosity is *negative* for $q \ll k_m$. This is a manifestation of the inverse cascade (Sec. 3.8.3, p. 74); it signals the breakdown of simple heuristic ideas about eddy viscosity for 2D flow.

7.3.1 Eddy viscosity for Hasegawa–Mima dynamics

To be specific, I sketch the analysis for HM dynamics. Chechkin et al. (1998) attempted to compute an eddy viscosity for this situation, following earlier work of Montgomery and Hatori (1984) for 2D NS flows. However, those authors considered the unrealistic situation of frozen short-wavelength statistics and obtained results that did not properly reduce to Kraichnan's 2D NS formula. Krommes and Kim (2000) reconsidered the problem allowing for self-consistent interactions between long and short scales, and I follow that work here (the work of Chechkin *et al.* is discussed in the appendix of that paper). For simplicity I shall assume that the spectrum is isotropic although this is not strictly correct in the presence of a diamagnetic frequency. [The assumption will be relaxed in Sec. 7.3.2

[232] The rate-of-strain tensor S is defined by Eq. (10a). The eddy turnover rate ν_{eddy} of an eddy of size d due to a sheared flow can be estimated as $d^{-1}(Sd) = S$ (note that S vanishes for rigid rotation). Smagorinsky's result is the only dimensionally correct diffusion coefficient that can be constructed from ν_{eddy} and d.

[233] I have changed the wave-number notation slightly to conform to current practice in plasma physics. Namely, following the work of Krommes and Kim (2000) and references therein, I use $\boldsymbol{q}$ for the small or resolved wave numbers and $\boldsymbol{k}$ for the large or unresolved wave numbers.

(p. 193).] Then the energy spectrum is $E(k) = \pi k U(k)$, where $U(k) \doteq (1 + k^2)\langle\delta\varphi^2\rangle(k)$. Nonlinear contributions to $E(k)$ give

$$\partial_t E(k) = T(k) = \frac{1}{2}\int_\Delta \mathrm{d}p\,\mathrm{d}q\,\mathcal{T}(k,p,q), \tag{385}$$

where $\mathcal{T}(k,p,q)$ is defined by Eq. (119). Now divide $T(k)$ into resolved plus unresolved contributions. The total transfer into the unresolved modes is

$$\mathcal{T}(k_m) = \int_{k_m}^{\infty}\mathrm{d}k\,T(k) = -\int_0^{k_m}\mathrm{d}q\,T(q\,|\,k_m), \tag{386}$$

where $T(q\,|\,k_m) \doteq \frac{1}{2}\int_\Delta' \mathrm{d}k\,\mathrm{d}p\,\mathcal{T}(q,k,p)$, the prime indicating that k and/or p lie above k_m. It is then consistent to write

$$\partial_t E(q) = (\text{resolved statistics}) + T(q\,|\,k_m) \tag{387}$$

and to rigorously define a generalized eddy viscosity $\nu(q\,|\,k_m)$ by

$$T(q\,|\,k_m) = -2\nu(q\,|\,k_m)q^2 E(q). \tag{388}$$

In a Markovian closure the desired triplet correlation is approximated by the right-hand side of Eq. (210). That expression can be transformed by eliminating the $M_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ in the $F_{\boldsymbol{k}}^{\mathrm{nl}}$ noise term with the aid of the detailed conservation property (184) (with $\sigma_{\boldsymbol{k}} = 1$ for energy dynamics). The final result is

$$T(q\,|\,k_m) = 2\pi q^3 \int_\Delta' \mathrm{d}k\,\mathrm{d}p\,b(q,k,p)[\theta_{qkq}U(k) - \theta_{kpq}U(q)]U(k), \tag{389a}$$

$$b(q,k,p) \doteq \frac{2|\sin(\boldsymbol{k},\boldsymbol{p})|k^2p^2(k^2-p^2)(q^2-p^2)}{q^2(1+k^2)(1+p^2)(1+q^2)}. \tag{389b}$$

[The θ_{kpq} here and in subsequent formulas must be evaluated from a renormalization that is random-Galilean-invariant (Kraichnan, 1976b; Krommes and Kim, 2000), such as the TFM discussed in Sec. 7.2.2 (p. 187).] This formula can be evaluated numerically (Kraichnan, 1976b) for any given spectrum $U(k)$ such as the Kolmogorov $k^{-5/3}$ law. A more explicit analytical result can be obtained by considering the limit $q \ll k_m$ (where the conventional notion of an eddy viscosity would be expected to apply). Then because of the restrictions that k and/or p be greater than k_m, one has $k \gg q$, $p \gg q$, and $|k - p| = O(q)$. This allows functions of p to be expanded around k, and the p integration to be performed. The integration domain is shown in Fig. 24 (p. 192); the final result, correct to lowest order in q^2, is [234]

$$\nu(q\,|\,k_m) = \frac{\pi}{4}\left(\frac{q^2}{1+q^2}\right)\int_{k_m}^{\infty}\mathrm{d}k\,\left(\frac{k^2}{1+k^2}\right)^2\theta_{qkk}\frac{\mathrm{d}W(k)}{\mathrm{d}k}, \tag{390}$$

[234] Many details of the algebra leading to this and similar results may be found in the comprehensive paper by Krommes and Kim (2000). The 2D NS limit studied by Kraichnan (1976b) can be recovered by taking all wave numbers to be much larger than 1; one then obtains Kraichnan's Eq. (4.6).

where $W(k) \doteq k^2 U(k)$ is the enstrophy density conserved by the polarization-drift nonlinearity [see Eqs. (50)]. This function is clearly negative if $W(k)$ decreases for $k > k_m$. More generally, it is negative if θ_{qkk} is sensibly independent of k, as can be shown by an integration by parts (Krommes and Kim, 2000). The appearance of W is consistent with the discussion by Lebedev et al. (1995), who showed that W is conserved when a drift-wave gas is modulated by a long-wavelength convective cell. A more general discussion of conservation properties was given by Smolyakov and Diamond (1999); see also the material in Sec. 12.7 (p. 248) and Appendix F (p. 286).

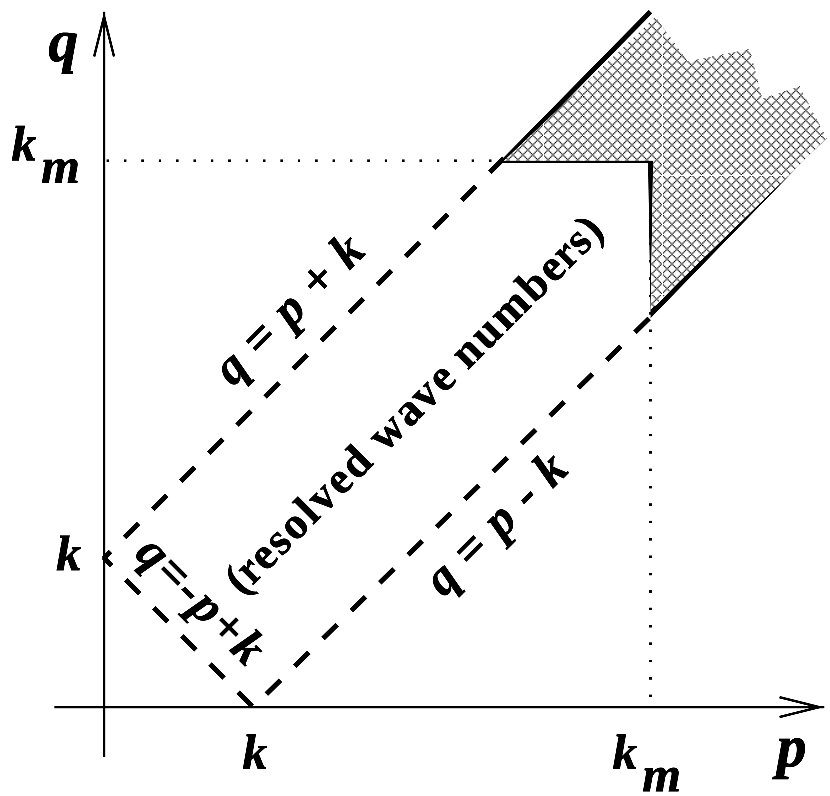

Fig. 24. Integration domain (shaded) for the contribution of the unresolved modes to $T(k \mid k_m)$.

Analogous calculations can be performed for enstrophy transfer. For that situation Kraichnan concluded that the concept of eddy viscosity was even less justified.

It should be noted that the eddy viscosity defined above is intended to be used in an equation for the statistically averaged spectrum. In a single realization, explicit scales are randomly excited by the subgrid scales (this effect, sometimes called *stochastic backscatter*, is due to the incoherent noise). Only on the average is that excitation appropriately represented by the statistical analysis above, as discussed with the aid of a model problem by Rose (1977). Nevertheless, Kraichnan's eddy viscosity has been used successfully in a number of simulations, as reviewed together with a number of other recent approaches by Lesieur and Métais (1996).

Smith (1997) and Smith and Hammett (1997) considered aspects of eddy viscosity for particular plasma problems of current relevance. Smith made numerical calculations of $\mu_{\rm eddy}$ by numerically computing the transfer across a cutoff wave number k_m interior to the domain of a very-well-resolved simulation; he verified the expected cusp in $\mu_{\rm eddy}(q)$ as $q \to k_m$. He then discussed parametrizations of hyperviscosity [damping $\propto (q/k_m)^{2p}$ for $p > 1$] suggested by the numerical simulations. Although instructive, such parametrizations are difficult because of the nonuniversal, anisotropic, and wavelike nature of typical plasma problems.

To conclude the discussion of eddy viscosity as applied to LES, I quote Kraichnan (1976b):

"We feel, therefore, that the theoretical basis for the use of simple eddy viscosities to represent subgrid scales is substantially insecure. Why then have they worked so well in practice? Apparently this is largely because the flow has built-in compensatory mechanisms. The effect of a crude and inaccurate term to represent the passage of energy or enstrophy through the boundary at k_m has the principal effect of distorting the flow in a relatively restricted wavenumber range below k_m. It remains to be seen whether the use of more accurate, and thereby more complicated, representations of subgrid scales pays off. For it to do so, the increased accuracy near k_m, and thereby the implied possibility of lowering k_m in a given calculation, must overbalance the added computational load of carrying the more complex subgrid-scale representation. "

This remark may be particularly relevant to the anisotropic situations characteristic of plasma turbulence. The complexity of a satisfactory analytical representation may be prohibitive although further research is required.

7.3.2 Energy conservation and the interaction of disparate scales

Now consider the consequences of eddy viscosity for the spectral balance equation rather than for LES. Energy input into the small q's due to large-k interactions must show up as a drain of energy from the large k's; it is instructive to verify that directly. Consider the evolution of $W^{>}(k)$ (where $>$ denotes large argument) due to interactions with small wave numbers of maximum size $q_{\max}$. The integration domain, depicted in Fig. 3 of Krommes and Kim (2000), consists of two subdomains: region A (q small); and region B (p small), which is found to contribute at higher order in q^2. Integration over region A leads at lowest order in k^2 to the enstrophy diffusion equation

$$\partial_t W^{>}(k) = \frac{1}{k}\frac{\partial}{\partial k}\left(k D_k \frac{\partial W^{>}}{\partial k}\right), \quad \text{where} \quad D_k \doteq k^6 \int_0^{q_{\max}} \mathrm{d}q\, q^2 W^{<}(q)\theta_{k,k,q}. \tag{391a,b}$$

Integration of Eq. (391a) over all k's with $k \geq k_m$ shows that the interaction with the small k's conserves the large-k enstrophy $W^{>}$ except for the boundary term: $\partial_t W^{>} = -2\pi k_m (k_m^2 D_{k_m})\partial_k W^{>}(k_m)$. That $W^{>}$ is conserved is consistent with the results of Lebedev et al. (1995) and Smolyakov and Diamond (1999). Energy, however, is not conserved. Upon dividing Eq. (391a) by k^2, one finds

$$\partial_t U^{>}(k) = \frac{1}{k^3}\frac{\partial}{\partial k}\left(k D_k \frac{\partial W^{>}}{\partial k}\right) = \frac{1}{k}\frac{\partial}{\partial k}\left(\frac{D_k}{k}\frac{\partial W^{>}}{\partial k}\right) + \frac{2D_k}{k^3}\frac{\partial W^{>}}{\partial k}, \tag{392a,b}$$

where the first term has been put into conservative form. Upon integrating this result over the large k's and recalling formulas (391b) and (390), one readily finds

$$\partial_t E^{>} = -2\pi k_m D_{k_m} \partial_k W^{>}(k_m) + 2\int_0^{q_{\max}} q\, \mathrm{d}q\, \nu(q \mid k_m) q^2 U^{<}(q); \tag{393}$$

the last term is precisely the required energy drain into the small k's.

The lowest-order result $\partial_t E(q) = -2\nu(q \mid k_m) q^2 E(q)$ is missing an incoherent-noise contribution involving a product such as $U^{>}(k)U^{>}(p) \approx [U^{>}(k)]^2$, the effect being of higher order in q^2. A direct higher-order calculation is extremely tedious, but the effect can be extracted simply by noting that for the contribution of the $U(k)U(p)$ term in Eq. (389a), formula (389b) can be symmetrized (Krommes and Kim, 2000). Further calculation (see the general anisotropic results below) shows that the energy input to the small q's is accounted for by the contribution from region B to the large-k evolution. Thus one has demonstrated that the interactions between the large and small scales are appropriately energy-conserving.

It is unnecessary to assume isotropic spectra; anisotropic generalizations can be obtained by expanding the general nonlinear contribution $\partial_t W_{\boldsymbol{q}} = \sum_{\boldsymbol{k},\boldsymbol{p}} \delta_{\boldsymbol{q}+\boldsymbol{k}+\boldsymbol{p}} K(\boldsymbol{q},\boldsymbol{k},\boldsymbol{p})$ (where K is a known function) in powers of the small wave vectors. One must be cautious, however. If one performs the sum over $\boldsymbol{p}$ to find the asymmetrical form $\partial_t W_{\boldsymbol{q}} = \sum_{\boldsymbol{k}} K(\boldsymbol{q}, -(\boldsymbol{q}+\boldsymbol{k}), \boldsymbol{k})$ and if $\boldsymbol{k}$ is considered to be large, then large-scale contributions arise from not only small $\boldsymbol{q}$ but also large $\boldsymbol{q}$ [i.e., small $\boldsymbol{p} = -(\boldsymbol{k}+\boldsymbol{q})$], and those contributions partially cancel. That cancellation is handled automatically in the isotropic formulas involving the integration $\sum_\Delta \mathrm{d}p\,\mathrm{d}q$ over the symmetrical domain Δ. In any event, the final results (Krommes and Kim, 2000) are

$$\partial_t W^>_{\boldsymbol{k}} = \frac{\partial}{\partial \boldsymbol{k}} \cdot \mathsf{D}_{\boldsymbol{k}} \cdot \frac{\partial W^>_{\boldsymbol{k}}}{\partial \boldsymbol{k}} - \frac{\partial}{\partial \boldsymbol{k}} \cdot (\boldsymbol{V}_{\boldsymbol{k}} W_{\boldsymbol{k}}), \qquad \partial_t W^<_{\boldsymbol{q}} = 2\gamma^{\mathrm{nl}}_{\boldsymbol{q}} W_{\boldsymbol{q}} + \dot{W}^{\mathrm{noise}}_{\boldsymbol{q}}, \tag{394a,b}$$

$$\mathsf{D}_{\boldsymbol{k}} \doteq \frac{1}{4} k^4 \sum_{\boldsymbol{q}} k_y^2 q^2 W_{\boldsymbol{q}} \theta^{(E)}_{\boldsymbol{k},-\boldsymbol{k},\boldsymbol{q}} (\widehat{\boldsymbol{q}}\,\widehat{\boldsymbol{q}}), \qquad V_{\boldsymbol{k}} \doteq \frac{1}{2}\Big(\sum_{\boldsymbol{q}} \widehat{\boldsymbol{q}}\, k_y^2 k_x q^4 \theta^{(E)}_{\boldsymbol{k},\boldsymbol{q},-\boldsymbol{k}}\Big) W_{\boldsymbol{k}}. \tag{394c,d}$$

$$\gamma^{\mathrm{nl}}_{\boldsymbol{q}} \doteq -\frac{1}{4} q^4 \sum_{\boldsymbol{k}} k_y^2 k_x \theta^{(E)}_{\boldsymbol{q},-\boldsymbol{k},\boldsymbol{k}} \widehat{\boldsymbol{q}} \cdot \frac{\partial W_{\boldsymbol{k}}}{\partial \boldsymbol{k}}, \qquad \dot{W}^{\mathrm{noise}}_{\boldsymbol{q}} \doteq q^2 q^4 \sum_{\boldsymbol{k}} \left(\frac{k_x^2 k_y^2}{\overline{k}^4}\right) \theta^{(E)}_{\boldsymbol{q},-\boldsymbol{k},\boldsymbol{k}} W^2_{\boldsymbol{k}}, \tag{394e,f}$$

where $\theta^{(E)}_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \doteq 8\,\mathrm{Re}\,\theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}/[(1+k^2)(1+p^2)(1+q^2)]$. These formulas reduce correctly to the isotropic results deduced above and obey the energy conservation theorems

$$-\sum_{\boldsymbol{k}} \frac{1}{k^2} \frac{\partial}{\partial \boldsymbol{k}} \cdot \left(\mathsf{D}_{\boldsymbol{k}} \cdot \frac{\partial W_{\boldsymbol{k}}}{\partial \boldsymbol{k}}\right) = \sum_{\boldsymbol{q}} \frac{1}{q^2} (2\gamma^{\mathrm{nl}}_{\boldsymbol{q}} W_{\boldsymbol{q}}), \quad \sum_{\boldsymbol{k}} \frac{1}{k^2} \frac{\partial}{\partial \boldsymbol{k}} \cdot (V_{\boldsymbol{k}} W_{\boldsymbol{k}}) = \sum_{\boldsymbol{q}} \frac{1}{q^2} \dot{W}^{\mathrm{noise}}_{\boldsymbol{q}}. \tag{395a,b}$$

So far both the large $\boldsymbol{k}$'s and small $\boldsymbol{q}$'s have been assumed to obey the same dynamics. The interesting generalization in which the long-wavelength fluctuations describe *convective cells* ($q_\parallel = 0$) is discussed in Sec. 12.7 (p. 248).

7.3.3 Functional methods and the use of wave kinetic equations

Although the algebraic reduction of Markovian closure formulas to obtain results such as Eqs. (394) is ultimately straightforward, and the final forms are suggestive, the intermediate details are tedious and without immediate physical interpretation. In the context of zonal flows, Diamond et al. (1998) suggested the use of a wave kinetic equation to describe the large-$\boldsymbol{k}$ evolution and also proposed a recipe for calculating $\gamma^{\mathrm{nl}}_{\boldsymbol{q}}$. Because those procedures were not derived from first principles, Krommes and Kim (2000) reexamined their foundations using the systematic methods of classical field theory described in Sec. 6 (p. 146). In a broad sense the intuitive basis of the procedures of Diamond *et al.* was upheld; however, a variety of detailed differences in both physical interpretation and mathematical description emerged. Although those cannot all be reviewed here, the analysis led to interesting connections between (i) the general methods of external sources and generating functionals that define the MSR procedure, and (ii) WKE's for slightly inhomogeneous systems. I shall now briefly discuss each of these topics.

If one renormalizes a nonlinear equation like that of HM under the assumption of homogeneous statistics, one is led to Markovian closure formulas like those described in Sec. 7.2.1 (p. 183). The nonlinear growth rate $\gamma_{\boldsymbol{q}}^{\mathrm{nl}} = -\operatorname{Re}\eta_{\boldsymbol{q}}^{\mathrm{nl}}$ thus defined contains terms of all orders in $\epsilon \doteq q/k$. Nevertheless, according to the MSR procedure this homogeneous growth rate is related to response in an inhomogeneous ensemble, since if the dynamical equation is written in the form $\partial_t\varphi + \mathrm{i}\widehat{\mathcal{L}}\varphi + N[\varphi] = 0$, one has

$$\Sigma^{\mathrm{nl}}(1,\overline{1}) = \left.\frac{\delta\langle\langle N\rangle\rangle(1)}{\delta P(\overline{1})}\right|_{P=0}, \tag{396}$$

where $P \doteq \langle\langle\varphi\rangle\rangle$. Now if one were to impress a long-wavelength, *statistically sharp* potential on the system, thereby creating a weak inhomogeneity, the short scales would respond according to the inhomogeneous WKE, as discussed in Appendix F (p. 286). The question is, how does one make those dynamics explicit, since in the actual turbulent system the potential is random and there is no inhomogeneity on the average? The technique adopted by Krommes and Kim (2000) was to perform a statistical average *conditional on the long-wavelength statistics*; this formalizes the intuition of Diamond et al. (1998). The resulting dynamics are inhomogeneous; the long-wavelength potential $\widetilde{\varphi}$ is effectively frozen and plays the role of the mean field in Eq. (396). A result for $\gamma_{\boldsymbol{q}}^{\mathrm{nl}}$ correct to lowest order in ϵ then emerges by evaluating (a Markovian version of) Eq. (396) with the aid of the appropriate WKE for the large $\boldsymbol{k}$'s. In practice, instead of using Eq. (396) directly it is most convenient to use an equivalent result that follows from energy conservation between the long and short scales:

$$\gamma_{\boldsymbol{q}}^{\mathrm{nl}} \propto -\frac{\delta^2\partial_T\mathcal{E}}{\delta P_{\boldsymbol{q}}\delta P_{\boldsymbol{q}}^*}, \tag{397}$$

where $\mathcal{E}$ is the short-wavelength energy. This follows from what can be described as a nonlinear statistical Poynting theorem. The derivation of this formula and examples of its use can be found in Krommes and Kim (2000).

Use of the WKE in this way has created considerable confusion in the literature. In particular, initial interpretations of $\gamma_{\boldsymbol{q}}^{\mathrm{nl}}$ in terms of wave-packet propagation (Diamond and Kim, 1991; Diamond et al., 1998) were shown to be incorrect by Krommes and Kim (2000). Instead, a key role is played by the first-order distension rate $\gamma_{\boldsymbol{k}}^{(1)} \doteq \boldsymbol{k}\cdot\boldsymbol{\nabla}\Omega_{\boldsymbol{k}}$, which describes the evolution of wave *number* under the long-wavelength modulation according to $\mathrm{d}\ln k^2/\mathrm{d}T = -2\gamma_{\boldsymbol{k}}^{(1)}$. A crucial question is, what Ω should be used? Krommes and Kim (2000) showed that the linear normal-mode frequency was irrelevant, and that instead one must use an appropriate nonlinear advection frequency $\widetilde{\Omega}$. For HM dynamics, that is the random variable $\widetilde{\Omega}_{\boldsymbol{k}} = \boldsymbol{k}\cdot\widetilde{\overline{\boldsymbol{V}}}k^2/(1+k^2)$, where $\overline{\boldsymbol{V}}$ is the long-wavelength velocity field. The resulting random rate $\widetilde{\gamma}_{\boldsymbol{k}}^{(1)}$ figures not only in the determination of $\gamma_{\boldsymbol{q}}^{\mathrm{nl}}$ but also in a Fokker–Planck evaluation of the D in Eq. (394a). Wave-packet propagation is relevant to the $\boldsymbol{V}_{\boldsymbol{k}}$ term in Eq. (394a), which is analogous to the polarization drag in classical kinetic theory (Sec. 2.3.2, p. 30). The discussion by Krommes and Kotschenreuther (1982) on analogies between renormalized and classical theory is relevant in this context.

7.4 Renormalization-group techniques

"What we want to emphasize is that *RG is not a 'magic formula.'* There is nothing inherent in the idea of successive integration, or looking for 'fixed points' in lowest-order recursion formulas, etc., which guarantees that the results will have some special validity. Always some analysis of the higher-order terms and sources of error must be made. Otherwise, the method is just an uncontrolled approximation, no better than naive use of low Reynolds number expansions or *ad hoc* closures." — *Eyink (1994).*

The theory of eddy viscosity is one example of a reduced description in which unwanted dynamical scales are eliminated. For example, if one is interested only in a description of long-wavelength fluctuations, one may attempt to eliminate the detailed dynamics of very short scales (large k's) in favor of a renormalized viscosity. The theory of the renormalization group (RG) provides an algorithm for eliminating such unwanted information that at least at first glance is quite appealing. The use of the RG in statistical field theories has a long and illustrious history that cannot be adequately surveyed here; some modern references are Zinn-Justin (1996), Binney et al. (1992), Goldenfeld (1992), and Chang et al. (1992). Historical and philosophical reflections on renormalization-group ideas in statistical physics and condensed-matter theory were made by Fisher (1998), who also provided many references. A review of various uses of the RG in turbulence theory was given by Zhou et al. (1997); see also the recent monograph by Adzhemyan et al. (1999).

The following discussion is a mild rewording and condensation of a brief review given by the present author (Krommes, 1997c). For details and many references to the original field-theoretic RG method, see Brézin et al. (1976). Kadanoff and then Wilson developed related methods originally motivated by problems of equilibrium critical phenomena (Domb and Green, 1976); for some general discussions, see Wilson and Kogut (1974) and Wilson (1975, 1983). Martin et al. (1973) noted that the general MSR formalism was amenable to treatment by RG; an example of such a calculation is the work of DeDominicis and Martin (1979).

In an important calculation that is an antecedent to many practical applications to turbulence, Forster et al. (1976, 1977) carried out an RG procedure for a randomly stirred fluid. The work of FNS is relatively clean because the random forcing is statistically specified. However, that is not true in common applications to turbulence. Although each scale is subjected to an internally induced forcing [modeled at the DIA level by the right-hand side of the Langevin equation (239)], the full statistics of that forcing are unknown and may be very delicate to calculate. Nevertheless, in one way or another, most popular implementations attempt to develop a recursion relation that describes the iterated effects of eliminating infinitesimal bands in k space. The work of Rose (1977) on Navier–Stokes eddy viscosity [see Sec. 7.3 (p. 189)] employed such a method. The approach of Yakhot and Orszag (1986)[235] explicitly involves an additive random forcing described by a controversial (Lam, 1992) *correspondence principle*. Other methods that extend the work of Ma and Mazenko (1975) do not explicitly introduce the random forcing; see Zhou et al. (1994) for further references. Eyink (1994) has clearly discussed the differences between various technical approaches to RG calculations.

In plasma physics a number of RG calculations have been done. Longcope and Sudan (1991) considered an RG analysis of reduced MHD equations. Camargo (1992) performed a related analysis

[235] For corrections and further discussion of the work of Yakhot and Orszag (1986), see Smith and Reynolds (1992).

of a complete MHD system; such works require analytical or symbolic calculations of hundreds of primitive wave-number integrals. Hamza and Sudan (1995) discussed an application to weakly ionized collisional plasma of interest to ionospheric research. Diamond and Hahm (1995) used the ideas in some considerations of self-organized criticality; see Sec. 12.4 (p. 241). Liang and Diamond (1993a) considered 2D MHD and concluded that RG analysis was not applicable; however, Kim and Yang (1999) reconsidered those calculations and arrived at the opposite conclusion.

Most RG calculations treat the approach as a well-specified algorithm. Indeed, because infinitesimal bands of k space are eliminated, there may be a tendency to believe that the resulting recursion relation is exact. That is not the case. Kraichnan (1982) provided a number of incisive arguments against the claims that RG analysis either adds rigor to qualitative analysis or is better justified than standard renormalized perturbation theory. A specific example is afforded by the EDQNM calculation of the long-wavelength statistics of the Model-B forced Burgers equation (Sec. 7.2.1, p. 183), which recovers the RG results of Hwa and Kardar (1992).

Later, Kraichnan attempted to interpret the YO version of the theory (Kraichnan, 1987b; see also Kraichnan, 1987a). He showed that the principle results could be obtained without recourse to the successive elimination of one infinitesimal shell at a time. Instead, he demonstrated that the YO RG algorithm was essentially equivalent to what he called the *distant-interaction approximation*, in which only wave-number triads with one leg very much smaller than the others are retained. It is troubling that the approximation is missing quasi-equilateral triads, which one might believe should be very important in a quasilocal inertial-range cascade. Kraichnan (1987b) suggested that the nonlinear dynamics may be somewhat forgiving because they possess certain self-regulating properties. In any event, it is clear that many deep questions relating to the justification of RG algorithms remain unanswered. An important and readable critique was given by Eyink (1994).

Finally, it must be noted that the conventional RG methods do not apply to common problems of plasma microturbulence in which there are no well-developed inertial ranges. The RG approach is not a panacea.

7.5 Statistical decimation

Statistical theories can be viewed as reduced descriptions of the wealth of information in the true turbulent dynamics. Reductions can be achieved in many different ways. In ambitious work Kraichnan (1985) attempted to provide a unified description of such procedures that he called *statistical decimation*. The method makes contact with the general philosophy of RG, renormalized perturbation theory, eddy viscosity, variational methods, and simulation techniques. Because virtually nothing has been done in this area on a specifically plasma-physics calculation, the following survey will be relatively brief. Nevertheless, the method has considerable appeal and further developments would be of great interest.

The general idea behind decimation is to treat a certain *sample set* of modes explicitly (either analytically or numerically) while representing the effects of the remaining modes in an approximate statistical way. Here are two examples: (i) The sample set might be the energy-containing modes simulated in large-eddy simulations (as in the usual RG applications). (ii) The sample modes could be dispersed throughout the turbulent spectrum (a few per octave, say). In either case the sample set could be simulated directly, or it could be used just as an intermediary for the development of evolution equations for statistical quantities, as in the usual closure approaches.

For definiteness, let us first pursue the latter possibility. In a statistical theory, the justification for retaining a small explicit sample set is that statistical quantities like covariances are smoothly varying functions of their wave-number arguments whereas the stochastic primitive amplitudes can vary wildly with $\boldsymbol{k}$. Consider a closure such as the DIA, initially written for a continuum of wave numbers. One way of reducing the continuum to a small set of sample $\boldsymbol{k}_i$'s is to first divide the $\boldsymbol{k}$ space into appropriately coarse-grained bins centered on $\boldsymbol{k}_i$, then average the continuum evolution equations over each bin. This method is described in more detail in Sec. 8.3.2 (p. 206). For closures such as the DIA, it introduces effective coupling terms that are the bin averages of the squares of the primitive mode-coupling coefficients.

The decimation method proceeds differently. It works directly with primitive amplitudes, not covariances. Instead of averaging the primitive amplitudes over a bin,[236] it seeks to represent the effects of the implicit modes by a random force q whose statistical properties are constrained by various symmetry conditions and realizability criteria. It is possible to envisage a sequence of such constraints such that in the limit the true statistics of the sample set are recovered.

To be more explicit, let us adopt Eq. (227b) as the fundamental dynamical equation and rewrite it as

$$G_0^{-1} u_\alpha(t) - \frac{1}{2} \sum_{\beta,\gamma\in\mathcal{S}} M_{\alpha\beta\gamma} u^*_\beta(t) u^*_\gamma(t) = q_\alpha(t) + f^{\text{ext}}_\alpha(t), \tag{398}$$

where the sample set $\mathcal{S}$ is assumed to contain S modes (S being assumed to be much smaller than the total number of modes) and where $q_\alpha(t) \doteq \frac{1}{2}\sum'_{\beta,\gamma} M_{\alpha\beta\gamma} u^*_\beta(t) u^*_\gamma(t)$, the prime on the summation indicating that at least one of β or γ is not in $\mathcal{S}$. This representation is exact, but not very useful. In decimation theory one expresses ignorance of the implicit modes not in $\mathcal{S}$ and treats only $\{u_\alpha, q_\alpha\}$ explicitly, imposing just a small number of statistical constraints on q. Those constraints may be of varied form, but are typically deduced from low-order statistical moments evaluated with the assumption of statistical symmetry.

Kraichnan (1985) illustrated the technique with a model system with few degrees of freedom; he showed how to numerically enforce constraints with the aid of stochastic Newton–Raphson iteration. Williams et al. (1987) discussed decimation procedures applied to the Betchov (1966) model of turbulence,[237] which has many degrees of freedom. Kraichnan and Chen (1989) applied decimation to a dynamical model of triad interactions that "imitates, in a very primitive way, the treelike structure of interactions present in the NS equation." They showed that the method was able to capture some aspects of intermittency effects with which the DIA could not cope.

One of the most important theoretical results that stems from the decimation procedure is the demonstration (Kraichnan, 1985) that random-Galilean-invariant statistical closures can be formed by the imposition of a particular constraint. If those are developed as the next step in a sequence of decimated approximations beginning with the DIA, their complexity is great (comparable to, but

[236] Bowman et al. (1996, 1999) proposed to do precisely that with their *spectral reduction* scheme, and demonstrated that certain statistics of the exact dynamics can be recovered with high accuracy. Nevertheless, fundamental questions remain about the general viability of this procedure, particularly for the calculation of high-order statistical moments.

[237] The Betchov model is $\dot{x}_i = \sum_{j,k}^N C_{ijk} x_j x_k \quad (i = 1, \ldots, N)$, where the C's are chosen randomly subject to $C_{ijk} + C_{jki} + C_{kij} = 0$. It is therefore closely related to the original random-coupling model of Kraichnan (1958c).

distinct from, the vertex renormalizations of MSR). It is not excluded that simpler RGI closures can be developed. None of those closures has been extensively studied; that would be an interesting task for the future.

8 MODERN DEVELOPMENTS IN THE STATISTICAL DESCRIPTION OF PLASMAS

"The conventional example of a DIA-based Markovian closure, the EDQNM, severely violates realizability in the presence of linear wave phenomena." — *Bowman et al. (1993).*

In Sec. 5 (p. 126) I reviewed the development, justification, and early plasma applications of Kraichnan's DIA [considerably extending the earlier review of Krommes (1984a)]. As of the middle 1980s the theoretical foundations of the plasma DIA were relatively well understood.

The present and next several sections are concerned with relatively modern developments of particular relevance to plasma physics taking place over approximately the period 1985–99 [for a time line, see Fig. 36 (p. 262)]. In the present section I focus on studies of computable Markovian closures. A major surprise was that one theoretically popular closure, the EDQNM (Sec. 7.2.1, p. 183), is *not realizable in the presence of waves* (which are ubiquitous in plasma physics). I describe one solution to this problem, Bowman's Realizable Markovian Closure (RMC), which is a particular modification of the EDQNM. Technical foundations are provided by the theory of realizability constraints reviewed in Sec. 3.5.3 (p. 63). A realizable version of the random-Galilean-invariant test-field model is also mentioned.

I then survey computational approaches to the DIA and RMC as well as the applications of those closures to paradigms important to nonlinear plasma physics. The most thoroughly studied such application is the system of two coupled fields known as the Hasegawa–Wakatani equations; that is discussed in detail. Suggestions for future studies are also given.

Simple models such as that of HW foster the conventional wisdom that turbulence is excited by linear instability. However, in some situations turbulence can exist even for a completely stable linear spectrum. An introduction to this topic of "submarginal" turbulence is given in Sec. 9 (p. 210).

Readily identifiable successes of the conventional statistical closures are surprisingly accurate quantitative predictions of wave-number spectra and advective fluxes, which are quantities of key experimental concern. However, higher-order statistics and the physics of intermittency are also of interest. Here the theory is very difficult and far from completely developed. Some related topics are described in Sec. 10 (p. 220).

8.1 Antecedents to the modern plasma developments

Before turning to the modern developments, one should note several important antecedents.

8.1.1 Miscellaneous practical applications

The 1980s were noteworthy for serious attempts to apply renormalization techniques to practical problems of tokamak microturbulence. Four representative examples are the effects of turbulent diffusion on collisionless tearing modes (Meiss et al., 1982) and the theories of turbulence driven by resistivity gradients (Garcia et al., 1985), ion temperature gradients (Lee and Diamond, 1986), and pressure gradients (Carreras et al., 1987). One significant technical difficulty with such calculations is the treatment of magnetic shear, which introduces nontrivial x-dependent inhomogeneity. Although many interesting insights into the features of the turbulence were obtained, and in some cases intriguing agreement between the closure calculations and direct numerical simulations was found (Carreras et al., 1987), it must be stressed that no *fully systematic* closure was studied: the nonlocal effects of inhomogeneity were not described in the integral representation natural to the DIA, for example (admittedly an extremely difficult calculation that even as of 2000 had not been done), and many other simplifying approximations were made as well [238] in order that one could proceed analytically. Therefore it is difficult to assess the fidelity of the resulting predictions. At the very least, numerical coefficients from such exercises must not be taken seriously. In some cases there were qualitative difficulties as well [see Sec. 4.4 (p. 119)].

In the spirit of this article on systematic techniques, I shall not pursue such practical calculations and the similar ones that followed. Instead, I will remark in Secs. 8.1.2 (p. 200) and 8.1.3 (p. 200) on work that more naturally bridges between the formal statements of the DIA and EDQNM closures and their detailed numerical solutions for tractable model problems.

8.1.2 Renormalization and mixing-length theory

Sudan and Pfirsch (1985) returned to the relationship between the DIA and mixing-length theory. [I have already commented in Sec. 5.7 (p. 140) on the seminal work by Kraichnan (1964c) on that topic.] Whereas Kraichnan was concerned with the theory of inhomogeneous turbulence and the evolution of mean fields, Sudan and Pfirsch were interested primarily in the cascade processes that lead to the development of inertial ranges in homogeneous turbulence (Kraichnan, 1966b, 1971b, 1973b, 1974). In addition to reviewing the general concepts, they made explicit calculations for the interesting problem of turbulence in the equatorial electrojet. In later related work by Rosenbluth and Sudan (1986), a diffusion equation in wave-number space was developed to describe spreading of an almost 2D spectrum (Albert et al., 1990) into the parallel direction. Some of that work was briefly reviewed by Sudan (1988).

8.1.3 Statistical closures for drift waves

Horton (1986) undertook direct numerical simulations of drift waves, using the model equations of Terry and Horton (1982). He also proposed a Markovian statistical closure and used approximate analytical methods to deduce from it various features in agreement with the DNS. He did not attempt a numerical solution of the closure equations, leaving the detailed fidelity of the closure in some doubt. Nevertheless, he was able to explain broad features of the simulations in terms of the analytical framework, including the observation that the turbulent line broadening can be much larger than a typical linear growth rate [see also Sudan and Keskinen (1977)]. Horton's work was summarized by Horton and Ichikawa (1996), who also provided additional background discussion of drift waves.

[238] For example, Meiss et al. (1982) claimed that they employed the (coherent) DIA (Krommes and Kleva, 1979). However, in reality they did not faithfully solve the specific integral equations of the DIA as described in Sec. 5 (p. 126). It seems best to reserve the phrase "direct-interaction approximation" to the specific closure, nonlocal in space and time, first obtained by Kraichnan (1959b) and generalized to arbitrary quadratically nonlinear systems by Martin et al. (1973).

8.2 Realizability and Markovian closures

In summary, by approximately the mid-1980s a variety of qualitatively plausible moment-based closures were available to plasma physicists. It was clear that numerical solutions would be required to quantitatively treat realistic many-mode spectra, but that had not been done. It was known that the DIA was realizable, but that approximation was particularly daunting for the anisotropic spectra characteristic of drift-wave models. Realizability of the plasma Markovian closures was not discussed; it was assumed that those would be as well behaved as their Navier–Stokes counterparts.

A dramatic demonstration of the breakdown of realizability in a plasma-physics application came when Bowman and Krommes attempted in 1989 to numerically integrate the EDQNM closure for model drift-wave equations of the Hasegawa–Mima variety. The development of catastrophically negative energy spectra in the presence of linear drift waves demonstrated an important difficulty with that closure that had been previously unappreciated.

8.2.1 Nonrealizability of the EDQNM

The straightforward derivation of the DIA-based EDQNM (henceforth simply called *EDQNM*) from the robust DIA, as outlined in Sec. 7.2.1 (p. 183), led to a long-standing belief that the Markovian approximation should itself be robust (discounting the known, relatively subtle difficulty with RGI). In fact, however, the theory has a serious deficiency that renders it virtually useless in its original form: *The EDQNM is nonrealizable in the presence of linear waves.* As an example, let us follow the work of Bowman (1992) and apply the EDQNM to a particular example of the three-wave example of Sec. 5.10.3 (p. 144). I consider the degenerate case $(M_K, M_P, M_Q) = (1, -1, 0)$ with $\Omega_Q = \gamma_Q = 0$, so mode Q does not evolve; the resulting system is amenable to analytical treatment (Kraichnan, 1963; Bowman, 1992). Gaussian initial conditions are assumed with covariances $(C_K, C_P, C_Q) = (0, 1, 2)$. In Fig. 25 I consider the case in which the complex linear frequencies are all set to zero. The behavior predicted here is reasonable; the modal intensities approach the correct equipartition values $C_K = C_P = C_Q = 1$. The approach is nonmonotonic in both the exact solution and the DIA; it is monotonic in the EDQNM, as could be expected from its Markovian nature. In Fig. 26 (p. 202) I show the evolution of mode K for the same parameters except that the complex frequencies are $(\omega_K, \omega_P, \omega_Q) = 3(1 + \mathrm{i}, 1 + \mathrm{i}, 0)$. In the DIA that mode correctly grows exponentially; for these parameters (growth rates all positive), no steady state is possible. However, in the EDQNM the energy of mode K goes negative at a finite time and subsequently explodes catastrophically to $-\infty$. This is an example of nonrealizable behavior. In more realistic situations in which some growth rates are negative and time-asymptotic steady states are expected, one infers that in the presence of linear waves it is not necessarily possible to achieve those steady states by integrating the EDQNM forward from realizable initial conditions. Note that neither the exact dynamics nor the DIA exhibit this difficulty; both are guaranteed to be realizable.

8.2.2 Langevin representation of the EDQNM

Insight into the catastrophic failure of the EDQNM in the presence of wave phenomena can be gained by considering a Langevin representation due to Leith (1971) and Kraichnan (1970a):

$$\partial_t \psi_{\boldsymbol{k}}(t) + \mathrm{i}\Omega_{\boldsymbol{k}}\psi_{\boldsymbol{k}} + \eta^{\mathrm{nl}}_{\boldsymbol{k}}(t)\psi_{\boldsymbol{k}} = \widetilde{f}_{\boldsymbol{k}}(t) \doteq \widetilde{w}(t) \sum_{\Delta} M_{\boldsymbol{kpq}} \sqrt{\theta_{\boldsymbol{kpq}}}\, \widetilde{\xi}^*_{\boldsymbol{p}}(t) \widetilde{\xi}^*_{\boldsymbol{q}}(t). \qquad \text{(399a,b)}$$

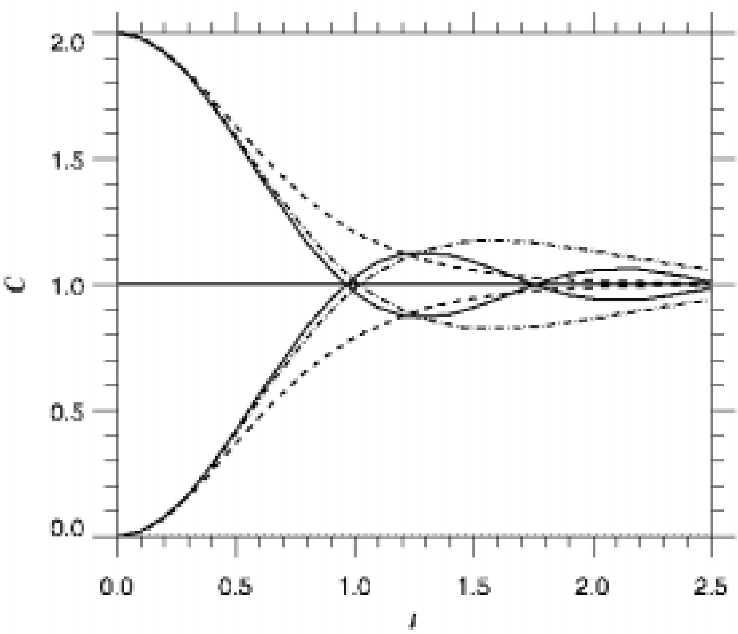

Fig. 25. Equal-time covariance C of the three-mode problem for a case in which all linear frequencies and growth rates vanish. Solid curves, DIA; dashed curves, EDQNM; chain-dotted curves, RMC (Sec. 8.2.3, p. 203).

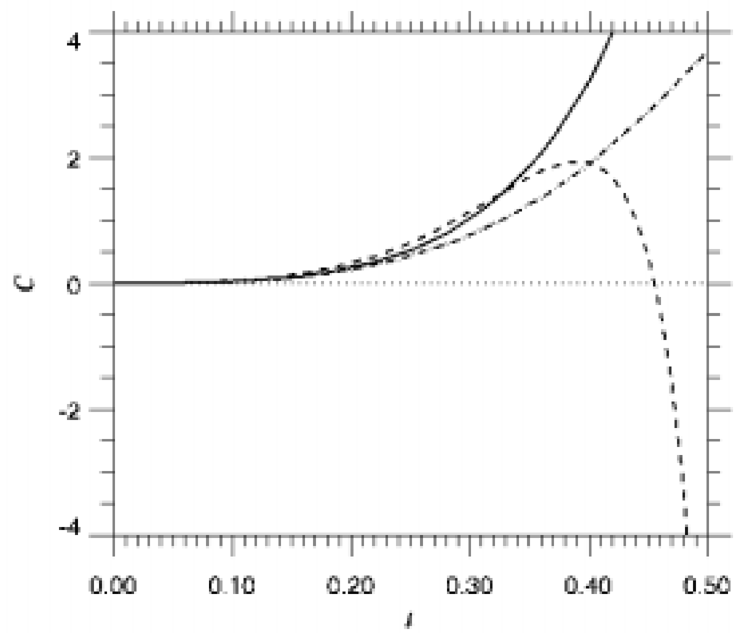

Fig. 26. Evolution of the covariance of mode K of the three-mode problem for a case in which modes K and P have positive frequencies and growth rates. Solid curve, DIA; dashed curve, EDQNM; chain-dotted curve, RMC. The EDQNM solution exhibits nonrealizable behavior.

Here $\widetilde{w}(t)$ is a Gaussian white-noise process of unit amplitude; its presence ensures the Markovian nature of the resulting approximation. The random auxiliary field $\widetilde{\xi}$ is, as in the Langevin representation (239) of the DIA, constrained to have variance identical to that of $\psi_{\boldsymbol{k}}$. The term $\sqrt{\theta}$ is necessary on dimensional grounds.[239] A good review of the Langevin representation was given by Orszag (1977).

It is easy to show that the second-order statistics of Eqs. (399) obey the EDQNM equations (367). First, it can be seen that the response function derived from Eq. (399a) obeys Eq. (365). Next, the covariance equation derived from Eq. (399a) reproduces the left-hand side of Eq. (367), with a right-hand side of $2\,\mathrm{Re}\,\langle\widetilde{f}_{\boldsymbol{k}}(t)\psi^*_{\boldsymbol{k}}(t)\rangle$. Finally, upon inserting the solution $\psi_{\boldsymbol{k}}(t) = \int_0^\infty d\overline{t}\, R_{\boldsymbol{k}}(t;\overline{t})\widetilde{f}_{\boldsymbol{k}}(\overline{t})$ and noting that $\langle\widetilde{f}_{\boldsymbol{k}}(t)\widetilde{f}^*_{\boldsymbol{k}}(t')\rangle = 2\delta(t-t')F_{\boldsymbol{k}}(t)$, one verifies the correspondence. However, this is in general a formal result only. In the presence of waves θ is complex, so the $\sqrt{\theta}$ in Eq. (399b) is ill defined. More heuristically, it is actually the real part of $\theta_{\boldsymbol{kpq}}$ that serves as the physical interaction time in Eq. (368b). For complex θ that real part can easily be negative. Not only is the meaning of a negative interaction time unclear, the resulting equations are obviously ill behaved, as demonstrated by Fig. 26.

The work of Bowman (1992) was not the first mention of the EDQNM in plasma-physics contexts; see, for example, Diamond and Biglari (1990) and Gang et al. (1991). Previous work focused on *qualitative* properties of *steady-state* solutions of the EDQNM. However, some qualitative arguments are insensitive to signs or numerical coefficients; such nuances can be revealed only by detailed quantitative solutions of the nonlinear closure equations.

In principle one can contemplate numerical solutions of the steady-state EDQNM equations, which comprise a nonlinear algebraic system in a large number of variables $C_{\boldsymbol{k}}$. Typical numerical procedures would accomplish this by an iterative procedure; frequently, algorithms that iterate to fixed points can be described as discretized integrations in a fictitious time. For an arbitrary initial guess, the stability of such iterations is unclear, and such solutions have not been attempted as far as the author knows. Integration of the causal time-dependent EDQNM equations to a time-asymptotic steady state seems both physically and mathematically to be more appropriate. But then one requires a cure for the breakdown of realizability.

8.2.3 Bowman's Realizable Markovian Closure

Since steady-state solutions are of principal interest, Bowman, Krommes, and Ottaviani, in a fruitful collaboration during the period 1989–91, adopted the point of view that the transient dynamics of the EDQNM should be modified to *guarantee* a realizable evolution to the steady state described by the time-independent EDQNM. This single criterion permits considerable leeway in the development of a satisfactory algorithm, which can be viewed as a specific example of the numerical techniques mentioned in the previous paragraph. One way of proceeding is to seek modifications of the $\theta_{\boldsymbol{kpq}}$ dynamics that (i) ensure $\mathrm{Re}\,\theta_{\boldsymbol{kpq}} \ge 0$, and (ii) preserve the EDQNM steady-state solution (370b). For a single scalar field it is not hard to invent such dynamics (Bowman, 1992). However, when one attempts to generalize such techniques to systems of multiple coupled fields, major technical difficulties are encountered. For example, θ becomes a sixth-rank tensor $\theta^{\alpha\beta\gamma}_{\alpha'\beta'\gamma'}$ in the field (species) label α (or a second-rank tensor in the extended index $s \doteq \{\alpha,\beta,\gamma\}$). Constraints on $\mathrm{Re}\,\theta$ generalize to constraints on positive definite matrices that are difficult to handle. More fundamentally, it proves difficult if not impossible to simultaneously retain the conservation of multiple nonlinear invariants. Some work in these directions was summarized by Ottaviani et al. (1991), Bowman (1992), and Bowman et al. (1993).

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[239] Because $\langle\widetilde{w}(t)\widetilde{w}(t')\rangle = \delta(t-t')$, dimensionally $[\widetilde{w}] = [t]^{-1/2}$. θ has the dimension of time.

The EDQNM emerges from the DIA with the aid of just two approximations: (i) the Markovian approximation (365); (ii) the fluctuation–dissipation Ansatz [FDA; Eq. (366)]. Ultimately, Bowman (1992) was led to focus on the FDA as the root of the difficulty. He showed that the form (366) leads to an incoherent noise function that is not intrinsically positive definite (with time playing the role of a continuum matrix index) since $C_{\boldsymbol{k}}(t)$ does not enter Eq. (366) symmetrically in t and t'.[240] Because in the DIA positive definiteness of F guarantees positive definiteness of the covariances for proper initial conditions, Bowman was led to restate the FDA as an equality between the *correlation coefficient* $c_{\boldsymbol{k}}(t,t') \doteq C_{\boldsymbol{k}}(t,t')/[C_{\boldsymbol{k}}^{1/2}(t)C_{\boldsymbol{k}}^{1/2}(t')]$ and the response function; thus[241] he assumed that

$$c_{\boldsymbol{k}}(t,t') = R_{\boldsymbol{k}}(t;t') \quad (t \geq t'). \tag{400}$$

He proved a variety of theorems relating to positive definiteness and was ultimately led to the following system of equations, which he called the *Realizable Markovian Closure* (RMC):

$$\partial_t C_{\boldsymbol{k}}(t) + 2\,\mathrm{Re}\,\eta_{\boldsymbol{k}}(t) C_{\boldsymbol{k}}(t) = 2F^{\mathrm{nl}}_{\boldsymbol{k}}(t), \tag{401a}$$

$$\eta_{\boldsymbol{k}}(t) \doteq \mathrm{i}\mathcal{L}_{\boldsymbol{k}} - \sum_{\Delta} M_{\boldsymbol{kpq}} M^*_{\boldsymbol{pqk}} \theta^{\mathrm{eff}*}_{\boldsymbol{pqk}}(t) C_{\boldsymbol{q}}(t), \tag{401b}$$

$$F^{\mathrm{nl}}_{\boldsymbol{k}}(t) \doteq \frac{1}{2}\sum_{\Delta} |M_{\boldsymbol{kpq}}|^2\, \mathrm{Re}\,\theta^{\mathrm{eff}}_{\boldsymbol{kpq}}(t) C_{\boldsymbol{p}}(t) C_{\boldsymbol{q}}(t), \tag{401c}$$

$$\theta^{\mathrm{eff}}_{\boldsymbol{kpq}} \doteq \Theta_{\boldsymbol{kpq}} C_{\boldsymbol{p}}^{-1/2} C_{\boldsymbol{q}}^{-1/2}, \tag{401d}$$

$$\partial_t \Theta_{\boldsymbol{kpq}} + [\eta_{\boldsymbol{k}} + \mathcal{P}(\eta_{\boldsymbol{p}}) + \mathcal{P}(\eta_{\boldsymbol{q}})]\Theta_{\boldsymbol{kpq}} = C_{\boldsymbol{p}}^{1/2}(t) C_{\boldsymbol{q}}^{1/2}(t), \tag{401e}$$

together with $\Theta_{\boldsymbol{kpq}}(0) = 0$ and a positive definite initial condition on $C_{\boldsymbol{k}}$. Here the $\mathcal{P}$ operator is defined by $\mathcal{P}(\eta) \doteq \mathrm{Re}(\eta) H(\mathrm{Re}\,\eta) + \mathrm{i}\,\mathrm{Im}\,\eta$; i.e., it annihilates any antidissipative effect in the temporal development of Θ. Note that although the original $\theta_{\boldsymbol{kpq}}$ was fully symmetric, $\Theta_{\boldsymbol{kpq}}$ and $\theta^{\mathrm{eff}}_{\boldsymbol{kpq}}$ are symmetrical only in their last two indices.

For the three-wave problem the prediction of the RMC is also displayed in Figs. 25 (p. 202) and 26 (p. 202). Figure 26 shows that the RMC makes sensible, realizable predictions. This result is extremely encouraging, so one is led to consider the application of the RMC to more complicated and physically relevant systems with many modes. That is discussed in Sec. 8.3 (p. 205).

The RMC inherits the difficulty with random Galilean invariance that plagues the DIA and EDQNM. The test-field model (Sec. 7.2.2, p. 187) corrects that problem, but Bowman and Krommes (1997) showed that in general the complex TFM is also not realizable in the presence of waves. They proposed a realizable modification that follows along the lines of the RMC. I shall not write it here for lack of space.

[240] One is not referring here to the non-Hermitian nature of the one-sided function $C_+(t,t')$ [see Eq. (366)]; the two-sided function $C(t,t') = C_+(t,t') + C_+^*(t',t)$ is always Hermitian no matter what form of the FDA is adopted.

[241] A fluctuation–dissipation relation involving $c_{\boldsymbol{k}}(t,t')$ previously appeared in the literature (Kraichnan, 1971a), but to the author's knowledge was stated only in the context of thermal equilibrium, for which the choice of t or t' for the arguments of the normalizing intensities in the denominator is irrelevant.

8.3 Numerical solution of the DIA and related closures in plasma physics

Each of the systematic closures presented so far has identical mode-coupling structure: for each mode $\boldsymbol{k}$, sums are required over wave numbers $\boldsymbol{p}$ and $\boldsymbol{q}$ such that $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}$. Although simple qualitative analysis can be done analytically, reliable quantitative information for systems with many modes must be obtained numerically. The numerical solution of statistical closures has a long history in neutral-fluid turbulence [see, for example, Kraichnan (1964b) and Herring and Kraichnan (1972)]. Corresponding attempts in plasma physics are much more recent. In the present section I briefly describe some of the numerical considerations. In Sec. 8.4 (p. 206) I review recent applications.

8.3.1 Scalings with computation time and number of modes

Let T be the total run time, N_T be the total number of discrete times to be evolved, and $N_{\boldsymbol{k}}$ be the total number of wave numbers to be retained in a numerical procedure. Let primes (no primes) denote, respectively, quantities for DNS (closures). Direct pseudospectral integrations of the primitive equations of motion require $O(N_T' N_{\boldsymbol{k}}' \log N_{\boldsymbol{k}}')$ operations. To employ fast Fourier transforms (FFT's), the $\boldsymbol{k}$'s must be uniformly distributed on a Cartesian lattice. Because individual realizations fluctuate rapidly in time, even in steady state, one must integrate primitive dynamics for many autocorrelation times past the point of saturation t_{sat} in order to accumulate reliable statistics, particularly for two-time correlation functions; closures, however, need only be integrated to t_{sat}. Because realizations fluctuate rapidly in $\boldsymbol{k}$ space whereas statistical quantities vary smoothly, one can work with $N_{\boldsymbol{k}} \ll N_{\boldsymbol{k}}'$.

For homogeneous, possibly anisotropic turbulence, the basic structure of the DIA is $\partial_t C_{\boldsymbol{k}}(t,t') + \int_0^t d\bar{t}\, \Sigma^{\text{nl}}_{\boldsymbol{k}}(t;\bar{t}) C_{\boldsymbol{k}}(\bar{t},t') + \cdots$, with $\Sigma^{\text{nl}}_{\boldsymbol{k}}(t;\bar{t})$ requiring the familiar sum over $\boldsymbol{p}$ and $\boldsymbol{q}$. For given $\Sigma^{\text{nl}}_{\boldsymbol{k}}$, calculation of $C_{\boldsymbol{k}}(t,t')$ for all $t' \le t$ requires on the order of $N_T^3 N_{\boldsymbol{k}}$ operations; for a general, nonuniform $\boldsymbol{k}$ lattice, calculation of $\Sigma^{\text{nl}}_{\boldsymbol{k}}(t;\bar{t})$ requires on the order of $N_T^2 N_{\boldsymbol{k}}^3$ operations. [Dannevik (1986) has stressed that one does not need to recompute $\Sigma^{\text{nl}}_{\boldsymbol{k}}(t;\bar{t})$ for each t' at fixed t.] The total DIA operation count is thus $O(N_T^3 N_{\boldsymbol{k}}) + O(N_T^2 N_{\boldsymbol{k}}^3)$. The highly unfavorable N_T^3 scaling is frequently cited as an argument against the practical use of the DIA; however, the $O(N_T^2)$ term can dominate in practice. If all fluctuations decayed on a characteristic timescale τ_{ac}, then the time-history integrals could be truncated and $N_T^3 \to N_T^2 N_{\tau_{\text{ac}}} = N_T^3(\tau_{\text{ac}}/T)$. However, in practice τ_{ac} depends on $\boldsymbol{k}$, and DIA runs are constrained by the time to integrate the slowest (generally longest-wavelength) mode through at least one but preferably several dynamical times.

For the Markovian closures, whose structure is $\partial_t C_{\boldsymbol{k}}(t) + \eta^{\text{nl}}_{\boldsymbol{k}}(t) C_{\boldsymbol{k}}(t) + \cdots$ and $\partial_t \theta_{\boldsymbol{kpq}} + \eta^{\text{nl}}_{\boldsymbol{k}} \theta_{\boldsymbol{kpq}} + \cdots$, $O(N_{\boldsymbol{k}}^2)$ operations are required to compute $\eta^{\text{nl}}_{\boldsymbol{k}}$ for given $\boldsymbol{k}$. Scaling is linear in N_T; however, $\theta_{\boldsymbol{kpq}}$ must be evolved for each of the $\frac{1}{6}N_{\boldsymbol{k}}^3$ distinct wave-number triads.[242] The final Markovian scaling is thus $O(N_T N_{\boldsymbol{k}}^5)$.

These considerations suggest that Markovian closures can compete favorably with direct numerical simulations, and this is born out in practice (Bowman and Krommes, 1997). In order to obtain DIA solutions, a technique that sometimes works successfully is to perform coarse-grained integrations with a Markovian closure, then use the resulting $\boldsymbol{k}$ spectrum to initialize a DIA run.

Direct calculations of the wave-number sums required for the nonlinear terms can be avoided by using FFT-based pseudospectral techniques,[243] at the price, however, of employing a uniform $\boldsymbol{k}$ lattice

[242] Sulem et al. (1975) proposed a simplification wherein the number of unknowns is proportional to $N_{\boldsymbol{k}}$, not $N_{\boldsymbol{k}}^3$, leading to a very favorable Markovian scaling $O(N_T N_{\boldsymbol{k}}^3)$.

[243] The suggestion that pseudospectral techniques might be useful in this context is apparently due to Orszag.

that is relatively densely populated. This technique was adopted by LoDestro et al. (1991) in their DIA studies of the HW equations.

8.3.2 Decimating smooth wave-number spectra

In order to implement the observation that statistical quantities vary smoothly in $\boldsymbol{k}$ space, one may divide the $\boldsymbol{k}$ space into coarse bins and compute representative values of $C_{\boldsymbol{k}}$ averaged over each bin. For homogeneous turbulence, for which angular integrations can be performed analytically, that was first done by Leith (1971) and Leith and Kraichnan (1972). The procedure introduces effective, bin-averaged mode-coupling coefficients. Leith (1971) called their computation a "complex exercise in solid geometry and computer logic"; however, Bowman (1992) found a simple and efficient analytical procedure [244] that also generalized naturally to anisotropic turbulence. In his practical calculations of 2D anisotropic plasma turbulence, Bowman used a cylindrical-polar $\boldsymbol{k}$-space geometry with bins spaced logarithmically in wave-number magnitude and linearly in wave-number angle. Quite coarse partitions seem to be satisfactory in order to resolve the energy-containing portion of the spectrum. Note that a logarithmic partition precludes straightforward use of the FFT approach to computing the closure terms.

A conceptual difficulty with a fully binned description of the $\boldsymbol{k}$ space is that the optimal angular orientation of the bins is uncertain. An alternate procedure is to use a truncated Fourier description of the angular dependence (Herring, 1975). That method is certainly appropriate for weak anisotropy. It should be explored further for plasma-physics problems in which the anisotropy is nominally of order unity.

8.4 Application: Statistical closures for the Hasegawa–Mima and Terry–Horton equations

"To sum up, we now have methods of closure which are not ridiculous. We don't know how good they are, nor do we know how to apply them to practical problems: these are the questions which must now be studied." — *Leslie (1973a).*

The goal in this and the next subsection is to describe how well second-order statistical closures

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Dannevik (1986) cites a preprint by Domaradzki and Orszag (1986).

[244] The typical convolution structure of a closure in the continuum representation is $\partial_t C(\boldsymbol{k}) = (2\pi)^{-d} \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{q}\, \delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}) f(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) S(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$, where f is a rapidly varying function determined by the square of the mode-coupling coefficients and S is slowly varying—for example, for the incoherent-noise term of the DIA, $f = \frac{1}{2}|M_{\boldsymbol{kpq}}|^2$ and $S = C(\boldsymbol{p})C(\boldsymbol{q})R^*(\boldsymbol{k})$. Define the average over the $\boldsymbol{l}$th wave-number bin of volume $\Delta_{\boldsymbol{l}}$ by $C_{\boldsymbol{l}} \doteq \Delta_{\boldsymbol{l}}^{-1} \int_{\Delta_{\boldsymbol{l}}} \mathrm{d}\boldsymbol{k}\, C(\boldsymbol{k})$. Approximate the $\boldsymbol{p}$ and $\boldsymbol{q}$ integrals by Riemann sums, average the $C(\boldsymbol{k})$ equation over the $\boldsymbol{l}$th bin, and evaluate S at the central wave numbers. Then $\partial_t C_{\boldsymbol{l}} = \sum_{\boldsymbol{m},\boldsymbol{n}} \mathcal{M}_{\boldsymbol{lmn}} S(\boldsymbol{k_l}, \boldsymbol{p_m}, \boldsymbol{q_n})$, with $\mathcal{M}_{\boldsymbol{lmn}} \doteq (2\pi)^{-d} \Delta_{\boldsymbol{l}}^{-1} \int_{\Delta_{\boldsymbol{l}}} \mathrm{d}\boldsymbol{k} \int_{\Delta_{\boldsymbol{m}}} \mathrm{d}\boldsymbol{p} \int_{\Delta_{\boldsymbol{n}}} \mathrm{d}\boldsymbol{n}\, \delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}) f(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$. In general, the integrations can be carried out partly analytically, partly numerically; they must be computed just once for each bin resolution. For isotropic statistics, for which the angular bins extend over the entire interval $[0, 2\pi)$, the angular integrations can be easily done analytically and one is quickly led to the results of Leith and Kraichnan (1972). In general, however, although the basic procedure is straightforward, practical details are nontrivial. A thorough discussion was given by Bowman (1996b); see also Bowman and Krommes (1997).

perform when applied to paradigms of interest to the physics of turbulent plasmas in strong magnetic fields. The Hasegawa–Mima equation (48) and Terry–Horton equation (43) are important examples of one-field models, so I begin with them. In Sec. 8.5 (p. 208) I consider the much richer two-field Hasegawa–Wakatani model.

8.4.1 Thermal equilibrium for Hasegawa–Mima dynamics

The primitive HM dynamics are conservative. As described in Sec. 3.7.2 (p. 68), the HME admits two quadratic invariants [Eqs. (50)]: the energy $\mathcal{E}$; and the enstrophy $\mathcal{W}$. Therefore any initial condition (truncated to a finite number of discrete wave numbers) will relax to the two-parameter Gibbs distribution discussed in Sec. 3.8.3 (p. 74). Furthermore, those two invariants are preserved by the DIA-based second-order closures. An important and nontrivial test (of the *nonlinear* structure of a code) is therefore to demonstrate that both the primitive dynamics and the closures relax to the proper two-parameter Gibbs distribution. Such relaxation experiments are now standard tools of the simulation repertoire. [245]

8.4.2 Transport in the Hasegawa–Mima equation

There is no transport in such thermal-equilibrium states. Indeed, particle transport vanishes even for nonequilibrium transient states of the HME because the electron response is adiabatic: $\Gamma_e \doteq \langle \delta V_{E,x} \delta n_e \rangle \propto -\langle (\partial_y \varphi)\varphi \rangle = -\frac{1}{2}\langle \partial_y(\varphi^2) \rangle = 0$, the last result following from periodic boundary conditions or statistical homogeneity. Since one has already seen that the particle fluxes are ambipolar, it follows that $\Gamma_i = 0$ as well.

Nevertheless, Connor–Taylor analysis applied to the HME demonstrates an intrinsic *scaling* of transport that will emerge when nonadiabatic effects permit nonzero flux, as in the Terry–Horton equation. The gyro-Bohm units described in Sec. 2.4.3 (p. 34) remove all parameters from the HME, so one is immediately led to conclude that the diffusion coefficient in physical units must be the gyro-Bohm coefficient (6). The fundamental physics underlying this result is $\boldsymbol{E} \times \boldsymbol{B}$ advection of ion polarization charge density. This physics is retained by even substantially more sophisticated models, which therefore also tend to exhibit gyro-Bohm scaling.

8.4.3 Forced Hasegawa–Mima equation and spectral cascades

In the presence of growth or dissipation, the appropriate physical equation to employ is that of Terry and Horton. However, as an intermediate step it is instructive to first consider a forced HM equation in which a linear growth or dissipation rate is included, but dissipation in the nonlinear term is ignored. Ottaviani and Krommes (1992) used Kolmogorov arguments to deduce criteria under which spectral domains lie in weak- or strong-turbulence regimes; both can occur simultaneously, suggesting the need for numerical work. Bowman and Krommes (1997) performed various simulations of the forced HME using the RMC, RTFM, and DIA. It was found that the predictions of those realizable closures were in reasonable agreement with each other and with those of DNS.

[245] For the closures the expected equilibria are computed for given initial conditions following the procedures of Sec. 3.7.2 (p. 68). They are then perturbed in a way that preserves the given invariants (Hu et al., 1997).

8.5 Application: Statistical closures for the Hasegawa–Wakatani equations

As I stressed in Sec. 3.8.4 (p. 76), realistic plasma-physics applications involve growth rates broadly distributed in $\boldsymbol{k}$ space. For practical work one could insert growth rates derived from kinetic theory into the THE and proceed with closure calculations. However, continuing in the spirit of the present article aimed at elucidating fundamentals, I now consider the two-field, 2D HW paradigm defined by Eqs. (52), for which a linear growth rate is built in. Aspects of the statistical dynamics of that system were discussed by Gang et al. (1991), who used a simplified form of the EDQNM.[246] It has already been noted that the system reduces to a forced, dissipative HME in the adiabatic regime $\alpha \gg 1$, so it is most interesting to focus on the hydrodynamic regime $\alpha \ll 1$. The interesting questions there become whether the vortices observed in the DNS influence the transport, and to what extent a low-order statistical closure can correctly predict the spectra and flux. Some discussion of these issues was given by Koniges and Craddock (1994).

The HW equations can be studied with either the DIA or the RMC. LoDestro et al. (1991) implemented the DIA, using FFT's to compute the wave-number sums. They discussed relaxation to thermal equilibrium, the approximate validity of the fluctuation–dissipation Ansatz in the energy-containing range, and the importance of off-diagonal correlations.

Such DIA calculations are numerically challenging. For rapid studies and parameter scans it is easier to use a Markovian approximation, as was done by Hu et al. (1995) and Hu et al. (1997). Let us continue to employ the same parameters as those of Koniges et al. (1992). In Fig. 1 (p. 41) are compared the particle flux $\Gamma(\alpha)$ from (i) DNS, (ii) Bowman's RMC, and (iii) QLT. The latter prediction is defined by inserting the linear phase relation between φ and n for the unstable branch into Eq. (3b), then taking the saturated spectrum $\langle \varphi^2 \rangle_{\boldsymbol{k}}$ from the DNS. It can be seen that the closure reproduces the actual flux very well, and that in the hydrodynamic regime the true flux is substantially reduced from the quasilinear prediction, indicating the importance of nonlinear effects.

Koniges et al. (1992) suggested that this depression from the quasilinear value was related to the "coherent [vortex] structures" that can be seen clearly in snapshots of the vorticity field. They offered no analysis in support of this hypothesis, and the closure calculations described here do not uphold it. As I have discussed in some detail, second-order closures of the kind used here do not capture the details of coherent structures, which cannot be reconstructed in real space from merely two-point statistics. Furthermore, the particle transport being measured depends directly not on the vorticity ω but rather on the velocity $V_{E,x}$, an integral of ω. Integration is a smoothing operation, and a qualitative application of the central limit theorem suggests that $V_{E,x}$ should be more nearly Gaussian than is ω. This is born out by the DNS, where one finds that while the kurtosis $K(\omega)$ rises to a highly non-Gaussian value in the saturated state, the kurtoses of n, φ, and $V_{E,x}$ remain relatively close to the Gaussian value 0. Snapshots of the saturated density and velocity fields reveal little evidence of the coherent vortices.

Nevertheless, the true flux is indeed substantially depressed from the quasilinear prediction. An explanation that does not involve coherent structures was given by Hu et al. (1995). The argument begins by estimating $\Gamma \sim \overline{V}_E^2 \tau_{\text{ac}}$, so the ratio of true to quasilinear flux is $\Gamma^{\text{true}}/\Gamma^{\text{QL}} = \tau_{\text{ac}}^{\text{true}}/\tau_{\text{ac}}^{\text{QL}}$. In general, for a wavelike spectrum one has $\tau_{\text{ac}}^{\text{QL}} \sim (\gamma/\omega)\omega^{-1}$ [see footnote 19 (p. 16)]; in the

[246] Gang et al. (1991) parametrized all time correlations by a single nonlinear damping independent of field index, so the reduced theory involved a scalar $\theta_{\boldsymbol{kpq}}$. That is a substantial simplification [independently suggested by Ottaviani et al. (1991)], at the cost of a less faithful representation of the nonlinear interactions.

hydrodynamic regime with $\gamma \sim \omega$, however, one estimates $\tau_{\mathrm{ac}}^{\mathrm{QL}} \sim \omega^{-1} \sim \gamma^{-1}$. In the remainder of this paragraph, I write $\mathrm{Re}\,\eta \to \eta$. A measure of the true τ_{ac} is η^{-1}. From the steady-state spectral balance, $\eta = \eta^{\mathrm{nl}} - \gamma = F^{\mathrm{nl}}/C > 0$. Calculations show that the inequality is well satisfied, so $\eta^{\mathrm{nl}} \gg \gamma$ and $\eta \sim \eta^{\mathrm{nl}}$. Thus $\tau_{\mathrm{ac}}^{\mathrm{true}}/\tau_{\mathrm{ac}}^{\mathrm{QL}} \sim \eta^{-1}/\gamma^{-1} \sim \gamma/\eta^{\mathrm{nl}} \ll 1$. That η^{nl} and F^{nl}/C can be separately much larger than γ was previously noted by Horton (1986).

Thus one concludes that the gross depression of the true hydrodynamic flux from the quasilinear estimate can be explained and satisfactorily predicted by a second-order closure like the RMC without explicit reference to coherent structures. This conclusion, however, is not definitive [247]; further analysis would be worthwhile.

It is interesting to verify that qualitative spectral transfer arguments are compatible with the calculated spectral levels. For a spectrum characterized by a single wave number $\overline{k}$, growth rate $\overline{\gamma}$, and rms velocity $\overline{V}$, forcing can be estimated as $\overline{\gamma}\overline{V}^2$, and transfer can be estimated as $\overline{V}^2(\tau_{\mathrm{ac}}/\tau_{\mathrm{eddy}})\tau_{\mathrm{eddy}}^{-1}$; balance gives $\overline{\gamma} \sim (\tau_{\mathrm{ac}}/\tau_{\mathrm{eddy}})\tau_{\mathrm{eddy}}^{-1}$. In the strong-turbulence (hydrodynamic) limit with $\tau_{\mathrm{ac}} \sim \tau_{\mathrm{eddy}}$, the conventional expression (122) for τ_{eddy} leads to

$$E(\overline{k}) \sim \overline{k}^{-3}\overline{\gamma}^2 = E^{\mathrm{ml}}(\overline{k})(\overline{\gamma}/\overline{\varpi})^2 \quad (\overline{\gamma}/\overline{\varpi} \geq 1), \tag{402}$$

where the result has been normalized to the mixing-length level $E^{\mathrm{ml}} \doteq \overline{k}^{-1}(\overline{\varpi}/\overline{k})^2$, at which $V_E \sim V_*$. In the weak-turbulence (adiabatic) limit, where $\gamma \ll \omega$, an argument based on the randomly directed propagation of wave packets leads (Hu et al., 1997) for moderately dispersive waves to $\tau_{\mathrm{ac}} \sim \omega^{-1}$ and to

$$E(\overline{k}) \sim E^{\mathrm{ml}}(\overline{k})(\overline{\gamma}/\overline{\varpi}) \quad (\overline{\gamma}/\overline{\varpi} \ll 1). \tag{403}$$

Hu et al. (1997) estimated these levels for the parameters of their simulations and found reasonable agreement with the detailed predictions of both the closure and DNS.

8.6 Conclusion: Systematic statistical closures in plasma physics

This concludes the discussion of modern studies of systematic second-order statistical closures in plasma physics. In summary, their development can be loosely viewed as consisting of several stages [Figs. 35 (p. 261) and 36 (p. 262)]: (i) the inception in the late 1950s of the DIA for the NSE; (ii) intense development during the 1960s of fluid closures, but virtual neglect of those techniques by plasma physicists; (iii) Dupree's qualitative ideas of resonance broadening (approximate treatment of coherent response) and clumps (approximate treatment of incoherent response) in the late 1960s and early 1970s); (iv) plasma-related development of formal renormalization procedures *a la* Martin et al. (1973) by DuBois, Krommes, and co-workers in the mid- to late 1970s; (v) early explorations, mostly

[247] Hu et al. (1997) remarked, "This conclusion does not necessarily contradict the conclusions of a recent analysis of transport in the HW model [using the methods of biorthogonal decomposition (BOD) and conditional averaging (de Wit et al., 1995)] that in the small-α limit the transport depression was due to the 'influence of large-scale and long-lived vortex structures' in the near-Gaussian density and potential field. We argue that in such a near-Gaussian field the main contributions of these vortex structures (quantified by the BOD method) to the second-order statistics, like the transport coefficient, have been captured and represented correctly by the second-order closure calculations."

analytical, of primitive Markovian closures by Waltz, Horton, Diamond, and others in the 1980s; (vi) the recognition by Bowman, Krommes, and Ottaviani (approximately 1990) that the standard EDQNM was nonrealizable in the presence of linear waves; (vii) the development of the realizable Markovian closure in the early 1990s; and (viii) numerical investigations of the RMC culminating in the work by Hu et al. (1997) that demonstrated excellent agreement between statistical closure and direct numerical simulations. For simple nonlinear paradigms, it seems fair to say that a sensible framework for using analytical theory to predict turbulent plasma spectra and transport is in place.

Although space precludes a detailed discussion, it is appropriate to close this section by mentioning the work of Chandran (1996), who in Chap. 5 of his dissertation described analytical and numerical applications of the DIA and RMC to the nonlinear MHD dynamo, which arises in discussions of the origin of the galactic magnetic field (Kulsrud, 1999). His discussion provides a useful summary of the formalism, conceptual ideas, and numerical issues.

9 SUBMARGINAL TURBULENCE

"One of the characteristic features of transport phenomena in confined plasmas is that the plasma inhomogeneity is the order parameter that governs the transport. The fluctuations are self-sustained: they can be driven through subcritical excitation, being independent of the linear stability of the confined plasma. ... We try ... to put forward the point of view that the plasma structure, fluctuations and turbulent transport are regulating each other ... ." — ***Itoh et al. (1999).***

The picture of steady-state turbulence as arising from the balance between linear forcing, nonlinear transfer, and linear dissipation has frequently been referred to in this article. In Sec. 1.2 (p. 10) the rich linear behavior of the plasma medium was emphasized. It is difficult to overstate the degree to which plasma-physics research has focused on linear theory, a point previously made by Montgomery (1977).

The conventional interpretation of linear forcing is that fluctuations arise from linear instability due to a linear growth rate $\gamma_{\boldsymbol{k}}^{\text{lin}}$ that is positive at one or more wave numbers $\boldsymbol{k}$, and there is a widespread belief in the plasma community that linear stability should lead to complete suppression of turbulence. Two important driving mechanisms for microinstabilities in magnetically confined plasmas are ion temperature gradients (Sec. 2.4.6, p. 42) and the toroidal precession drift of magnetically trapped particles. In configurations with reversed magnetic shear, the sign of the precession drift is reversed, theoretically improving the linear stability of trapped-particle modes. It can also be argued that such configurations should also reduce the ITG drive. For these and other reasons, it was predicted (Kessel et al., 1994) that reversed-shear configurations should lead to important improvements in confinement, and this idea was given dramatic practical support by experiments on so-called enhanced-reversed-shear operating regimes for tokamaks (Levinton et al., 1995). Direct measurements of fluctuation spectra showed strong suppression of fluctuations in the core region of reversed shear, well correlated with enhanced confinement.

Nevertheless, the link between positive linear growth rates and turbulence (and, presumably, transport) is not entirely well founded. It is theoretically possible that linearly *stable* systems can exhibit large-amplitude, steady-state turbulence (called *submarginal* as opposed to the linearly unstable *supermarginal* variety), and this phenomenon has been observed in a variety of neutral-fluid experiments and plasma-physics computer simulations. Indeed, Itoh et al. (1999) have suggested, by analogy with pipe flow,[248] that submarginal turbulence may be the *generic* state of inhomogeneous confined plasmas. This view may be too extreme since there is no lack of linearly unstable processes in the toroidal systems typically used for magnetic confinement. Nevertheless, it presents an interesting challenge. At the present time, detailed understanding of the mechanisms underlying submarginal turbulence is quite incomplete; accordingly, the following discussion is introductory in nature and describes topics substantially more immature than most of the others reviewed in this article. It therefore identifies an interesting and fertile area for future work.

I shall briefly discuss several general mechanisms for submarginal turbulence: (i) transition to turbulence via subcritical bifurcations; (ii) a "mostly linear" bootstrap mechanism; and (iii) a more specific "roll–streak–roll" scenario for shear flows that builds on the insights provided by item (ii). First, however, it is useful to identify situations for which submarginal turbulence is impossible.

9.1 Energy stability vs linear instability

Consider a nonlinear equation depending on a dimensionless parameter λ (for example, the Reynolds number $\mathcal{R}$ or normalized density gradient κ). Assume that one has derived the balance equation for an energylike quantity $\mathcal{E}$ quadratic in the fluctuations. As discussed in Sec. 2.1.1 (p. 23) for the Navier–Stokes paradigm, the general form of such an equation is

$$\partial_t \mathcal{E}(\boldsymbol{x}, t) = \mathcal{P}(\boldsymbol{x}, t) - \mathcal{T}(\boldsymbol{x}, t) - \mathcal{D}(\boldsymbol{x}, t), \tag{404}$$

where $\mathcal{P}$ is a *production* term (involving a source of free energy), $\mathcal{T}$ is a *transfer* term (involving the divergence of a triplet correlation function), and $\mathcal{D}$ is a *dissipation* term. If $\mathcal{E}$ is chosen appropriately, it is usually possible to employ global conservation laws and boundary conditions to annihilate the transfer term by an appropriate spatial average that I shall denote by an overline. Then

$$\partial_t \overline{\mathcal{E}}(t) = \overline{\mathcal{P}}(t) - \overline{\mathcal{D}}(t). \tag{405}$$

An *energy stability* boundary or threshold λ_E is defined (Joseph, 1976) by the λ_E that satisfies

$$1 = \min(\overline{\mathcal{D}}_{\lambda_E} / \overline{\mathcal{P}}_{\lambda_E}), \tag{406}$$

the variational problem being conducted in the space of all fields that satisfy the boundary conditions and kinematic constraints such as $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. If $\overline{\mathcal{D}}/\overline{\mathcal{P}}$ monotonically increases as λ decreases,[249] then for $\lambda < \lambda_E$ the energy of perturbations of arbitrary amplitude monotonically decreases with time. Fluctuations damp to zero, so turbulent steady states are impossible.

Linear stability theory defines another characteristic value λ_c, the threshold for linear instability, which must necessarily obey $\lambda_c \geq \lambda_E$. The regime $\lambda_E < \lambda_c$, if it exists, may support submarginal turbulence for $\lambda \geq \lambda_s$, for some λ_s lying in the range $\lambda_E < \lambda_s < \lambda_c$. Note that $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$ are constructed from the linear dynamics. Therefore if no information is lost in the transition from the

[248] It is believed that flow in a circular pipe is linearly stable at all Reynolds numbers; see, for example, Landau and Lifshitz (1987).

[249] In the Navier–Stokes case treated by Joseph (1976), $\overline{\mathcal{D}}$ is multiplied by $\mathcal{R}^{-1}$; in characteristic plasma problems, $\overline{\mathcal{P}}$ is multiplied by a profile gradient κ. In such cases the monotonicity condition is satisfied.

primitive amplitude equation to the energy stability problem, then λ_c and λ_E coincide. That is the case if the linear operator is *symmetric* [see the discussions by Waleffe (1995, Sec. IV) and Grossmann (1996)]. Accordingly, submarginal turbulence can occur only for nonsymmetric linear operators. Such operators are common in the problems of multiple coupled fields ubiquitous in plasma physics. In fluid problems the nonsymmetry frequently arises because of boundary-condition constraints.

Henningstone and Reddy (1994) proved a stronger condition: submarginal turbulence is possible only if the linear operator is *non-normal*. A non-normal operator L obeys $LL^\dagger \neq L^\dagger L$; alternatively, it is an operator whose eigenvectors are not orthogonal.[250] A simple example is given in Sec. 9.5 (p. 215). In plasma physics, non-normality frequently arises from diamagnetic terms; see the $\kappa\,\partial_y\varphi$ term in the HW density equation (52b).

9.2 Evidence for submarginal turbulence

It has long been known that in particular geometries fluid flows are linearly stable at all Reynolds numbers (Landau and Lifshitz, 1987); examples include planar Couette flow and Hagen–Poiseuille (pipe) flow. Since they are experimentally observed to be turbulent at finite Reynolds numbers, those flows are submarginal. Many references to original work can be found in the various papers of Waleffe and co-workers cited below; see also the short account by Grossmann (1996) and the colloquium by Grossmann (2000).

In plasma physics, evidence for submarginal turbulence has accumulated on the basis of computer simulations of MHD equations (Waltz, 1985), drift waves (Scott, 1992; Drake et al., 1995), resistive pressure-gradient-driven modes (Carreras et al., 1996b), and current-diffusive interchange modes (Itoh et al., 1996). Additional theoretical analysis of some models, using simple nonlinear dispersion relations [see the discussions in Sec. 4.3.5 (p. 114) and footnote 150 (p. 115)], was described by Itoh et al. (1999) and Yoshizawa et al. (2001). Scott (1992) proposed a qualitative explanation involving inverse cascade of φ, direct cascade of n and T, and the role of magnetic shear; Drake et al. (1995) discussed a more specific mechanism involving vortex peeling that will be elaborated in Sec. 9.6 (p. 216). One should also mention the earlier simulations of Berman et al. (1982), interpreted by Tetreault (1983) as a nonlinear clump instability. Tetreault (1988) discussed an MHD variant. Although some of that work is quite physically motivated and mathematically detailed, none of those discussions provided a solid, systematic analysis of the underlying nonlinear dynamical processes that would satisfy an expert in dynamical systems theory. Progress in that direction is described in Sec. 9.6 (p. 216).

9.3 Introduction to bifurcation theory

The bifurcation theory of ordinary differential equations has been studied extensively; see, for example, Guckenheimer and Holmes (1983) and Holmes et al. (1996). The following introductory

[250] It is a common misconception that normal operators are self-adjoint. A simple counterexample is $\mathsf{L} = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$, which is not self-adjoint with the usual complex-valued scalar product but is normal: $\mathsf{L}\cdot\mathsf{L}^\dagger = \mathsf{L}^\dagger\cdot\mathsf{L} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The complete set of orthogonal eigenfunctions is $\boldsymbol{e}_\pm = 2^{-1/2}(1, \pm 1)^T$; the associated eigenvalues are $\lambda_\pm = 1 \pm \mathrm{i}$.

discussion is highly condensed and incomplete. Consider a scalar amplitude variable A depending on a parameter λ, and let A obey the prototypical dynamical equation

$$\partial_t A = (\lambda - \lambda_c)A - \beta A^3 = (\lambda - \lambda_c - \beta A^2)A, \tag{407}$$

where λ_c and (positive) β are constants. For $\lambda < \lambda_c$, A decays to zero; the system is linearly (and nonlinearly) stable. For $\lambda > \lambda_c$, the state $A = 0$ remains an equilibrium solution but is linearly unstable; however, a stable steady-state nonlinear equilibrium is achieved at $A = \pm[(\lambda - \lambda_c)/\beta]^{1/2}$. The equilibrium solutions are diagrammed in Fig. 27 (p. 213). The appearance of a second equilibrium solution at the critical point $\lambda = \lambda_c$ is called a *bifurcation*; Eq. (407) provides an example of a *supercritical bifurcation.* An early calculation of a supercritical bifurcation in plasma physics was by Hinton and Horton (1971).

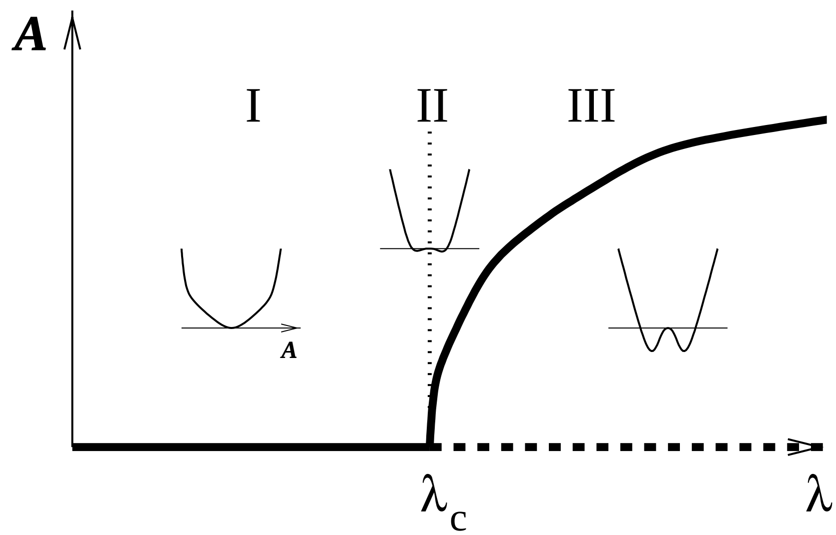

Fig. 27. Supercritical bifurcation. In regime I, the zero-amplitude solution is absolutely stable (solid line). In regime III, the zero-amplitude solution is linearly unstable (dashed line) whereas a nonzero-amplitude branch is stable. (A symmetrical branch $A \to -A$ is not shown.) The bifurcation point is at ($\lambda = \lambda_c, A = 0$). Also shown are the qualitative shapes of the effective potentials $V_\lambda(A)$, where $\partial_t A = -\partial V_\lambda/\partial A$.

The nonlinear structure of Eq. (407) is insufficiently rich to support turbulence; there are no fluctuations in the steady state. In more realistic physical systems additional bifurcations ultimately terminating in chaos may occur as λ is increased farther beyond λ_c.[251] Nevertheless, the behavior of Eq. (407) fosters the prevailing intuition that linear instability is responsible for nonzero-amplitude steady states.

However, *subcritical* bifurcations are also possible. Consider, for example,

$$\partial_t A = (\lambda - \lambda_c + \beta A^2 - cA^4)A, \tag{408}$$

[251] There is an extensive literature on the transition to turbulence, a topic that would carry us far beyond the bounds of this article. In addition to Guckenheimer and Holmes (1983), see, for example, Martin (1982) and more introductory discussions by Eckmann (1981), Ott (1981), Lichtenberg and Lieberman (1992), and Ott (1993).

where β and c are positive constants. The bifurcation diagram for the equilibria of Eq. (408) is shown in Fig. 28 (p. 214). Although the zero-amplitude equilibrium is again linearly unstable for $\lambda > \lambda_c$, a nonzero-amplitude stable equilibrium exists for $\lambda > \lambda_s \doteq \lambda_c - \beta^2/4c$. The regime $\lambda_s \leq \lambda < \lambda_c$ is linearly stable but nonlinearly unstable. This can be seen most vividly by writing $\partial_t A = -\partial V_\lambda(A)/\partial A$ and examining the qualitative shape of the potential V as λ is varied. Representative potentials are also shown in Figs. 27 and 28.

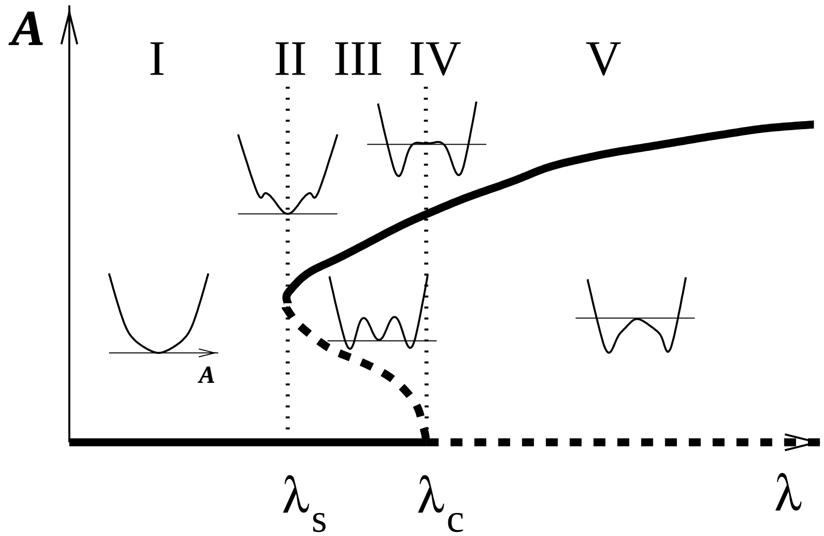

Fig. 28. Subcritical bifurcation. In regime III, stable nonzero-amplitude solutions are possible even though the zero-amplitude solution is linearly stable. (A symmetrical branch $A \to -A$ is not shown.)

I again remark that the simple model (408) does not support turbulence; the solid curves in Fig. 28 describe stable attracting states. Nevertheless, its behavior strongly suggests that submarginal turbulence should be possible. For example, in important and elegant pioneering work McLaughlin and Martin (1975) [see also McLaughlin (1974)] showed that the subcritical-bifurcation route to chaos is realized by the famous Lorenz system of equations (Lorenz, 1963). More detailed and modern discussion of that problem was given, for example, by Ott (1993).

Looking ahead to a later discussion of statistical descriptions, one can give another interpretation of Figs. 27 and 28. Instead of interpreting A as a dynamical amplitude, let it be a statistically averaged *fluctuation intensity*. Then the diagrams no longer describe the detailed dynamical bifurcations, but offer a usefully coarser-grained description if there is, in fact, underlying chaos. In particular, Fig. 28 with A interpreted as intensity shows the expected dependence on λ of a closure prediction for the steady-state fluctuation level.

9.4 Digression: Plasma turbulence and marginal stability

The marginal-stability scenario of Manheimer et al. (1976) and Manheimer and Boris (1977) can be interpreted in terms of bifurcations and the transition to turbulence. The hypothesis states that turbulent transport adjusts its magnitude to bring the background plasma profiles to linear marginal stability ($\lambda \approx \lambda_c$). If true, this would allow turbulent transport coefficients to be found from solely linear calculations, a compelling simplification. However, if the transition to turbulence is supercritical (an assumption made implicitly in Manheimer's analysis), then a plasma hovering precisely at marginal stability cannot be turbulent, so the scenario is *prima facie* inconsistent. Of course, if one assumes that the series of supercritical bifurcations to turbulence are compressed into a very narrow region just above λ_c, there may be little difference in parameter space between the points of marginal stability and fully developed turbulence.

If the transition to turbulence is subcritical, then either marginal stability has no relevance or the subcritical regime is so narrow that its left-hand edge practically coincides with λ_c.

Even if the transitional regime is very narrow (which must ultimately be proven by detailed analysis), the marginal-stability hypothesis need not apply. Consider a diffusion equation in which the diffusion coefficient turns on only when the background gradient exceeds a critical value; for definiteness, let the equation be forced to carry a fixed amount of turbulent flux Γ injected at the left-hand boundary. The solution of such an equation can for sufficiently large Γ be driven arbitrarily far from the critical profile. The larger the intrinsic diffusion, the closer the profile will stay to critical, but the issue is a quantitative one. For example, detailed numerical solutions of ITG equations for realistic tokamak parameters [see, for example, Kotschenreuther et al. (1995)] predict profiles that are nearly marginal in the core but unambiguously supermarginal toward the edge. For some further discussion, see Krommes (1997c) and the analysis by Krommes (1997b) of the role of instability thresholds in simple stochastic models.

In summary, the marginal-stability hypothesis camouflages important nonlinear considerations about the transition to turbulence and assumes that the turbulent diffusion above threshold is essentially infinitely large. The fidelity of the hypothesis must be confirmed by detailed nonlinear analysis of each particular situation, which determines the dependence of the turbulent flux on the driving profiles. The existence of submarginal turbulence, for which the linear stability threshold is irrelevant, shows that the hypothesis cannot be generally true as a governing principle for turbulent plasmas.

9.5 An "almost-linear" route to submarginal turbulence

I now return to dynamical possibilities for submarginal fluctuations. In addition to subcritical bifurcations, another possible mechanism for nonlinearly self-sustained turbulence was discussed by Trefethen et al. (1993), Baggett et al. (1995), and others. It relies on the ability of non-normal linear operators to transiently amplify perturbations to large levels even though they ultimately decay. The simplest nontrivial example of such an operator L (which would appear in the dynamical equation $\partial_t \psi = \mathsf{L}\psi + \text{n.l. terms}$) is $\mathsf{L} = \begin{pmatrix} -\alpha & 0 \\ \kappa & -\beta \end{pmatrix}$, whose linear eigenvalues[252] are $-\alpha$ and $-\beta$. Even for positive α and β (stable linear spectrum), the Green's function

$$\exp(\mathsf{L}t) = \begin{pmatrix} e^{-\alpha t} & 0 \\ \kappa(e^{-\beta t} - e^{-\alpha t})/(\alpha - \beta) & e^{-\beta t} \end{pmatrix} \tag{409}$$

exhibits transient growth in the off-diagonal component even for distinct α and β, the maximum occurring at $t_* = \ln(\alpha/\beta)/(\alpha - \beta)$. If the nonlinearity can feed back a portion of that transient sufficiently rapidly and in the correct phase, nonlinear self-sustainment can result. A three-mode

[252] The eigenvectors corresponding to $(-\alpha, -\beta)$ are $\boldsymbol{e}_{-\alpha} = (1,\, \kappa/(\beta - \alpha))^T$ and $\boldsymbol{e}_{-\beta} = (0,\, 1)^T$. Since these cannot be made orthogonal for $\kappa \neq 0$, L is non-normal as claimed. Alternatively, one can directly verify that $\mathsf{L} \cdot \mathsf{L}^\dagger \neq \mathsf{L}^\dagger \cdot \mathsf{L}$.

dynamical model that illustrates the possibility was studied in detail by Baggett et al. (1995). A pedagogical discussion was given by Grossmann (2000).

A frequent source of non-normality in models of magnetized plasma is the $\boldsymbol{E} \times \boldsymbol{B}$ advection of a mean variable ψ (e.g., $\psi = n$ or T), which couples fluctuations $\delta\psi$ to the fluctuating potential $\delta\varphi$; this usually shows up as an off-diagonal effect. For example, the HW equations are non-normal, as discussed by Camargo et al. (1998). Transient amplification significantly exceeding the linear growth rate was found in the hydrodynamic regime. (In the adiabatic regime the HW equations approach the one-field HM model; one-field equations are trivially normal, so do not exhibit such amplification.) Properties of an electromagnetic extension of that model were discussed by Camargo et al. (2000); it was found that electromagnetic effects enhance the non-normality.

In considering the relevance of the almost-linear mechanism, note that the issue is not the non-normality of the linear operator *per se*, as that is always required for submarginal turbulence. The question is, is the scenario of transient amplification *plus feedback* relevant to self-sustainment? Waleffe (1995) emphasized that it is crucial that the fed-back signal contain projections onto the transiently unstable directions. He argued that the model proposed by Baggett et al. (1995), which involved a specific assumed form for the nonlinearity, failed to capture important properties of the Navier–Stokes operator, and he was in general relatively critical of the almost-linear mechanism for realistic neutral-fluid shear flows. Clearly considerable additional work must be done to elucidate the relevance of that scenario for any particular physical problem. A specific alternative is discussed in the next section.

9.6 The roll–streak–roll scenario and its generalization to drift-wave turbulence

In the present section I deviate from the focus on statistical methods in order to elaborate one plausible physical mechanism for nonlinear self-sustainment. Such discussion is useful because clearly a statistical description of submarginal turbulence should incorporate salient features of the physical dynamics.

In a beautiful synthesis of numerical and analytical work, Hamilton et al. (1995) isolated a "roll–streak–roll" scenario for self-sustaining planar Couette flow that did not rely on the specific feedback mechanism proposed by Baggett et al. (1995). Their work continued a long series of studies of so-called *streaks* in shear flows, both experimental (Kline et al., 1967; Kim et al., 1970) and analytical (Landahl and Mollo-Christensen, 1992). One works in a planar geometry in which[253] $(x, y, z) =$ (inhomogenity, spanwise, streamwise). The basic idea is that even though the steady-state velocity profile $\boldsymbol{U} = U'x\,\hat{\boldsymbol{z}}$ is linearly stable at all Reynolds numbers, streamwise rolls (vortices) with $k_z = 0$ are linearly *unstable* to $k_z \neq 0$ perturbations; the resulting fluctuations then beat together to regenerate the original rolls. Waleffe (1997) discussed the process in detail, and proposed and analyzed various low-dimensional dynamical models that seemed to capture the essence of the mechanism.

Coincidentally in the same year as the work of Hamilton *et al.*, Drake et al. (1995) performed numerical simulations of 3D drift-wave turbulence linearly stabilized by magnetic shear. They showed that sufficiently large levels of fluctuations persisted in the face of the shear stabilization, and suggested a simple analytical model of a nonlinear instability that they physically described as *vortex peeling* (Drake et al., 1992).

[253] As discussed in footnote 12 (p. 14), the convention in the fluids literature is instead $(x, y, z) =$ (streamwise, inhomogenity, spanwise); that is, the fluid coordinates are the plasma ones cyclically permuted by $1 \bmod 3$.

Krommes (1999a) argued that the physical mechanisms described by Hamilton et al. (1995) and Drake et al. (1995) are closely related. If for plasmas one replaces "streamwise" (z) by "magnetic-field direction," then both scenarios begin with z-aligned, $k_z = 0$ vortices that are subsequently unstable to $k_z \neq 0$ perturbations. The details of those instabilities differ substantially for the fluid and the plasma situations because of the nature of the collective modes (drift waves for the plasmas) supported by the two mediums. Nevertheless, in both cases $k_z \neq 0$ fluctuations can beat together to regenerate the original $k_z = 0$ vortices.

I shall sketch the self-sustainment scenario in the context of the HW equations in the absence of magnetic shear. (More appropriately, one should study the equations written by Drake et al. (1995), which include sound-wave propagation and magnetic shear; however, that analysis is very difficult and has not yet been done.) In the absence of magnetic shear, those equations already possess a linear instability. Even if that is artificially suppressed, however, the analysis to follow predicts nonlinearly self-sustained fluctuations. To study those, one begins with the equation for total density $N = \langle n \rangle + \delta n$:

$$\partial_t N + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} N = \widehat{\alpha}(\varphi - N) + D_{\mathrm{cl}} \nabla_\perp^2 N. \tag{410}$$

Postulate that a $k_z = 0$ potential $\overline{\varphi}(x, y)$ is somehow excited, and assume that its advective effect is dominant; i.e., $\boldsymbol{V}_E(\boldsymbol{x}) \approx \overline{\boldsymbol{V}}_E(x, y)$. Then Eq. (410) can be averaged over z:

$$\partial_t \overline{N} + \overline{\boldsymbol{V}}_E \cdot \boldsymbol{\nabla} \overline{N} = D_{\mathrm{cl}} \nabla_\perp^2 \overline{N}. \tag{411}$$

(The $\widehat{\alpha}$ term rigorously disappears under this average.) Equation (411) is a passive advection–diffusion equation that can be solved if the advecting potential $\overline{\varphi}$ is given. (I shall solve it in steady state, even though that assumption is really valid only for statistical averages.) The idea is to postulate a plausible form for $\overline{\varphi}$, then show that it can be appropriately regenerated. To determine $\overline{\varphi}$, consider the z average of the vorticity equation:

$$\partial_t \overline{\varpi} + \overline{\boldsymbol{V}_E \cdot \boldsymbol{\nabla} \omega} = \mu_{\mathrm{cl}} \nabla_\perp^2 \overline{\varpi}. \tag{412}$$

This equation is rigorous; again the $\widehat{\alpha}$ term has disappeared. Following Waleffe, one chooses $\overline{\varphi}$ to be the lowest-order eigenfunction of the system $\nabla_\perp^2 \overline{\varpi} = \lambda \overline{\varpi}$, $\overline{\varpi} = \nabla_\perp^2 \overline{\varphi}$ with no-slip boundary conditions. The solution is given by Eq. (7) of Krommes (1999a). Then Eq. (411) can be solved in steady state with the boundary conditions $N(x = 1) = 0$, $N(x = -1) = 2$. In the absence of the advection term, such boundary conditions lead to a solution with a gradient solely in the x direction. The advection rotates that into a y-dependent solution constrained by the classical diffusion, as shown in Fig. 29 (p. 218).

Note that the x-averaged y dependence $\overline{N}(y)$ has a characteristically triangular shape. Such a shape also figured importantly in the model proposed by Drake et al. (1995). Those authors, who omitted classical dissipation altogether, suggested that it would arise because of turbulent diffusion and did not attempt to calculate the shape precisely; here the shape is pinned to classical transport and a more serious calculation is possible. Given the shape, the next step is to examine the linear stability of this advected density profile. That can be done numerically (Krommes, 1999a). Drake et al. (1995) essentially used $\overline{N}(y)$ to introduce a diamagnetic term $V_{E,y} \partial_y \overline{N}(y)$ in place of the conventional $V_{E,x} \partial_x \overline{N}(x)$; the new term drives unstable drift waves propagating in the x direction (orthogonal to the y direction associated with the usual drift waves driven by gradients in the x direction). Note that although these fluctuations are explicitly driven by $\overline{N}(y)$, they are implicitly driven by the original

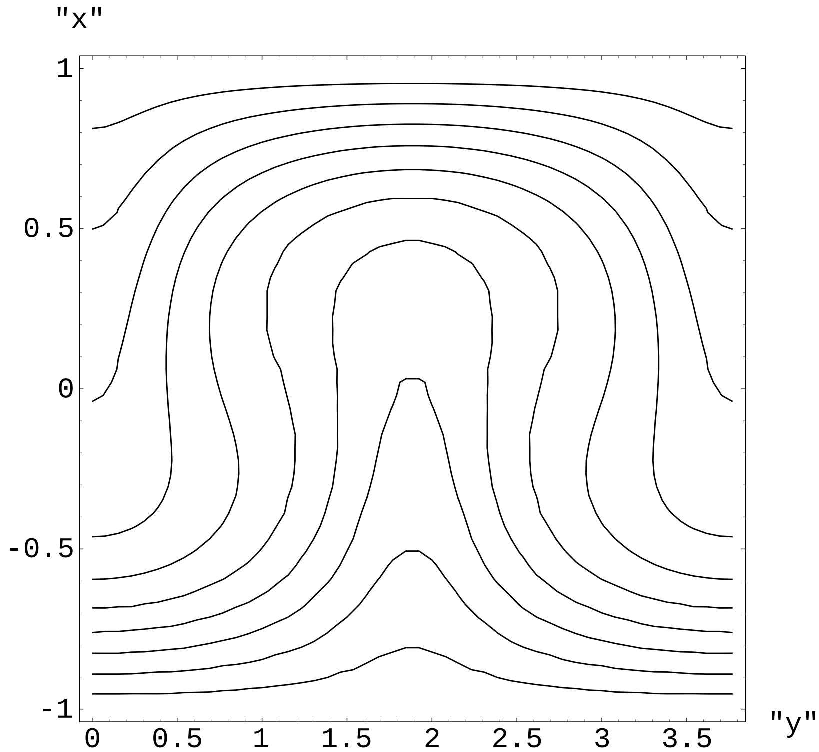

Fig. 29. Solution of the steady-state advection–diffusion equation (411) [after Fig. 2 of Waleffe (1997) and Fig. 1(a) of Krommes (1999a), used with permission]. Note the generation of y dependence.

x-directed gradient imposed by the boundary conditions.[254] This is where the source of free energy (linear forcing or non-normality) enters the formalism, even though the x gradient may be linearly stable.

Finally, one attempts to close the loop by using the unstable linear eigenfunctions, labeled by superscript 1, as a source on the right-hand side of the steady-state version of Eq. (412), namely,

$$\mu_{\rm cl}\nabla_\perp^2\overline{\varpi} = \overline{\boldsymbol{V}_E^{(1)}\cdot\boldsymbol{\nabla}\omega^{(1)}}. \tag{413}$$

One must show that the solution of this equation for $\overline{\varpi}$ has a shape similar and sign identical to those of the original assumed potential. The result of this exercise is shown in Fig. 30 (p. 219); the agreement is quite acceptable.

These calculations demonstrate a plausible self-sustainment mechanism, although they neither constitute a rigorous proof that these particular interactions are dominant nor accomplish a proper steady-state statistical analysis. Additional insights can be gained by using Galerkin truncation to derive a low-dimensional model that can be studied in its own right with numerical computation and the techniques of dynamical systems theory. A step in this direction was made by Son (1998), who derived a nine-dimensional generalization of the six-dimensional model of Drake et al. (1995). Because that calculation is ongoing, I do not discuss the details here. However, preliminary results show that Lorentz-type chaotic dynamics can spontaneously arise for sufficiently large density gradients, adding

[254] This important result was obscured in the model of Drake et al. (1995), which contained no x gradient but postulated the existence of the y dependence.

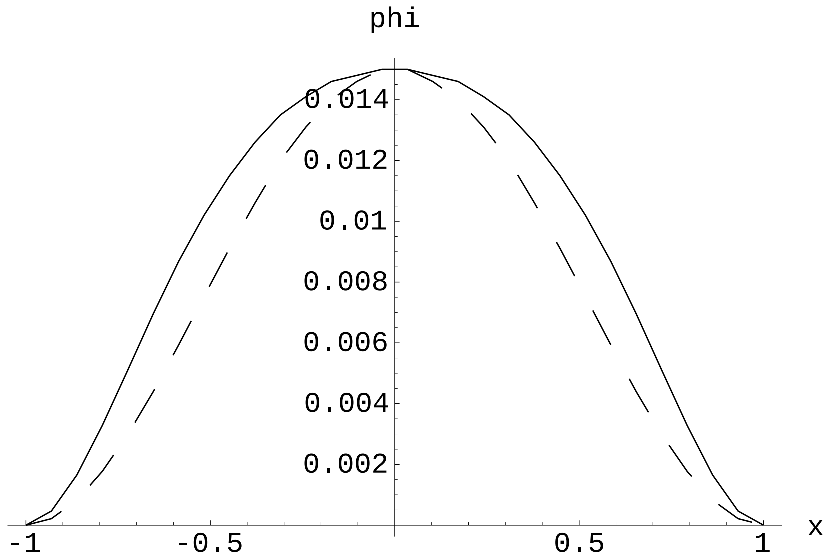

Fig. 30. Solid curve, steady-state solution of Eq. (412) with nonlinear forcing determined by the most unstable eigenfunction; dashed curve, original potential. After Fig. 1(b) of Krommes (1999a), used with permission.

further support to the belief that the generalized roll–streak–roll scenario is relevant to nonlinear plasma self-sustainment.

It seems clear that some sort of model at this level of physical complexity is necessary to explain submarginal turbulence in continuum systems with advective nonlinearity. However, it has been shown that certain discrete sandpile models can also exhibit submarginal profiles. For further discussion, see Sec. 12.4 (p. 241).

9.7 Bifurcations and statistical closures

There is a large gap between the elementary ideas of subcritical bifurcations or transient amplification—both of which describe detailed dynamical properties of primitive amplitude equations—and observations or predictions of submarginal turbulent states. The simplest statistical measure of turbulence is (time- or ensemble-averaged) fluctuation intensity. Because averaging washes out many fine-grained dynamical details, it may be that the simplest analytical route to the determination of submarginal regimes is *via* statistical procedures.

The questions of whether (practical) statistical closures adequately represent any kind of transition to turbulence or capture essential features of submarginal turbulence are difficult. In general, the complicated nonlinear structures of the DIA and similar closures are daunting. Such nonlinear equations may exhibit their own bifurcations. Kraichnan (1964d) noted that the DIA may support static covariance solutions[255] that presumably become unstable at sufficiently large Reynolds number. Additional discussion in this vein was given by Herring (1969).

Unfortunately, the presence or absence of bifurcations in a statistical closure may have nothing to

[255] Specifically, $C_{\boldsymbol{k},\omega} = 2\pi\delta(\omega)$; $R_{\boldsymbol{k},\omega}$ contains nonzero ω components. It can be shown that static solutions are impossible for forced, three-mode, energy-conserving models. In general, the stability condition for static solutions can be reduced to the study of the eigenvalues of a complicated integral operator in frequency space and a matrix operator in the wave numbers.

do with the behavior of the original primitive amplitude equations. Very little in the way of systematic study has been done in this area. McLaughlin (1974) studied the DIA for the Lorenz equations in detail; he concluded that the DIA was not a particularly faithful theory and did not capture the subcritical onset of the strange attractor.

Nevertheless, experience has shown that second-order statistical closures are quite robust; a closure-based prediction of submarginal turbulence must not be dismissed out of hand. Recently K. Itoh, S. Itoh, and co-workers have explored in a series of papers, discussed by Itoh et al. (1999) and summarized by Yoshizawa et al. (2001), the predictions of a Dupree-style diffusive renormalization of equations characteristic of current-diffusive interchange modes and similar fluctuations. Regimes of submarginal turbulence are predicted quite generally, and agreement between theory and computer simulation is found. It remains a major challenge of modern statistical plasma theory to justify the several substantial approximations [256] and embed the formalism in a systematic theory of nonlinearly self-sustained fluctuations.

Submarginal states may be relevant to the theory of self-organized criticality. For further discussion, see Sec. 12.4 (p. 241).

10 HIGHER-ORDER STATISTICS, INTERMITTENCY, AND COHERENT STRUCTURES

The article thus far has strongly focused on second-order statistics and transport. I now turn to issues related to higher-order statistics and intermittency. In neutral-fluid theory exciting progress is presently being made in this area, some of which was reviewed by Frisch (1995) [see also the earlier and shorter discussion by Frisch (1980)]. In particular, the "anomalous" (non-Kolmogorov) inertial-range scaling of higher-order structure functions is slowly yielding to concerted analytic attacks. Although I shall mention some of that work briefly, an extensive review of it is out of place here since very little has been done on this topic in the context of plasma physics. In general, the topic of higher-order statistics in plasma problems is poorly developed, partly because a compelling "need to know" has not been established from the toroidal confinement experiments. A representative early work in which experimental data were unsuccessfully searched for coherent structures was by Zweben (1985). More recently, intermittency has been observed in tokamak configurations (Jha et al., 1992; Carreras et al., 1999a), and there have been suggestions that simple random-walk estimates for confinement may require modification, although this has not yet been convincingly demonstrated. In any event, the theoretical issues are challenging in their own right, and one can expect substantial further work in this area.

[256] In brief: The predictions are based on a very approximate theory of the infinitesimal response matrix for systems involving multiple coupled fields. A Markovian approximation is made, only the diagonal components of the nonlinear damping are considered, and those are approximated as turbulent diffusion operators that renormalize the classical dissipation. In the simplest version of the theory the nonlinear noise is neglected, leading to a nonlinear dispersion relation that in simple cases can be approximately solved algebraically. A general issue is to what extent consideration of incoherent noise changes the picture. Yoshizawa et al. (2001) reviewed attempts to include that noise. They concluded that it does not qualitatively change the results obtained from the nonlinear dispersion relation. From properties of the noise they also derived a PDF that can be used to predict the rate of transitions between various submarginal states.

10.1 Introductory remarks on non-Gaussian PDF's

It was emphasized in Sec. 3.5.2 (p. 59) that nonlinear dynamical evolution precludes Gaussianity. The spatially differential nature of the nonlinear advection–diffusion equations typical in practice makes prediction of the inherently non-Gaussian statistics very difficult; this is the central mathematical problem of turbulence theory. But nontrivial PDF's can arise in even much simpler contexts.

Consider, for example, a generalized flux $\Gamma \doteq \langle xy \rangle$, where x and y are jointly Gaussian with zero means. With $\boldsymbol{x} \doteq (x,\, y)^T$ the joint PDF $P(\boldsymbol{x})$ is fully characterized by the correlation matrix $\mathsf{C} \doteq \langle \boldsymbol{x}\,\boldsymbol{x}^T \rangle = \begin{pmatrix} \sigma_x^2 & \sigma_x\sigma_y\rho \\ \sigma_x\sigma_y\rho & \sigma_y^2 \end{pmatrix}$, where ρ is the correlation coefficient ($0 \le |\rho| \le 1$). The PDF of Γ follows readily from the formula (Sec. 3.5.1, p. 59) $P(\Gamma) = \langle \delta(\Gamma - \widetilde{x}\widetilde{y}) \rangle$, or upon normalizing x and y to σ_x and σ_y, respectively,

$$P(\Gamma) = \frac{1}{\pi(1-\rho^2)^{1/2}} \exp\left[\left(\frac{\rho}{1-\rho^2}\right)\Gamma\right] K_0\left(\frac{|\Gamma|}{1-\rho^2}\right), \tag{414}$$

where K_0 is the modified Bessel function of the second kind. $P(\Gamma)$ exhibits exponential tails as $|\Gamma| \to \infty$:

$$P(\Gamma) \to \frac{1}{(2\pi|\Gamma|)^{1/2}} e^{-|\Gamma|/(1\pm\rho)} \quad (\Gamma \to \pm\infty), \tag{415}$$

a common signature of non-Gaussian PDF's signifying a relatively enhanced probability of large-amplitude events and quantified by a non-Gaussian kurtosis [Eq. (96b)]; for $\rho = 0$, $K = 6$. As ρ increases from 0 to 1, $P(\Gamma)$ becomes increasingly skewed [257] (necessary for positive flux). Representative PDF's are displayed in Fig. 31 (p. 222).

The PDF (414) was used by Carreras et al. (1996a) [258] to interpret various experimental data. It must be cautioned, however, that in practice one or the other of x and y is generally non-Gaussian, so the above analysis based on jointly Gaussian variables does not apply. [For an example involving passive advection, see Krommes and Ottaviani (1999).] In the next sections I consider techniques for predicting the non-Gaussianity arising from nonlinear dynamical evolution.

10.2 The DIA kurtosis

The distinction between PDF methods and moment-based closures for nonlinear dynamical equations was drawn in Sec. 3.5 (p. 59). In the present section I consider the possibility that moment-based closures can sensibly predict the kurtosis, a typical fourth-order statistic. [259]

[257] In the limit $\rho \to 1$ (in which x and y become perfectly correlated, so that $\Gamma = \langle x^2 \rangle$), $P(\Gamma)$ vanishes for $\Gamma < 0$ and according to Eq. (415) properly approaches the χ^2 PDF $P(z) = H(z)(2\pi z)^{-1/2} e^{-\frac{1}{2}z}$, where $z \doteq x^2$.

[258] When comparing Eq. (414) to Eq. (7) of Carreras et al. (1996a), note that the latter authors use the definitions $W = (1-\rho^2)^{1/2}\sigma$ and $\gamma = -\rho$.

[259] For quadratically nonlinear equations, fourth-order statistics have special significance because knowledge of many-time moments through fourth order is sufficient for one to determine whether the dynamical equation

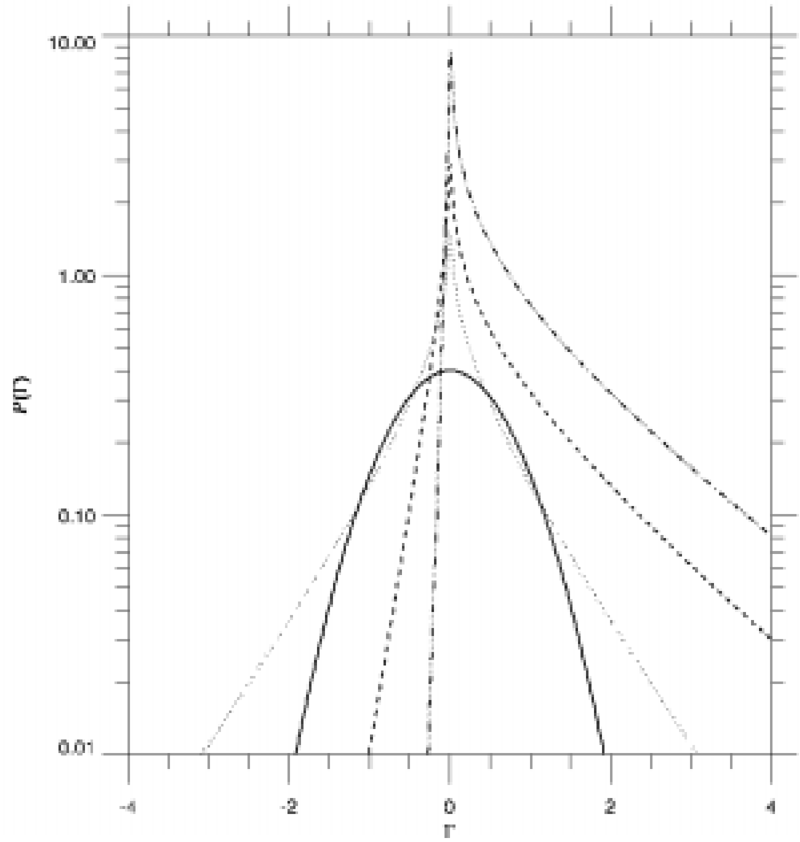

Fig. 31. Comparison of the non-Gaussian flux PDF's $P(\Gamma;\rho)$, for various correlation coefficients ρ, with a reference Gaussian (solid line). Dotted line, $\rho = 0$; dashed line, $\rho = 0.75$; dash-dotted line, $\rho = 0.95$. The logarithmic singularity $K_0(z) \sim -\ln z$ as $z \to 0$ is rendered finite by the graphics.

In familiar moment closures such as the DIA, triplet correlations are approximated in terms of two-point ones; closed equations for the first- and second-order cumulants result. From this point of view, it may seem that nothing can be said about fourth-order cumulants. Nevertheless, there are two arguments to the contrary: (i) The MSR formalism shows (Sec. 6.2.1, p. 153) that the n-point cumulant is the functional derivative of the cumulant of order $n-1$. (ii) The DIA is the exact statistical description of the random-coupling model (Sec. 5.2, p. 131). Although the latter observation is usually used to argue that the second-order statistics are realizable, realizable statistics of all orders can be predicted from the random-coupling amplitude equation. I shall now argue, following Krommes (1996), that these two points are intimately related.

Consider the generalized fourth-order statistic

$$Z(\underline{1}, \overline{\underline{1}}, t) \doteq \langle z(\underline{1}, t) z(\overline{\underline{1}}, t) \rangle = Z^G + Z^c, \tag{416}$$

where $z(1) \doteq \lambda(1, 2, 3)\psi(2)\psi(3)$, λ is a specified coefficient (not necessarily the mode-coupling coefficient in a dynamical evolution equation), and Z^G and Z^c are the Gaussian and cumulant

is satisfied in mean square (Kraichnan, 1958c). This observation was used by Kraichnan (1979) to motivate a computational scheme based on realizability inequalities. Further discussion of fourth-order statistics and related issues was given by Dubin (1984b).

contributions to Z, respectively. Chen et al. (1989a) used the RCM to find a formula for Z^c (which is basically an unnormalized flatness). One finds $Z^c = \sum_{\boldsymbol{k}} Z^c_{\boldsymbol{k}}$, where

$$Z^{\rm c}_{{\rm DIA},\boldsymbol{k}}(t) = 4\sum_{\boldsymbol{p},\boldsymbol{q}}\sum_{\overline{\boldsymbol{p}},\overline{\boldsymbol{q}}} d_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}\Bigg[d^*_{\boldsymbol{k},\overline{\boldsymbol{p}},\overline{\boldsymbol{q}}}\int_0^t {\rm d}t'\int_0^t {\rm d}\overline{t}'\, R_{\boldsymbol{p}}(t;t')C_{\boldsymbol{q}}(t;t')C^*_{\boldsymbol{k}}(t';\overline{t}')R^*_{\overline{\boldsymbol{p}}}(t;\overline{t}')C^*_{\overline{\boldsymbol{q}}}(t;\overline{t}') + \left(c^*_{\boldsymbol{k},\overline{\boldsymbol{p}},\overline{\boldsymbol{q}}}\int_0^t {\rm d}t'\int_0^{t'} {\rm d}\overline{t}'\, R_{\boldsymbol{p}}(t;t')C_{\boldsymbol{q}}(t;t')R^*_{\boldsymbol{k}}(t';\overline{t}')C^*_{\overline{\boldsymbol{p}}}(t;\overline{t}')C^*_{\overline{\boldsymbol{q}}}(t;\overline{t}')\right)^H\Bigg], \tag{417}$$

$c_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \doteq \lambda_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} M^*_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$, $d_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \doteq \lambda_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} M^*_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}}$, and $A^H_{\boldsymbol{p},\boldsymbol{q};\overline{\boldsymbol{p}},\overline{\boldsymbol{q}}} \doteq \frac{1}{2}(A_{\boldsymbol{p},\boldsymbol{q};\overline{\boldsymbol{p}},\overline{\boldsymbol{q}}} + A^*_{\overline{\boldsymbol{p}},\overline{\boldsymbol{q}};\boldsymbol{p},\boldsymbol{q}})$. Formula (417) can be computed from the usual second-order closure information (R and C). A Markovian version can also be derived (Krommes, 1996).

Krommes (1996) showed how this result follows more succinctly from the MSR formalism, which permits ready generalization to inhomogeneous and multiple-field situations. The basic result [see Appendix A of Martin et al. (1973)] is that the four-point cumulant is approximated in the DIA by the singly connected term represented by the first diagram of G_4 in Fig. 19 (p. 158). $Z^{\rm c}_{\rm DIA}$ then follows by contracting each two-point end of G_4 with λ and summing over all internal spinor indices, as shown in Fig. 32.

Fig. 32. The DIA for the cumulant part of the fourth-order statistic Z. The triangles represent the coupling coefficient λ; other notation is as in Fig. 20 (p. 161).

Formula (417) would appear to make a nontrivial prediction for non-Gaussian fourth-order statistics. In some cases it is at least qualitatively successful; for example, the DIA correctly captures a numerically observed *depression of nonlinearity* (Kraichnan and Panda, 1988; Kraichnan and Chen, 1989). Unfortunately, in several important situations Eq. (417) can be shown to vanish identically by virtue of symmetry. Chen et al. (1989a) considered $\langle|\boldsymbol{\omega}|^4\rangle$, $\langle\delta\varepsilon^2\rangle$, and other measures of isotropic, incompressible Navier–Stokes turbulence ($\boldsymbol{\omega}$ is the vorticity and ε is the dissipation); Krommes (1996) discussed a kurtosis statistic for a solvable system of three coupled modes. Such difficulties led Chen *et al.* to argue for the necessity of PDF methods; see Sec. 10.4 (p. 224).

10.3 The α^2 effect

The preceding section focused on *predictions* of fourth-order statistics. It is also of interest to understand how such statistics affect the values of lower-order quantities such as transport coefficients. One interesting application is the α^2 *effect* discussed by Kraichnan (1976a).

As reviewed in Sec. 5.8 (p. 141), the mean field of the kinematic dynamo evolves according to Eq. (253), in which the α effect vanishes for nonhelical but isotropic turbulence. Even then, however, there are helicity fluctuations that may be expected to play some role in the mean square. Indeed, Kraichnan argued that such fluctuations tended to reduce the value of the magnetic diffusivity μ_m, perhaps even to the point of making it negative. Unfortunately, this α^2 effect involves a particular fourth-order cumulant that is not captured by the DIA.

Kraichnan (1976a) proposed two possible solutions to this difficulty. First, he discussed a double-averaging procedure in which equations of the form (253) are first obtained for ensembles with locally coherent helicity fluctuations, then the resulting equation with stochastic α is treated with the DIA. Second, he showed that the first vertex correction [Secs. 3.9.8 (p. 85) and 6.2.3 (p. 159)] does capture the effect. These results help one to interpret the meaning and necessity of vertex renormalization. More importantly, they make a good case for physical understanding. Vertex renormalizations are very complicated, so probably would not be explored unless absolutely necessary; understanding the consequences of helicity provide such a motivation. Alternatively, the double-averaging procedure may provide a useful means of circumventing the tedious mechanics of vertex renormalization in problems other than the kinematic dynamo.

10.4 PDF methods

"If closure at the level of fourth-order moments cannot deal successfully with [the intermittency of vorticity and dissipation], then it may be necessary to seek theories in which partial probability distributions play an irreducible role." — *Chen et al. (1989a).*

We learned in Sec. 10.2 (p. 221) that standard moment-based closures are incapable of capturing important aspects of higher-order statistics and intermittency. Even though convergent sequences of moment closures can be envisaged [Kraichnan (1985); see Sec. 7.5 (p. 197)], practical implementation may be difficult or impossible. For many questions relating to the shapes of partial PDF's, it seems more natural to develop approximation schemes that work directly on the functional form of the PDF.

10.4.1 The Liouville equation for a PDF

For definiteness, consider turbulence described by a single random variable $\psi(\boldsymbol{x}, t)$ that obeys

$$\dot{\psi} \equiv \partial_t \psi(\boldsymbol{x}, t) + \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi = F(\psi, \boldsymbol{\nabla}\psi, \nabla^2\psi, \dots), \tag{418}$$

with $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. One may inquire about the PDF $P(\psi, \boldsymbol{x}, t)$ (independent of $\boldsymbol{x}$ for homogeneous statistics and independent of t in a steady state). Clearly $P(\psi)$ is not the complete (fully multivariate in space and time) probability density functional of the turbulence, but it is experimentally accessible yet surprisingly difficult to predict analytically. For homogeneous statistics one can show that $P(\psi)$ obeys the Liouville equation (Pope, 1985)

$$\frac{\partial}{\partial t} P(\psi) + \frac{\partial}{\partial \psi}\Big(\langle \dot{\psi} \mid \psi \rangle P\Big) = 0. \tag{419}$$

Here $\langle \dot{\psi} \mid \psi \rangle$ denotes the average of $\dot{\psi}$ *conditional on* observing the value ψ. A particularly clear version of the derivation was given by Gotoh and Kraichnan (1993). The procedure [260] is to manipulate the expression for the time evolution of the average of an arbitrary test function $g(\psi)$. It is readily

[260] On the one hand, $\partial_t \langle g(\psi) \rangle = \int \mathrm{d}\psi\, g(\psi) \partial_t P(\psi)$. On the other hand,

$$\left\langle \frac{\partial g}{\partial t} \right\rangle = \left\langle \frac{\partial g}{\partial \psi} \frac{\partial \psi}{\partial t} \right\rangle = \left\langle \frac{\partial g}{\partial \psi}(-\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi + F) \right\rangle. \tag{f-22a,b}$$

generalized to joint PDF's such as $P(\psi, \boldsymbol{\nabla}\psi)$. When $\boldsymbol{\nabla} \cdot \boldsymbol{u} \neq 0$ a compressibility term appears on the right-hand side of Eq. (419).

Equation (419) presents a closure problem that is even more fundamental than the one encountered in moment-based procedures. Whereas Eq. (419) displays no closure problem for either an advective nonlinearity or any nonlinear function of ψ itself, to the extent that $\dot{\psi} \equiv \mathcal{V}$ depends on *gradients* of ψ the conditional average $\langle \mathcal{V} \mid \psi \rangle$ is not known in terms of ψ. This difficulty exists even at *linear* order—for example, $\mathcal{V} = \mu_{\mathrm{cl}} \nabla^2 \psi$. The technical problem is that an operator such as $\nabla^2 \psi$ couples the statistics of $\psi(\boldsymbol{x})$ to those of neighboring points whereas one is attempting to restrict attention to $P(\psi, \boldsymbol{x}, t)$ at a single space point.

10.4.2 Mapping closure

Some early attempts at the closure problem for PDF's were reviewed by Pope (1985). A major advance was made by Chen et al. (1989b), who introduced the technique now known as *mapping closure*. Representative papers in which the method was developed include those by Kraichnan (1990), Gotoh and Kraichnan (1993), and Kimura and Kraichnan (1993). For a recent review, see Kraichnan (1991). The following brief discussion is taken, in part almost verbatim, from Krommes (1997c).

In the simplest version of the mapping closure, it is assumed that at any point $\boldsymbol{x}$ and time t the field $\psi(\boldsymbol{x}, t)$ can be represented by a *surrogate field* mapped via a nonlinear *function* (not a functional) from a *Gaussian reference field* $\psi_0(\boldsymbol{x}, t)$ (at the same point in space-time):

$$\psi(\boldsymbol{x}, t) = F(\psi_0(\boldsymbol{x}, t), t). \tag{420}$$

Given an F, conditional expectations such as $\langle \nabla^2 \psi \mid \psi \rangle$ can then be evaluated explicitly. The mapping function $F(\psi_0, t)$ itself is determined such that the evolution of the ψ statistics from t to $t + \Delta t$ are treated exactly in terms of the estimated conditional dissipation.

The mapping technique is a very bold Ansatz because the statistics at a point depend in a complicated way on the detailed dynamics for all previous times and all space points. The Liouville equation closed with Eq. (420) is not exact because the *multivariate* statistics of the surrogate field differ from those of the exact field; this means that the conditional dissipation $\langle \nabla^2 \psi \mid \psi \rangle$ is not represented faithfully. A clear discussion was given by Kimura and Kraichnan (1993). Nevertheless, in the context of Navier–Stokes turbulence Kraichnan (1991) argued in favor of the approximation by suggesting, "Perhaps only the final stretching, the one that narrows a vortex to the point where viscosity acts strongly on it, is actually important in determining the asymptotic skirts of the PDF. It may be that the details of the path in parameter space followed by a fluid element are relatively unimportant in this respect and, instead, the Gaussianly distributed initial gradient and the final viscous relaxation are what count."

Although the theory of mapping closures is still in its infancy, it is already clear that the technique can capture both qualitative and quantitative features of highly non-Gaussian PDF's. Chen et al. (1989b) considered the PDF of a 1D chemical reaction; Gotoh and Kraichnan (1993) studied the

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But by chain differentiation, $\langle (\partial_\psi g)(-\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi) \rangle = \langle -\boldsymbol{u} \cdot \boldsymbol{\nabla} g \rangle = -\boldsymbol{\nabla} \cdot \langle \boldsymbol{u}\, g \rangle = 0$, the last following by homogeneity. Also

$$\left\langle \frac{\partial g}{\partial \psi} F \right\rangle = \int \mathrm{d}\psi\, \frac{\partial g}{\partial \psi} \langle F \mid \psi \rangle P(\psi) = -\int \mathrm{d}\psi\, g(\psi) \frac{\partial}{\partial \psi} [\langle F \mid \psi \rangle P(\psi)]. \tag{f-23a,b}$$

Burgers equation; Kimura and Kraichnan (1993) focused on passive advection. All of those works demonstrated strikingly successful agreement with direct numerical simulations. A heuristic model of the PDF of the transverse velocity gradient for Navier–Stokes turbulence based on these ideas was very encouraging (Kraichnan, 1990). That work was pursued by She (1991a) and She and Orszag (1991); for a summary and further discussion, see She (1991b).

In plasma physics the first published work on mapping closures is due to Das and Kaw (1995). Motivated by simulations of Crotinger and Dupree (1992), they applied a mapping technique to the HM equation (48). In particular, they considered the PDF for the vorticity $\omega = \nabla_\perp^2 \varphi$. A full-blown, self-consistent mapping closure for this problem is extremely difficult and was not attempted. However, by making various heuristic approximations, they were able to argue that the effects of advective stretching lead to a qualitative model similar to the one studied by Chen et al. (1989b). There resulted a prediction for non-Gaussian vorticity kurtosis in crude qualitative agreement with some of the numerical observations.

As Das and Kaw admitted, considerable further work must be done before such theory can make quantitatively acceptable predictions. For example, the effects of anisotropy, wave propagation, and linear forcing must be considered. It is probably best to do this in the context of self-consistent models such as that of Hasegawa and Wakatani. However, straightforward application of the technique introduces (Krommes, 1994) the algebraic complexity of a 2D mapping function $X(\omega, n)$, uncertainties in the modeling of the nonlocal relationship $\varphi = \nabla^{-2}\omega$, and probably the need for working with the joint PDF of the fields and their gradients. As of early 2000 the latter calculation had not yet been done for even a single scalar field.

Various mechanisms can be responsible for intermittency in equations like those of HW. One obvious candidate is the formation of coherent structures in self-consistent turbulence; see Sec. 10.5 (p. 228). But intermittency can arise even in passive advection. [In the hydrodynamic regime $\alpha \ll 1$ of the HW equations, the density field is essentially passive, as discussed in Sec. 2.4.5 (p. 38).] Kimura and Kraichnan (1993) discussed two mechanisms for inducing non-Gaussian behavior: (i) advection of fluid elements in a nonlinear mean profile; and (ii) generation (through advection) and subsequent relaxation (by viscosity) of spatially intermittent temperature gradients. Since many plasma models such as that of HW are developed from a two-scale approximation in which the logarithmic derivative of the mean profile is taken to be constant and the background profile is not evolved, the second mechanism is particularly relevant. In more detail, Kimura and Kraichnan explained that (i) either advection or classical diffusion separately leave an initially homogeneous and Gaussian field Gaussian; (ii) in a homogeneous Gaussian field, the field and its gradient are statistically independent at a point [261]; (iii) the independence continues to hold under advection acting alone; but (iv) highly strained regions diffuse more strongly than others, so the fluctuations in those regions decay rapidly. This causes statistical dependence between the field and its gradient, so the resulting PDF's must be non-Gaussian. Such behavior was studied in a limited way by Krommes and Hu (1995).

[261] Consider a Gaussian field T with homogeneous statistics. The gradient of a Gaussian field is also Gaussian. One has $\langle T(\boldsymbol{x}) \boldsymbol{\nabla} T(\boldsymbol{x}) \rangle = \boldsymbol{\nabla} \langle \frac{1}{2} T^2 \rangle = 0$ by homogeneity, so T and $\boldsymbol{\nabla} T$ are uncorrelated. But for Gaussian variables, uncorrelated implies independent.

10.4.3 Generating functional techniques

The Burgers equation (19) is a popular model for studies of intermittency. Although it describes physics very different from that of the NSE, one expects that appropriate analytical approximations that work well in the Burgers context may be generalizable to more physical contexts as well.

This is not the place to review the multitude of approximation procedures that have been applied to the Burgers equation. The mapping-closure work of Gotoh and Kraichnan (1993) has already been mentioned. Here I shall briefly introduce the recent work of Polyakov (1995) and Boldyrev (1997, 1998, 1999), who have made impressive progress on the velocity-difference and velocity-gradient PDF's by using an operator product expansion and general symmetry considerations.

Let the goal be to calculate the *velocity-difference PDF* $P(\Delta u, y)$, where $\Delta u \doteq u(x+y,t)-u(x,t)$, for homogeneous, stationary statistics and very large Reynolds number ($\mu_{\rm cl} \to 0$). (This P is obviously a very special case of the fully multivariate space-time PDF.) $P(\Delta u, y, t)$ is the inverse Fourier transform with respect to η of the characteristic function

$$Z(\eta, y, t) \doteq \langle e^{\eta \Delta u(y,t)} \rangle. \tag{421}$$

It is not hard to show that Z evolves according to

$$\frac{\partial Z}{\partial t} + \left(\frac{\partial}{\partial \eta} - \frac{2}{\eta} \right) \frac{\partial Z}{\partial y} - [F^{\rm ext}(0) - F^{\rm ext}(y)]\eta^2 Z = \mu_{\rm cl} \eta \langle \Delta u_{xx}(y) e^{\eta \Delta u} \rangle \equiv \mathcal{D}(\eta). \tag{422}$$

Note that the advective term and external forcing are represented exactly. As discussed in Sec. 10.4.1 (p. 224), the closure problem for such a PDF arises from the differential nature of the classical dissipation.

The dissipation $\mathcal{D}$ [the right-hand side of Eq. (422)] remains nonzero as $\mu_{\rm cl} \to 0$. To approximate it in that limit, Polyakov (1995) asserted that

$$\mathcal{D}(\eta) \to [a(\eta) + \beta(\eta)\eta^{-1}\partial_y] Z. \tag{423}$$

Here the undetermined functions $a(\eta)$ and $\beta(\eta)$ are called the *anomalies*. Equation (423) is a special case of the so-called *operator product expansion*. The basic idea is that the coefficient $\mathcal{D}/\mu_{\rm cl}$, divergent as $\mu_{\rm cl} \to 0$, should be expanded in terms of finite operators (with divergent coefficients) already in the theory; this is an unproven conjecture. Symmetry considerations show that β is a constant.

If Eq. (423) is granted, then one has in steady state

$$\left(\frac{\partial}{\partial \eta} - \frac{2b}{\eta} \right) \frac{\partial Z}{\partial y} - [F^{\rm ext}(0) - F^{\rm ext}(y)]\eta^2 Z = a(\eta) Z, \tag{424}$$

where $b \doteq 1 + \beta$. This equation must be solved subject to the requirements that the PDF is positive, finite, and normalizable. This was done by Boldyrev (1997, 1998) for various α's, where $F^{\rm ext}(y) \sim 1 - y^\alpha$; he proved that those conditions determine a one-parameter family of solutions $a = a(b)$. Particular choices of b led to truly excellent agreement with high-quality simulations of Yakhot and Chekhlov (1996).

Because no theory that determines the value of b has been given, this approach is obviously incomplete. But the tantalizing agreement with simulations shows that this area is a challenging and probably fruitful one for future research.

10.5 Coherent structures

It is well known that nonlinear equations can support coherent as well as stochastic solutions. Soliton solutions of fluid equations [see, for example, Drazin and Johnson (1988)] are one example. A useful discussion of solitons in the context of plasmas was by Horton and Ichikawa (1996). Bernstein et al. (1957) showed how to construct arbitrary nonlinear solutions of the Vlasov equation (BGK modes); some of those can be stable. Dupree (1982) used maximum entropy methods and the statistics proposed by Lynden-Bell (1967) to discuss Vlasov *holes*, phase-space depressions that can be viewed as particular cases of BGK modes.

Although considerable work on coherent structures in plasmas has been done, the following discussion of the *statistical* description of such structures will be very brief, as relatively little is known and a formalism that is both truly *systematic* as well as workable is lacking.[262] Development of such a formalism would be a significant contribution to the general theory of turbulence.

10.5.1 Coherent solutions and intermittency

Berman et al. (1983) performed simulations of both decaying and forced Vlasov turbulence. For decaying turbulence they observed strong skewness and intermittency of the phase-space PDF that they interpreted in terms of phase-space holes. At approximately the same time, McWilliams (1984) performed simulations of decaying 2D NS turbulence that showed the emergence from a sea of random initial conditions of highly intermittent states with isolated vortices. A heuristic criterion was that vortices formed when fluctuations raised the local enstrophy above the mean-square shear. For more detailed and recent simulations and diagnostics, see Benzi et al. (1988). Leith (1984) employed a variational method, based on the selective-decay hypothesis of Matthaeus and Montgomery (1980), to predict the structure of the localized vortex solutions. See also more recent discussion by Terry et al. (1992).

Coherent solutions can make nontrivial contributions to fluctuation spectra and higher-order statistics. Meiss and Horton (1982) considered a gas of noninteracting drift-wave solitons, each of which being the solution of a nonlinear equation due to Petviashvili (1977). They calculated fluctuation spectra that were broad and peaked above the diamagnetic frequency ω_*; note that conventional drift waves satisfy $\omega < \omega_*$.

The degree to which noninteracting limits are sensible depends on the *packing fraction* p, with $p \ll \frac{1}{2}$ describing a highly intermittent state. The fate of coherent nonlinear structures for $p \approx \frac{1}{2}$ is far from clear. An analogy is to KAM theory and phase-space trapping; crudely, when primary islands overlap, stochasticity ensues (Appendix D, p. 279); analogous effects can be expected for BGK modes and similar structures.

Uncertainty in the physics of interacting coherent structures shows up in formal statistical theory. If the final state is fully stochastic, DIA-based descriptions may be adequate. For example, Fedutenko (1996) has used a two-scale DIA analysis of the Charney–Hasegawa–Mima equation to discuss the nonlinear generation of large-scale mean flows that he identified with ordered structures; that work

[262] Spineanu and Vlad (2000) have recently proposed a direct attack on the problem of mixed states of coherent structures and drift-wave turbulence by combining approximations to the MSR generating functional (Sec. 6.2, p. 153) with the inverse scattering method of soliton theory; they performed detailed calculations on a specific model. The fidelity of this approach for predicting higher-order statistics remains to be assessed.

is closely related to the discussions of eddy viscosity in Sec. 7.3 (p. 189) and zonal flows in Sec. 12.7 (p. 248). Nevertheless, Dupree has argued that self-trapping effects cannot be properly described at the DIA level.[263] If not, then vertex renormalization must be considered, a formidable task. A formal program would be to examine the Bethe–Salpeter equation (Sec. 6.2.4, p. 162) in, say, the first vertex renormalization and to show some relation to Dupree's intuitive notions about self-binding. That has not been attempted to the author's knowledge. Because of the focus of the present article on systematic procedures, I shall not attempt to review the detailed (and often qualitatively successful) attempts of Dupree and co-workers to include the effects of coherent phase-space structures in statistical spectral balances. For a cogent discussion, see Berman et al. (1983); see also Boutros-Ghali and Dupree (1981).

10.5.2 Statistics and the identification of coherent structures

Modern approaches to the identification of coherent structures frequently rely on the *proper orthogonal decomposition* (POD), or *Karhunen–Loève expansion*, championed by Lumley (1967, 1970, 1981), Berkooz et al. (1993), and Holmes et al. (1996) for problems of fluid turbulence, especially inhomogeneous flows. The idea is to expand the random flow in terms of orthogonal deterministic functions ϕ_i labeled by eigenvalues λ_i in such a way that the modal energies decreases monotonically with i; the energetically most significant modes are then identified with coherent structures. When the decomposition is performed in space at fixed time, the ϕ_i's are eigenfunctions of the covariance $C(\boldsymbol{x}, \boldsymbol{x}')$. More generally, a symmetrical *biorthogonal decomposition* can be performed in space-time (Aubry et al., 1991; Aubry, 1991). Usually experimentally determined covariances are used. A theoretical proposal to self-consistently determine the eigenfunctions was made by Kraichnan (1988a), but has not been implemented to the author's knowledge.

The purpose of the POD is not merely to identify coherent structures. One wishes to understand how they interact and to qualitatively link observed intermittent behavior to those interactions. Holmes et al. (1996) discussed the current state of an ambitious program to first obtain low-order systems of coupled, nonlinear ODE's for the time-dependent amplitudes of the coherent structures, then to analyze those equations with the powerful tools of nonlinear dynamics and bifurcation theory. A relatively short recent review is by Holmes et al. (1997). One difficulty of this approach is that it is frequently impossible to expand the entire flow because the resulting system of ODE's would be too large and intractable. Instead, a restricted region near a wall is analyzed. However, the interface to the interior of the fluid then introduces unusual random boundary conditions whose effect must be somehow modeled. Since only the most energetic modes are retained, one must also introduce some sort of eddy-viscosity mechanism to model energy dissipation by the small, unresolved scales.

Some additional discussion of the identification and description of coherent structures in plasma turbulence can be found in Horton (1996) and in the other articles in the collection edited by Benkadda et al. (1996). Beyer et al. (2000) used POD analysis to find the appropriate low-dimensional dynamical system for a plasma model in the presence of magnetic shear. Aspects of statistical closure in the face of intermittency were considered by Herring and McWilliams (1985).

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[263] The argument (Berman et al., 1983) is that self-trapping corresponds to the preferred sign $\delta f < 0$ (Dupree, 1983); however, conventional second-order closures are built from the covariance $\langle \delta f \, \delta f \rangle$, which is invariant to the transformation $\delta f \to -\delta f$.

11 RIGOROUS BOUNDS ON TRANSPORT

"The optimum theory represents an alternative approach [to statistical moment closures] in which the lack of information about properties of the fluctuating ... field is reflected by the theoretical results. Instead of a theoretical prediction of the physically realized average properties, bounds on those properties are obtained. No assumptions are introduced in the theory ... ." — *Busse (1978).*

The closure techniques discussed so far represent attempts to predict various quantities (such as turbulent fluxes) as precisely as possible. A philosophically appealing alternative is the bounding or "optimum"[264] method, due to workers such as Malkus, Howard, Joseph, and Busse. The basic formalism and neutral-fluid applications (to the Navier–Stokes or Boussinesq equations in various bounded geometries) through 1978 were reviewed by Busse (1978), who gave an extensive list of original references that will mostly not be repeated here. A more recent but much shorter account was given by Busse (1996). The method was also briefly discussed by Kraichnan (1988a,b). Krommes and Smith (1987) extended the formalism to treat problems of passive advection, and further analyses of passive models were done by Kim and Krommes (1988). The results of Krommes and co-workers were briefly reviewed by Krommes and Kim (1990). Practical applications to plasma problems are briefly mentioned in Sec. 11.4 (p. 235).

11.1 Overview of the variational approach

Consider the problem of finding an upper bound for a turbulent flux Γ. Such fluxes are bilinear functionals of two fields. For example, if Γ were the particle flux in a magnetized plasma problem, it would be given by Eq. (3a); i.e., Γ would be a bilinear functional of potential φ and density n. Now clearly an unconstrained maximum of such a functional taken over all possible fields is infinite, thus useless. Therefore in the bounding method Γ is maximized *subject to one or more additional rigorous constraints*. Typically, those are balance equations appropriately selected from the infinity of moment constraints that follow from the fundamental equation of motion. However, they can also be kinematic constraints such as the requirement that the Navier–Stokes velocity field be solenoidal. One can envisage a sequence of constraints, of increasing number and/or complexity, such that the sequence of resulting bounds approaches the true answer from above.

The bounding method is not a panacea. An inaptly chosen constraint need not lower the bound (though at least it cannot raise it). Certain subsequences of constraints may converge, but to an answer larger than the true result and possibly with different parameter scaling. The simplest version of the method works with inhomogeneous mean fields; homogeneous problems are much more difficult and have not yet been satisfactorily addressed. Finally, constraints of richness sufficient to ensure convergence to the true answer may be of prohibitive complexity.

It is also possible to seek lower bounds on the transport; however, there are technical difficulties since a rigorous lower bound is zero. The present discussion is restricted to upper bounds.

[264] Carefully distinguish *optimum*, implying a variational procedure, from *optimal*, implying the best of all possible worlds (von Leibniz, 1720; Voltaire, 1759). The case for the latter interpretation has not yet been made by any extant analytical theory of turbulence.

11.2 The basic upper bound

Original uses of the optimum method for neutral-fluid applications were reviewed by Busse (1978). Those applications all involved self-consistent dynamics; the simplest cases (e.g., Navier–Stokes pipe or channel flow) involved a single dimensionless parameter such as the Reynolds number R. [In Sec. 11 I write R and K rather than $\mathcal{R}$ and $\mathcal{K}$ in order to accommodate a useful notational scheme for the mathematical statement of the variational principles; see Eqs. (438).]

As I discussed in Sec. 3.4 (p. 56), problems of passive advection are in a certain sense richer than self-consistent ones since they also involve the Kubo number K as an additional independent parameter [see the discussion of Fig. 2 (p. 58)]. In the context of the bounding theory, this point was made by Krommes and Smith (1987), who discussed features and difficulties of the method with the aid of model problems of passive advection. The following discussion draws heavily from that work.

The general method, either self-consistent or passive, as employed to date invokes a common formalism that Krommes and Smith called the *basic* [upper] *bound*. To derive the basic bound, one returns to the general energy balance (404). Its steady-state version obviously provides a constraint on the flux; however, it is inconvenient in that it (i) involves a triplet correlation function (the transfer term $\mathcal{T}$), which is both unknown and of indefinite sign; and (ii) contains in the production term $\mathcal{P}$ the gradient of the mean field, which is also unknown. However, as in Sec. 9.1 (p. 211), the transfer term can be averaged away, giving rise to a steady-state constraint that rigorously relates the volume-averaged production $\overline{\mathcal{P}}$ to the volume-averaged dissipation $\overline{\mathcal{D}}$:

$$\overline{\mathcal{P}} = \overline{\mathcal{D}}. \tag{425}$$

Equation (425) is the *basic constraint*. It contains the unknown mean gradient. However, if the mean field is allowed to evolve solely under the influence of the turbulence and boundary conditions (no sources or other constraints), its gradient can be eliminated in terms of terms quadratic in the fluctuations by using the exact equation for the mean field. Then if $\mathcal{E}$ has been chosen appropriately, it is usually possible to prove that an appropriately defined[265] volume-averaged flux $\overline{\Gamma}$ is positive definite. One is then led to the *basic variational principle*:

$$\text{Maximize } \overline{\Gamma} \text{ subject to the constraint } \overline{\mathcal{P}} = \overline{\mathcal{D}}. \tag{426}$$

To see how this works in practice, consider the *generalized reference model* of Krommes and Smith (1987), namely, Eqs. (87) with $\mu_{\mathrm{cl}} \to R^{-1}$ (a problem of passive advection). Upon defining $\Gamma(x,t) \doteq \langle \delta u(t) \delta T(x,t) \rangle$, one finds that the mean field obeys the continuity equation

$$\partial_t \langle T \rangle(x,t) + \partial_x \Gamma(x,t) - R^{-1} \langle T \rangle''(x,t) = 0. \tag{427}$$

In steady state the total flux (advective plus classical) is conserved:

$$\Gamma_{\mathrm{tot}}(x) = \Gamma(x) + \Gamma_{\mathrm{cl}}(x) = \text{const}, \tag{428}$$

where $\Gamma_{\mathrm{cl}}(x) \doteq -R^{-1} \partial_x \langle T \rangle(x)$. The constant of integration in Eq. (428) (the conserved total flux) can be related to the desired volume average $\overline{\Gamma}$ by averaging Eq. (428) over x and using the boundary

[265] As Busse (1978) has pointed out, the flux that naturally appears in the variational principle need not be the most interesting or experimentally accessible one.

conditions (87b). Specifically, for arbitrary $A(x)$ let us define $\overline{A} \doteq \int_0^1 \mathrm{d}x\, \langle A\rangle(x)$. (Of course, if the quantity A has been previously averaged, the angular brackets are redundant.) Then one finds

$$\Gamma_{\mathrm{tot}} = \overline{\Gamma} + \overline{\Gamma}_{\mathrm{cl}}, \tag{429}$$

where $\overline{\Gamma}_{\mathrm{cl}} = R^{-1}$. Equation (429) can be used in Eq. (428) to express the unknown gradient of the mean field in terms of quadratic averages:

$$-\partial_x\langle T\rangle(x) = 1 - R\Delta\Gamma(x), \tag{430}$$

where $\Delta A(x) \doteq A(x) - \overline{A}$.

Now consider the exact equation for the fluctuations,

$$\partial_t\delta T(x,t) + \delta u(t)\partial_x\langle T\rangle + \partial_x(\delta u\,\delta T - \langle\delta u\,\delta T\rangle) - R^{-1}\delta T'' = 0, \tag{431}$$

and define an "energy" by $\mathcal{E} \doteq \frac{1}{2}\langle\delta T^2\rangle$. [A perhaps better interpretation of this quantity is in terms of a quadratic approximation to information-theoretic entropy; see Sec. 12.2 (p. 238).] If one further defines the production by

$$\mathcal{P} \doteq -\Gamma(x)\partial_x\langle T\rangle(x), \tag{432}$$

the transfer by $\mathcal{T} \doteq \frac{1}{2}\partial_x\langle\delta u\,\delta T^2\rangle$, and the dissipation by $\mathcal{D} \doteq -R^{-1}\langle\delta T\,\delta T''\rangle$, one arrives at precisely Eq. (404).

The steady-state balance

$$\mathcal{P}(x) = \mathcal{T}(x) + \mathcal{D}(x) \tag{433}$$

exhibits the closure problem, since $\mathcal{T}$ is unknown. $\mathcal{T}$ can be eliminated, however, and the basic constraint (425) obtained by barring Eq. (433), noting that $\mathcal{T}$ is a spatial derivative, and using the statistically sharp boundary conditions that δT vanishes at the boundaries. One can express $\overline{\mathcal{P}}$ entirely in terms of fluctuating quantities by inserting Eq. (430) into Eq. (432) and using the identity $\overline{\Gamma(x)\Delta\Gamma(x)} = \overline{\Delta\Gamma^2(x)}$, which holds because $\overline{\Delta\Gamma} = 0$:

$$\overline{\mathcal{P}} = \overline{\Gamma} - R\,\overline{\Delta\Gamma^2}. \tag{434}$$

For $\overline{D}$ one can integrate by parts to find[266]

$$\overline{\mathcal{D}} = R^{-1}\overline{(\partial_x\delta T)^2}. \tag{435}$$

Upon inserting Eqs. (434) and (435) into Eq. (425) and rearranging the result, one finds

$$\overline{\Gamma} = R^{-1}\overline{(\delta T')^2} + R\,\overline{\Delta\Gamma^2}. \tag{436}$$

[266] In Krommes and Smith (1987) the last line of Eq. (2.19) is incorrect. The bracket average alone does not permit integration by parts in x because of the inhomogeneous boundary conditions. Nevertheless, the subsequent equations involving $\overline{\mathcal{D}}$ are correct.

This alternate form of the basic constraint proves that the volume-averaged flux is positive definite. For any given system of equations, if such a proof is not forthcoming, then the flux function whose bound is to be sought has been chosen inappropriately.[267]

The basic variational principle (426) can be formulated as the problem of determining the unconstrained maximum γ of the functional

$$\mathcal{G}[\delta T;\zeta] \doteq \overline{\Gamma} + \zeta[\overline{\Gamma} - R^{-1}\overline{(\delta T')^2} - R\overline{\Delta\Gamma^2}], \tag{437}$$

where the Lagrange multiplier ζ must be determined such that Eq. (436) is satisfied. In an important technical advance, Howard (1963) showed how to reformulate the problem as a homogeneous *minimum* variational principle in which a Lagrange multiplier does not appear explicitly. The basic idea is to invert the bounding curve $\gamma(R)$ and to seek the minimum R for given γ. A principle that accomplishes this is

$$\text{Minimize } \mathcal{R}(\gamma) \text{ subject to } \overline{\Gamma} = \gamma, \tag{438a}$$

where

$$\mathcal{R}(\gamma) \equiv \mathcal{R}[\delta T;\gamma] \doteq \gamma\left(\frac{\overline{(\delta T')^2} + R^2\overline{\Delta\Gamma^2}}{\overline{\Gamma}^2}\right). \tag{438b}$$

The normalization constraint $\overline{\Gamma} = \gamma$ reduces Eq. (438b) to Eq. (436), and inverting the relation $R = \mathcal{R}_{\min}(\gamma)$ gives the bounding curve $\gamma(R)$ describing the maximum flux γ for fixed R. The advantage of the minimum principle is that the functional $\mathcal{R}[\delta T]$ is homogeneous in δT; because of that, the implicit Lagrange multiplier that would implement the normalization constraint turns out to vanish. Further discussion was given by Krommes and Smith (1987); see also Busse (1978). An interpretation of the minimizing principle and a necessary consistency check follow by noting that at $\gamma = 0$ one has $\Delta\Gamma = 0$ and Eq. (438a) reduces to the variational calculation (406) for the energy-stability threshold R_E. Thus the variational principles discussed in the present section are nonlinear generalizations of the linear energy-stability calculation.

For the present model the results of the variational calculation (Krommes and Smith, 1987) are shown in Fig. 33 (p. 234), where the basic bound $\overline{\Gamma}(R)$ is compared with both the exact solution and the prediction of the DIA. For this particular application the DIA is perceptibly closer to the exact solution than is the basic bound although it involves considerable more labor to compute. Interestingly, the DIA lies below the true solution.

For self-consistent problems the structure of the Euler–Lagrange equations is more complicated because the flow velocity as well as δT (or its analog) must be varied. Busse (1969) has shown that the optimizing solution possesses multiple, nested boundary layers that can be analyzed asymptotically,[268] and exhibits bifurcations because the number of contributing modes depends on R. The agreement between the bound and the true solution is sometimes remarkably close, suggesting (Busse, 1972) that the physically realized turbulence tends to extremalize the transport. This hints at the possible existence of an underlying variational description of the flow itself, such as a principle of minimum

[267] It is presumed that there are no sources; those have not yet been satisfactorily incorporated into the formalism.
[268] For the details, see the discussion and references in Busse (1978) and Joseph (1976, Chap. XII).

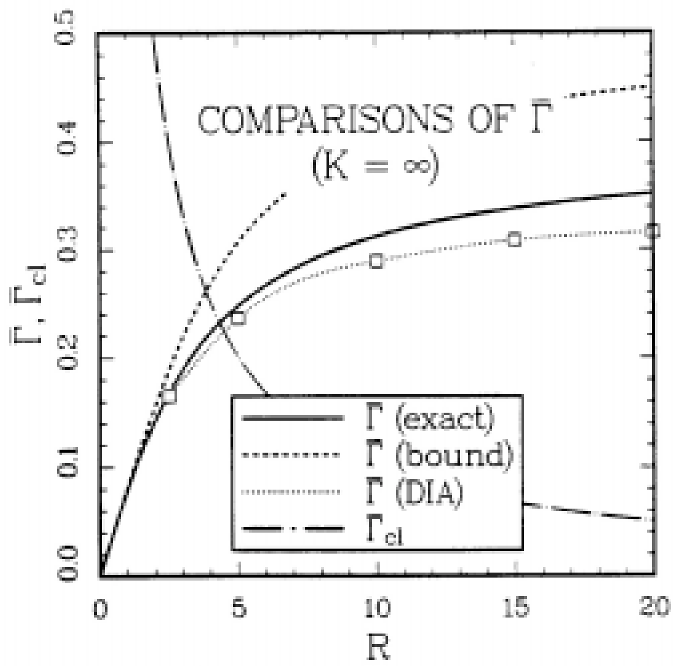

Fig. 33. For the generalized reference model (87), comparison of the exact solution for the mean turbulent flux as a function of Reynolds number R (solid line) with the basic bound (dashed line) and the direct-interaction approximation (dotted line). From Fig. 7 of Krommes and Smith (1987), used with permission.

dissipation (Montgomery and Phillips, 1988). It should be stressed, however, that the principles that have been articulated to date are approximate whereas the bounding method is exact.

11.3 Two-time constraints

An important and undesirable property of the basic-bound procedure as applied to problems of passive advection is that the basic bound is completely independent of the Kubo number K. This follows quite generally because only equal-time correlation functions appear in the steady-state energy balance, so no multiple-time correlation functions that could involve the (linear, or imposed) autocorrelation time appear in the basic constraint or variational principle. For self-consistent problems a similar argument shows that the basic principle cannot take account of a wide variety of potentially important dynamical processes, including, for example, the effects of propagating waves or streaming particles. Thus neither the basic bound nor more highly constrained bounds that invoke only one-time constraints properly reduce to the quasilinear scaling (Sec. 3.4.3, p. 57) $\overline{\Gamma} \propto K$ as $K \to 0$.

In order to recover accurate results as $K \to 0$, Krommes and Smith (1987) argued that it is necessary to include in the formalism constraints involving *two-time correlations*. That requires consideration of *time-dependent Lagrange multipliers* and time-dependent Euler–Lagrange equations, significant technical complications. Nevertheless, for one particular two-time constraint they were able to carry through the formalism for the generalized reference model (87). They found a bound of the appealing form

$$\gamma^{-1}(R,K) = \gamma_\infty^{-1}(R) + \gamma_{\rm q}^{-1}(K), \tag{439}$$

where $\gamma_\infty(R)$ is the basic bound and $\gamma_{\rm q}$ is the quasilinear flux. Formula (439) has the desired limits

$$\gamma(R,K) \sim \begin{cases} K & (K \to 0) \\ \gamma_\infty(R) & (K \to \infty). \end{cases} \tag{440}$$

11.4 Plasma-physics applications of the optimum method

An application of considerable practical importance is turbulence in reversed-field pinches (Bodin and Newton, 1980), which is adequately described by MHD equations (including Ohm's law $\boldsymbol{E} + c^{-1}\boldsymbol{u} \times \boldsymbol{B} = \eta_{\rm cl}\boldsymbol{j}$, where $\eta_{\rm cl}$ is the classical resistivity). In a typical experimental situation, the axial current is specified. Then Ohm's law written in the form $E_z = -c^{-1}\overline{\widehat{\boldsymbol{z}} \cdot (\boldsymbol{u} \times \boldsymbol{B})} + \eta_{\rm cl} j_z \equiv \eta_{\rm tot} j_z$ introduces the *turbulent electromotive force* $\overline{\varepsilon} \doteq -c^{-1}\overline{\widehat{\boldsymbol{z}} \cdot (\delta\boldsymbol{u} \times \delta\boldsymbol{B})}$, and one can show that this is the natural quantity to maximize. That problem was considered by Kim (1989) and Kim and Krommes (1990); a brief overview of those latter works was given by Krommes and Kim (1990). The topic was pursued by Wang et al. (1991).[269]

Very recently, Kim and Choi (1996) applied the variational method to the problem of transport due to ITG fluctuations. However, I shall not describe here the detailed results that emerged from either of the practical calculations just mentioned, partly for lack of space but also because considerable additional technical work remains to be done. In particular, the analyses (already very tedious and involved) were restricted to a single mode, and nested boundary-layer analysis was not performed. There is considerable room here for further systematic work on the nonlinear behavior of plasma transport models.

12 MISCELLANEOUS TOPICS IN STATISTICAL PLASMA THEORY

In this section I briefly discuss various miscellaneous topics that rely on statistical formalisms but do not fit naturally into the previous discussions of statistical closure techniques. I consider the possibility of Onsager symmetries for turbulence (Sec. 12.1, p. 235), the interpretation of entropy balances (Sec. 12.2, p. 238), experimental determination of mode-coupling coefficients (Sec. 12.3, p. 241), self-organized criticality (Sec. 12.4, p. 241), percolation theory (Sec. 12.5, p. 245), a potpourri of results on mean-field dynamics (Sec. 12.6, p. 245), and (briefly) the statistical treatment of convective cells and other long-wavelength fluctuations (Sec. 12.7, p. 248).

12.1 Onsager symmetries for turbulence

Onsager's theorem (Onsager, 1931a, 1931b) imposes important constraints (the *Onsager symmetries*) on the transport matrix describing the hydrodynamic relaxation of perturbations of

[269] For some related discussion of the significance of such optimal states, see Hameiri and Bhattacharjee (1987).

a steady state. Onsager's original application was to perturbations of absolute thermal equilibrium. However, it is also of considerable interest to ask whether systems of steady-state turbulence, which are far from thermal equilibrium, may also exhibit Onsager-like theorems.

The literature is highly confused on this point. Many references to literature on both sides of the issue were listed by Krommes and Hu (1993), from whose work most of the following discussion is drawn. On the one hand, people who believe in thermodynamic descriptions of macroscopic systems often argue in favor of Onsager symmetries. On the other hand, those who think in terms of the statistical-mechanical PDF may feel that an Onsager theorem for turbulence is miraculous if it exists at all, since the PDF for forced, dissipative steady-state turbulence bears no resemblance to the Gibbs distribution of thermal equilibrium. The correct answer probably lies somewhere in-between.

12.1.1 Onsager's original theorem

The statement of the classical Onsager theorem is as follows. Consider a system in absolute thermal equilibrium (so that the appropriate PDF is the Gibbs distribution). Introduce a set of *state variables* represented as the column vector $\boldsymbol{X} \equiv (X^1, \ldots, X^n)^T$. These variables will fluctuate and possess the two-time correlation matrix $\mathsf{C}(\tau) \doteq \langle \delta\boldsymbol{X}(t+\tau)\delta\boldsymbol{X}^T(t)\rangle$. Assume that

$$\partial_\tau \mathsf{C}(\tau) = -\mathsf{M}\cdot\mathsf{C}(\tau) \quad (\tau > 0). \tag{441}$$

M is called the *regression matrix*. It provides a Markovian description of the dynamics and is appropriate for long-wavelength, low-frequency fluctuations. Define the *Onsager matrix*

$$\mathsf{L} \doteq \mathsf{M}\cdot\mathsf{C}, \tag{442}$$

where $\mathsf{C} \equiv \mathsf{C}(\tau = 0)$. Assume that under time reversal $\boldsymbol{X}$ is transformed to a new set $\overline{\boldsymbol{X}} = \boldsymbol{\epsilon}\cdot\boldsymbol{B}$, where $\boldsymbol{\epsilon}$ is called the *parity matrix*. It is then a consequence of the microscopic time-reversibility properties of $\boldsymbol{X}$ that

$$\mathcal{L}(\boldsymbol{B}) = \mathcal{L}^\dagger(-\boldsymbol{B}), \tag{443}$$

where $\mathcal{L} \doteq \boldsymbol{\epsilon}\cdot\mathsf{L}$. Equation (443) is the most general form of the classical Onsager theorem. In the absence of a magnetic field and for a set of variables that are all even under time reversal, the theorem reduces to the statement that the Onsager matrix L is symmetric. The generalization to include magnetic fields and variables odd under time reversal is due to Onsager (1931b); for a more thorough discussion, see Casimir (1945).

Actually, as the theorem was originally and is still usually stated, M is taken to be a property not of the decay of the two-time correlation function of the finite-amplitude but microscopically derived fluctuations, but rather of the regression of *mean infinitesimal perturbations* $\langle\Delta\boldsymbol{X}\rangle$ away from the equilibrium state. In thermal equilibrium the equivalence of the two descriptions is guaranteed by the fluctuation–dissipation theorem (Martin, 1968), one form of which is (Kraichnan, 1959a; Krommes, 1993b) $\mathsf{C}_+(\tau) = \mathsf{R}(\tau)\cdot\mathsf{C}(0)$, R being the mean infinitesimal response matrix introduced in Sec. 3 (p. 46) and discussed in Secs. 5 (p. 126) and 6 (p. 146). For further discussion, see Krommes and Hu (1993).

It is a point of considerable confusion in the literature that Onsager's theorem applies *not* to the regression matrix M itself but rather to the Onsager matrix $\mathsf{L} \doteq \mathsf{M}\cdot\mathsf{C}$. M describes the fate of (nonequilibrium) perturbations *away from* the steady state; L additionally includes through C information about the fluctuation intensity *in* the steady state. In thermal equilibrium and for special choices of variables, C can be reduced to a form proportional to $T\mathsf{I}$, where T is the equilibrium temperature and I is the identity matrix. For such special situations the symmetry properties of L are the same as those of M. However, that is certainly not true in general.

In fact, the presence of C in the definition of the Onsager matrix is just what is needed to develop a *covariant* formulation of Onsager symmetry. For example, Grabert and Green (1979) argued, on the basis of representation independence of the thermodynamic entropy, that thermodynamic forces naturally transform like a covariant vector and that the Onsager matrix is naturally a contravariant tensor. For many related references, see Krommes and Hu (1993). A family of covariant descriptions can be parametrized by the choice of fundamental tensor g. Whereas Grabert and Green chose $g^{ij} = L^{ij}$, Krommes and Hu argued for the choice

$$g^{ij} = C^{ij} \equiv \langle \delta X^i \, \delta X^j \rangle, \tag{444}$$

with $g_{ij} = \mathsf{C}^{-1}_{ij}$. Indices are raised and lowered in the conventional fashion—for example, $X_i = g_{ij} X^j$. Note that covariant regression laws are properly written as

$$\partial_t \Delta X^i = -M^i{}_j \Delta X^j, \tag{445}$$

so the abstract regression tensor M most naturally appears in the *mixed* representation $M^i{}_j$. However, Onsager's theorem is a statement about the *fully contravariant* matrix L^{ij}. By definition, one has

$$L^{ij} = M^i{}_k C^{kj} \quad \text{or} \quad L^{ij} = M^{ij}, \tag{446a,b}$$

the latter following by virtue of Eq. (444). That is, the abstract tensors L and M are identical in a properly covariant formalism. However, one must be sure to not mix representations, since $L^{ij} \neq L^i{}_j$ except in special cases. Thus the standard operational definition of the regression matrix in terms of incremental perturbations to equilibrium states defines the mixed matrix components $M^i{}_j$ according to Eq. (445). However, an independent measurement of the steady-state fluctuation level is needed in order to deduce the fundamental tensor g [Eq. (444)] that is required to raise the indices and calculate L^{ij} according to Eqs. (446).

The existence of a covariant description renders moot the recurring question of what the appropriate set of variables that exhibits Onsager symmetry is. In fact, there is no such preferred set; any set of independent variables may be employed. Onsager's symmetry, being covariant, holds for any set related by linear transformation to the original set. Linear transformations are sufficient because Onsager's theorem is a statement about *infinitesimal* perturbations.

12.1.2 The generalized Onsager theorem

Now consider *nonequilibrium* steady states. It is no longer obvious that Onsager's theorem holds, since it was derived from specific detailed-balance properties (Haken, 1975) of absolute thermal equilibrium. In the case of forced, dissipative turbulence the steady-state PDF bears no resemblance to that of Gibbs. For the general case Dufty and Rubí (1987) showed that a *generalized Onsager theorem* holds: $\mathcal{L}(\boldsymbol{B}) = \overline{\mathcal{L}}^{\dagger}(-\boldsymbol{B})$. Here $\overline{\mathcal{L}}$ is a property of the time-reversed steady state. Unfortunately, the form of a PDF for an equilibrium held steady by the balance between forcing and dissipation need not be invariant under time reversal, significantly complicating the practical use of the generalized theorem. Some examples and further discussion were given by Krommes and Hu (1993).

12.1.3 Onsager symmetries for turbulence

By definition, turbulence describes statistical disequilibrium. If an appropriate set of state variables that express a sensible Markovian dynamics can be identified, then the generalized Onsager theorem must hold. With regard to analytical descriptions of turbulence, the existence of *realizable, Markovian Langevin representations* with *built-in fluctuation–dissipation relations* is key. It would appear that the generalized theorem should hold for the resulting closures.[270]

Krommes and Hu considered a model two-variable Langevin system for which nonvanishing cross correlations of the Langevin forces broke the original Onsager symmetry; such cross correlations are to be expected in the general case. In any event, it is vital to identify the appropriate matrix to study—it is L, not M. Thoul et al. (1994) made an ambitious attempt to consider Onsager symmetry for drift waves described by a crude version of the DIA. Unfortunately, their conclusion that the DIA grossly violates Onsager symmetry was incorrectly based on M (Krommes and Hu, 1993). In preliminary analysis of simulation and closure calculations for HW dynamics, Hu (1995) found a mild violation of Onsager symmetry. Further analysis of Onsager symmetries for turbulence would be very desirable.

12.2 Entropy balances

The evolution of various entropies or entropylike quantities has frequently figured in the description of statistical-mechanical systems, including plasmas. Martin et al. (1973) discussed a generalized entropy functional in terms of the MSR formalism, closely following earlier work by de Dominicis and Martin (1964a).

Entropy figures prominently in the discussion of Onsager relations (Sec. 12.1, p. 235). Pattern-recognition algorithms, relevant to the diagnosis of turbulent systems, frequently employ a principle of maximum entropy (Jaynes, 1982).

In plasma physics early and partially unpublished important work on entropy balances for microturbulence was done by Nevins (1979a,b, 1980). More recently, particular versions of such entropy balances were considered by Lee and Tang (1988), Rath and Lee (1992), Lee et al. (1992), and Rath and Lee (1993). This and other literature was summarized and discussed by Krommes and Hu (1994).

12.2.1 The Entropy Paradox

In most of the works cited in the last paragraph, a quadratic approximation to the information-theoretic entropy (Shannon and Weaver, 1972) was employed. For example, consider a kinetic theory described by the PDF $f = \langle f \rangle + \delta f$. Through second order the conventional information-theoretic

[270] The time-reversibility properties of Langevin equations as descriptions of microscopic dynamics can be confusing because of the dissipative damping on the left-hand side. See Krommes and Hu (1993, Sec. IV B) for a thorough discussion.

entropy $\overline{\mathcal{S}}$ (relative to a reference PDF f_0) is (Jaynes, 1968)

$$\overline{\mathcal{S}} \propto -\int \mathrm{d}z\, f \ln(f/f_0) \approx -\frac{1}{2}\int \mathrm{d}z\,(\delta f^2/f_0). \tag{447}$$

The quantity $\overline{\mathcal{F}} \doteq -\overline{\mathcal{S}}$ was measured in various computer simulations (Rath and Lee, 1992; Lee et al., 1992; Rath and Lee, 1993). It was observed that $\overline{\mathcal{F}}$ increased without bound in collisionless particle simulations even after the turbulent fluxes appeared to have achieved saturated steady states. This behavior was called the *entropy paradox* by Krommes and Hu (1994) because in a true statistical steady state all statistical observables must be independent of time. Attempts to reconcile this paradox led Krommes and Hu (1994) to perform analyses of various quadratic balance equations.

Quite generally, the production term $\mathcal{P}$ in the appropriate turbulent energy balance is related to the flux of some quantity; it often has the simple form $\overline{\mathcal{P}} = \kappa\overline{\Gamma}$, where κ is the inverse of a gradient scale length (which represents the source of free energy for the turbulence) and Γ is the turbulent flux. Then one has

$$\partial_t\overline{\mathcal{F}} = \kappa\overline{\Gamma} - \overline{\mathcal{D}}, \tag{448}$$

where $\overline{\mathcal{F}}$ is the mean-square average of a certain fluctuating field and $\overline{\mathcal{D}}$ is a positive definite dissipation functional. For example, the spatially averaged energy balance (14) for the NSE has the form (448), as do the equations (55c) for fluid energy and (55d) for mean-squared ion density in the HW system. To the extent that dissipation is negligible and flux is positive, Eq. (448) suggests an indefinite increase of $\overline{\mathcal{F}}$, the behavior observed by Rath and Lee. If a steady state is actually achieved, however, Eq. (448) shows that dissipation can never be negligible, since the steady-state balance

$$\kappa\overline{\Gamma} = \overline{\mathcal{D}} \tag{449}$$

must hold. This equality raises the question of whether the value of Γ is actually *determined* by the dissipation processes. Krommes and Hu (1994) argued to the contrary. Motivated by the well-known paradigm of NS turbulence (Sec. 3.8.2, p. 73), they argued that detailed nonlinear dynamical processes not apparent in the macroscopic balance (449) determine the steady-state value of Γ; then the net dissipation is determined by Eq. (449). The analog to the Kolmogorov picture of NS turbulence is that the dissipation wave number adjusts to accommodate the amount of long-wavelength forcing.

For an example that fits well with the emphasis of the present article on model equations and statistical paradigms, I follow Krommes and Hu (1994) in considering steady states of the system composed of (i) the Terry–Horton equation (43) for the ions, and (ii) an entirely *linear* kinetic equation for the electrons that supports linear *growth* but also includes collisional effects. Recall that the THE is defined by the complex susceptibility χ, which determines the linear growth γ from the iδ model. Let an ion damping $\eta^{\mathrm{lin}}_{\boldsymbol{k}}$ be added by hand. To obtain the cleanest model, take $T_i \to 0$; then there is no ion Landau damping, so η^{lin} must be purely collisional. Recall that the model possesses the single quadratic invariant $\mathcal{Z}$ defined by Eq. (44). The balance equation is

$$\partial_t\mathcal{Z} = 2\sum_{\boldsymbol{k}}(\gamma^{\mathrm{lin}}_{\boldsymbol{k}} - \eta^{\mathrm{lin}}_{\boldsymbol{k}})\mathcal{Z}_{\boldsymbol{k}}. \tag{450}$$

From Eq. (3b) for the electron particle flux Γ_e, and with the iδ model for χ, one can show (Krommes and Hu, 1994) that the γ^{lin} term in Eq. (450) is just $\kappa\Gamma_e$:

$$\partial_t \mathcal{Z} = \kappa\Gamma_e - 2\sum_{\boldsymbol{k}} \eta^{\text{lin}}_{\boldsymbol{k}} \mathcal{Z}_{\boldsymbol{k}}. \tag{451}$$

The fundamental steady-state scenario for fluid-level quantities is as follows: (i) First, fluctuations are excited by inverse electron Landau damping. (ii) Next, fluid nonlinearities transfer energy to nonresonant ion sloshing motion. (iii) Finally, that motion is stabilized by ion collisional dissipation. Krommes and Hu showed that this fluid scenario is compatible with direct calculations at the kinetic level. Note, however, that if one were to look in detail at the particle distributions, the electrons would cool and the ions would heat; the fluid scenario does not appear to be compatible with a kinetic steady-state balance. The resolution is that a true steady state is finally achieved by ion–electron collisional coupling of the ions back to the electrons. To restate the philosophy of the last paragraph, one does not need to calculate the latter dissipation explicitly; it adjusts as necessary to ensure the steady-state flow of $Z_{\boldsymbol{k}}$ through the $\boldsymbol{k}$ space.

12.2.2 Thermostats

Evidently, collisional dissipation is important even in the limit that the collision frequency ν approaches zero. If so, the limit $\nu \to 0$ is *singular*; the limits $t \to \infty$ and $\nu \to 0$ must not be interchanged. Krommes and Hu (1994) discussed a solvable stochastic-oscillator model that demonstrated this behavior explicitly.

The existence of a singular limit poses a problem for collisionless particle simulations because it implies that purely collisionless algorithms are never correct in principle. Although they may make valid predictions for low-order statistical moments such as the turbulent flux, they are likely to fail for long simulation times because successively finer and finer scales in phase space will be excited and statistical noise may become unacceptably large. One can attempt to model collisions explicitly [for a discussion of techniques and original references, see Brunner et al. (1999)]. Alternatively, one can use of *thermostats.*

Thermostats are a technique originally developed for use in nonequilibrium molecular dynamics (NEMD). In NEMD one formulates *homogeneous* algorithms for the calculation of transport coefficients. Instead of driving, say, thermal flux by imposing a temperature gradient, which would relax inhomogeneously and give trouble near simulation boundaries, one imposes a fictitious force that interacts homogeneously with the particles and drives a flux that agrees with linear response theory (the Green–Kubo formulas). Nevertheless, such a driven system will heat, complicating the interpretation of diagnostic measurements. To ensure steady state, a dynamical damping or *thermostat* $\widetilde{\zeta}$ is added to the equations of motion according to $\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i - \widetilde{\zeta}(t)\boldsymbol{p}_i$. The value of $\widetilde{\zeta}$ is chosen at each time step to freeze the kinetic temperature, either exactly or on the time average. The technique was reviewed by Evans and Morris (1984, 1990), and Hoover (1991). Krommes (1999b) advocated the use of a generalized thermostat for collisionless particle simulations of plasmas. Nevertheless, as he discussed, the justification of a thermostat for simulations of steady-state turbulence is substantially less secure than is the one for NEMD simulations. Because heating is a quadratic effect, the thermostat does not disturb near-equilibrium linear response. Nevertheless, turbulent transport coefficients do not obey linear response theory, and the proper fluctuation level at which to stabilize the system is unknown *a priori.* Krommes suggested a certain extrapolation procedure [271] for predicting the collisionless limit of the turbulent flux, but further work is called for.

12.3 Statistical method for experimental determination of mode-coupling coefficients

In this article I have stressed the analytical statistical analysis of primitive amplitude equations of the form (223) with *specified* linear operator $\mathcal{L}_{\boldsymbol{k}}$ and nonlinear mode-coupling coefficient $M_{\boldsymbol{kpq}}$. In an interesting inversion of the procedure, Ritz and Powers (1986) and Ritz et al. (1989) suggested that statistical analysis of experimental power spectra can be used to *deduce* $\mathcal{L}_{\boldsymbol{k}}$ and $M_{\boldsymbol{kpq}}$. In brief, the technique is to (i) assume that the quadratically nonlinear, scalar field equation (223) holds; (ii) write the exact equations for the equal-time two- and three-point correlation functions; (iii) apply the quasinormal approximation to the undetermined four-point function [Ritz *et al.* cite Millionshtchikov (1941a)] ; (iv) insert measured two-point spectra; and (v) infer $\mathcal{L}_{\boldsymbol{k}}$ and $M_{\boldsymbol{kpq}}$.

Ritz *et al.* showed that this procedure could be used successfully if the original Ansatz (223) provides an adequate description of the turbulent dynamics. [272] Mynick and Parker (1995b,a) applied the method to the analysis of data from gyrokinetic particle simulations. They showed that a modest number of dynamical amplitudes sufficed to describe the turbulence, and they were able to explain various features of the simulations in terms of an HW paradigm.

Nevertheless, questions remain. In general, fluid descriptions of plasma phenomena are best described by multifield models that do not naturally reduce to the scalar paradigm (223). (This point was noted by Ritz *et al.*) For strong turbulence the proper statistical description should also be self-consistently renormalized; see the triad interaction time in the Markovian theories. How much the uncertainties upset the interpretation of the data is unknown, and more work is needed. However, the method is an instructive application of basic statistical theory and is probably sufficiently robust that broad conclusions about experimental data can be believed.

12.4 Self-organized criticality (SOC)

"We argue and demonstrate numerically that dynamical systems with extended spatial degrees of freedom naturally evolve into self-organized critical structures of states which are barely stable. We suggest that this self-organized criticality is the common underlying mechanism behind [$1/f$ noise and other self-similar phenomena]. . . . We believe that the new concept . . . can be taken much further and might be *the* underlying concept for temporal and spatial scaling in a wide class of dissipative systems with extended degrees of freedom." — *Bak et al. (1987).*

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[271] Krommes's work was aimed at the so-called δf simulations (Appendix C.1.7, p. 276), in which an important role is played by the *particle weight* w, a measure of δf. The analog of the kinetic temperature in NEMD is $W \doteq \langle w^2 \rangle$. The procedure involves calculating the flux Γ for sample W's, then extrapolating $\Gamma(W)$ to the limit $W \to 0$.

[272] Ritz *et al.* pointed out that in some situations cubic or higher nonlinearities may need to be considered. That may be correct, but it is misleading to call Eq. (223) a "three-wave coupling equation," as Ritz *et al.* did. If the weak-turbulence statistical theory of the quadratically nonlinear Eq. (223) is carried beyond second order in the intensity expansion, n-wave coupling processes emerge for all $n \geq 3$.

Self-organized criticality (SOC) refers to the states achieved by certain driven nonlinear systems that exhibit no preferred length or time scales. Unlike the familiar models of equilibrium critical phenomena (Binney et al., 1992; Goldenfeld, 1992), in which long-ranged correlations emerge only at a particular critical point in parameter space that can be reached by adjusting an external parameter such as the temperature, self-organizing systems *self-tune* to the critical state; they determine that state self-consistently. [273]

12.4.1 Sandpile dynamics

The concept of self-organized criticality was introduced by Bak et al. (1987, 1988) with the aid of a simple discrete nonlinear dynamics for the slope of a model sandpile. [274] Their motivation was to provide a dynamical explanation for the $1/f$ noise [275] observed in many natural phenomena and physical systems. Because of the relative tractability of the discrete dynamical models and the goal of a universal mechanism, an explosion of literature followed. Some representative works are by Kadanoff et al. (1989), Carlson et al. (1990), Carlson and Swindle (1995), and Vespignani and Zapperi (1998). Bak (1996) gave a nontechnical discussion of the elementary ideas. A pedagogical synthesis of the literature through the mid-1990s was given by Jensen (1998).

A typical sandpile model gives the dynamics for a discrete height variable h defined on a d-dimensional lattice. Discrete units of height ("grains of sand") are added one at a time to random lattice sites. After each addition the new height is compared with a specified *toppling threshold*. If the threshold is exceeded, N units are redistributed to neighboring sites. Those sites, in turn, may themselves be destabilized and topple. Thus an *avalanche* may be initiated. In the classic sandpile the avalanche is allowed to go to completion before the next grain is added. In the absence of any characteristic length except for the macroscopic lattice size L, avalanches may be of any spatial extent up to L. In the long-time limit the statistical distributions of the avalanches saturate and a steady-state profile emerges. Various time series can be constructed from the avalanches (such as measuring the number of grains crossing an edge of the lattice) and related to spectra of the form $\omega^{-\alpha}$.

Sandpile dynamics are interesting in part because they demonstrate one mechanism by which local perturbations (forcing or fluctuations) can generate nonlocal effects (avalanches). [It is worth noting here that even the classical diffusion equation exhibits such phenomena. However, sandpile models can propagate perturbations faster than diffusive, even approaching *ballistic* speeds ($x \propto t$ rather than $x \propto \sqrt{t}$).] Another interesting feature of many sandpile models is that the mean profiles can be *submarginal* [defined in Sec. 9 (p. 210)]. For example, when the toppling condition is based on

[273] In fact, various control parameters for SOC have been identified, including the forcing rate (Vespignani and Zapperi, 1998) and the *local rigidity* (Cafiero et al., 1995), a measure of the importance of the threshold condition for toppling or linear growth. Pure SOC emerges in the limit that the forcing rate is tuned to zero and the local rigidity is tuned to infinity. Vespignani and Zapperi (1998) argued, "SOC models appear to be nonequilibrium systems with steady states, reaching criticality by the fine tuning of control parameters." Nevertheless, they acknowledged that SOC systems are distinct from those of ordinary critical phenomena because of the limiting procedure required for the former.

[274] Real sand need not behave as does the theorists' reference sandpile model (Nagel, 1992). Nevertheless, that observation does not vitiate the utility of the sandpile model for studies of and insights about general nonlinear behavior.

[275] That is, the power spectrum of an observed time series varies as ω^{-1} ($\omega \doteq 2\pi f$) over an interesting range of frequencies. More generally, spectra of the form $\omega^{-\alpha}$ are relevant, where α need not be precisely 1.

the local slope $s \doteq h_{i+1} - h_i$ [see, for example, Kadanoff et al. (1989)], the mean slope may saturate a finite amount below the threshold slope S for toppling. An illustrative 1D model that has been used in various plasma discussions (Newman et al., 1996; Carreras et al., 1996c) is defined by requiring that when $s \geq S$, then N grains are redistributed according to

$$h_n \to h_n - N, \quad h_{n+1} \to h_{n+1} + N. \tag{452a,b}$$

For $N = 1$ the steady-state profile is marginal ($\langle s \rangle = S$), but for $N > 1$ it is submarginal. Of course, the detailed dynamics of a discrete submarginal sandpile are ridiculously simpler than those of a nonlinearly self-sustained advective continuum system such as discussed in Sec. 9.6 (p. 216). Nevertheless, one can at least inquire whether all such systems share certain universal characteristics. The answer is far from clear because nonlinear systems may live in various universality classes.

12.4.2 Continuum dynamics and SOC

Certain perturbation experiments in tokamak plasmas (Gentle et al., 1997) have suggested to some that nonlocal effects or superdiffusive propagation may be operating. In some situations it may also be that anomalous transport is observed even for linearly stable profiles, although that is not very clear. Noting that sandpile dynamics might provide an explanation, Diamond and Hahm (1995) suggested that concepts of SOC might be applicable to the behavior and determination of steady-state profiles (of temperature, say) in plasma confinement devices. Their qualitative ideas have been subsequently reiterated and explored by various authors; representative references include Newman et al. (1996), Carreras et al. (1996b,c), and Newman (1999).

Although applications to tokamak phenomenology are beyond the scope of this review, the technical methods are not. Diamond and Hahm reiterated a result from the work of Hwa and Kardar (1992), who attempted to argue that a relatively universal continuum hydrodynamic model of self-organizing behavior is the 1D forced Burgers equation (19). Once such an equation is postulated, standard renormalization techniques can be applied. For the Burgers equation, I have already referred to the works of Forster et al. (1976, 1977) [Secs. 7.2.1 (p. 183) and 7.4 (p. 196)] as well as Medina et al. (1989) and Hwa and Kardar (1992; Sec. 7.2.1, p. 183). In Sec. 7.2.1 it was shown that in one dimension and for nonconservative random forcing, renormalized pulses exhibit *on the average* a *ballistic scaling* with $\eta^{\mathrm{nl}}_{\boldsymbol{k}} = |k|\overline{V}$ rather than $k^2 D$. Diamond and Hahm (1995) also showed that such behavior could be suppressed by the imposition of mean flow shear. Aspects of that work were reconsidered by Krommes (2000a).

Space does not permit a complete analysis of the various, sometimes contradictory literature on self-organization in continuum systems; for some early discussion, see Krommes (1997c). One central issue, however, does not seem to have received adequate attention in at least the plasma-physics literature. That is the nature of the forcing in the nonlinear continuum model. The rain of sand on a model sandpile is clearly represented by an *additive* forcing. The Burgers equation is additively forced as well. But suppose the goal is to assess the role of microturbulence on steady-state mean temperature profiles. Such microturbulence would *advect* the temperature, leading to a *multiplicative* forcing; i.e., temperature fluctuations would evolve according to $\partial_t \delta T + \widetilde{\boldsymbol{V}}(\boldsymbol{x}, t) \cdot \boldsymbol{\nabla} \delta T + \cdots = 0$, where $\widetilde{\boldsymbol{V}}$ represents the random velocity field of the microturbulence. But nonlinear advection equations with multiplicative statistics are precisely the ones treated in the standard statistical closures described earlier in this article. In particular, to the extent that the microturbulence is short ranged in space and/or time (the usual assumption), standard coarse-graining arguments lead to *regular* diffusion equations for the mean fields, not the singular diffusion equations evinced by self-organizing systems (Carlson et al., 1990).[276]

Of course, an advective nonlinearity can always be placed on the right-hand side of the dynamical equation, where it then plays the role of an additive forcing. However, that is not useful because (i) the statistics of such a term are not arbitrary and specifically cannot be Gaussian (even for passive advection), and (ii) such forcing would be conservative (Model A of FNS). But according to the results of FNS, Models A and B belong to different universality classes.

I have noted that steady-state SOC profiles can be submarginal. Krommes (1997b) analyzed a simple stochastic model with multiplicative forcing that led to submarginal profiles. However, that model was simply postulated, not derived. As indicated in Sec. 9 (p. 210) on the general problem of submarginal turbulence, much further work on the nonlinear dynamics of such systems remains to be done.

12.4.3 Long-time tails and SOC

Do real confinement devices like tokamaks follow principles of SOC? Carreras et al. (1998b) [see also Carreras et al. (1998a)] attempted to answer that question by analyzing experimental data in search of long-time correlations—in particular, algebraic tails on two-time *Eulerian* correlation functions of the form $C(\tau) \sim |\tau|^{-\beta}$ $(0 < \beta < 1)$. [Intuitively, one expects that long-ranged, scale-invariant avalanches should give rise to long-lived, scale-invariant time correlations, as argued in the first paper by Bak et al. (1987), although I will question the uniqueness of this interpretation shortly.] A related measure is the *Hurst exponent* (Hurst et al., 1965; Mandelbrot, 1998) $H \doteq 1 - \frac{1}{2}\beta$. Measurements of edge density fluctuations in a variety of tokamaks were reported to give rise to tails with $0.62 \pm 0.01 < H < 0.72 \pm 0.07$. It was suggested (Carreras et al., 1998b) that such results are consistent with ideas of SOC. It was also asserted that if fluctuation data possess such long-time tails, then the "standard transport paradigm" might have to be abandoned.

Exactly what is the standard transport paradigm is subject to some debate. From the point of view of this article on fundamental statistical methods, the DIA is a standard paradigm (which makes no fundamental distinctions between long and short scales and and copes with nonlocal phenomena in space and time). However, most people would probably consider the standard paradigm to be one in which Markovian approximations hold and transport coefficients can be calculated from random-walk estimates like $D \sim L_{\rm ac}^2/\tau_{\rm ac}$ or $D \sim \overline{v}^2\tau_{\rm ac}$ for microscopic correlation lengths and times. If one recalls the Lagrangian representation (Taylor, 1921) $D = \int_0^\infty d\tau\, C_L(\tau)$, and if $C_L(\tau) \sim \tau^{-\beta}$, then D does not exist for $\beta \le 1$. This raises the possibility of anomalous diffusion processes Sec. 3.2.3, p. 52 for which $\langle x^2 \rangle \propto t^\alpha$ with $\alpha \ne 1$. Various authors have attempted to describe such motion with fractional Fokker–Planck operators (Metzler et al., 1999; Carreras et al., 1999b).

That Lagrangian correlations enter into the definitions of transport coefficients is crucial. It does not follow that long-time tails on Eulerian correlation functions—e.g., the equal-space-point density autocorrelation function $\langle \delta n(\boldsymbol{x}, t + \tau)\delta n(\boldsymbol{x}, t)\rangle$—necessarily lead to divergent transport coefficients, because additional decorrelation effects in the Lagrangian variable $\boldsymbol{x}(t)$ can superimpose a short decay

276 It has been suggested that the presence of linear instabilities with nonzero thresholds may vitiate this argument. Krommes (1997b) discussed some consequences of thresholds for various stochastic models; although certain assumptions about the behavior near threshold were relevant to the existence of submarginal profiles, the existence of thresholds did not interfere with the derivation of regular (if nonlinear) diffusion equations.

on a long-time envelope. [A simple illustration of this remark is provided by passive quasilinear theory Sec. 4.1.2, p. 91, in which stochasticity (orbit diffusion) introduces a slow (although not algebraic) nonlinear envelope but for which the Lagrangian force correlation decays on the short $\tau_{\rm ac}^{\rm lin}$ timescale.] Krommes and Ottaviani (1999) discussed a specific stochastic model, with an advecting velocity that was assigned short $L_{\rm ac}$ and $\tau_{\rm ac}$, in which the Eulerian correlations possessed a well-developed $\tau^{-1/2}$ tail, yet the transport could be adequately estimated by a quasilinear approximation. Those authors also attempted to give general arguments based on the spectral balance equation (286). They showed that if long-wavelength fluctuations developed (due to inverse cascade, say), then one may expect long-time Eulerian tails. Both a simple estimate based on HM dynamics as well as a numerical simulation of the HW system (neither of which involved notions of SOC) led to predictions for Hurst exponents in the range $\frac{1}{2} \leq H \leq \frac{3}{4}$, in reasonable agreement with the experimental observations. That may be fortuitous, but it shows that there is not an inevitable connection between long-time tails and SOC. Further discussion of these and related issues was given by Krommes (2000c).

The previous discussion merely provides an introduction to a large literature. Self-organization is an interesting and challenging topic in its own right and deserves a separate review article. To the extent that the Burgers equation is a reasonable model (a controversial point), one can apply techniques described elsewhere in the present article, such as the DIA Sec. 5, p. 126, the renormalization group Sec. 7.4, p. 196, or theories of intermittent PDF's Sec. 10.4.3, p. 227. Further discussion of the relevance of SOC to plasma physics was given by Krommes (1997c).

12.5 Percolation theory

A vast area of research concerns the development of the methods of statistical topography to describe transport in random mediums. The subject is closely related to classical percolation theory. An excellent review of these issues (with many references) was given by Isichenko (1992), so it is unnecessary to develop them here. One interesting result is a nontrivial dependence of the effective diffusivity D of 2D passive advection on the Kubo number $\mathcal{K}$: $D \sim \mathcal{K}^{-3/10}$ (Gruzinov et al., 1990). (The simple strong-turbulence estimates described in this article merely suggest $D \sim \mathcal{K}^0$.) Ottaviani (1992) discussed difficult attempts to verify this prediction numerically.

12.6 Inhomogeneities and mean fields

The formal moment-based closures emphasized in this article effect their approximations at the second-order level, i.e., on the fluctuations; the equation for the mean field is retained exactly. This is true both in the general MSR formalism [Sec. 6 (p. 146); see Eq. (276)] and in the DIA Sec. 5, p. 126. The fluctuation-induced contributions to the mean-field dynamics generalize the Navier–Stokes Reynolds stress $\boldsymbol{\tau} \doteq -\rho_m \langle \delta \boldsymbol{u}\, \delta \boldsymbol{u} \rangle$ introduced in Sec. 2.1.1 (p. 23).

In this article mean fields have appeared in a variety of contexts: (i) fluid equations, like that of Navier–Stokes, in inhomogeneous geometries; (ii) the one-particle PDF f, the average of the Klimontovich microdensity Sec. 2.2.2, p. 27; (iii) frozen backgrounds in drift-wave equations such as those of HM [Eq. (48)], TH [Eq. (43)], or HW [Eq. (52)]; (iv) the theory of eddy diffusivity Sec. 5.7, p. 140; (v) mixing-length estimates of saturation level; (vi) upper bounds on transport in inhomogeneous systems Sec. 11, p. 230; and (vii) self-organized criticality Sec. 12.4, p. 241. The practical importance of mean fields is evident. Nevertheless, for the purposes of this article it is necessary to make a clean distinction between formal, systematic theories and the rich potpourri of practical applications; the latter cannot be reviewed here although a few entry points to the literature are given.

12.6.1 K–ε models

In a classical K–ε model[277] the form of the Reynolds stress is restricted by the use of symmetry properties and gradient expansions, and the undetermined coefficients are represented by dimensionally correct combinations of gross fluctuation properties such as the mean fluctuation energy K and the energy dissipation rate ε. Similar procedures applied to the evolution equations for K and ε lead to closure approximations generically called K–ε models (Bradshaw et al., 1981). There are many variants; see, for example, Besnard et al. (1996). A recent review of this topic and the more sophisticated *second-order modeling* is by Yoshizawa et al. (2001).

It is possible to *derive* K–ε models from second-order closures. For example, Yoshizawa has proposed and studied in detail a *two-scale DIA* [Yoshizawa (1984) and references therein; Yoshizawa et al. (2001)]. The technique was pursued by Sugama et al. (1993) for resistive MHD.

12.6.2 L–H transitions

A typical K–ε model is a system of equations in which quadratic measures of fluctuation intensity are coupled to mean fields. Such systems exhibit bifurcations Sec. 9.3, p. 212 and phaselike transitions as functions of external control parameters. This possibility is of considerable interest for magnetic confinement of plasmas because of the experimentally observed "low–high" (L–H) transition (Wagner et al., 1982). Diamond et al. (1994) introduced a particular system of two coupled ODE's for density fluctuation level and flow shear that exhibited a plausible bifurcation. Considerable research has been expended on the physics of the transition; recent reviews are by Burrell (1997) and Connor and Wilson (2000). Spatschek (1999) emphasized the general importance of low-dimensional descriptions and provided many references.

An example of an attempt to develop a systematic L–H model is the work by Horton et al. (1996). They considered a 6-ODE model that generalized earlier work of Sugama and Horton (1995), who used three "thermodynamic" functions to characterize the turbulence. Note, however, that a fully systematic derivation of a low-degree-of-freedom thermodynamic description of turbulence is highly nontrivial and has not yet been accomplished. For example, Markovian statistical closures for a system of n real ODE's involves without further approximation $\frac{1}{2}n(n+1)$ independent correlation functions; in the absence of symmetry considerations, there is as yet no simple way of deciding in advance whether any of those is negligible.[278] Also note that for highly nonequilibrium systems "thermodynamics" is an ill-defined concept (although the MSR formalism can be described as a kind of functional thermodynamics).

12.6.3 The effects of mean shear

Unquestionably, models with homogeneous statistics are the easiest to analyze. Nevertheless, in a variety of important physical situations fluctuations are profoundly affected by inhomogeneities. Those

[277] This discussion paraphrases a section of Krommes (1997c).

[278] LoDestro et al. (1991) noted that cross correlations can be at least quantitatively important.

enter the formalism as coefficients depending on one or more of the independent variables such as x. In this case two-point functions must depend separately on x and x'. This general dependence is at the very least a technical annoyance, but there are important associated physical effects as well. Frequently the strength of an inhomogeneity is measured by a generalized *shear*. Given an inhomogeneous coefficient $V(x)$, the shear is measured by the first Taylor coefficient: $V(x) \approx V(0) + V'(0)x$. The unnormalized shear is V'. (If x and V are literally position and velocity, V' is a *shearing rate*.) Given a reference spatial scale L and a reference velocity $\overline{V}$ [possibly either $V(0)$ or, if that vanishes, an rms fluctuation level], a normalized measure of shear is $S \doteq LV'/\overline{V}$.

Here are some examples in which background inhomogeneities play a role: (i) *Vlasov plasma.* The streaming term $\boldsymbol{v} \cdot \boldsymbol{\nabla}$ is inhomogeneous in velocity space, so the particle propagator $g(\boldsymbol{x}, \boldsymbol{v}, t; \boldsymbol{x}', \boldsymbol{v}', t')$ depends separately on $\boldsymbol{v}$ and $\boldsymbol{v}'$. The background PDF (often taken to be Maxwellian) also varies with $\boldsymbol{v}$, so the full infinitesimal response function R Sec. 6.5.2, p. 172 would depend separately on $\boldsymbol{v}$ and $\boldsymbol{v}'$ even if the streaming term were absent. (ii) *Sheared magnetic fields.* Magnetic fields with radially varying winding numbers (*safety factors* or *inverse rotational transforms*) $q(r)$ are characterized by the shear parameter $\widehat{s} \doteq \mathrm{d}\ln q/\mathrm{d}\ln r$. In a slab geometry periodic in the y and z directions, fluctuations with given values of k_y and k_z have a parallel wave number $k_\parallel$ (in the direction of the magnetic field) that varies with x: $k_\parallel(x) = k_z + k_y x/L_s$, where L_s is the shear length (inversely proportional to $\widehat{s}$). In microturbulence problems the spatial variation of $k_\parallel$ can lead to complete linear stabilization; see footnote 60 (p. 41). (iii) *Mean velocity shear.* In fluid problems mean (background) flows $\boldsymbol{U}$ enter the linearized velocity equation as $\boldsymbol{U}(x) \cdot \boldsymbol{\nabla}\delta\boldsymbol{u}$, where x labels the inhomogeneity direction in which boundary conditions are applied. A spatially independent part of $\boldsymbol{U}$ can be transformed away by a Galilean transformation. Sometimes a pure shear flow, $\boldsymbol{U}(x) = U'x\,\widehat{\boldsymbol{y}}$, is an exact equilibrium, as in planar Couette flow. As mentioned in the previous section, it is believed that mean flow shear (in the form of sheared poloidal $\boldsymbol{E} \times \boldsymbol{B}$ flow arising from radial electric fields) is important in the physics of the L–H transition (Terry, 1999).

An important physical effect is the interaction of shear with diffusion. The resulting behavior can be illustrated with the 1D Vlasov particle propagator g renormalized as in resonance-broadening theory to include a velocity-space diffusion term. The form of g is discussed in Appendix E.1.2 (p. 284). The most important feature of the results (E.10) and (E.13) is the appearance of spatial dispersion proportional to $D_v\tau^3$. If $\overline{k}$ is a characteristic spatial scale, the normalized dispersion (proportional to $\overline{k}^2 D_v\tau^3$) defines the characteristic diffusion frequency $\omega_d \doteq (\overline{k}^2 D_v)^{1/3} \equiv \tau_d^{-1}$. By introducing a characteristic velocity scale Δv, one has

$$\omega_d = \omega_s^{2/3}\omega_D^{1/3}, \tag{453}$$

where $\omega_s \doteq \overline{k}\Delta v$ is the shearing rate and $\omega_D \doteq D_v/\Delta v^2$ is the velocity diffusion rate. One can see that ω_d is a hybrid frequency comprising two-thirds shearing and one-third diffusion.

One significant application of formula (453) arises in the context of sheared magnetic fields. When a test particle initially streaming along a field line experiences a small amount of radial diffusion, it moves to a neighboring field line with a different rotational transform; it then streams along the new field line, rapidly separating azimuthally from its original position. This effect was analyzed in great detail by Krommes et al. (1983) in their studies of transport in stochastic magnetic fields. Of course, a completely analogous interpretation of the Vlasov motion can be given in terms of the appropriate phase-space characteristic trajectories, as was done by Dupree (1972b).

The results in Appendix E.1.2 (p. 284) that lead to formula (453) are based on the diffusive propagator described by Eq. (E.9), an approximate description based on a Gaussian white-noise hypothesis. However, in particular circumstances the result (453) is more general. In the context of velocity-space diffusion, consider the equation

$$g^{-1}\psi(x,v,t) = S(x,v,t), \tag{454}$$

where S is some given source. The Green's function g is assumed to obey a passive advection equation such as Eq. (E.9), but the nonlinearity is represented quite generally by a wave-number- and frequency-dependent mass operator $\Sigma^{\mathrm{nl}}_{k,\omega}$. The driven solution is

$$\psi_k(v,t) = \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}v'\, g_k(v,t;v',t') S_k(v',t') = \int_0^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}v'\, g_k(v,\tau;v') S(v',t-\tau), \tag{455a,b}$$

or

$$\psi_k(v,\infty) = \int_{-\infty}^{\infty} \mathrm{d}v' \left(\int_0^{\infty} \mathrm{d}\tau\, g_k(v,\tau;v') \right) S_k(v',\infty) = \int_{-\infty}^{\infty} \mathrm{d}v'\, g_{k,\omega=0}(v;v') S_k(v',\infty). \tag{456a,b}$$

This result shows how the $\omega = 0$ Fourier amplitude rigorously enters the long-time solution. If S_k is concentrated at a characteristic wave number $\overline{k}$, then it is $\Sigma^{\mathrm{nl}}_{\overline{k},\omega=0}$ that enters the equation for g; for small $\overline{k}$, $\Sigma^{\mathrm{nl}}_{\overline{k},\omega=0}$ can be adequately approximated by a diffusive operator, as in Eq. (E.9). Finally, suppose that S is independent of v and that ψ is desired at $v = 0$. Then [see the derivation of Eq. (E.15)]

$$\psi_{\overline{k}}(v=0,\infty) = \int_0^{\infty} \mathrm{d}\tau\, \exp(-\tfrac{1}{3}k^2 D_v \tau^3) S_{\overline{k}}(\infty) = [3^{-2/3}\Gamma(\tfrac{1}{3})\tau_d] S_{\overline{k}}(\infty). \tag{457a,b}$$

With these specific approximations, the effective value of the operator g in the formal solution $\psi = gS$ is seen to be the hybrid diffusion time τ_d to within a known numerical coefficient. Clearly this simple result does not hold more generally, although in some cases it may still provide a useful dimensional estimate.

Biglari et al. (1990) employed arguments related to formula (453) to discuss the influence of sheared poloidal rotation on the fluctuation and transport levels in edge turbulence. Although their detailed mathematics based on the clump formalism is suspect [see the discussion in Sec. 4.4 (p. 119)], their qualitative conclusions are plausible. Burrell (1999) presented experimental evidence that such sheared flows do affect turbulence and transport.

12.7 Convective cells, zonal flows, and streamers

Convective cells are defined to be fluctuations with $k_z = 0$ (with $\boldsymbol{B} \propto \widehat{\boldsymbol{z}}$). The nonlinear generation of convective cells by drift-wave interactions was originally discussed by Cheng and Okuda (1977); for more discussion, see Sagdeev et al. (1978). Note that convective cells are already present in homogeneous, isotropic turbulence. There they are not preferred in any way, though, because of isotropy; for every Fourier amplitude $(k_x, k_y, 0)$, there is another statistically equivalent one $(0, k_y, k_z)$. Symmetry is broken by a strong magnetic field. For $\boldsymbol{B} \neq 0$, modes with $k_z = 0$ are special because they do not experience Landau damping or (in a slab [279]) collisional damping. Furthermore, modes with

[279] In toroidal geometry the situation is more complicated; see Rosenbluth and Hinton (1998).

vanishing k_y [i.e., dependent only on x; $\boldsymbol{k} = (k_x, 0, 0)$] have vanishing diamagnetic frequency ω_*, so they cannot be linearly driven by background gradients. Potentials $\varphi(x)$ with such variation generate purely y-directed $\boldsymbol{E} \times \boldsymbol{B}$ velocities $V_{E,y}(x)$, creating random shear layers in poloidal zones; hence they are called *zonal flows*. Zonal flows have long been known to be important in rotating systems such as planets; two representative references are Busse (1994) and Marcus et al. (2000). Accumulated evidence from many years of computer simulations has shown that they are important constituents of plasma microturbulence as well. This could be expected from the close analogy between Rossby waves and drift waves (Horton and Hasegawa, 1994), and indeed was anticipated by Hasegawa and Mima (1978). Zonal flows are theoretically interesting because they are nonlinearly driven and very weakly damped, and their self-generated (random) shear can be expected to play a role in the dynamics of the modes (which will be called *drift waves* for short) that drive them.

Another extreme limit of convective cells is the case of purely y-dependent potentials [$\boldsymbol{k} = (0, k_y, 0)$]; the resulting x-directed $\boldsymbol{E} \times \boldsymbol{B}$ flows are called *streamers*. Because streamers provide a mechanism for direct advection along the background gradient, they can be important in enhanced transport. Drake et al. (1988) have observed streamers in computer simulations.

As this section of the article was being completed (early 2000), the statistical dynamics of long-wavelength fluctuations, including zonal flows and streamers, were a subject of active investigation. Some of that work has been conducted in the context of toroidal magnetic systems, the details of which are outside the scope of this review [see, for example, Beyer et al. (2000)]. Nevertheless, the general problem provides interesting and subtle applications of the general statistical theory described in this article. I shall comment very briefly on some of the conceptual pitfalls.

The interactions between convective cells and drift waves of arbitrary perpendicular wavelengths are difficult to analyze analytically. Nevertheless, energy conservation provides an important constraint on the structure of the theory. Consider for definiteness a one-field model (e.g., forced HM). The spectral evolution equation is

$$\partial_t C_{\boldsymbol{k}} - 2\gamma^{\mathrm{lin}}_{\boldsymbol{k}} C_{\boldsymbol{k}} + 2\,\mathrm{Re}\,\eta^{\mathrm{nl}}_{\boldsymbol{k}} C_{\boldsymbol{k}} = 2F^{\mathrm{nl}}_{\boldsymbol{k}}, \tag{458}$$

and one knows that the total energy $\mathcal{E} \doteq \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}$ is conserved by the nonlinear terms $N_{\boldsymbol{k}}$ described by $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$ and $F^{\mathrm{nl}}_{\boldsymbol{k}}$. Let $\boldsymbol{k}_*$ denote the drift waves (abbreviated by d) and $\overline{\boldsymbol{k}}$ denote the convective cells (abbreviated by c). The total vanishing wave-number sum over the nonlinear terms, $0 = \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} N_{\boldsymbol{k}} = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}) \ldots$, can be broken into eight terms, as follows:

$$\begin{aligned} 0 = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} = \; & \underbrace{\sum_{\boldsymbol{k}_*,\boldsymbol{p}_*,\boldsymbol{q}_*}}_{\text{(a): d} \leftarrow \text{d}+\text{d}} + \underbrace{\sum_{\boldsymbol{k}_*,\boldsymbol{p}_*,\overline{\boldsymbol{q}}}}_{\text{(b): d} \leftarrow \text{d}+\text{c}} + \underbrace{\sum_{\boldsymbol{k}_*,\overline{\boldsymbol{p}},\boldsymbol{q}_*}}_{\text{(c): d} \leftarrow \text{c}+\text{d}} + \underbrace{\sum_{\boldsymbol{k}_*,\overline{\boldsymbol{p}},\overline{\boldsymbol{q}}}}_{\text{(d): d} \leftarrow \text{c}+\text{c}} \\ & + \underbrace{\sum_{\overline{\boldsymbol{k}},\boldsymbol{p}_*,\overline{\boldsymbol{q}}}}_{\text{(e): c} \leftarrow \text{d}+\text{c}} + \underbrace{\sum_{\overline{\boldsymbol{k}},\overline{\boldsymbol{p}},\boldsymbol{q}_*}}_{\text{(f): c} \leftarrow \text{c}+\text{d}} + \underbrace{\sum_{\overline{\boldsymbol{k}},\boldsymbol{p}_*,\boldsymbol{q}_*}}_{\text{(g): c} \leftarrow \text{d}+\text{d}} + \underbrace{\sum_{\overline{\boldsymbol{k}},\overline{\boldsymbol{p}},\overline{\boldsymbol{q}}}}_{\text{(h): c} \leftarrow \text{c}+\text{c}} . \end{aligned} \tag{459}$$

Terms (a) and (h) separately vanish because of internal energy conservation. Terms (d), (e), and (f) vanish because the interaction of two convective cells cannot generate a nonzero k_z. The basic energy-conserving structure of the drift-wave–convective-cell system is therefore

$$\partial_t \mathcal{E}_* = 2\gamma_* \mathcal{E}_* - \underbrace{\dot{\mathcal{E}}^{\rm nl}}_{\text{(b) + (c)}}, \quad \partial_t \overline{\mathcal{E}} = -2\overline{\gamma}\overline{\mathcal{E}} + \underbrace{\dot{\mathcal{E}}^{\rm nl}}_{\text{(g)}}, \tag{460a,b}$$

where γ_* is a typical linear growth rate for the drift waves, $\overline{\gamma}$ is a typical damping rate for the convective cells, and the energy transfer $\dot{\mathcal{E}}^{\rm nl}$ can heuristically be argued to be positive (because it is the drift waves that explicitly extract free energy from the background gradients).

Further insights emerge by calculating $\dot{\mathcal{E}}^{\rm nl}$ under the assumption of disparate scales, as discussed in Sec. 7.3.2 (p. 193). If one ignores incoherent noise, then one finds

$$\partial_t \mathcal{E}_* = 2\gamma_* \mathcal{E}_* - \underbrace{2a\overline{\mathcal{E}}\mathcal{E}_*}_{\approx \text{(b) + (c)}}, \quad \partial_t \overline{\mathcal{E}} = -2\overline{\gamma}\overline{\mathcal{E}} + \underbrace{2a\mathcal{E}_*\overline{\mathcal{E}}}_{\approx \text{(g)}}, \tag{461a,b}$$

where $a > 0$. The stable steady-state solutions of Eqs. (461) are $\mathcal{E}_* = \overline{\gamma}/a$ and $\overline{\mathcal{E}} = \gamma_*/a$. These results appear to suggest that the drift-wave fluctuation level and, presumably, transport level as well are proportional to the damping rate of the convective cells, as concluded by Diamond et al. (1998) for the special case of zonal flows.

Suppose, however, that the convective cells had not been separated out for special attention. The analog of Eq. (461a) for the total system energy would be $\partial_t \mathcal{E} = 2\gamma\mathcal{E}$, which for a nontrivial steady state requires $\gamma = 0$ and provides no information about the actual saturation level. The basic difficulty is that so far no distinction has been made between positive and negative growth rates. The exact definition[280] of γ is $\gamma \doteq \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} \gamma_{\boldsymbol{k}}^{\rm lin} C_{\boldsymbol{k}}/\mathcal{E}$. The constraint $\gamma = 0$ provides some information on the relative fluctuation levels for different $\boldsymbol{k}$'s, but says nothing about the absolute spectral level. A further conceptual problem for systems with well-defined inertial ranges is that the procedure of encapsulating all growth and damping processes into a single γ does not distinguish between the positive $\gamma_{\boldsymbol{k}}$'s, which excite fluctuations (frequently) locally in $\boldsymbol{k}$ space, and the dissipation rate, which in the presence of an inertial range absorbs cascaded energy (or possibly enstrophy). First consider 3D Navier–Stokes turbulence. A better way of writing the total energy conservation law is

$$\partial_t \mathcal{E} = 2\gamma^+ \mathcal{E} - \mathcal{D}, \tag{462}$$

where γ^+ represents macroscopic forcing and a dissipation term $\mathcal{D}$ is now included explicitly. Nevertheless, this more explicit equation does not determine the value of $\mathcal{E}$ either. According to the discussion of the entropy paradox in Sec. 12.2.1 (p. 238), the dissipation rate is determined by the saturation level, not *vice versa*. Different scalings of $\mathcal{E}$ as a function of γ^+ emerge depending on whether the weak- or strong-turbulence limit is appropriate; however, the necessary information (the dispersion properties of the waves) is absent from the total energy balance.

The perpendicular nonlinear dynamics of the drift-wave–convective-cell system are 2D, not 3D; that a is positive in Eq. (461b) is a reflection of the inverse energy cascade (Kraichnan, 1976b). Equations (461) should be augmented by dissipation terms to absorb direct enstrophy transfer; however, specific predictions about saturation levels are then lost. Diamond et al. (1998) attempted to introduce more dynamical information by focusing specifically on the zonal flows. Their formulas

[280] In Eqs. (461) it must be true that $\gamma_*\mathcal{E}_* - \overline{\gamma}\overline{\mathcal{E}} = \gamma\mathcal{E}$, as follows immediately from the definitions of the various $\gamma\mathcal{E}$ terms as partial $\boldsymbol{k}$ sums over $\gamma_{\boldsymbol{k}}^{\rm lin} C_{\boldsymbol{k}}$. See the explicit example in Appendix J (p. 297).

for the nonlinear amplification of zonal flows and associated $\boldsymbol{k}$-space enstrophy diffusion for the drift waves are related to the formulas derived in Sec. 7.3.2 (p. 193) with one important difference: because $k_\parallel = 0$ fluctuations exhibit highly nonadiabatic response, Poisson's equation must be modified for those modes, as explained in Sec. 2.4.4 (p. 38). A consequence is that for generalized HM dynamics enstrophy is no longer conserved; instead, the appropriate invariant is $\mathcal{Z} \doteq (1 + k^2)^2 \langle |\varphi_{\boldsymbol{k}}|^2 \rangle$, as discussed by Lebedev et al. (1995), Smolyakov and Diamond (1999), and Krommes and Kim (2000). The last reference gives a unified treatment of both pure and generalized HM dynamics from the point of view of the interactions of disparate scales. It employs many of the techniques referred to in the present article, including Markovian closures, MSR generating functionals, wave kinetic equations, and spectral balance equations, and it presents heuristic derivations of the nonlinear long-wavelength growth rate that differ from and contradict ones in previous literature. The analysis is valid for arbitrary $\boldsymbol{q} \ll \boldsymbol{k}$; special assumptions about zonal flows or streamers are not required.

Even given a formula for the nonlinear growth rate $\gamma_{\boldsymbol{q}}^{\mathrm{nl}}$, the analysis is not complete because the details of the nonlinear couplings into the $k_z \neq 0$ modes, which ultimately provide the principal dissipation, have not been elucidated. Also note that one cannot cavalierly conclude that the ultimate drift-wave fluctuation level is controlled by the rms shear generated by the zonal flows, because the rapid-change model discussed in Sec. 4.4.4 (p. 123) provides a counterexample (Krommes, 2000b). Detailed information about saturation is provided by the closures discussed in the present article; however, to date no systematic statistical calculation that extends work such as described in Sec. 8.5 (p. 208) to include convective cells has yet been reported. This is an interesting topic for future research.

13 DISCUSSION

A wealth of information has been presented in this article. It can be summarized in various ways.

13.1 Time lines of principal research papers

A chronology of some of the principal research papers on systematic, analytical, statistical turbulence theory is given in Figs. 34 (p. 260) through 36 (p. 262), where specifically plasma-physics works are listed in the right-hand column and everything else is listed in the left-hand column. Numerical superscripts preceding an author's name correlate related items in the left- and right-hand columns. A historian of science will note the substantially derivative nature of fundamental plasma turbulence methods, which are deeply indebted to pioneers in statistical physics, quantum field theory, and neutral fluids.

13.2 The state of statistical plasma turbulence theory 35 years after Kadomtsev (1965)

In this article I have described the status of the statistical approach to plasma turbulence as it exists some 35 years after the seminal monograph by Kadomtsev (1965). It is useful to assess the progress in light of the extended quotation from that reference given in Sec. 1.3.4 (p. 17).

"Many forms of noise and oscillation arise spontaneously in the plasma." The presence of spontaneous or intrinsic noise is the *raison detre* of the statistical approach. Through the theory of nonlinear dynamics [not reviewed here; see, for example, Lichtenberg and Lieberman (1992)], we now have a relatively deep understanding of the origins of that noise. The statistical methods provide the most direct way of quantitatively describing the resulting processes of turbulent diffusion. They generalize the long-known methods of Langevin, Fokker, and Planck [see, for example, the articles reprinted by Wax (1954)], cornerstones of many disciplines of physics, to situations in which time and/or length scales are not cleanly separated. Modern approaches based on the DIA and its Markovian relatives properly incorporate the effects of self-consistency.

"The theoretical consideration of a weakly turbulent state is considerably facilitated by the possibility of applying perturbation theory . . . ." Weak-turbulence theory was already well developed in the 1960s. The present article adds little to its practical understanding. Instead, I stressed that the structure of the (unrenormalized) weak-turbulence wave kinetic equation persists in the fully renormalized DIA-like descriptions. It is useful to note that one now understands how to recover the equations of weak-turbulence theory, to any order, from the fully renormalized equations of MSR, which lead to a nonperturbative definition of a nonlinear dielectric function (DuBois and Espedal, 1978).

"For the case of very small amplitude, . . . one can use the so-called quasi-linear approximation . . . ." In the early 1960s QLT existed as a well-specified mathematical algorithm. In the interim its foundations in nonlinear dynamics have been solidified by insights into the Chirikov criterion for stochasticity and the smoothing engendered by the onset of stochastic diffusion [see Appendix D (p. 279)]. In the simplest cases that diffusion can be described as a resonance broadening, an observation that motivated the resonance-broadening theory of Dupree (1966, 1967).

"In numerous practical cases one is faced not by weak but by strong turbulence." The advances focused on by this article have been very largely in the *systematic* development of the statistical description of strong turbulence. Enormous progress has been made. The early semiheuristic descriptions based on resonance broadening have matured into highly developed methods based on renormalized field theory (Martin et al., 1973). The intimate links between nonlinear plasma physics, neutral fluids, and quantum mechanics become particularly evident in the light of that unifying formalism.

"Strong turbulence is related to an anomalous diffusion of the plasma across the magnetic field." Because $\boldsymbol{E} \times \boldsymbol{B}$ motions dominate cross-field transport, one can work with quasi-2D models. Fortunately, those have structure quite similar to the intensively studied 2D NSE, from which much can be learned, including the theory of two-parameter Gibbs ensembles and dual cascades.

"To determine the fluctuation spectrum in a strongly turbulent plasma and the effect of these fluctuations on the averaged quantities, it is sometimes possible to use the analogy with ordinary hydrodynamics and, in particular, to apply a phenomenological [mixing-length] description of the turbulent motion. However, in a plasma other strongly turbulent motions which are different from the eddy motion of an ordinary fluid may develop. It is therefore desirable to have available more systematic methods for describing strong turbulence. In our view, such a method may be the weak coupling approximation . . . ." By the "weak-coupling approximation," Kadomtsev meant (DuBois and Pesme, 1985) what is now universally referred to as Kraichnan's direct-interaction approximation. Kadomtsev's remarks heralded a long development of the DIA for plasma physics, to which this article devoted considerable space. The DIA quantifies the notion of mixing length [281] for strongly turbulent fluids (Kraichnan, 1964c; Sudan and Pfirsch, 1985). The seminal observations

[281] Caveats and limitations on the use of mixing-length estimates in practical situations were given by Diamond and Carreras (1987). Note that their criticism that the mixing-length level does not depend on dissipative parameters must be viewed with caution. In some of the more elaborate plasma theories,

of Orszag and Kraichnan (1967) about realizability of random-coupling Vlasov models were followed by the unification and derivation by DuBois and Espedal (1978) and Krommes (1978) of various of the simpler renormalizations from the DIA and the MSR formalism as well as a complete formal description of the renormalized dielectric function.

"It has now become evident, however, that the coefficient of turbulent diffusion cannot be obtained without a detailed investigation of the instability of an inhomogeneous plasma and in particular of its drift instability." In the last three decades a huge literature on both the theory and numerical simulation of drift and related instabilities has blossomed with the increasing appreciation of their relevance to turbulent transport in magnetically confined fusion plasmas. From the involved practical details of realistic devices have been distilled some simple generic models such as those of Hasegawa and Mima (1978), Terry and Horton (1982), and Hasegawa and Wakatani (1983). The statistical theory of strong turbulence has been proven capable of making quantitatively accurate predictions for some key features of such nonlinear models, including the steady-state turbulent flux and spectral shape. Recent representative work is by Bowman and Krommes (1997) and Hu et al. (1997).

It is clear that Kadomtsev's insights were seminal. There were, of course, omissions from his 1965 monograph. He did not comment in any detail on the topics of intermittency and submarginal turbulence although those were known to occur in neutral fluids. He did not anticipate the possibility of determining rigorous bounds on turbulent plasma transport Sec. 11, p. 230; the seminal paper on the variational method for fluids (Howard, 1963) had appeared just a few years earlier. The dual cascade, now frequently invoked in heuristic interpretations of 2D turbulence phenomena, was not discussed until several years later (Kraichnan, 1967). The beautiful and systematic results of nonlinear dynamics had not yet reached the plasma mainstream (Smith and Kaufman, 1975; Karney, 1978; Treve, 1978). Nonlinear gyrokinetics had not yet been invented (Frieman and Chen, 1982); gyrofluid closures came still later (Hammett and Perkins, 1990). Advanced computation for plasma turbulence was in its infancy. Nevertheless, his ideas, either explicit or implicit, of generalizing and applying known techniques of hydrodynamic turbulence to the magnetized plasma problem were remarkably apt and prescient.

13.3 Summary of original research and principle conceptual points in the article

Although this is primarily a review article, some research topics appeared here for the first time. Those include (i) discussion of the relationships between kinetic and fluid resonance-broadening theory, and extraction of the proper polarization-drift nonlinearity from kinetic renormalization Sec. 6.5.5, p. 178; (ii) a generating-function approach to the efficient generation of fluid moment equations Appendix C.2.1, p. 277; and (iii) the class structure of the `DIA` code Appendix I, p. 295. The pedagogical discussion in Sec. 6.1.2 (p. 150) of the appearance of anomalous exponents in the stochastic oscillator model is also new, as are some of the remarks in Sec. 12.7 (p. 248) about the interactions between convective cells and drift waves.

Significant conceptual points discussed in the article include the following: (i) Nonlinear gyrokinetics Appendix C.1, p. 267 provides the best route to the derivation of equations for the low-frequency dynamics of magnetized plasma. (ii) The core of the statistical turbulence problem is the need for renormalization. (iii) Renormalization embraces much more than the Feynman-like

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dissipation enters the saturation level through the clump lifetime τ_{cl}. But the discussion in Sec. 4.4.4 (p. 123) shows that τ_{cl} should not arise in a proper description of the energy-containing scales.

diagrams introduced in Sec. 3.9.5 (p. 81), and can be accomplished [Secs. 6.2 (p. 153) through 6.4 (p. 166)] without ever introducing such diagrams. (iv) An exact definition of the (renormalized) plasma dielectric function can be given Sec. 6.5, p. 170 [it does not reduce to the algorithm of resonance-broadening theory Sec. 4.3, p. 108]. (v) Renormalization techniques provide an efficient and systematic way of discussing the statistical interactions of disparate scales Sec. 7.3, p. 189 and the nonlinear generation of long-wavelength fluctuations Sec. 12.7, p. 248. (vi) Statistical realizability Sec. 3.5.3, p. 63 provides a key constraint on the nature of permissible approximations Sec. 8.2, p. 201. (vii) The DIA Sec. 5, p. 126 is a profoundly unifying approximation to the formal developments. (viii) Linear waves violate the realizability of the EDQNM Sec. 8.2.1, p. 201. (ix) A tentative solution to the latter problem, apparently successful in practice, is Bowman's Realizable Markovian Closure Sec. 8.2.3, p. 203. (x) Submarginal turbulence Sec. 9, p. 210 provides an interesting challenge for marrying nonlinear dynamics with statistical closure methods. (xi) PDF methods Sec. 10.4, p. 224 have the potential for treating difficult problems of intermittency. (xii) Rigorous bounds on turbulent transport can be formulated Sec. 11, p. 230.

13.4 Retrospective on statistical methods

Statistical closure has its limitations. It cannot easily deal with instances of divided phase space, multiple and macroscopically distinct basins of attraction, *etc.* Nevertheless, for the many physical systems of interest that are known to exhibit extreme sensitivity to initial conditions, it is not a foolish approach. Even when the entire phase space is accessible to the dynamical trajectories, issues of phase mixing, mode coupling, and self-consistency arise that demand a systematic treatment. A key advance, both philosophically and technically, was Kraichnan's version of perturbation theory, in which the effects of a primitive interaction are assessed by *removing* it perturbatively from the sea of all interactions rather than by adding it to a small number of other primitive ones. The latter procedure leads to the unrenormalized equations of weak-turbulence theory; the former leads to the more general self-consistent renormalized equations of strong-turbulence theory, the DIA being the prime example.

Both the similarities and differences between the works of Kraichnan (1959b) and Dupree (1966) are striking. Dupree's test waves are loosely analogous to Kraichnan's primitive wave-number triads. Unfortunately, in the application to Vlasov dynamics the waves live in $\boldsymbol{x}$ space whereas the particles live in $\boldsymbol{v}$ space. Because of that extra level of complexity and for other reasons, Dupree was led to mishandle in detail the effects of self-consistency; his resonance-broadening theory has more conceptual relevance to problems of passive advection. For self-consistent turbulence RBT is best viewed as a merely qualitative description of extremely complex mode-coupling processes in phase space. Such incomplete description can nevertheless be very useful. It may provide the only practical way of isolating the dominant effects in situations complicated by such practical effects as magnetic shear. Nevertheless, it cannot reliably provide numerical coefficients or the spectral details of dynamical mode coupling.

Important goals for this article were to draw the lines between heuristic and systematic statistical descriptions of plasma turbulence more clearly and to emphasize that some truly systematic steps can, in fact, be taken. The lowest-order approximation to the formally exact renormalized equations of Martin, Siggia, and Rose (1973) is the DIA. That approximation is robust, realizable, properly self-consistent, and quantitatively accurate for energy-containing modes; it can be applied to both kinetic and fluid descriptions of both homogeneous and inhomogeneous situations. The formal theory of the nonlinear dielectric function (DuBois and Espedal, 1978) considerably helps with the understanding of the structure of the plasma-physics DIA. One now has precise decompositions into nonlinear noise and dielectric shielding, which help one to avoid egregious examples of double- or undercounting of mode-coupling processes.

On the other hand, the systematic developments are by no means complete. The MSR equations are not practically useful beyond lowest order. The DIA is not invariant to random Galilean transformations (Kraichnan, 1964e) and cannot capture key aspects of intermittency (Chen et al., 1989a). The DIA can be embedded in a convergent sequence of ever-more-constrained approximations (Kraichnan, 1985), but the higher-order approximations are complicated and have been little studied. PDF methods (Chen et al., 1989b) are promising (Kraichnan, 1991), but are in their infancy.

At the practical level, self-consistent Markovian closures seem to be faithful enough to capture the broad features of turbulent transport due to energy-containing modes. A key advance was made with the realization (Bowman, 1992) that the evolution of such closures in the presence of waves need not be realizable. Bowman's Realizable Markovian Closure (Bowman, 1992; Bowman et al., 1993) generalizes the EDQNM approximation in a way that ensures realizability. The RMC is clearly not the last word in the development of realizable Markovian closures, but it stands as a practically useful algorithm (Bowman and Krommes, 1997; Hu et al., 1997) and perhaps suggests paths of further development.

One significant area where heuristic and systematic approaches to plasma turbulence have historically collided is the so-called "two-point" analysis [see footnote 154 (p. 122)] of the spectral balance equation, which can be traced back to Dupree's 1972 discussion of kinetic phase-space granulation. Although no detailed discussion of Dupree's technique for kinetic problems was offered in the present article, it has become clear [most recently with the discussion of an exactly solvable statistical model by Krommes (1997a)] that at least for saturation levels in fluid problems the focus on the dynamics of the very smallest scales, which lies at the heart of the clump algorithm, is misplaced Sec. 4.4, p. 119. It fosters a qualitatively incorrect picture of the dynamics responsible for saturation, and it leads to a spurious dependence on the clump lifetime. A consequence with practical implications is that the issue of whether fluctuation levels are controlled by the rms self-generated shear is more complicated than had previously been believed (Krommes, 2000b).

13.5 Basic quantities and concepts of practical significance

The article touched only incidently on the more practical uses to which plasma turbulence theory is frequently put. Most experimentalists are probably not interested in subtleties of, say, vertex renormalizations in quantum field theory; more likely, they want to know whether their data are consistent with simple theoretical estimates of turbulent transport. Introductions to some of the concepts useful in describing or analyzing inhomogeneous plasmas in magnetic fields have been given by Itoh et al. (1999) and (more briefly) by Yoshizawa et al. (2001). I did not attempt to review the vast area of correlations (or lack thereof) between theory and experiment; some references include Liewer (1985), Haas and Thyagaraja (1986), Robinson (1987), Surko (1987), Ritz et al. (1988), Wootton et al. (1990), and Burrell (1997). Here I shall merely enumerate a few of the more fundamental quantities or concepts mentioned in those articles, and briefly comment on the principal conceptual points made about them in the present paper.

13.5.1 The gyrokinetic description

The gyrokinetic description Appendix C.1, p. 267, based on the unusual gyrokinetic Poisson equation that includes the effects of the ion polarization drift, represents a fundamental advance in one's ability to concisely treat either analytically or numerically low-frequency fluctuations in magnetized plasmas. Sadly, it has been neglected in even very recent textbooks; it deserves better.

13.5.2 The turbulent diffusion coefficient D

As noted in Sec. 1.3.1 (p. 13), turbulent fluxes can be obtained from equal-time Eulerian cross correlations. Those are the natural dependent variables of the systematic statistical closures discussed in this article, including the multifield DIA and the related Markovian approximations. That formalisms exist that make any sensible predictions at all for those cross correlations in the face of strong nonlinearity must be seen as significant intellectual progress; one has come a long way from the early 1960s.

13.5.3 Dimensional analysis

It is worth repeating with Connor and Taylor (1977) that given a nonlinear equation [see examples in Sec. 2 (p. 22)], many (and sometimes all) predictions about parameter dependence of turbulent fluxes or transport coefficients follow by general scaling arguments Appendix B, p. 264. As applied to transport, the roles of a statistical closure are therefore to calculate numerical coefficients and/or detailed functional dependences on dimensionless parameters ϵ—heavy burdens, indeed. Nevertheless, important ϵ-dependent scaling laws frequently follow from the generally robust structure of the closure; cf. the appearance of anomalous dimensions, discussed in the next section.

13.5.4 Renormalization

As invoked in the more practical plasma theories, many of which have been strongly influenced by Dupree's 1966–67 RBT, renormalization can be said to be an approximate technique for broadening linear resonance functions by appropriately adding a turbulent diffusion coefficient. If that is done literally, however, as Dupree did in his early works, ill-behaved formalisms result if one pursues predictions beyond those of dimensional analysis; as examples, energy is not properly conserved and wave-number dependence is misrepresented. Systematic renormalization techniques for classical statistical dynamics lie at the heart of this article. Although renormalization is frequently introduced through partial diagrammatic summations, as was done in Secs. 3.9.5 (p. 81) through 3.9.8 (p. 85), the functional approach of MSR shows that the underlying concept is much deeper. The appearance of anomalous dimensions Sec. 6.1.2, p. 150 is profound, and the related problem in intermediate asymptotics (Barenblatt, 1996) provides an elegant way of unifying essential concepts in critical phenomena, quantum field theory, and classical turbulence theory. Anomalous dimensions also have practical significance: for example, turbulent transport due to $\boldsymbol{E} \times \boldsymbol{B}$ velocities leads in the purely 2D case to the anomalous scaling B^{-1} (instead of the classical result B^{-2}) and the Bohm diffusion coefficient (Taylor and McNamara, 1971).

13.5.5 Clumps

Perhaps no technique in plasma turbulence theory has been as confusing as the clump algorithm initiated by Dupree (1972b). It has been alleged to be a useful, if approximate, statistical closure. Nevertheless, it was not derived in the same mathematically clean fashion that, say, the DIA was, and it has become clear that the way in which it has been applied at least to fluid problems is qualitatively (and quantitatively) incorrect, as discussed in Sec. 4.4 (p. 119). One cannot argue with the predictions of the exactly solvable model discussed by Krommes (1997a), which except for one very special case are not in agreement with those of the clump algorithm.

13.5.6 Saturation mechanisms

The formal statistical theory discussed in this article provides a general framework for thinking about and calculating the properties of saturated steady states; it quantifies the balance between (i) the tendency for the nonlinear system to seek statistical equilibrium [see the discussion of Gibbs ensembles in Sec. 3.7.2 (p. 68)], and (ii) the disequilibrium induced by forcing and dissipation. The spectral balance equation [Secs. 5.4 (p. 133) and 6.2.2 (p. 155)] is the mathematical backbone of the formalism, and statistical closures such as the DIA Sec. 5, p. 126 or the RMC Sec. 8.2.3, p. 203 provide specific formulas for the nonlinear damping and incoherent noise. Solutions of the nonlinear, wave-number-dependent, coupled equations for modal covariances for specific applications were largely not discussed in this article although aspects of the important Hasegawa–Mima and Hasegawa–Wakatani paradigms of drift-wave turbulence were mentioned briefly in Secs. 8.4 (p. 206) and 8.5 (p. 208). The statistical dynamics of zonal flows Sec. 12.7, p. 248, apparently key constituents of such turbulence, were under active investigation at the time of writing.

13.6 The future, and concluding remarks

"Closures, as a broad subject, have told us very little about turbulence that we did not know first from other means." — *Montgomery (1989).*

"I will not rule out that renormalization methods will have a lot to say about turbulence once they are applied to the right objects, which has not been the case so far." — *Frisch (1993).*

Some aspects of the future of statistical plasma turbulence theory are easy to predict; others are not. A variety of unfinished research threads identified in the article are ripe for further development, including (i) detailed numerical solutions of statistical closures for more practical situations than have been considered to date, for both weak turbulence Sec. 4.2, p. 98 and strong turbulence [Secs. 5 (p. 126) and 8.2.3 (p. 203)] and for both fluid and kinetic models; (ii) a high-quality numerical solution of the DIA for a 1D Vlasov model, which would add greatly to one's understanding Sec. 6.5.6, p. 180 of turbulent trapping and the roles of self-consistency and non-Gaussian statistics; (iii) detailed mechanisms for submarginal turbulence Sec. 9, p. 210, and the integration of such mechanisms into a systematic statistical closure apparatus; (iv) PDF methods Sec. 10.4, p. 224 for strong intermittency (including studies of more realistic models that involve multiple coupled fields and linear wave effects); and (v) a systematic quantitative description of the statistical dynamics of interacting zonal flows and drift waves, including the role of self-generated random flow shear. A workable theory that systematically incorporates coherent structures and turbulence on equal footing is very desirable, but also very challenging.

What is the status of the provocative quote from Montgomery (1989) given at the beginning of this section? In its defense, one must emphasize that basic ideas such as energy conservation, cascade direction, or dynamical mechanisms for submarginal turbulence do not require closure; indeed, one must work hard in order to ensure that closures are consistent with those properties. The rigorous bounding approach to transport intentionally avoids closure. Nevertheless, the historical emphasis on closure does not appear to have been misplaced. The analytical formalism identifies the natural entities and concepts useful for theoretical description, including turbulent damping coefficients, incoherent noise, statistical transfer, renormalized vertices, *etc.* Detailed considerations of the structure of systematically renormalized closures elucidate the difficulties of the clump algorithm. The mapping-closure approach to PDF's has motivated plausible heuristic descriptions of intermittent turbulence. The method of Polyakov and Boldyrev, also a closure, leads to excellent quantitative agreement with numerical experiment; without such a framework, it would be very difficult to sort out competing qualitative ideas. Finally, as the concluding quote from Frisch (1993) emphasizes, the subject is certainly not closed. Although it is difficult to foresee just what sorts of new but also practically useful statistical techniques will emerge, one general possibility is a hybrid method that in some clever way combines rapid numerical computation with a theoretical superstructure such as a path-integral representation Sec. 6.4, p. 166 of the statistical dynamics.

In conclusion, it is hoped that this broad survey of theoretical techniques for predicting the statistical behavior of turbulent plasmas may somewhat ameliorate any feelings that the plasma turbulence problem is just too complicated to treat at any level more sophisticated than simple random-walk ideas and dimensional analysis. It *is* complicated, but it has yielded—and will yield further, if slowly—to systematic analysis. In so doing, plasma turbulence theory should come to be seen as just one more interesting example of the statistical description of nonlinear dynamical systems rather than as an arcane subspecialty of interest to just a few devoted practitioners. The benefits of such integration into the wider world of modern physics are considerable. If this article has aided in elucidating at least some of the appropriate foundations, its goals will have been met.

Acknowledgements

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Finally, I wish to acknowledge the outstanding library facilities of Princeton University and PPPL, and especially the diligent and expert work of the librarians in promptly tracking down some very obscure references.

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	— Neutral fluids/general physics —	— Plasmas —
< 1900	[1]**Boltzmann (1896)**—Boltzmann collision operator	
1900–04		
1905–09	**Einstein (1905)**—Brownian motion	
1910–14	[2]**Buckingham (1914)**—dimensional arguments	
1915–19		
1920–24	**Taylor (1921)**—D as Lagrangian integral	
1925–29	**Prandtl (1925)**—concept of mixing length	
·	**Dirac (1927)**—quantum theory of radiation	
1930–34	**Uhlenbeck and Ornstein (1930)**—Brownian motion	
·	[3]**Onsager (1931a)**—symmetry of transport matrix	
1935–39		[1]**Landau (1936)**—plasma collision operator
1940–44	**Kolmogorov (1941)**—energy spectrum[282] $E(k) \sim k^{-5/3}$	
·	**Feynman (1942)**—principle of least action (dissertation)	
45		
6		
7		
8	**Feynman (1948b)**—space-time approach to QM	
9	**Feynman (1949)**—space-time approach to QED	**Bohm (1949)**—$D_\perp \sim cT/eB$
·	**Dyson (1949)**—synthesis of quantum radiation theories	
1950		
1	**Schwinger (1951a)**—sources in QFT	
2		
3	[4]**Batchelor (1953)**—*Theory of Homogeneous Turbulence*	
4		
55		
6		
7	**Tatsumi (1957)**—theoretical studies of quasinormal approx.	
·	**Kraichnan (1957)**—insights on fourth-order moments	
1958	**— DIA era begins; see Fig. 35 (p. 261) —**	**— Continued in Fig. 35 (p. 261) —**

Fig. 34. Chronology of selected research papers discussed in the present article, for the pre-DIA period earlier than 1958. The relative youth of plasma-physics research is evident. Limited space precludes completeness. At most, two references per year and per column are given. Significant conceptual correlations between items in the left- and right-hand columns [spanning all of Figs. 34–36] are indicated by matching superscripts preceding the author's name, with at least one entry in each of the columns. Lines connecting such superscripts would slope from upper left to lower right; thus the flow of information is from neutral fluids and general physics to plasmas.

[282] Actually, Kolmogorov worked in x space. The Fourier spectrum was given by Oboukhov (1941).

	— Neutral fluids/general physics —	— Plasmas —
1958	**Kraichnan (1958c), Kraichnan (1958b)**—first accounts of DIA	
·	**Kraichnan (1958a)**—higher order interactions	
9	**Kraichnan (1959a)**—classical FDT	
·	[5]**Kraichnan (1959b)**—DIA	
1960		**Balescu (1960), Lenard (1960)**—BL operator
·		**Spitzer (1960)**—turbulent diffusion in a magnetic field
1	**Wyld (1961)**—diagrammatic formulation of NS turbulence	**Taylor (1961)**—Bohm diffusion is maximum cross-$\boldsymbol{B}$ transport
·	**Kraichnan (1961)**—stochastic osc.; random-coupling model	
2	**Kubo (1962a)**—cumulant expansions	**Drummond and Pines (1962)**—quasilinear theory
·	**Bourret (1962)**—Bourret closure approximation	**Vedenov et al. (1962)**—QLT
3	[6]**Kraichnan (1963)**—three-mode DIA	
·	**Howard (1963)**—variational bound on thermal convection	
4	**Kraichnan (1964b)**—integrations of DIA	**Rostoker (1964a)**—test-particle superposition principle
·	**Kraichnan (1964e)**—random Galilean invariance	
65	[7]**Herring (1965)**—self-consistent field approximation	[4]**Kadomtsev (1965)**—monograph on *Plasma Turbulence*
·	[8]**Kraichnan (1965a)**—Lagrangian History DIA	
6		**Dupree (1966)**—unmagnetized RBT
·		
7	**Kraichnan (1967)**—dual cascade	[5]**Orszag and Kraichnan (1967)**—Vlasov DIA
·		**Dupree (1967)**—RBT for $\boldsymbol{E} \times \boldsymbol{B}$ motions
8		**Rogister and Oberman (1968)**—WTT with Klimontovich
·		**Weinstock (1968)**—turbulent diffusion in $\boldsymbol{B}$
9	**Chirikov (1969)**—PhD thesis on stochasticity	**Weinstock (1969)**—projection operators and RBT
·		[8]**Orszag (1969)**—stochastic acceleration
1970	**Kraichnan (1970a)**—Langevin representation of DIA	**Dupree (1970)**—collisionless resistivity
·		**Rudakov and Tsytovich (1971)**—critique of RBT
1	[9]**Kraichnan (1971a)**—test-field model	**Taylor and McNamara (1971)**—plasma diffusion in 2D
·		**Dawson et al. (1971)**—simulation/theory of 2D convective cells
2	**Herring and Kraichnan (1972)**—closures compared	**Benford and Thomson (1972)**—probabilistic model of RBT
·	**Candlestickmaker (1972)**—accurate determination of e	**Dupree (1972b)**—phase-space granulation and clumps
3	[10]**Martin et al. (1973)**—renormalized classical field theory	**Taylor and Thompson (1973)**—statistical mechanics of GC plasma
·		
4		**Dupree (1974)**—2D turbulence
·		**Taylor (1974a)**—guiding-center dielectric
75	**Kraichnan (1975b)**—statistical dynamics of 2D flow	**Misguich and Balescu (1975)**—renormalized QLT
·		**Orszag (1975)**—comments on convergence
6	**Kraichnan (1976b)**—eddy viscosity	[7]**Krommes and Oberman (1976a)**—DIA/SCF from BBGKY
·	**Forster et al. (1976)**—randomly stirred fluid and RG	**Krommes and Oberman (1976b)**—stat. theory of conv. cells
7	[11]**Orszag (1977)**—thorough discussion of EDQNM	[2]**Connor and Taylor (1977)**—scaling laws and transport
·		**Sudan and Keskinen (1977)**—DIA for 2D convection
8	[12]**Busse (1978)**—review of bounding theories	**Dupree and Tetreault (1978)**—energy nonconservation in RBT
·	**Fournier and Frisch (1978)**—*d*-dimensional turbulence	[10]**DuBois and Espedal (1978), Krommes (1978)**—plasma DIA
9	**Rose (1979)**—renormalization of many-particle systems	**Krommes and Kleva (1979)**—nonlinear dielectric
·		**Adam et al. (1979)**—reconsideration of QLT
1980		**Krommes and Similon (1980)**—guiding-center dielectric redux
·		
1	**Jensen (1981)**—synthesis of path-integrals and MSR	**Boutros-Ghali and Dupree (1981)**—two-point correlations
·	**Carnevale et al. (1981)**—H theorems for closures	**DuBois (1981)**—renormalized quasiparticles and plasma turb.
2	**Kraichnan (1982)**—renormalized pert. theory vs RG	**Terry and Horton (1982)**—RPA for three drift waves
·	[13]**Carnevale and Martin (1982)**—reduction of DIA to Markovian form	[6]**Krommes (1982)**—DIA for three-mode Terry–Horton eqns.
3		**Waltz (1983)**—low-resolution Markovian closure
·		
4		
·		
1985	**— Continued in Fig. 36 (p. 262) —**	**— Continued in Fig. 36 (p. 262) —**

Fig. 35. Chronology of selected research papers on statistical methods (DIA to 1984). Citations are mostly limited to papers on systematic analytical statistical turbulence theory.

	— Neutral fluids/general physics —	— Plasmas —
85	**Kraichnan (1985)**—statistical decimation	**Sudan and Pfirsch (1985)**—mixing length and DIA
·	**Rose (1985)**—cumulant-update DIA	**DuBois and Pesme (1985)**—Vlasov DIA and Kadomtsev
6		**Krommes (1986a)**—critique of clump formalism
·		**Horton (1986)**—statistical dynamics of drift waves
7	**Kraichnan (1987b)**—distant-interaction approximation	[12]**Krommes and Smith (1987)**—passive upper bounds
·	**Bak et al. (1987)**—self-organized criticality	
8	[14]**Chen et al. (1989a)**—RCM and non-Gaussian statistics	**Krommes and Kim (1988)**—further analysis of clump theory
·		
9	[15]**Chen et al. (1989b)**—mapping closure for PDF's	
·		
1990		**Hammett and Perkins (1990)**—Landau-fluid closures
·		[12]**Kim and Krommes (1990)**—rigorous bound for turbulent EMF
1	**Hunt et al. (1991)**—50-year retrospective on Kolmogorov	**Koniges et al. (1991)**—4-parameter Gibbs PDF for HW
·	**She (1991b)**—intermittency and non-Gaussian statistics	
2		
·		
3	**Gotoh and Kraichnan (1993)**—mapping clos. (Burgers)	[11,13]**Bowman et al. (1993)**—Realizable Markovian Closure
·	**Kimura and Kraichnan (1993)**—mapping closure (passive)	[3]**Krommes and Hu (1993)**—Onsager symmetries for turbulence
4	[16]**Kraichnan (1994)**—rapid-change model & structure functions	**Krommes and Hu (1994)**—dissipation and the entropy paradox
·		
95	[17]**Hamilton et al. (1995)**—regeneration in submarginal turb.	[15]**Das and Kaw (1995)**—mapping closure for plasma turbulence
·	**Polyakov (1995)**—turbulence without pressure	[17]**Drake et al. (1995)**—nonlinearly self-sustained d.w. turbulence
6	**Holmes et al. (1996)**—turbulence, coherent structures,	[14]**Krommes (1996)**—non-Gaussian statistics revisited
·	and dynamical systems	
7	[18]**Waleffe (1997)**—self-sustainment in submarginal shear flows	[16]**Krommes (1997a)**—clump lifetime revisited
·	**Boldyrev (1997)**—PDF's for Burgers turbulence	[9]**Bowman and Krommes (1997)**—realizable test-field model
8		**Camargo et al. (1998)**—non-normality of resistive drift waves
·		
9		[18]**Krommes (1999a)**—Waleffe's mechanism for submarginal d. w.'s
·		**Krommes and Ottaviani (1999)**—tails, SOC, and transport
2000		[13]**Krommes and Kim (2000)**—interactions of disparate scales
·	—***The present***—	—***The present***—

Fig. 36. Chronology of selected research papers on statistical methods [1985 to 2000; see footnote 1 (p. 7)]. Citations are mostly limited to papers on systematic analytical statistical turbulence theory.

A FOURIER TRANSFORM CONVENTIONS

Space-time quantities are taken to vary as $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega t)$. The spatial discrete Fourier series represents a function $A(\boldsymbol{x})$ periodic in a d-dimensional box of side L:

$$A_{\boldsymbol{k}} = \frac{1}{L^d}\int_{-L/2}^{L/2} \mathrm{d}\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} A(\boldsymbol{x}), \quad A(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} A_{\boldsymbol{k}}. \tag{A1a,b}$$

Each Cartesian wave-number component satisfies $k = n\,\delta k$, where $\delta k \doteq 2\pi/L$ is the mode spacing and fundamental wave number. The convolution theorem is $[A(\boldsymbol{x})B(\boldsymbol{x})]_{\boldsymbol{k}} = \sum_{\boldsymbol{q}} A_{\boldsymbol{k}-\boldsymbol{q}} B_{\boldsymbol{q}}$; for real fields this can be written as

$$[A\,B]_{\boldsymbol{k}} = \sum_{\Delta(\boldsymbol{k};\boldsymbol{p},\boldsymbol{q})} A^*_{\boldsymbol{p}} B^*_{\boldsymbol{q}}, \quad \sum_{\Delta(\boldsymbol{k};\boldsymbol{p},\boldsymbol{q})} \equiv \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}}. \tag{A2a,b}$$

Frequently $\sum_{\Delta(\boldsymbol{k};\boldsymbol{p},\boldsymbol{q})}$ is abbreviated as $\sum_\Delta$ when there is no confusion about the relevant wave numbers. The spectra of statistically homogeneous processes obey $\langle A_{\boldsymbol{k}} B_{\boldsymbol{k}'}\rangle = \langle A\,B\rangle_{\boldsymbol{k}} \delta_{\boldsymbol{k}+\boldsymbol{k}'}$ and $\langle |A_{\boldsymbol{k}}|^2\rangle = \langle A\,A\rangle_{\boldsymbol{k}}$, where $\langle A\,B\rangle_{\boldsymbol{k}}$ means the transform with respect to $\boldsymbol{\rho}$ of the two-point correlation function $C_{AB}(\boldsymbol{\rho}) \doteq \langle A(\boldsymbol{x}+\boldsymbol{\rho})B(\boldsymbol{x})\rangle$ (which is $\boldsymbol{x}$-independent for homogeneous processes).

In the limit $L \to \infty$ the Fourier series becomes the continuous Fourier transform with the replacements $\sum_{\boldsymbol{k}} \to \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k}/\delta k^d$, $A_{\boldsymbol{k}} \to A(\boldsymbol{k})/L^d$, and $\delta_{\boldsymbol{k}} \to \delta k^d \delta(\boldsymbol{k})$. Thus the integral transform conventions consistent with Eqs. (A.1) are

$$A(\boldsymbol{k}) = \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} A(\boldsymbol{x}), \quad A(\boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} A(\boldsymbol{k}). \tag{A3a,b}$$

The convolution theorem for real fields is

$$[AB](\boldsymbol{k}) = \frac{1}{(2\pi)^d} \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{q}\, \delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}) A^*(\boldsymbol{p}) B^*(\boldsymbol{q}). \tag{A.4}$$

For homogeneous processes one has

$$\langle A(\boldsymbol{k}) B(\boldsymbol{k}') \rangle = (2\pi)^d \langle A\, B \rangle(\boldsymbol{k}) \delta(\boldsymbol{k}+\boldsymbol{k}'), \quad \langle |A(\boldsymbol{k})|^2 \rangle = L^d \langle A\, A \rangle(\boldsymbol{k}). \tag{A5a,b}$$

Equation (A5b) follows from Eq. (A5a) with the formal rule $\delta(\boldsymbol{k}=0) = \delta k^{-d}$.

Consider the case for which A and B are isotropic, i.e., dependent only on wave-number magnitudes. Then the angular integrations over the delta function can be performed in Eq. (A.4), giving rise to

$$\delta k^d \sum_{\Delta} \to \int \mathrm{d}\boldsymbol{p}\, \mathrm{d}\boldsymbol{q}\, \delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}) = \int_{\Delta} \mathrm{d}p\, \mathrm{d}q\, \mathcal{J}_d(k,p,q), \tag{A.6}$$

where the integration is over the domain Δ, shown in Fig. A.1, of magnitudes p and q compatible with a triangle whose third side is k. For the important case of 2D, $\mathcal{J}_2 = 2|\sin(\boldsymbol{p},\boldsymbol{q})|^{-1}$.

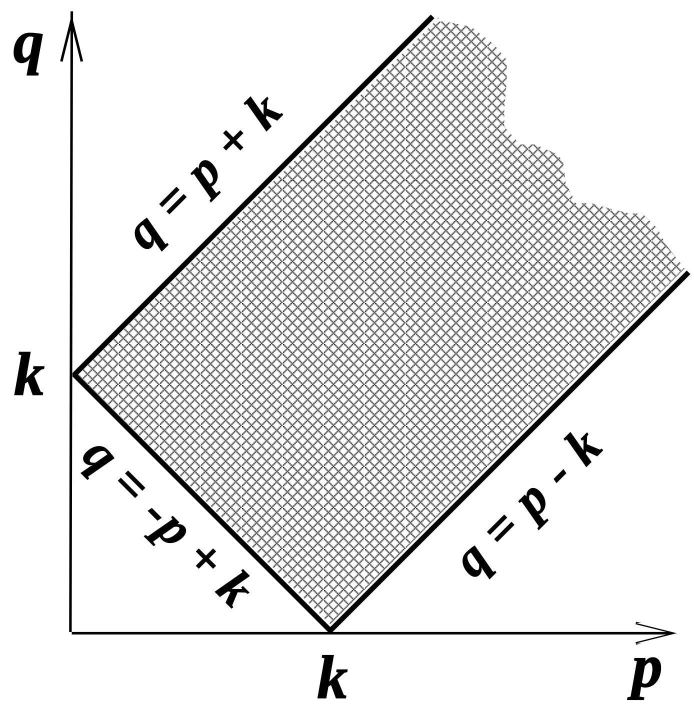

Fig. A.1. Integration domain Δ for the variable sides of a triangle of fixed side k.

B DIMENSIONAL AND SCALING ANALYSIS

"Even though the necessary non-linear theory may be quite intractable, the mere fact that a scaling law is, in principle, derivable from certain basic equations already provides information about that scaling law." — *Connor and Taylor (1977).*

I shall very briefly review the theory of dimensional and scaling analysis, following the lucid presentation by Barenblatt (1996). The method exploits the observation that the form of the governing system of equations should remain invariant under all possible rescalings of the dependent variables, independent variables, and physical parameters. In plasma physics, scaling analyses are frequently associated with the work of Connor and Taylor (1977) [see Connor (1988) for a review] although the techniques go back at least to Buckingham (1914) and Lamb (1932). A useful discussion was given in Appendix A of Smith (1986).

Let an abstract variable z measure some physical quantity Q. For example, if z is a spatial coordinate x, the measurable quantity is length. If the measuring system is changed by the rescaling $z = Zz'$, Z is called the *dimension* of Q; one writes $[Q] = Z$. More generally, the dimension of any physical quantity can be proven to be a power-law monomial (the dimension function) in a set of scaling factors Z_i. Closely following the definition of Barenblatt (1996), quantities $a_1, \ldots, a_k$ are said to have *independent dimensions* if none of them has a dimension function that can be expressed as a product of powers of the dimensions of the remaining quantities. For example, consider the set $\{\text{length, time, velocity}\} = \{x, t, v\}$. With $[x] = L$, $[t] = T$, and $[v] = V$, any two of the three variables can be taken to have independent dimensions. If x and t are chosen as fundamental, then velocity is clearly not independent because a rescaling of the fundamental definition $v = \mathrm{d}x/\mathrm{d}t$ leads to $Vv' = (L/T)\mathrm{d}x'/\mathrm{d}t'$. Since this definition must be covariant (its form must remain unchanged by the choice of units), one finds $V = L/T$.

Now suppose that a physical problem is specified by a set of quantities $a_1, \ldots, a_k, b_1, \ldots, b_m$, where the a's have independent dimensions and the b's have dimensions dependent on the a's. One wishes to determine the functional dependence of some quantity [284] b on the given quantities according to

$$b = f(a_1, \ldots, a_k, b_1, \ldots, b_m). \tag{B.1}$$

By definition of the b's, one must have $[b_i] = [a_1]^{p_i} \ldots [a_k]^{r_i}$ for certain powers $p_i, \ldots, r_i$; such a result also holds for b by dropping the i subscripts. It is now natural to introduce the m *dimensionless parameters*

$$\Pi_i \doteq \frac{b_i}{a_1^{p_i} \ldots a_k^{r_i}}; \tag{B.2}$$

these are unchanged by rescalings of the independent quantities. Then it can be shown that Eq. (B.1) can be expressed as

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_m), \tag{B.3}$$

[284] Barenblatt (1996) called the determined quantity a rather than b. The present notation is more economical because results that hold for the b_i's hold also for b if the subscript is dropped.

where Φ is an undetermined function. The key result is that Φ depends on m rather than $k+m$ arguments; it does not depend on the k quantities with independent dimensions.

It must be emphasized that *dimensionless* is being used here in a more generalized sense than in introductory discussions of systems of physical units. In the cgs system of measurement, the unit of length is cm; if length is arbitrarily rescaled as above, then the unit of length becomes cm$/L$. Such lengths, involving a physical unit of measurement, are frequently said to be dimensional. But suppose one works with the variable $\overline{x} \doteq x/\rho_s$, where ρ_s is some spatial distance also measured in cm. One frequently calls $\overline{x}$ "dimensionless," but it is dimensional in the present theory because one can still perform the rescaling $\overline{x} = L\overline{x}'$, so $[\overline{x}] = L$. That is, the unit of length is the pure number L^{-1}. Only if a quantity is completely invariant under rescaling should it properly be called dimensionless. I shall call the physicist's "dimensionless" variables *normalized*.

To illustrate these considerations, consider the Hasegawa–Wakatani equations (52) in the constant-$k_\parallel$ approximation $\widehat{\alpha} = \alpha = \text{const}$. The equations are already written in normalized variables, whose dimensions are therefore pure numbers. Let us assume that the classical dissipation coefficients are small and in the spirit of Kolmogorov analysis will not appear in the final formulas for macroscopic quantities such as particle flux Γ. Then the complete set of quantities on which any physical quantity can depend is $\{x, y, t, \varphi, n, \alpha, \kappa\}$. To identify the quantities with independent dimensions, one demands that the form of the equations remains invariant under the rescalings

$$x = L_x x', \quad y = L_y y', \quad t = Tt', \quad \varphi = P\varphi', \quad n = Nn', \quad \alpha = A\alpha', \quad \kappa = K\kappa'. \quad \text{(B4a,b,c,d,e,f,g)}$$

Invariance of the Laplacian in $\omega = \nabla^2\varphi$ requires $L_x = L_y = L$, and invariance of the $\varphi - n$ term requires $P = N$. The remaining constraints can readily be found to be

$$TP/L^2 = 1, \quad L^2TA = 1, \quad TA = 1, \quad TK/L = 1. \quad \text{(B5a,b,c,d)}$$

Since there are $k = 4$ (independent) constraints in the $k + m = 5$ unknowns $\{L, T, P, A, K\}$, there is precisely $m = 1$ independent quantity. That can be chosen arbitrarily, but it is most efficient to choose it to be one of the physical parameters $\{\alpha, \kappa\}$ rather than one of the independent variables; it is conventional to choose α. (Note that the number of physical parameters is not necessarily equal to the number of independent quantities.) Equations (B5b) and (B5c) require that $L = 1$ (i.e., the spatial variables are truly dimensionless whereas the other normalized variables are dimensional), whereupon one finds that

$$T = 1/A, \quad K = A, \quad P = N = A. \quad \text{(B6a,b,c)}$$

Now consider the time-averaged particle flux $\Gamma = \overline{\delta V_{E,x}\,\delta n}$. That must ultimately depend on the physical parameters: $\Gamma = f(\alpha, \kappa)$. Now $[\Gamma] = [\delta V_{E,x}][\delta n] = (L/T)N = A^2$. Therefore the dimensionless flux is $\Pi \doteq \alpha^{-2}\Gamma = f\big(\alpha, (\kappa/\alpha)\alpha\big) = \Phi(\alpha, \Pi_1) = \Phi(\Pi_1)$ (independent of the independent quantity α), where $\Pi_1 \doteq \kappa/\alpha$. Thus dimensional analysis proves that the HW flux must have the form

$$\Gamma = \alpha^2\Phi(\kappa/\alpha), \quad \text{(B.7)}$$

where the function Φ is undetermined. One can readily check that the gyro-Bohm scaling (6) is equivalent to $\Phi(\Pi) = \Pi^2$ (by Fick's law, Γ contains one extra power of κ relative to D), but serious statistical analysis is required in order to deduce the proper form of Φ. In fact, gyro-Bohm scaling does not hold for HW dynamics at general α.

As Connor and Taylor have emphasized (Connor and Taylor, 1977; Connor, 1988), further approximation of the dynamics can provide further constraints on the final form of the result. For example, consider the asymptotic limit $\alpha \to 0$. Then the α term can be neglected in Eq. (52b). [It cannot be neglected in Eq. (52a) because it provides the only coupling between φ and n.] That removes the covariance constraint (B5c) from consideration and implies that there are $5 - 3 = 2$ independent quantities; those can conveniently be taken to be α and κ. The solutions to Eqs. (B5a), (B5b), and (B5d) are

$$L = A^{-1/3}K^{1/3}, \quad T = A^{-1/3}K^{-2/3}, \quad P = N = A^{-1/3}K^{4/3}. \tag{B8a,b,c}$$

Since $[\Gamma] = (L/T)N = A^{-1/3}K^{7/3}$, one must have $\alpha^{1/3}\kappa^{-7/3}\Gamma = f(\alpha, \kappa) = C$, where C (a specialization of Φ) is a constant since both α and κ are independent. Therefore

$$\Gamma = C\alpha^{-1/3}\kappa^{7/3}. \tag{B.9}$$

In this asymptotic limit only the constant C remains to be determined by statistical theory or numerical simulation. Notice that the form (B.9) is consistent with Eq. (B.7) for $\Phi(\Pi) = \Pi^{7/3}$.

Frequently a dimensionless parameter, say, Π_m is close to 0 or ∞. It is generally believed that the Φ function of Eq. (B.3) has a finite limit as $\Pi_m \to 0$ or $\Pi_m \to \infty$; however, that is by no means always true. Let Eq. (B.3) be augmented to include an additional dimensionless parameter ϵ:

$$\Pi = \Phi(\Pi_1, \Pi_2, \ldots, \Pi_m; \epsilon). \tag{B.10}$$

It is assumed that at $\epsilon = 0$ Φ has a finite limit as $\Pi_m \to 0$, so for small Π_m Φ can be approximated by $\Phi_1(\Pi_1, \Pi_2, \ldots, \Pi_{m-1})$. In this case a small Π_m disappears from consideration. However, for $\epsilon \neq 0$ and Π_m small the most general situation is that

$$\Pi = \Pi_m^{\alpha_m}\Phi_1\left(\frac{\Pi_1}{\Pi_m^{\alpha_1}}, \ldots, \frac{\Pi_{m-1}}{\Pi_m^{\alpha_{m-1}}}; \epsilon\right), \tag{B.11}$$

where the α's (called *anomalous exponents* or *anomalous dimensions*) are definite functions of ϵ. The important result is that *the anomalous dimensions cannot be determined from dimensional analysis*. They can, however, be calculated by using additional dynamical information about the system; as discussed by Barenblatt, they frequently obey nonlinear eigenvalue conditions. The behavior (B.11) is called *incomplete similarity asymptotics* or *self-similarity of the second kind*. Barenblatt emphasized that such behavior is not pathological but rather normal and widespread, and he discussed a variety of examples drawn from physical applications. In particular, the appearance of anomalous exponents lies at the heart of renormalized field theory and critical phenomena; for further discussion and a simple example, see Sec. 6.1 (p. 147).

The possibility of anomalous exponents resolves a fundamental paradox that arises in the application of dimensional analysis to physical problems. Although in some limits the problem can apparently be reduced unambiguously to the determination of a single dimensionless constant C [see, for example, Eq. (B.9) for the HW small-α limit discussed above], the predicted scaling behavior is not necessarily observed. The resolution is that *C can be infinite*. A familiar example is gyrokinetic analysis [Sec. 2.3.1 (p. 28) and Appendix C.1 (p. 267)] based on spatially periodic boundary conditions. Dimensional analysis leads one to predict gyro-Bohm scaling [Eq. (6)], yet both experimental (Perkins et al., 1993) and numerical (Nevins, 2000) observations sometimes display Bohm-like scaling [Eq. (5)]. Such scalings can emerge if the turbulent flux depends on the macroscopic box size [285] L; the small quantity $\epsilon \doteq \rho/L$ (ρ being the gyroradius) is ignored in naive analyses.

C DERIVATIONS OF GYROKINETIC AND GYROFLUID EQUATIONS

In this appendix I survey the derivations of the nonlinear gyrokinetic and gyrofluid equations whose solutions form the focus of much of the modern research on magnetized plasma turbulence. For gyrokinetics I concentrate on modern techniques utilizing Hamiltonian, Lie, and symplectic techniques. The discussion is brief and incomplete, but many references to the literature are provided. Lie methods are the formal implementations of nonresonant averaging procedures, which establishes a connection to the statistical methods discussed in the body of the article [see the introductory discussion in Sec. 1.5 (p. 20)]. For gyrofluids I discuss the difficulties of fluid closure from the perspective of general statistical theory, then briefly describe some of the practical implementations of fluid closures based on the gyrokinetic equation.

C.1 Gyrokinetics

The fundamental problem of describing turbulence in magnetized plasmas, whether analytically or computationally, is the vast disparity between the very short timescale for gyration of a charged particle around a magnetic field line and the much longer timescales characteristics of profile-gradient-driven microinstabilities. For example, for $k_y\rho_\mathrm{s} \lesssim 1$ one has $\omega_{ci}/\omega_* = \omega_{ci}/[(k_y\rho_\mathrm{s})(c_\mathrm{s}/L_n)] \gtrsim L_n/\rho_\mathrm{s}$; for TFTR parameters this ratio was several hundred.

C.1.1 Adiabatic invariants and charged-particle motion

It was recognized very early that the problem of charged-particle motion in a magnetic field defines a central problem of asymptotics (Kruskal, 1965), with the formal expansion parameter being ω_c^{-1}. Kruskal (1962) developed a general theory suitable for many systems whose lowest-order solutions were periodic. Much early work oriented toward the adiabatic invariants of charged-particle motion was summarized and unified by Northrop (1963b); for a shorter account, see Northrop (1963a). The work of Hastie et al. (1967) should also be mentioned.

Kruskal's work applied to very general dynamical systems. Frequently, however, the dynamics are Hamiltonian; then special, more efficient techniques can be employed. Perturbation techniques based on canonical transformations and mixed-variable generating functions (Goldstein, 1951) are well known (Treve, 1978); however, the resulting algebra can be messy and cumbersome. Significant qualitative advances in the mathematical technology were made in a series of seminal papers by

[285] Nevins (2000) found such a dependence in simulations of ITG turbulence by numerically measuring slowly decaying algebraic tails in the PDF of heat flux Γ. Note that although the $P(\Gamma)$ based on jointly normal variables [Eq. (414)] asymptotically contains the algebraic prefactor $|\Gamma|^{-1/2}$, that is dominated by exponential decay. Nevins's observations, lacking such an exponential factor, therefore correspond to extremely non-Gaussian statistics, the significance of which is not understood.

Littlejohn, who argued for the use of noncanonical Hamiltonian mechanics and Lie perturbation theory. Littlejohn (1979) pointed out that it is unnecessary to develop Hamiltonian perturbation theory in canonical coordinates; it is merely required that the *symplectic structure* (Arnold, 1978, Appendix 3) be maintained. However, he showed with the aid of a theorem of Darboux that the theory could be brought to a *semicanonical* form in which the rapidly varying, periodic coordinate (the gyro-angle, in this case) and its associated adiabatic invariant [286] $\overline{\mu}$ were canonical to lowest order; that expedited the application of perturbation theory. The Lie version of perturbation theory that was actually employed was reviewed by Cary (1981); see also Lichtenberg and Lieberman (1992).

Dubin and Krommes (1982) used these methods to discuss (i) the problem of superadiabatic invariance in guiding-center theory; and (ii) the nonlinear, stochastic interaction between the first and second adiabatic invariants $\overline{\mu}$ and $\overline{J}$. They also remarked in mathematical detail on a feature that would later turn out to be of great significance for the development of nonlinear gyrokinetic–Poisson systems: the Hamiltonian equations for guiding centers, in coordinates for which μ is conserved, do *not* contain the polarization drift; instead, the effect is buried in the change of coordinates from those of the original laboratory (particle) frame.

The basic picture is summarized in Fig. C.1 (p. 269). Consider for simplicity the case of constant $\boldsymbol{B} = B\widehat{\boldsymbol{b}}$. At any spatial point, vectors can be resolved onto the right-handed, orthogonal coordinate system $(\widehat{\boldsymbol{e}}_1, \widehat{\boldsymbol{e}}_2, \widehat{\boldsymbol{b}})$, where the overall orientation of $\widehat{\boldsymbol{e}}_1$ and $\widehat{\boldsymbol{e}}_2$ is arbitrary; for constant $\boldsymbol{B}$ one may take $\widehat{\boldsymbol{e}}_1 = \widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{e}}_2 = \widehat{\boldsymbol{y}}$. Also introduce the triad $(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \widehat{\boldsymbol{c}})$, where the gyroradius vector is $\boldsymbol{\rho} = \rho\widehat{\boldsymbol{a}}$ $(\rho \doteq v_\perp/\omega_c)$ and the perpendicular vector velocity is $\boldsymbol{v}_\perp = v_\perp\widehat{\boldsymbol{c}}$. With θ being the gyro-angle (increasing clockwise for a positively charged particle), one has

$$\widehat{\boldsymbol{a}} = \cos\theta\,\widehat{\boldsymbol{e}}_1 - \sin\theta\,\widehat{\boldsymbol{e}}_2, \quad \widehat{\boldsymbol{c}} = -\sin\theta\,\widehat{\boldsymbol{e}}_1 - \cos\theta\,\widehat{\boldsymbol{e}}_2; \tag{C1a,b}$$

see Fig. C.1(a). This purely circular motion trivially conserves $\mu \doteq v_\perp^2/2\omega_c$.

Now add a constant electric field $\boldsymbol{E}$ in the $-\widehat{\boldsymbol{x}}$ direction. The motion in the presence of the resulting $\boldsymbol{E}\times\boldsymbol{B}$ drift is sketched in Fig. C.1(b). μ is no longer conserved, but a Galilean transformation to the drifting frame shows that

$$\overline{\mu} \doteq |\boldsymbol{v}_\perp - \boldsymbol{V}_E|^2/2\omega_c \tag{C.2}$$

is conserved. Locate the instantaneous gyrocenter by $\overline{\boldsymbol{R}}(t)$. The gyro-angle $\overline{\theta}$ in the drifting frame differs from the angle θ, defined relative to $\overline{\boldsymbol{R}}(0) \equiv \boldsymbol{R}(0)$, by a first-order amount proportional to $\epsilon \doteq V_E/v_\perp$. Specifically, one has $-\overline{v}_\perp \sin\overline{\theta} = (\boldsymbol{v}_\perp - \boldsymbol{V}_E)\cdot\widehat{\boldsymbol{e}}_1$. Upon noting that $\overline{v}_\perp = |\boldsymbol{v}_\perp - \boldsymbol{V}_E| \approx v_\perp - \widehat{\boldsymbol{c}}\cdot\boldsymbol{V}_E$, it is straightforward to find that $\overline{\theta} = \theta - (c/B)\widehat{\boldsymbol{c}}\cdot\boldsymbol{E}/v_\perp + O(\epsilon^2)$. Similarly, it follows from Eq. (C.2) that $\overline{\mu} = \mu - \rho(c/B)\widehat{\boldsymbol{a}}\cdot\boldsymbol{E} + O(\epsilon^2)$.

Now let $\boldsymbol{E}$ evolve slowly in time with a characteristic timescale much longer than a gyroperiod. $\overline{\mu}$ remains conserved through first order, so μ has first-order variation (as does θ). Upon differentiating the instantaneous relation $\boldsymbol{x} = \overline{\boldsymbol{R}} + \rho\widehat{\boldsymbol{a}}$ with respect to t, one finds after some straightforward algebra that

$$\frac{\mathrm{d}\boldsymbol{x}_\perp}{\mathrm{d}t} = \boldsymbol{v}_\perp + \boldsymbol{V}_E + \frac{1}{\omega_c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c\boldsymbol{E}_\perp}{B}\right) + O(\epsilon^3), \tag{C.3}$$

[286] The overline notation $\overline{\mu}$, although not common in general physics writing, is used here to be compatible with my subsequent discussion of changes of variable and the development of $\overline{\mu}$ through all orders in slow variations.

the last explicit term of which is the *polarization drift*. Now the constraint of adiabatic $\overline{\mu}$ conservation can be systematically enforced through all orders, leading to an asymptotic series $\overline{\mu}(\epsilon)$ and similar series for the gyrocenter $\overline{\boldsymbol{R}}$ and gyro-angle $\overline{\theta}$. One sees that the constraint of $\overline{\mu}$ conservation *requires* that the polarization drift does not appear in the formula for $\overline{\boldsymbol{R}}$ but rather describes the second-order discrepancy between the gyrocenter and true particle position. If $\boldsymbol{E}$ increases secularly, $\boldsymbol{x}$ deviates secularly from $\overline{\boldsymbol{R}}$.

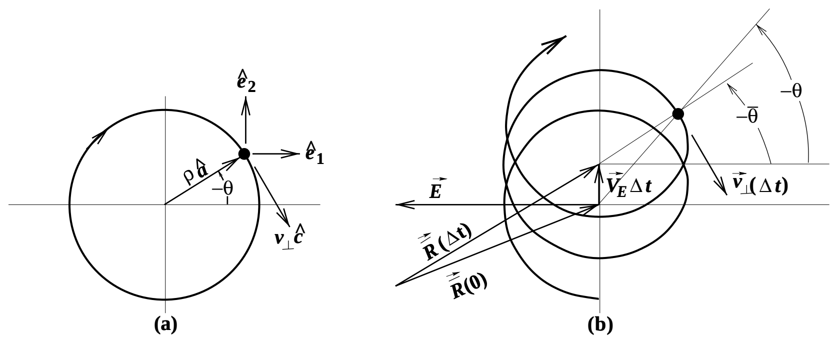

Fig. C.1. Geometry of a gyrospiraling particle in a constant magnetic field $\boldsymbol{B} = B\widehat{\boldsymbol{b}}$. (a) Pure circular motion ($\boldsymbol{E} = 0$). (b) Motion including $\boldsymbol{E} \times \boldsymbol{B}$ drift, and the gyrocenter position after a time Δt such that $\omega_c \Delta t \gg 1$.

The Darboux methods are straightforward though somewhat inelegant. Littlejohn (1981) showed that the calculations could be further compacted by performing perturbation calculations directly on differential one-forms. Those methods will be further discussed in Sec. C.1.4 (p. 271) below.

C.1.2 Early derivations of the gyrokinetic equation

By gyrokinetics, one means a low-frequency ($\omega \ll \omega_c$) kinetic description in which gyration has been systematically averaged away. Early derivations of linear GK equations (GKE's) were by Rutherford and Frieman (1968) and Taylor and Hastie (1968). The calculations exploited the smallness of one or more of the following parameters: ϵ_ω, the ratio of the characteristic frequency ω to the gyrofrequency ($\epsilon_\omega \doteq \omega/\omega_{ci}$); ϵ_x, the ratio of the gyroradius ρ to the scale length $L_B \sim R$ of variation of the background magnetic field ($\epsilon_x \doteq \rho/L_B$); and ϵ_δ, a measure of the size of the fluctuations (for example, $\delta f/f$, $e\delta\varphi/T_e$, or $\delta B/B$). Linear GKE's in general geometry were considered by Antonsen and Lane (1980) and Catto et al. (1981).

The first derivation of a *nonlinear* GKE was given by Frieman and Chen (1982). Those authors used the conventional and well-developed approach of breaking the Vlasov PDF into its mean and fluctuating pieces, then applying multiple-scale, secular perturbation theory to each piece. They employed an optimal ordering in which $\epsilon \sim \epsilon_\omega \sim \epsilon_x \sim \epsilon_\delta$ and $k_\perp \rho = O(1)$; the last ordering is the trademark of a GK (as opposed to a drift-kinetic) theory. While suitable for analytical work, the resulting equations were not in an obvious characteristic form and did not exploit the Hamiltonian nature of the dynamics.

Lee (1983) considered the problem of developing a suitable particle-simulation approach to GKE's. In that method the characteristic particle trajectories are directly advanced in time, the charge density is calculated by an appropriately coarse-grained sum over the particles, then a Poisson equation is solved to find the fields. Because the characteristics are properties of the full kinetic equation not split into mean and fluctuating parts, Lee found the equations of Frieman and Chen (1982) unsuitable, and he attempted an alternate "recursive" derivation [for more discussion, see Appendix A of Dubin and Krommes (1982)]. A vital feature of the result (already present mathematically, if not fully emphasized, in earlier derivations of GKE's) was the appearance of the ion polarization-drift effect not as a velocity in the characteristic equations of motion but rather in the form of a *polarization charge density* in the GK Poisson equation. This could have been anticipated from the work of Dubin and Krommes (1982), for the GK Poisson equation is essentially Poisson's original equation (couched in particle coordinates) restated in the gyrokinetic variables. If the true, adiabatically conserved $\overline{\mu}$ is taken to be one of those variables, then the polarization drift cannot appear in the characteristic equations of motion; the effect must appear as a polarization contribution to the charge density.

The simplest heuristic derivation of a GK Poisson equation is as follows. For simplicity, consider low-frequency fluctuations ($\omega \ll \omega_{ci}$), for which the quasineutrality condition is appropriate ($k\lambda_D \ll 1$). In particle coordinates Poisson's equation is therefore $n_e = n_i$. In gyrokinetics, however, the description is instead developed in terms of gyrocenter PDF's F_s, hence gyrocenter densities n_s^{G}. For electrons $n_e \approx n_e^{\mathrm{G}}$ since the electron polarization is negligible ($k_\perp \rho_e \to 0$). For ions, however, one has $n_i = n_i^{\mathrm{G}} + n_i^{\mathrm{pol}}$, where the ion polarization density n_i^{pol} can be calculated from the continuity equation $\partial_t n_i^{\mathrm{pol}} + \boldsymbol{\nabla}_\perp \cdot (\boldsymbol{V}_i^{\mathrm{pol}} n_i) = 0$, or through linear order, $\partial_t(\delta n_i^{\mathrm{pol}}/\overline{n}_i) = -\boldsymbol{\nabla}_\perp \cdot \boldsymbol{V}_i^{\mathrm{pol}}$. The polarization drift velocity $\boldsymbol{V}^{\mathrm{pol}}$ (Chandrasekhar, 1960) is defined by Eq. (30), and its divergence is readily calculated. Because time derivatives then appear on both the left- and right-hand sides of the polarization density equation, it can readily be integrated in time; upon normalizing φ to T_e/e, one finds $\delta n_i^{\mathrm{pol}}/\overline{n}_i = \rho_{\mathrm{s}}^2 \nabla_\perp^2 \delta\varphi$. A simple GK Poisson equation is therefore [287]

$$\rho_{\mathrm{s}}^2 \nabla_\perp^2 \varphi = \delta n_i^{\mathrm{G}}/\overline{n}_i - \delta n_e/\overline{n}_e. \tag{C.4}$$

In essence, Lee's derivation of nonlinear gyrokinetics was successful, and he was able to perform pioneering GK particle simulations of his "full-F" gyrokinetic equation. Nevertheless, certain technical difficulties were also apparent in his approach. In particular, as in the approach of Frieman and Chen (1982), the Hamiltonian nature of the dynamics was obscured or lost, so certain conservation laws were difficult to discern and treat accurately.

C.1.3 Hamiltonian formulation of gyrokinetics

Dubin et al. (1983) argued that it was important and even technically simpler in the long run to retain the Hamiltonian structure of the equations. For the simplest example of constant magnetic field in slab geometry, they employed the then-current Darboux method of Littlejohn (1979) to find the appropriate GK change of variables, and they showed how rigorous transformation theory led naturally to a full-F gyrokinetic equation directly from the Vlasov equation. The same transformation theory led to natural FLR generalizations of the GK Poisson equation. Dubin *et al.* also showed how various simplifications of the resulting system led to important familiar fluid equations, including the HM equation (48), and the weak-turbulence wave kinetic equation of Sagdeev and Galeev (1969).

Hamiltonian methods have been used in virtually all subsequent GK calculations. Immediate successors to Dubin *et al.* included Hagan and Frieman (1985) and Yang and Choi (1985). Following the work of Littlejohn (1982), Cary and Littlejohn (1983) published a definitive paper on noncanonical perturbation theory using differential forms that appeared approximately at the same time as the work of Dubin et al. (1983). It took about five years for the implications of that work to be properly

[287] For $k_\parallel = 0$ this relation must be modified to take account of the highly nonadiabatic electron response; see Sec. 2.4.4 (p. 38).

appreciated, but then development was rapid. Hahm et al. (1988), Hahm (1988), and Brizard (1989) used the one-form technique to derive GKE's for general geometries and magnetic fluctuations. Hahm (1996) revisited the derivation of the GKE for modified orderings more appropriate to regimes of enhanced confinement observed in modern tokamaks. Much information about the modern tools and advances, not all of which are described here, can be found in the dissertations by Brizard (1990) and Qin (1998).

C.1.4 Differential geometry and n-forms

Here I briefly describe aspects of the method of Cary and Littlejohn (1983). A convenient starting point is the variational approach to classical mechanics (Lanczos, 1949). A well-known action variational principle that leads to Hamilton's equations of motion (upon assuming that $\boldsymbol{p}$ and $\boldsymbol{q}$ are independent) is $\delta\int\mathcal{L}\,\mathrm{d}t = 0$, where $\mathcal{L}$ is the Lagrangian: $\mathcal{L} \doteq \sum_{i=1}^{N}\boldsymbol{p}_i\cdot\boldsymbol{v}^i - H(\boldsymbol{p},\boldsymbol{q})$, $H = \sum_{i=1}^{N}\frac{1}{2}m_i v_i^2 + e\varphi(\boldsymbol{q},t)$, and $\boldsymbol{v} = m^{-1}(\boldsymbol{p} - e\boldsymbol{A}/c)$ ($\boldsymbol{A}$ being the vector potential). The variational principle can be written most symmetrically as $\delta\int\gamma = 0$, where $\gamma \equiv \sum_{i=1}^{N}\boldsymbol{p}_i\cdot\mathrm{d}\boldsymbol{q}^i - H\,\mathrm{d}t$ is the *Poincaré–Cartan one-form*.

A more general representation is to arrange the phase-space coordinates according to

$$z^i = \{\boldsymbol{q}^1,\ldots,\boldsymbol{q}^N,\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N\} \quad (i = 1,\ldots,2Nd), \tag{C.5}$$

then to introduce the extended phase-space coordinates $z^\mu = \{t, z^i\}$. [It is conventional to use Greek indices in the extended $(2Nd+1)$-D phase space and Roman ones in the ordinary $2Nd$-D phase space.] Correspondingly, one introduces

$$\gamma_\mu = \{-H,\boldsymbol{p}_1,\ldots,\boldsymbol{p}_N,\boldsymbol{0},\ldots,\boldsymbol{0}\}. \tag{C.6}$$

Then the Poincaré–Cartan one-form is

$$\gamma = \gamma_\mu \mathrm{d}z^\mu. \tag{C.7}$$

The (covariant) coefficients γ_μ are the components of the abstract differential one-form γ. For a beautiful discussion of the geometric significance of differential forms, see Misner et al. (1973).

In the form (C.7) there is no longer any reference to the particular representations (C.5) and (C.6). Therefore if one treats γ_μ as a covariant vector, then γ can be expressed in *any* set of coordinates (including noncanonical ones). Thus $\gamma \to \overline{\gamma} = \overline{\gamma}_\mu \mathrm{d}\overline{z}^\mu$, where $\overline{\gamma}_\mu = (\partial z^\nu/\partial\overline{z}^\mu)\,\gamma_\nu$. The freedom to deal with virtually any variables at all is a great technical advantage.

One important example for plasma-physics applications involves the distinction between the canonical coordinate $\boldsymbol{p}$ and the noncanonical coordinate $\boldsymbol{v}$ in the presence of a magnetic field. In the canonical coordinates, $\gamma = \boldsymbol{p}\cdot\mathrm{d}\boldsymbol{q} - [(\boldsymbol{p} - e\boldsymbol{A}/c)^2/2m + e\varphi]\mathrm{d}t$. Consider instead the transformation $(\boldsymbol{x},\boldsymbol{p}) \to (\overline{\boldsymbol{x}},\overline{\boldsymbol{v}})$, defined by $\boldsymbol{x} = \overline{\boldsymbol{x}}$ and $\boldsymbol{p} = m[\overline{\boldsymbol{v}} + e\boldsymbol{A}(\overline{\boldsymbol{x}})/c]$. Then the change of coordinates is explicitly (in 1D for simplicity)

$$\overline{\gamma}_x = (\partial x/\partial\overline{x})\,\gamma_x + (\partial p/\partial\overline{x})\,\gamma_p = \gamma_x = p = m[v + eA(\overline{v})/c], \tag{C8a,b,c,d}$$

$$\overline{\gamma}_v = (\partial x/\partial\overline{v})\,\gamma_x + (\partial p/\partial\overline{v})\,\gamma_p = \gamma_p = 0, \tag{C8e,f,g}$$

and the one-form in the new coordinates is

$$\overline{\gamma} = m[\overline{\boldsymbol{v}} + e\boldsymbol{A}(\overline{\boldsymbol{x}})/c]\cdot\mathrm{d}\overline{\boldsymbol{x}} - [\tfrac{1}{2}m\overline{v}^2 + e\varphi(\overline{\boldsymbol{x}})]\mathrm{d}t. \tag{C.9}$$

(Note that in this example the time variable was not transformed; that is typical in practice.)

If one parametrizes the paths in the action principle by a parameter λ (typically the time t), then one is led with Eq. (C.7) to the general equations of motion in the form

$$\omega_{\mu\nu}\frac{\mathrm{d}z^\nu}{\mathrm{d}\lambda} = 0, \quad \omega_{\mu\nu} \doteq \frac{\partial\gamma_\nu}{\partial z^\mu} - \frac{\partial\gamma_\mu}{\partial z^\nu}; \tag{C10a,b}$$

ω is called the *fundamental two-form.* In a canonical coordinate system,

$$\omega = \begin{pmatrix} 0 & \partial H/\partial q^i & \partial H/\partial p^i \\ -\partial H/\partial q^i & 0 & -\mathsf{I} \\ -\partial H/\partial p^i & \mathsf{I} & 0 \end{pmatrix}. \tag{C.11}$$

The antisymmetric, covariant tensor ω is the *exterior derivative*[288] of the one-form γ. Equation (C10a) states that the flow "velocity" is the null eigenvector of ω.

The phase-space components of $\omega_{\mu\nu}$ define a submatrix $\widehat{\omega}_{ij}$ that is called the *Lagrange tensor.* Equations (C10a) can be written as

$$\widehat{\omega}_{ij}\frac{\mathrm{d}z^j}{\mathrm{d}z^0} = -\omega_{i0} = \frac{\partial\gamma_i}{\partial z^0} - \frac{\partial\gamma_0}{\partial z^i}. \tag{C.12}$$

It is generally convenient to leave the time untransformed: $z^0 = t$. Then $\gamma_0 = -H$ and

$$\widehat{\omega}_{ij}\frac{\mathrm{d}z^j}{\mathrm{d}t} = \frac{\partial H}{\partial z^i} + \frac{\partial\gamma_i}{\partial t}. \tag{C.13}$$

Upon introducing the *Poisson tensor* J as the (contravariant) inverse of $\widehat{\omega}$, one finds the explicit equations of motion to be

$$\frac{\mathrm{d}z^i}{\mathrm{d}z^0} = J^{ij}\left(\frac{\partial\gamma_j}{\partial z^0} - \frac{\partial\gamma_0}{\partial z^j}\right), \tag{C.14}$$

or when time is not transformed,

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = J^{ij}\left(\frac{\partial H}{\partial z^j} + \frac{\partial\gamma_j}{\partial t}\right). \tag{C.15}$$

In a canonical coordinate system one has $\partial\gamma_j/\partial t = 0$ and $J = \begin{pmatrix} 0 & \mathsf{I} \\ -\mathsf{I} & 0 \end{pmatrix}$, giving rise to Hamilton's equations in the conventional form.

Note that the equations of motion are unaffected by the presence of a perfect derivative in the action principle: $\delta\int(\gamma + \mathrm{d}S) = 0$. The transformation $\gamma_\mu \to \gamma_\mu + \partial S/\partial z^\mu$ is called a *gauge transformation*; S is called a *gauge scalar*. Gauge scalars can be used to great advantage in simplifying transformed equations of motion, as we will see.

It is well known that symmetries of the Lagrangian are related to conservation properties. In the present context, *Noether's theorem* states (Cary and Littlejohn, 1983) that if all of the γ_μ are

[288] For discussion of exterior derivatives, see Cary and Littlejohn (1983, Appendix A), Misner et al. (1973), or Arnold (1978).

independent of some particular coordinate z^α, then γ_α is conserved. An important use of Noether's theorem is in determining the adiabatic invariant μ, the momentum variable canonically conjugate to the gyrophase ϕ. A familiar result is that $\mu \approx \frac{1}{2}mv_\perp^2/\omega_c$ to *lowest order* in the gyroradius ρ. To find its form valid through all orders in ρ, one can search for a series of variable transformations (not necessarily canonical) such that $\overline{\gamma} = \overline{\gamma}_{(\nu)}\mathrm{d}\overline{z}^{(\nu)} + \overline{\mu}\,\mathrm{d}\overline{\phi}$ (the parentheses around ν indicate a sum over all variables except $\overline{\phi}$). If one can arrange for $\partial\overline{\gamma}_\nu/\partial\overline{\phi} = 0$, then $\overline{\mu}$ will be conserved. Note that $\overline{\mu} = \overline{\mu}(\mu, \phi)$ and $\overline{\phi} = \overline{\phi}(\mu, \phi)$ (dependences on other variables being suppressed).

C.1.5 Lie perturbation theory

"After all, what is a Lie [transform]? 'Tis but the truth in masquerade" — *Lord Byron, Don Juan (1823), quoted by Kaufman (1978).*

In traditional canonical perturbation theory one effects a transformation to new (barred) variables by a generating function such as $S(\overline{\boldsymbol{p}}, \boldsymbol{q}) \equiv F_2(\overline{\boldsymbol{p}}, \boldsymbol{q})$, viz., $\overline{\boldsymbol{q}} = \partial S/\partial\overline{\boldsymbol{p}}$, $\boldsymbol{p} = \partial S/\partial\boldsymbol{q}$. The technical problem with this approach is that the transformation appears in mixed form, so one must untangle the equations to get $\overline{\boldsymbol{p}} = \overline{\boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{q})$ and $\overline{\boldsymbol{q}} = \overline{\boldsymbol{p}}(\boldsymbol{p}, \boldsymbol{q})$. When one works to high order in perturbation theory, this untangling becomes messy and opaque.

In the Lie approach the transformation is explicitly determined from the beginning; i.e., one has $\overline{z}^\mu = Tz^\mu$. Here the operator T is constructed from z and ∂_z, so it produces a function $\overline{z} = \overline{z}(z)$. To be more explicit about the distinction between functions and values of functions [see, for example, Appendix B of Cary and Littlejohn (1983)], one sometimes introduces the forward and backward transformation functions according to $\overline{z}^\mu = Z_f^\mu(z)$ and $z^\mu = Z_b^\mu(\overline{z})$. In this notation the statement $z = Z_b\big(Z_f(z)\big)$ is equivalent to the operator identity $TT^{-1} = 1$.

In general, T depends on a parameter ϵ. The trick to developing Lie perturbation theory is to consider flows that "evolve" in ϵ instead of in the time variable t. Thus consider transformations of the form $\partial_\epsilon\overline{z}^\mu(z, \epsilon) = g^\mu(\overline{z})$ (note that g is assumed to be independent of ϵ). More explicitly, $\partial_\epsilon Z_f^\mu(z, \epsilon) = g^\mu\big(Z_f(z, \epsilon)\big)$. We will see shortly that the corresponding transformation operator is

$$T = \exp(\epsilon L_g), \tag{C.16}$$

where $L_g\psi(z) \doteq g^\nu(z)\partial\psi/\partial z^\nu$. Here g is called the *generating function* of the flow. In perturbation theory the strategy is to determine a set of generating functions g_i and a corresponding compound transformation

$$T = \ldots\exp(\epsilon^3 L_3)\exp(\epsilon^2 L_2)\exp(\epsilon L_1), \tag{C.17}$$

where $L_n \equiv L_{g_n}$, such that the representation of the dynamics is simplified in some appropriate way through some desired order in ϵ.

The proof of Eq. (C.16) exploits the transformation properties of a scalar field $s(z, \epsilon)$. By definition, the value of such a scalar is unchanged under a coordinate transformation. However, the *functional form* is changed. Thus one writes

$$\overline{s}(\overline{z}, \epsilon) = s(z, \epsilon). \tag{C.18}$$

Now $\overline{s}(\overline{z}, \epsilon) = \overline{s}\big(Z_f(z, \epsilon), \epsilon\big) \equiv T\overline{s}(z, \epsilon)$. Upon differentiating the last two members of this equivalence, one finds

$$\frac{\partial \overline{s}(\overline{z},\epsilon)}{\partial \epsilon} + g(\overline{z}) \frac{\partial \overline{s}(\overline{z},\epsilon)}{\partial \overline{z}} = \frac{\partial T}{\partial \epsilon} \overline{s}(z,\epsilon) + T \frac{\partial \overline{s}(z,\epsilon)}{\partial \epsilon}, \tag{C.19}$$

where in all cases $\overline{z}$ is evaluated at $Z_f(z,\epsilon)$. Because $T\partial_\epsilon \overline{s}(z,\epsilon) = \partial_\epsilon \overline{s}(\overline{z},\epsilon)$, the first and last terms of Eq. (C.19) cancel. Furthermore, upon replacing functions of $\overline{z}$ by T acting on those same functions, one obtains $TL_g\overline{s}(z,\epsilon) = \partial_\epsilon T\overline{s}(z,\epsilon)$. Because each term now depends on the variable z, this is the operator equation $\partial_\epsilon T = TL_g$, whose solution is just Eq. (C.16).

Since Eq. (C.18) can be written as $T\overline{s}(z) = s(z)$, one finds that the functional form of a scalar field changes under the variable transformation according to

$$\overline{s}(z) = T^{-1}s(z). \tag{C.20}$$

The one-form γ is one such field, so

$$\overline{\gamma} = T^{-1}\gamma + \mathrm{d}S \tag{C.21}$$

(in any coordinate system). It can be shown that the action of L_g on γ is $(L_g\gamma)_\mu = g^\sigma \omega_{\sigma\mu}$. Upon writing out Eq. (C.21) in perturbation theory with the aid of $T^{-1} = \exp(-\epsilon L_1)\exp(-\epsilon L_2)\ldots$, one obtains through second order

$$\overline{\gamma}_0 = \gamma_0 + \mathrm{d}S_0, \tag{C22a}$$
$$\overline{\gamma}_1 = \gamma_1 - L_1\gamma_0 + \mathrm{d}S_1, \tag{C22b}$$
$$\overline{\gamma}_2 = \gamma_2 - L_1\gamma_1 + (\tfrac{1}{2}L_1^2 - L_2)\gamma_0 + \mathrm{d}S_2. \tag{C22c}$$

The g_i's and S_i's can be chosen to satisfy various desiderata for the transformation. In particular, if the $\overline{\gamma}_\nu(\overline{z})$ are arranged to be independent of $\overline{\phi}$, then the true magnetic moment $\overline{\mu}$ will be conserved. A good, relatively straightforward example is the electrostatic case including magnetic drifts (Hahm, 1988).

C.1.6 The gyrokinetic and Poisson system of equations

An appropriate GK transformation begins with a preparatory transformation from the Cartesian particle coordinates $(\boldsymbol{x}, \boldsymbol{v})$ to lowest-order gyrocenter coordinates $z \doteq \{\boldsymbol{R}, \mu, \phi, v_\parallel\}$, with $\boldsymbol{x} = \boldsymbol{R} + \boldsymbol{\rho}(\mu, \phi)$. The potentials in Eq. (C.9) are then expanded perturbatively, and Eqs. (C22c) are used to determine $\overline{z} = Tz$ such that $\overline{\mu}$ is conserved. With a coordinate transformation in hand, one can derive a kinetic equation for the PDF F of gyrocenters. Let the Vlasov particle PDF f be expressed in the various sets of variables as $f(\boldsymbol{x}, \boldsymbol{v}) = f'(z) = \overline{f}(\overline{z})$. According to Eq. (C.20), $f'(\overline{z}) = T\overline{f}(\overline{z})$. Because the form of the Liouville equation is coordinate independent, one has

$$\frac{\partial \overline{f}(\overline{z})}{\partial t} + \sum_i \frac{\partial}{\partial \overline{z}^i}\left(\frac{\mathrm{d}\overline{z}^i}{\mathrm{d}t}\overline{f}\right) = 0. \tag{C.23}$$

The *gyrocenter* PDF is defined to be the gyro-angle average of $\overline{f}$ at fixed gyrocenter $\overline{\boldsymbol{R}}$:

$$\overline{F}(\overline{z}) \doteq \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\overline{\phi}\, \overline{f}(\overline{z}). \tag{C.24}$$

Because the γ_μ's were constructed to be independent of $\overline{\phi}$, the velocities determined by Eq. (C.15) are similarly independent, so Eq. (C.23) can trivially be averaged over $\overline{\phi}$, giving rise to the GKE

$$\frac{\partial \overline{F}(\overline{z})}{\partial t} + \sum_{i \neq \overline{\phi}, \overline{\mu}} \frac{\partial}{\partial \overline{z}^i} \left(\frac{\mathrm{d}\overline{z}^i}{\mathrm{d}t} \overline{F} \right) = 0. \tag{C.25}$$

Note that Liouville's theorem is preserved by the coordinate transformation, so the generalized velocities $\dot{\overline{z}}$ can be (simultaneously) moved outside of the gradients if desired.

To complete the description, one must relate the potentials appearing implicitly in the velocities to $\overline{F}$; i.e., in the electrostatic approximation (considered here for simplicity[289]) one must calculate the charge density, then solve Poisson's equation. That equation is conventionally couched in *particle* coordinates $\boldsymbol{x}$ whereas gyrokinetics is best expressed in the barred coordinates $\overline{z}$, so one must be careful. Now

$$n(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{v}' \, \delta(\boldsymbol{x} - \boldsymbol{x}') f(\boldsymbol{x}', \boldsymbol{v}') = \int J \, \mathrm{d}z' \, \delta(\boldsymbol{x} - \boldsymbol{x}') f'(z') \tag{C26a,b}$$

$$= \int J \, \mathrm{d}\overline{z}' \, \delta(\boldsymbol{x} - \overline{\boldsymbol{x}}') f'(\overline{z}') = \int J \, \mathrm{d}\overline{z}' \, \delta(\boldsymbol{x} - \overline{\boldsymbol{x}}') T(\overline{z}') \overline{f}(\overline{z}'). \tag{C26c,d}$$

In Eq. (C26a) a spatial delta function was inserted in order that one could integrate over the full phase space $(\boldsymbol{x}', \boldsymbol{v}')$. In Eq. (C26b) one changed variables to z', so the Jacobian J between $(\boldsymbol{x}', \boldsymbol{v}')$ and z' appeared. In Eq. (C26c) one renamed the dummy integration variable z' to $\overline{z}'$ in order that in Eq. (C26d) one could according to Eq. (C.20) introduce the transformation operator T relating f' to $\overline{f}$. Here $\overline{\boldsymbol{x}} \doteq \overline{\boldsymbol{R}} + \overline{\boldsymbol{\rho}}(\overline{\mu}, \overline{\varphi})$. To this point the analysis is exact, as the density is still expressed in terms of the particle PDF. The GK approximation is now to *ignore high-frequency dynamics* by writing $\overline{f}(\overline{z}) = \overline{F}(\overline{z}) + \delta \overline{f}(\overline{z})$, then *neglecting* $\delta \overline{f}$. This closure approximation is analogous to the neglect of the $\langle \delta \boldsymbol{E} \, \delta f \rangle$ term in deriving the mean-field Vlasov equation from the exact Klimontovich equation; it loses information about linear normal modes for $\omega > \omega_{ci}$, and it neglects quadratic beats of high-frequency noise that could in principle affect the low-frequency gyrocenter motion. Let us further write $T(\overline{z}) = 1 + \delta T(\overline{z})$. Then the GK Poisson equation is $\nabla^2 \phi = -4\pi(q_i n_i - q_e n_e)$, where

$$n(\boldsymbol{x}) = n^{\mathrm{G}}(\boldsymbol{x}) + n^{\mathrm{pol}}(\boldsymbol{x}), \tag{C.27}$$

$n^{\mathrm{G}}(\boldsymbol{x})$ is the gyrocenter density, and

$$n^{\mathrm{pol}}(\boldsymbol{x}) = \int J \, \mathrm{d}\overline{z}' \, \delta(\boldsymbol{x} - \overline{\boldsymbol{x}}') \delta T(\overline{z}') \overline{F}(\overline{z}') \tag{C.28}$$

provides the formal definition of the polarization density. Note that although $n(\boldsymbol{x})$ is nominally determined at the particle position $\boldsymbol{x}$, Eq. (C.25) requires the potential evaluated at the gyrocenter position $\overline{\boldsymbol{R}}$. It is a source of frequent confusion that Eq. (C.25) is almost always written without bars.

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[289] Finite-β effects were considered by Hahm et al. (1988); see also Qin (1998). One interesting issue of both conceptual and practical importance is whether to use the canonical momentum p_z or the velocity v_z as a fundamental coordinate. Hahm et al. (1988) and Krommes and Kim (1988) argued in favor of p_z because of certain covariance properties of low-β electromagnetics. That appears to be the better choice for numerical implementations as well.

When n^{pol} is evaluated for a space-independent Maxwellian through linear order in the gyrocenter transformation, it reduces to

$$n^{\mathrm{pol}}_{\boldsymbol{k}}/\overline{n} \approx -(\overline{k}\rho_{\mathrm{s}})^2\overline{\varphi}_{\boldsymbol{k}}, \quad (\overline{k}\rho_{\mathrm{s}})^2 \doteq \tau[1-\Gamma(\boldsymbol{k})], \tag{C29a,b}$$

where $\tau \doteq T_e/T_i$ and $\Gamma(\boldsymbol{k}) \doteq I_0(b)\mathrm{e}^{-b}$ ($b \doteq k_\perp^2\rho_i^2$); this is the generalization of Eq. (C.4) to scales of order ρ_{s}. The lowest-order GK Poisson equation can then be written as

$$\lambda_{De}^2(\nabla^2\overline{\varphi} + \widehat{\epsilon}_\perp\nabla_\perp^2\overline{\varphi}) = -(n_i^{\mathrm{G}}/\overline{n}_i - n_e/\overline{n}_e), \tag{C.30}$$

where the dielectric constant (actually an operator in $\boldsymbol{x}$ space) of the *gyrokinetic vacuum* [a phrase introduced by Krommes (1993c)] is

$$\epsilon_\perp(\boldsymbol{k}) \doteq (\overline{k}^2\rho_{\mathrm{s}}^2)/(k_\perp^2\lambda_{De}^2) \stackrel{k_\perp\to 0}{\longrightarrow} \rho_{\mathrm{s}}^2/\lambda_{De}^2 = \omega_{pi}^2/\omega_{ci}^2. \tag{C.31}$$

Extended discussion of $\widehat{\epsilon}_\perp$ and other related dielectric functions was given by Krommes (1993c). In practice the lowest-order equation is used in both analytical and numerical work. [290]

C.1.7 Modern simulations

A difficulty of particle simulations that work with the complete ("full") particle distribution is that most of the particles are merely used to resolve the background PDF. Works by Dimits and Lee (1993) and Kotschenreuther (1991) on so-called low-noise δf methods (simulations of just the fluctuating part of the PDF) culminated in the fully nonlinear particle-weighting scheme of Parker and Lee (1993). Additional interpretation and generalization of that method were given by Hu and Krommes (1994), who also made analytical calculations of the reduced noise level in such schemes, and by Krommes (1999b). In the original applications the background PDF was fixed in time; recently, however, Brunner et al. (1999) have shown how to implement the scheme with a background evolving on the transport timescale.

Even in thermal equilibrium, the statistical properties of gyrokinetic plasmas are interesting and subtle. Studies of the GK FDT were done by Krommes et al. (1986) for electrostatics and by Krommes (1993a,c) for low-β electromagnetics; the latter work was summarized by Krommes (1993d). The results have been used to partially test GK simulation codes (Lee, 2000).

The present discussion summarizes some of the key features of gyrokinetics related to averaging procedures and statistics. A rather orthogonal review oriented toward the present state of the art with regard to computational algorithms and physics results is by Lee (2000).

C.2 Gyrofluids

Gyrokinetic simulations are conceptually direct; in principle they contain all physics appropriate to low-frequency fluctuations in magnetized plasmas, including both linear and nonlinear kinetic effects.

[290] The WKE of Sagdeev and Galeev (1969) follows (Dubin et al., 1983) from Eqs. (C.25) and (C.30). One can note an interesting inconsistency: According to the discussion of Sec. 4.2.6 (p. 103) and Appendix G.3 (p. 291), the WKE contains contributions up to third order in the potential. In principle, therefore, second- and third-order contributions to n^{pol} should be retained. It is believed that those corrections do not significantly affect the physical content of the final result, but they have not been worked out in detail.

However, they employ a 5D phase space (three positions, μ, and $v_\parallel$). There is a powerful motivation to consider fluid descriptions of low-frequency plasma turbulence.

C.2.1 The fluid closure problem

Velocity moments of the Vlasov or GK equation lead to a coupled moment hierarchy analogous to the BBGKY hierarchy of many-body kinetic theory. The analogy to the statistical closure problem for passive advection was presented in Sec. 2.4.1 (p. 33). There are subtleties with that analogy, however, that must be discussed.

To begin, it is useful to have an efficient procedure for deriving the fluid moment hierarchy. The time- and space-varying density is defined by $n(\boldsymbol{x},t) = \overline{n}\int\mathrm{d}\boldsymbol{v}\, f(\boldsymbol{x},\boldsymbol{v},t)$. To write this as as average of a known PDF, write $f = \chi F_0$, where F_0 is presumed to be a known function—either the Maxwellian, say, or the steady-state solution to a statistically averaged kinetic equation. It is assumed that F_0 is normalized, $\int\mathrm{d}\boldsymbol{v}\, F_0 = 1$, so F_0 can play the role of a PDF for the random variable $\boldsymbol{v}$. Thus $n = \langle\overline{n}\chi\rangle$.

It is conventional to normalize higher velocity moments to n. For example, the fluid velocity $\boldsymbol{u}$ is defined by $n\boldsymbol{u} = \int\mathrm{d}\boldsymbol{v}\,\overline{n}\boldsymbol{v}f$, or $\boldsymbol{u} = \langle\boldsymbol{v}\chi\rangle/\langle\chi\rangle$. If an equation for $\partial_t(n\boldsymbol{u})$ is derived by taking the $\boldsymbol{v}$ moment of the kinetic equation, the equation for $\partial_t\boldsymbol{u}$ itself must be derived by subtracting the lower-order equation for $\partial_t n$. This is, of course, the usual behavior of a moment-based hierarchy; the subtraction process becomes cumbersome at higher order. Fortunately, it can be circumvented by a cumulant representation. A generating function for velocity cumulants is

$$\mathcal{N}(\boldsymbol{\lambda}) = \ln[\langle\chi\mathrm{e}^{\boldsymbol{\lambda}\cdot\boldsymbol{v}}\rangle/\langle\mathrm{e}^{\boldsymbol{\lambda}\cdot\boldsymbol{v}}\rangle] \equiv [\![1]\!]_{\boldsymbol{\lambda}}; \tag{C.32}$$

at $\boldsymbol{\lambda} = \boldsymbol{0}$ one has $\mathcal{N} = \ln(n/\overline{n})$. Analogously to Eqs. (263), define

$$C_1 \equiv [\![v_1]\!] = \partial\mathcal{N}(\boldsymbol{\lambda})/\partial\lambda_1, \quad C_{12\ldots n} \equiv [\![v_1 v_2 \ldots v_n]\!] = \partial C_{12\ldots n-1}/\partial\lambda_n, \tag{C33a,b}$$

where the subscripts denote Cartesian indices. Explicitly,

$$[\![\boldsymbol{v}]\!] = \frac{\langle\boldsymbol{v}\chi\mathrm{e}^{\boldsymbol{\lambda}\cdot\boldsymbol{v}}\rangle}{\langle\chi\mathrm{e}^{\boldsymbol{\lambda}\cdot\boldsymbol{v}}\rangle} - \frac{\langle\boldsymbol{v}\mathrm{e}^{\boldsymbol{\lambda}\cdot\boldsymbol{v}}\rangle}{\langle\mathrm{e}^{\boldsymbol{\lambda}\cdot\boldsymbol{v}}\rangle} \stackrel{\lambda=0}{\to} \boldsymbol{u} - \boldsymbol{u}_0 \equiv \delta\boldsymbol{u}. \tag{C34a,b}$$

Similarly, in 1D one has $[\![\frac{1}{2}mv^2]\!] = \delta T$, the fluctuation in temperature away from the background.

For definiteness, consider the GKE in the absence of FLR effects and magnetic drifts:

$$\partial_t F + v_\parallel\partial_z F + \boldsymbol{V}_E\cdot\boldsymbol{\nabla}F + E_\parallel\partial_\parallel F = 0. \tag{C.35}$$

To derive an equation for $\mathcal{N}$, multiply Eq. (C.35) by $\mathrm{e}^{\boldsymbol{\lambda}\cdot\boldsymbol{v}}$ and integrate over $\boldsymbol{v}$. One finds

$$\frac{\mathrm{D}\mathcal{N}}{\mathrm{D}t} = -\frac{\partial u}{\partial z} - \frac{\mathrm{D}\ln n_0}{\mathrm{D}t} + \frac{q}{m}E_\parallel\lambda_\parallel, \tag{C.36}$$

where $\mathrm{D}/\mathrm{D}t \doteq \partial_t + u\,\partial_z + \boldsymbol{V}_E\cdot\boldsymbol{\nabla}$ and $u \equiv u_\parallel = u_0 + [\![v_\parallel]\!]$. Partial derivatives of Eq. (C.36) with respect to $\boldsymbol{\lambda}$ directly generate the appropriate velocity cumulant equations. (One must remember the u dependence of $\mathrm{D}/\mathrm{D}t$.) For example, with $T = m\,\partial u/\partial\lambda_\parallel$ one finds

$$\frac{\mathrm{D}\delta u}{\mathrm{D}t} = \frac{q}{m}E_\parallel - \left(\frac{T}{m}\right)\frac{\partial\mathcal{N}}{\partial z} - \frac{\partial}{\partial z}\left(\frac{T}{m}\right) - u\frac{\partial u_0}{\partial z} - \frac{T}{m}\frac{\partial}{\partial z}\ln n_0, \tag{C.37}$$

which at $\boldsymbol{\lambda} = \mathbf{0}$, where the last term vanishes, can be seen to be a representation of the fluctuating momentum equation. [Note that the right-hand term $u\,\partial_z u_0$ combines with the left-hand term $u\,\partial_x \delta u$ to give the proper advective term $u\,\partial_z u$, and that $T\partial_z \ln n + \partial_z T = n^{-1}\partial_x(nT)$.] Of course, T remains undetermined; that is the fluid closure problem.

Upon noting that the undetermined cumulants always enter in combination with ∂_z, it is clear that truncations of the velocity cumulant hierarchy at any order generates successively more refined nonresonant thermal corrections; the expansion assumes the fluid limit $\omega/k_\parallel v_t \gg 1$. As Oberman (1960) pointed out, Landau damping is lost in this procedure because that resonant effect is asymptotically beyond all orders in the small parameter $\epsilon \doteq k_\parallel v_t/\omega$. Retaining the dissipative effects of Landau damping is the principal goal of a plasma fluid approximation, hence the nomenclature *Landau-fluid closure*.

C.2.2 Landau-fluid closures

"Our fluid models of kinetic resonances should improve the accuracy of future nonlinear calculations of ITG and other microinstability turbulence." — *Hammett and Perkins (1990).*

If the dynamics are *primarily* fluid ($\epsilon \ll 1$), as is true for ordinary drift waves (but not ITG modes near the linear stability threshold), a reasonable procedure might be to calculate the undetermined velocity cumulants from *linear* kinetic theory, which is well understood. That is essentially the method underlying the generalized Chapman–Enskog procedure of Chang and Callen (1992a). However, such a calculation best proceeds in the frequency domain, and it leads to ω-dependent closure coefficients. Such equations are nonlocal in the time domain and are not suitable for numerical integration.

One possibility would be to evaluate the closure coefficients at the linear eigenfrequency Ω^{lin}. However, that may be difficult to determine in general geometry, and Ω^{lin} may evolve with time or may be shifted by nonlinear effects. Of course, a sharp frequency is not even defined nonlinearly.

Instead, Hammett and Perkins (1990) suggested that the form of the fluid equations should be retained in terms of frequency-independent closure coefficients whose values are determined in such a way that linear theory is well reproduced. When more than one closure coefficient is involved, their values are not uniquely determined. Discussions of the methods used in practice were given by Hammett et al. (1992) and Hammett et al. (1993). Further remarks were made by Smith (1997). A closely related closure procedure was discussed by Bendib and Bendib (1999), who employed a projection-operator technique.

The Hammett–Perkins method makes a reasonable approximation to the linear drive and damping, and furthermore retains the dominant $\boldsymbol{E} \times \boldsymbol{B}$ nonlinearity in the form of the fluid equations that are integrated nonlinearly. It is therefore expected that saturation processes involving fluid mode coupling are well represented, and this appears to be born out in practice. For some comparisons between GK and GF simulations, see Parker et al. (1994).

Unfortunately, although linear theory is modeled accurately, there is no guarantee that nonlinear processes involving wave–wave–particle interactions are described correctly. As a special case, Mattor (1992) compared the kinetic and Landau-fluid descriptions of Compton scattering. He concluded that the Landau-fluid approximation adequately represented Compton scattering for ordinary drift waves but not for deeply resonant ITG modes near threshold. This may point to a significant practical difficulty in bringing kinetic and fluid simulations into precise agreement. For recent work on this difficult problem, see Mattor and Parker (1997) and Mattor (1998, 1999).

C.2.3 Formal theory of fluid closure

If only the linear streaming term is retained in Eq. (C.35), the Fourier-transformed equation $\partial_t F + \mathrm{i}k_\parallel v_\parallel F = 0$ is identical to the stochastic oscillator equation (74) at infinite $\mathcal{K}$. A formal attempt at closure might begin with the DIA. With $n(t) = R(t)n_0$, that approximation is $\partial_\tau R(\tau) + k_\parallel^2 v_t^2 \int_0^t \mathrm{d}\tau\, R(\tau)R(t-\tau) = \delta(\tau)$. The phase-mixing decay of R on the timescale $(k_\parallel v_t)^{-1}$ describes Landau damping in this context. As described in Sec. 3.3.1 (p. 54), the correct result, $R(\tau) = \exp(-\frac{1}{2}k_\parallel^2 v_t^2 \tau^2)$, is reasonably well reproduced by the DIA. Note that the almost-Markovian system

$$\partial_\tau R + \eta^{\mathrm{nl}}(\tau)R(\tau) = \delta(\tau), \quad \eta^{\mathrm{nl}}(\tau) \doteq k_\parallel^2 v_t^2 \theta(\tau), \quad \partial_\tau \theta = 1 \tag{C38a,b,c}$$

precisely reproduces the correct result. A Hammett–Perkins-type closure that produces the correct $\omega = 0$ response results by taking $\eta^{\mathrm{nl}} = (2/\pi)^{1/2} k_\parallel v_t$ (of course, the predicted exponential form of the decay in τ is not correct). Similar closures could be made by truncating at higher order.

Unfortunately, in the presence of nonlinear effects the analogy to the passive-advection problem partly breaks down. The difficulty is that $\boldsymbol{V}_E$ depends on the potential φ, which by Poisson's equation is proportional to $\int \mathrm{d}\boldsymbol{v}\, \overline{n} F$ (represented as $\langle \overline{n}\chi \rangle$ in the present formalism). That mean field *responds under perturbations*; the nonlinear problem includes self-consistent response. This significantly complicates formal attempts at systematic closure, which are in substantially preliminary stages of development.

D STOCHASTICITY CRITERIA

Here I derive the Chirikov criterion for stochasticity of the 1D electrostatic field assumed in the passive quasilinear theory of Sec. 4.1.2 (p. 91), then justify the usual continuum wave-number representation of the quasilinear diffusion coefficient. Finally, I discuss spatial trapping and the stochasticity criterion for $\boldsymbol{E} \times \boldsymbol{B}$ motion.

D.1 Stochasticity criterion for a one-dimensional electrostatic wave field

For definiteness and simplicity, consider test-particle motion in the one-dimensional wave field $E(x,t) = \frac{1}{2}\sum_k E_k \exp[\mathrm{i}(kx - \Omega_{\boldsymbol{k}} t)]$. The complex wave amplitudes E_k are assumed to be constant in time. One has the reality conditions $E_k^* = E_{-k}$ and $\Omega_{-k} = -\Omega_{\boldsymbol{k}}$. Assume that the modes are quantized in a box of side L, so the wave-number spacing is $\delta k = 2\pi/L$ and the wave numbers are integer multiples of δk: $k = n\,\delta k$. Also assume there are N (positive) k's in the spectrum, and define the spectral width $\Delta k \doteq N\,\delta k$.

Let us write $E_k = |E_k|\mathrm{e}^{\mathrm{i}\beta_k}$, where the β's are given, *fixed* phases. [In studies of wave–wave coupling Sec. 4.2, p. 98, one allows the wave phases to be random. Here, however, one is concerned only with particle stochasticity.] Then the equations of motion are

$$\dot{x} = v, \quad \dot{v} = \frac{1}{2}\left(\frac{q}{m}\right)\sum_k |E_k| \mathrm{e}^{\mathrm{i}(kx - \Omega_{\boldsymbol{k}} t + \beta_k)}. \tag{D1a,b}$$

There are resonances where the phase varies slowly: $k\dot{x} - \Omega_{\boldsymbol{k}} = 0$, or $v = \Omega_{\boldsymbol{k}}/k \doteq v_{\mathrm{ph}}$. If one assumes that $\delta k/k \ll 1$, the spacing in velocity between adjacent resonances is

$$\delta v = \left(\frac{\Omega_{\boldsymbol{k}}}{k}\right)\bigg|_{k+\delta k} - \left(\frac{\Omega_{\boldsymbol{k}}}{k}\right)\bigg|_{k} \approx \delta k \frac{\partial}{\partial k}\left(\frac{\Omega_{\boldsymbol{k}}}{k}\right), \tag{D2a,b}$$

or

$$|\delta v| = \Delta v_{\text{gr}}(\delta k/k), \quad \Delta v_{\text{gr}} \doteq |v_{\text{gr}} - v_{\text{ph}}|. \tag{D3a,b}$$

The stochasticity criterion is found (Zaslavskiĭ and Chirikov, 1972; Lichtenberg and Lieberman, 1992) by determining the width of just one resonance, then comparing that to the spacing between resonances. The Hamiltonian for one resonance (including its complex conjugate) is $H_k = p^2/2m + e|\varphi_k|\cos(kx - \Omega_{\boldsymbol{k}}t + \beta_k)$, which in the wave frame $x_w \doteq x - (\Omega_{\boldsymbol{k}}/k)t$ is the well-known pendulum Hamiltonian. Particles with energies $H \le e|\varphi_k|$ are trapped; the island or trapping width is $\Delta v_k = 4v_{\text{tr}}$, where the *trapping velocity* v_{tr} and associated *trapping frequency* ω_{tr} are

$$v_{\text{tr}} \doteq (e|\varphi_k|/m)^{1/2}, \quad \omega_{\text{tr}} \doteq k v_{\text{tr}}. \tag{D4a,b}$$

Then the Chirikov *stochasticity parameter* is

$$S \doteq \Delta v_k/\delta v = (4/\pi)(\omega_{\text{tr}}\tau_{\text{ac}}^{\text{lin}})N, \tag{D5a,b}$$

where Eq. (152) was used. Let us also introduce the *recurrence time* $\tau_r \doteq 2\pi/(\delta k\,\Delta v_{\text{gr}}) = L/\Delta v_{\text{gr}}$, which is the time it takes for the particle to sense the periodicity of the wave packet (which is periodic in space with period L). Note that $\tau_r/\tau_{\text{ac}} = 2N \gg 1$. Then the *Chirikov criterion* for stochasticity, $S \gtrsim 1$, can very simply be stated as $\omega_{\text{tr}}\tau_r \gtrsim 1$. This criterion is a lower limit on the wave amplitudes; it is generally very easy to satisfy because δk is small.

D.2 Justification of the continuum approximation

The envelope decay due to the nonlinear *orbit diffusion* (sometimes called *resonance broadening*) smooths the behavior of the correlation function. Due to the discreteness of $k = n\,\delta k$, the linear phase changes by approximately the first-order variation of $kv - \Omega_{\boldsymbol{k}}$, namely, $(v_{\text{ph}} - v_{\text{gr}})\delta k$, when k is incremented by δk in the wave-number sum. If the nonlinear frequency $\nu_d \doteq \tau_d^{-1}$ [formula (159) evaluated at a typical wave number $\overline{k}$] is greater than this phase change, then the discreteness is not really seen and one will get essentially the same answer by integrating rather than summing over k. Thus the continuum limit is justified if $|v_{\text{ph}} - v_{\text{gr}}|\delta k < \nu_d$. Upon cubing this inequality and using the formula for $\tau_{\text{ac}}^{\text{lin}}$, one gets

$$(1/N\tau_{\text{ac}}^{\text{lin}})^3 < k^2 D_v. \tag{D.6}$$

But one has [Eq. (151)]

$$D_v = (q/m)^2 \mathcal{E} \tau_{\text{ac}}^{\text{lin}}, \tag{D.7}$$

where $\mathcal{E}$ is the total fluctuation intensity. If one approximates $\mathcal{E} = \langle \delta E^2 \rangle \approx N\langle \delta E_k^2 \rangle$, where δE_k is the strength of one harmonic in the spectrum, then one can rewrite D_v in terms of the trapping frequency $\omega_{\rm tr}$ for a typical harmonic, $\omega_{\rm tr} = (ek\,\delta E_k/m)^{1/2}$. With the aid of the last two results, Eq. (D.7) can be written as $D_v = (N\omega_{\rm tr}^4/k^2)\tau_{\rm ac}^{\rm lin}$. Equation (D.6) then becomes $1 < N^4\tau_{\rm ac}^{\rm lin4}\omega_{\rm tr}^4 = S^4$, where one used the result (D5b). Upon taking the fourth root of of this inequality, one finds that the continuum limit is justified when $S \gtrsim 1$. This inequality is just the Chirikov criterion, so there is a pleasing consistency: The presence of stochasticity (manifested in the nonlinear terms) is just enough to justify the continuum limit. The situation is illustrated in Fig. 12 (p. 94).

D.3 $\boldsymbol{E} \times \boldsymbol{B}$ motion and stochasticity

Analogous considerations pertain to cross-field transport due to $\boldsymbol{E} \times \boldsymbol{B}$ motions although the dynamics differ substantially from the field-free case. Ching (1973) considered the potential $\varphi(\boldsymbol{x}, t) = \varphi_0 \cos(k_x x)\cos(k_y y - \Omega_{\boldsymbol{k}} t)$, which is stationary in the wave frame $y_w = y - (\Omega_{\boldsymbol{k}}/k_y)t$, and showed that *spatial trapping* occurs for $|k_x(c\varphi_0/B)| > |\Omega_{\boldsymbol{k}}/k_y|$ [a condition on the y component of the $\boldsymbol{E} \times \boldsymbol{B}$ velocity, thereby substantially refining an earlier argument of Dupree (1967)]. When parallel motion is admitted, $\Omega_{\boldsymbol{k}} \to \overline{\Omega}_{\boldsymbol{k}} \doteq \Omega_{\boldsymbol{k}} - k_\parallel v_\parallel$. Hirshman (1980) showed that the resulting dynamics are derivable from a Hamiltonian $H(x, y_w) = (c\varphi_0/B)\cos(k_x x)\cos(k_y y_w) - (\overline{\Omega}_{\boldsymbol{k}}/k_y)x$, with x playing the role of a momentum conjugate to y_w, and gave an elegant discussion of the conditions for trapping. Precisely resonant particles ($\overline{\Omega}_{\boldsymbol{k}} = 0$) circulate with the trapping frequency

$$\omega_{\rm tr} = (c\varphi_0/B)k_x k_y = k_y V_{E,y}; \tag{D.8}$$

thus the trapping condition is $\omega_{\rm tr} > \overline{\Omega}_{\boldsymbol{k}}$. Further discussion of spatial trapping was given by Smith et al. (1985).

Horton (1981) showed that the separatrices of the spatial islands are stochastically destroyed by a small-amplitude secondary wave; for more recent work, see Isichenko et al. (1992). In general, $\boldsymbol{E} \times \boldsymbol{B}$ stochasticity occurs for a spectrum of waves when the trapping frequency based on the rms $\boldsymbol{E} \times \boldsymbol{B}$ velocity exceeds $\overline{\Omega}_{\boldsymbol{k}}$. This agrees with the picture suggested by Dupree (1967).[291] Various research on $\boldsymbol{E} \times \boldsymbol{B}$ motion and stochasticity was summarized by Horton and Ichikawa (1996).

E SOME FORMAL ASPECTS OF RESONANCE-BROADENING THEORY

Dupree (1966) introduced his resonance-broadening theory with a considerable amount of formal trappings. In fact, his test wave and random particle propagator techniques have proven to be somewhat of a dead end, as they are unable to naturally handle self-consistency. Nevertheless, it is very instructive to understand something of what Dupree tried to do. In this appendix I describe some of the more formal manipulations related to the random particle propagator $\widetilde{U}$ and discuss the relationship between $\widetilde{U}$ and the infinitesimal response function R employed by Kraichnan Sec. 5, p. 126 and MSR Sec. 6.2, p. 153. Some early related discussion was by Birmingham and Bornatici (1971).

[291] Characteristically, in his early nomenclature Dupree did not distinguish between trapping and stochasticity although the latter was clearly implied. In the context of plasma turbulence theory, explicit reference to stochastic motion was finally made by Dupree and Tetreault (1978) more than a decade after the inception of the RBT.

E.1 The random particle propagator and passive diffusion

As was described in Sec. 4.3 (p. 108), Dupree's fundamental insight was that the turbulent fluctuations will diffuse particles away from their free-streaming trajectories, thereby introducing the diffusion time τ_d [see Eq. (159)] into the theory. Let us attempt to make that more quantitative.

E.1.1 Random particle propagator and the method of characteristics

Consider the Vlasov equation [more precisely, the Klimontovich equation (23) for $\epsilon_p = 0$] $\partial_t f + v\,\partial_x f + a\,\partial_v f = 0$, where $a \doteq (q/m)E$. This equation simply states that the particle trajectories obey Newton's laws of motion $\dot{x} = v$ and $\dot{v} = a(x,t)$. Now suppose that a is random. To be definite, it is convenient to assume that $a(t) \equiv a\big(x(t),t\big)$ is *Gaussian white noise*. Then from the elementary theory of Langevin equations Sec. 3.2, p. 48, one knows that the particle dispersions grow like $\langle \delta v^2 \rangle = 2D_v t$ and $\langle \delta x^2 \rangle = \frac{2}{3} D_v t^3$.

The Gaussian assumption is problematic. The plasma dynamics are highly nonlinear, and in the general case the particles and fields are coupled self-consistently. There is no reason why Gaussian statistics should emerge [indeed, as was discussed in Sec. 3.8.1 (p. 71), nonlinearity precludes the possibility of precisely Gaussian statistics], let alone white noise. Nevertheless, only in the Gaussian case can one really make analytical progress with the present techniques.

Fokker–Planck theory now allows one to assert that the mean PDF obeys $\partial_t f + v\,\partial_x f - \partial_v D_v(v)\partial_v f = 0$. Here, as usual,

$$D_v = \int_0^\infty \mathrm{d}\tau\, \langle \delta a\big(x(\tau),\tau\big)\delta a\big(x(0),0\big)\rangle = \int_0^\infty \mathrm{d}\tau\, \langle \delta a\big(x(0),0\big)\delta a\big(x(-\tau),-\tau\big)\rangle. \qquad \text{(E1a,b)}$$

These two formulas are equivalent because one assumes that the fluctuations are time stationary.

The difficulty with general evaluation of a formula such as Eq. (E1b) is its Lagrangian dependence on the actual turbulent trajectory $x(\tau)$. In order to focus on that trajectory, let us introduce a *random particle propagator* $\widetilde{U}$ that uniquely captures the details of the motion:

$$\widetilde{U}(z,t;z',t') \doteq \delta\big(z - \widetilde{z}_f(t;x',t')\big). \qquad \text{(E.2)}$$

Here $\widetilde{z}_f(t;z',t')$ is the phase-space location at time t of a particle that passed through point z' at time t' (the f stands for *forward*). Now what one wants to do is write something like $\delta a\big(x(-\tau)\big) = \int \mathrm{d}z'\, \widetilde{U}(z,t;z',t-\tau)\delta a(x')$. However, the utility of this form is a bit obscure since the function $\widetilde{x}_f(t;z',t')$ that appears in Eq. (E.2) is a highly nonlinear function of z'. What one really wants is a form in which x' appears explicitly. Physically, one need merely note that the change in position between x' and x can be parametrized by either the initial time t' or the final time t. Thus one will simply write $\widetilde{x}_f(t;z',t') - x' \doteq \delta\widetilde{x}_f(t;z',t') = \delta\widetilde{x}_b(t';z,t)$ (the b stands for *backward*). It is useful to see how this works in detail.

First note that $\widetilde{U}$ obeys $\partial_t \widetilde{U}(z,t;z',t') + v\,\partial_x \widetilde{U} + a(x,t)\partial_v \widetilde{U} = 0$, with initial condition $\widetilde{U}(z,t';z',t') = \delta(z - z')$. Now recall the method of characteristics for the solution of first-order PDE's of the form $\partial_t \psi(x,t) + V(x,t)\partial_x \psi = 0$. [For the present application, V is really the vector $(v,\, a)^T$.] The method exploits the fact that the initial conditions on a trajectory are constants of the motion, so it is convenient to change variables to those initial conditions. Formally, one defines the characteristic problem $\mathrm{d}\widetilde{x}(\widetilde{t})/\mathrm{d}\widetilde{t} = V\big(\widetilde{x}(\widetilde{t}),\widetilde{t}\big)$, where $\widetilde{x}(t) = x$ and $\widetilde{t}$ parametrizes the trajectory; one must carefully distinguish the independent variables (x,t) of ψ from the time-dependent characteristic orbit $\widetilde{x}(\widetilde{t}; x, t)$. The formal solution is $\widetilde{x}(\widetilde{t}; x, t) = x + \int_t^{\widetilde{t}} \mathrm{d}\widehat{t}\, V(\widetilde{x}(\widehat{t}), \widehat{t})$. Define the new variable $\overline{x} = \widetilde{x}(t')$. This is the position at time t' of the orbit that passes through x at time t. Thus $\overline{x} = x - \delta\widetilde{x}_b(t'; x, t)$, where $\delta\widetilde{x}_b(t'; x, t) \doteq \int_{t'}^{t} \mathrm{d}\overline{t}\, V(\widetilde{x}(\overline{t}; x, t), \overline{t})$. Now change variables according to $(x, t) \to (\overline{x}, \overline{t})$, with $\overline{t} = t$. One then readily finds that $\overline{\psi}$ is conserved along the $\overline{t}$ trajectories: $\overline{\psi}(\overline{x}, \overline{t}) = \overline{\psi}(\overline{x}, t')$. Upon applying this procedure to $\widetilde{U}$, one finds $\overline{U}(\overline{z}, \overline{t}; z', t') = \widetilde{U}(z, t; z', t') = \delta(\overline{z} - z')$. More explicitly, $\widetilde{U}(z, t; z', t') = \delta(z - \delta\widetilde{x}_b(t'; z, t) - z')$. This form is just what one wants for expressing the earlier motion in terms of $\widetilde{U}$. Thus

$$\delta a(x(t'; z, t), t') = \int \mathrm{d}z'\, \widetilde{U}(z, t; z', t')\delta a(x', t'). \tag{E.3}$$

Usually it is more convenient to introduce $\tau \doteq t - t'$ as a variable. Let us adopt the notation

$$\delta\widetilde{x}_f(t; z', t') \equiv \delta\widetilde{x}_f(\tau \mid z', t'), \quad \delta\widetilde{x}_b(t'; z, t) \equiv \delta\widetilde{x}_b(-\tau \mid z, t). \tag{E4a,b}$$

Then (after trivially integrating out the delta function in v')

$$\delta a(x(t'; z, t), t') = \int \mathrm{d}x'\, \delta(x - \delta\widetilde{x}_b(-\tau \mid z, t) - x')\delta a(x', t'). \tag{E.5}$$

Thus one has shown that one can write

$$D_v = \int_0^\infty \mathrm{d}\tau \int_{-\infty}^\infty \mathrm{d}\overline{v} \int_{-\infty}^\infty \mathrm{d}\overline{x}\, \langle \delta a(x, t)\widetilde{U}(x, v, t; \overline{x}, \overline{v}, t - \tau)\delta a(\overline{x}, t - \tau)\rangle. \tag{E.6}$$

Now in the general case it is still very difficult to systematically perform the ensemble average required in Eq. (E.6) because the Eulerian accelerations will still be random variables on which $\widetilde{U}$ depends. (They are proportional to the Fourier coefficients of the waves.) However, in the special case in which the Hamiltonian consists of a given (passive) collection of fixed (in both amplitude and phase) Fourier components, then the only randomness in the δa's comes from the initial particle position (which in practice one takes to be homogeneously distributed). Thus the ensemble average may be factored and one obtains

$$D_v = \int_0^\infty \mathrm{d}\tau\, \langle \delta a(x, t)\, U(\tau)\, \delta a(\overline{x}, t - \tau)\rangle, \tag{E.7}$$

where $U \doteq \langle \widetilde{U} \rangle$ and I follow Dupree's convention of not explicitly writing integrations over the phase-space coordinates. For passive problems U is identical to the passive response function R. Equation (E.7) thus has the schematic form $\int_0^\infty \mathrm{d}\tau\, R(\tau)\Upsilon(\tau)$ already seen in the introductory discussion of passive statistical closures [cf. Eq. (143b)]. Formulas of the form (E.7) are characteristic of the RBT. They emerge as well from projection-operator formalism Sec. 3.9.11, p. 88, as shown by Weinstock (1969) and Weinstock (1970). Note that the RBT school typically arrives at *Markovian* rather than nonlocal expressions for transport and response. Thus for the stochastic oscillator model the resonance-broadening approximation would be $\partial_\tau R + [\int_0^\infty \mathrm{d}\overline{\tau}\, R(\overline{\tau})C(\overline{\tau})]R(\tau) = \delta(\tau)$ rather than the properly nonlocal DIA equation (143a). Analogous remarks hold for the spatial variables; RBT calculations typically do not involve the detailed mode-coupling forms characteristic of DIA-like approximations.

Instead of working with U itself, it is generally more convenient to work with Fourier amplitudes. For statistically homogeneous and stationary turbulence with $\rho \doteq x - x'$, define $U_k(v, \tau; v') \doteq$

$\int d\rho\, e^{-ik\rho}\, U(\rho, v, \tau; v')$, where $U(\rho, v, \tau; v') \doteq \langle \delta(x - \delta\widetilde{x}_b(-\tau \mid z, t) - x')\delta(v - \delta v_b(-\tau \mid z, t) - v') \rangle$. One finds

$$U_k(v, \tau; v') = \langle e^{-ik\delta\widetilde{x}_b(-\tau)} \delta(v - \delta v_b(-t) - v') \rangle. \tag{E.8}$$

The function $U_k(v, \tau) \doteq \int dv'\, U_k(v, \tau; v') = \langle e^{-ik\delta\widetilde{x}_b(-\tau)} \rangle$, which sometimes arises in practice, is in principle accessible to cumulant expansion [Eqs. (93)]. Benford and Thomson (1972) discussed the more complete form (E.8); they observed that certain of Dupree's approximations of that quantity were flawed because x and v are correlated. How this works can explicitly be seen in the white-noise limit, discussed next.

E.1.2 Particle propagator in the white-noise limit

Resonance-broadening theory frequently exploits the Gaussian white-noise hypothesis. In that case formula (E.8) can be worked out explicitly. It is a rigorous result that Fokker–Planck theory is exact in that limit [see Sec. H.3 (p. 294)], so g obeys

$$\partial_t g(x, v, t; x', v', t') + v\, \partial_x g - \partial_v D_v \partial_v g = \delta(t - t')\delta(x - x')\delta(v - v'). \tag{E.9}$$

In general, D_v depends on v; however, let us assume that $D_v = \text{const}$ so one can proceed analytically. The most physical and direct way of solving Eq. (E.9) for constant D_v is to recognize the equivalence of Eq. (E.9) to the classical Langevin equations (66). Since the random variables x and v are known to be (jointly) Gaussian, the form of g is guaranteed to be Gaussian with the means and variances as given by the short-time (collisionless) limits of Eqs. (68) [see Table 1 (p. 50)] with initial conditions x' and v'. One finds

$$g(\rho, v, \tau; v') = \left[\frac{1}{(2\pi\sigma_X^2)^{1/2}} \exp\left(-\frac{\delta X^2}{2\sigma_X^2} \right) \right] \left[\frac{1}{(2\pi\sigma_v^2)^{1/2}} \exp\left(-\frac{\delta v^2}{2\sigma_v^2} \right) \right], \tag{E.10}$$

where

$$\delta X \doteq \rho - V\tau, \quad \rho \doteq x - x', \quad V \doteq \tfrac{1}{2}(v + v'), \quad \delta v \doteq v - v', \tag{E11a,b,c,d}$$

and

$$\sigma_X^2 \doteq \tfrac{1}{6} D_v \tau^3, \quad \sigma_v^2 \doteq 2 D_v \tau. \tag{E12a,b}$$

Equation (E.10) shows the well-known fact that the variables δX and δv are statistically independent (Benford and Thomson, 1972). A consequence is that the dispersion of δX (namely, $\frac{1}{6} D_v \tau^3$) differs from that of $\delta x \equiv \rho$ (namely, $\frac{2}{3} D_v \tau^3$).

Another solution method is to Fourier transform with respect to ρ, eliminate the streaming term *via* the transformation $g_k(v, \tau; v') = e^{-ikv\tau} h_k(v - v', \tau) H(\tau)$ and the initial condition $h_k(v - v', 0) = \delta(v - v')$, Fourier-transform the resulting equation for h with respect to $v - v'$ (using conjugate variable λ), integrate in τ, then invert the Fourier transformations. One finds

$$h_{k,\lambda}(\tau) = \exp(-\lambda^2 D_v \tau + k\lambda D_v \tau^2 - \tfrac{1}{3} k^2 D_v \tau^3). \tag{E.13}$$

At $k = 0$ one recognizes in the $\lambda^2 D_v$ term the Fourier transform of velocity-space diffusion; at $\lambda = 0$ one recognizes the signature of the associated spatial dispersion involving the diffusion time

$\tau_d = (k^2 D_v)^{-1/3}$. The cross term in $k\lambda$ reflects the correlated nature of x and v. Upon inverting the λ dependence, one finds

$$h_k(v - v', \tau) = \mathrm{e}^{\frac{1}{2}\mathrm{i}k(v-v')\tau}\mathrm{e}^{-\frac{1}{12}k^2 D_v \tau^3}\Phi(v - v'; 2D_v\tau), \tag{E.14}$$

where $\Phi(\delta v; \sigma)$ is a Maxwellian distribution with variance σ [i.e., $\Phi(\delta v; 2D_v\tau)$ is just the usual diffusion Green's function]. Upon multiplying by $\mathrm{e}^{\mathrm{i}kv\tau}$ to obtain g from h (note that $\mathrm{e}^{-\mathrm{i}kv\tau}\mathrm{e}^{\frac{1}{2}\mathrm{i}\delta v\tau} = \mathrm{e}^{-\mathrm{i}kV\tau}$), one is readily led to Eq. (E.10) by inverting the spatial transform.

As a check on the formulas (E.10) and (E.14), consider their integral over v' (or, equivalently, over δv for fixed v). Proceeding from Eq. (E.14) is slightly simpler. One has $\int \mathrm{d}\delta v\, h_k(\delta v, \tau) = \mathrm{e}^{-\frac{1}{3}k^2 D_v \tau^3}$ (the integral is the Fourier transform of a Gaussian; $\frac{1}{3} = \frac{1}{12} + \frac{1}{4}$); then

$$g(\rho, v, \tau) = \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{\mathrm{i}k(\rho - v\tau)} e^{-\frac{1}{3}k^2 D_v \tau^3} = \Phi(\rho - v\tau; \sigma_x^2). \tag{E.15}$$

The same result is, of course, obtained by integrating Eq. (E.10) over δv after writing $V = v - \frac{1}{2}\delta v$.

It should be noted that the solution of Eq. (E.9) for *nonconstant* $D_v(v)$ is *not* a jointly Gaussian distribution, and the slope $\mathrm{d}\langle v^2\rangle(t)/\mathrm{d}t$ is not a straight line. Failure to recognize this obvious fact has led to some confusion in the literature, as discussed by vanden Eijnden (1997).

Further interpretation of the white-noise propagator (E.10) is given in Sec. 12.6.3 (p. 246).

E.2 Random particle propagator vs infinitesimal response function

For passive problems it is easy to see that the infinitesimal response function $\widetilde{R}$ and the random particle propagator $\widetilde{U}$ are identical. They differ, however, for self-consistent problems, and substantial technical difficulties impede attempts to express the mean response function R in terms of U. Consider self-consistent Vlasov response. $\widetilde{R}$ obeys $\partial_t \widetilde{R} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\widetilde{R} + \widetilde{\boldsymbol{E}} \cdot \boldsymbol{\partial}\widetilde{R} + \boldsymbol{\partial}\widetilde{f} \cdot \widehat{\boldsymbol{\mathcal{E}}}\widetilde{R} = \mathsf{I}$, where self-consistency enters through the last term on the left. Formally, one may attempt to proceed as in the solution of the linearized Vlasov equation by using the method of characteristics and the definition of $\widetilde{U}$ to write $\widetilde{R} = \widetilde{U} - \widetilde{U}\boldsymbol{\partial}\widetilde{f} \cdot \widehat{\boldsymbol{\mathcal{E}}}\widetilde{R}$; upon formally solving for $\widehat{\boldsymbol{\mathcal{E}}}\widetilde{R}$, one finds

$$\widehat{\boldsymbol{\mathcal{E}}}\widetilde{R} = (\mathsf{I} + \widehat{\boldsymbol{\mathcal{E}}}\widetilde{U}\boldsymbol{\partial}\widetilde{f})^{-1} \cdot \widehat{\boldsymbol{\mathcal{E}}}\widetilde{U}. \tag{E.16}$$

Now although the *form* of Eq. (E.16) is identical to that of Eq. (343c), Eq. (E.16) is useless as it stands since the concept of a random dielectric function is not defined. Furthermore, how to perform the statistical average of Eq. (E.16) is not apparent since random variables appear in both the numerator ($\widetilde{U}$) and denominator ($\widetilde{U}$ and $\widetilde{f}$). Thus

$$\widehat{\boldsymbol{\mathcal{E}}}R = \langle(\mathsf{I} + \widehat{\boldsymbol{\mathcal{E}}}\widetilde{U}\boldsymbol{\partial}\widetilde{f})^{-1} \cdot \widehat{\boldsymbol{\mathcal{E}}}\widetilde{U}\rangle \neq (\mathsf{I} + \widehat{\boldsymbol{\mathcal{E}}}U\boldsymbol{\partial}f)^{-1} \cdot \widehat{\boldsymbol{\mathcal{E}}}U. \tag{E17a,b}$$

Although Eq. (E17b) resembles Eq. (343c), the forms differ in two important ways: (i) The passive propagator U is not equal to the self-consistent particle propagator g (the latter contains the extra term $\Sigma^{(p)}$). (ii) The nonlinear correction $\delta\overline{f}$ is missing in Eq. (E17b).

Fundamentally, attempts to usefully represent R in terms of U fail because self-consistency enters at the same order as the statistical effects of passive advection. The advantages of the statistical formalisms advocated by Kraichnan and MSR is that self-consistency is dealt with from the outset; passive dynamics are never misleadingly brought to the fore.

F SPECTRAL BALANCE EQUATIONS FOR WEAK INHOMOGENEITY

In a classic paper by Carnevale and Martin (1982, CM), the reduction of the general Dyson equations to Markovian form in the presence of weakly inhomogeneous backgrounds was addressed. They considered the specific application to Rossby waves [in the present context, one may instead think of the HM model introduced in Sec. 2.4.3 (p. 34)] and found reasonable conservation laws for both energy and enstrophy. Nevertheless, a recent derivation of such laws by Smolyakov and Diamond (1999) using a (superficially) different method was shown by Krommes and Kim (2000, KK) to imply a kinetic equation in disagreement with that of CM. Closer inspection revealed algebraic errors in the derivation of the CM conservation laws and to the realization that CM had made two compensating errors. Krommes and Kim pinpointed the flaw in the logic; when corrected, the final kinetic equation agrees with the one implied by the work of Smolyakov and Diamond.

Let $\widehat{A}$ and $\widehat{B}$ be the abstract operators corresponding to the two-point functions $A(1,1')$ and $B(1,1')$. Operator products are realized as space-time convolutions: $(\widehat{A}\widehat{B})(1,1') = (A\star B)(1,1') \doteq \int d\overline{1}\, A(1,\overline{1})B(\overline{1},1')$. The general procedure is to approximate the convolution in the limit of weak inhomogeneity. To that end, introduce the sum and difference coordinates

$$\boldsymbol{\rho} \doteq \boldsymbol{x} - \boldsymbol{x}', \quad \boldsymbol{X} \doteq \tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{x}'), \quad \tau \doteq t - t', \quad T \doteq \tfrac{1}{2}(t + t') \qquad \text{(F1a,b,c,d)}$$

and write $A(\boldsymbol{x}, t, \boldsymbol{x}', t') = A(\boldsymbol{\rho}, \tau \mid \boldsymbol{X}, T)$. It is assumed that the dependence on $\boldsymbol{\rho}$ and τ introduces a short autocorrelation length and time (usually one of the functions is a covariance matrix), so Taylor expansion in ρ/X and τ/T can be performed. Upon truncating at first order and Fourier-transforming with respect to $\boldsymbol{\rho}$ and τ, one is led to

$$(\widehat{A}\widehat{B})_{\boldsymbol{k},\omega}(\boldsymbol{X}, T) \approx A_{\boldsymbol{k},\omega}(\boldsymbol{X}, T)B_{\boldsymbol{k},\omega}(\boldsymbol{X}, T) + \tfrac{1}{2}i\{A, B\}. \qquad \text{(F.2)}$$

Here the braces denote the *Poisson bracket*, defined for two functions with arguments $(\boldsymbol{k}, \omega)$ and $(\boldsymbol{X}, T)$ as

$$\{A, B\} \doteq \left(\frac{\partial A}{\partial \boldsymbol{X}} \cdot \frac{\partial B}{\partial \boldsymbol{k}} - \frac{\partial A}{\partial \boldsymbol{k}} \cdot \frac{\partial B}{\partial \boldsymbol{X}}\right) - [(\boldsymbol{X}, \boldsymbol{k}) \Leftrightarrow (T, \omega)]. \qquad \text{(F.3)}$$

Equation (F.2) is the well-known result [292] rederived by CM and used to simplify the Dyson equations. Suppose, for example, one considers the linear dynamics

$$\partial_t \varphi(\boldsymbol{x}, t) + \mathrm{i} \int d\overline{x}\, \mathcal{L}(\boldsymbol{x}, \overline{\boldsymbol{x}})\varphi(\overline{\boldsymbol{x}}, t) = 0, \qquad \text{(F.4)}$$

[292] Carnevale and Martin cited prior work in quantum field theory. A classically oriented review that focuses on the derivations of wave kinetic equations in the eikonal approximation is by McDonald (1988). He showed that some manipulations can be formalized through all orders in the inhomogeneity by use of the *Weyl calculus*; see also McDonald and Kaufman (1985). The quantity $A_{\boldsymbol{k},\omega}(\boldsymbol{X}, T)$ is known as the *Weyl symbol* of the operator $\widehat{A}$ and can be shown to possess certain desirable properties that argue in favor of the use of the centered representation based on Eq. (F.1). Equation (F.2) is the first-order limit of Eq. (4.29) of McDonald (1988).

where $\widehat{\mathcal{L}} = \widehat{\Omega} + \mathrm{i}\widehat{\gamma}$ is weakly inhomogeneous (and dependent only on space for simplicity). A wave kinetic equation for $C_{\boldsymbol{k}}(\boldsymbol{X}, T) \doteq \langle|\varphi_{\boldsymbol{k}}|^2\rangle = (2\pi)^{-1}\int_{-\infty}^{\infty}\mathrm{d}\omega\, C_{\boldsymbol{k},\omega}(\boldsymbol{X}, T)$ then follows by identifying $A_{\boldsymbol{k},\omega}(\boldsymbol{X}, T) = -\mathrm{i}[\omega - \Omega_{\boldsymbol{k}}(\boldsymbol{X})]$ and $B = C_{\boldsymbol{k},\omega}(\boldsymbol{X}, T)$, taking the real part of Eq. (F.2), and integrating over all ω. One obtains

$$\partial_T C_{\boldsymbol{k}}(\boldsymbol{X}, T) - \{\Omega_{\boldsymbol{k}}, C_{\boldsymbol{k}}\} - 2\gamma_{\boldsymbol{k}} C_{\boldsymbol{k}} = 0. \tag{F.5}$$

Here

$$-\{\Omega_{\boldsymbol{k}}, C_{\boldsymbol{k}}\} = \frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{k}}\cdot\frac{\partial C_{\boldsymbol{k}}}{\partial\boldsymbol{X}} - \frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{X}}\cdot\frac{\partial C_{\boldsymbol{k}}}{\partial\boldsymbol{k}} \tag{F.6}$$

defines characteristics

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{k}}, \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\frac{\partial\Omega_{\boldsymbol{k}}}{\partial\boldsymbol{X}} \tag{F7a,b}$$

that are the usual ray equations of geometric optics. As is well known (Bernstein, 1975; Stix, 1992), the rays are Hamiltonian equations of motion with Ω playing the role of the Hamiltonian.

There is a subtle but important assumption in the derivation of Eq. (F.5), which is that the function $\mathcal{L}_{\boldsymbol{k}}(\boldsymbol{X})$ is of *zeroth order* in the large-scale gradient. This is frequently not true. Following KK, consider the example $\Omega(\boldsymbol{x}, \boldsymbol{x}') = -f(\boldsymbol{x})\nabla_{\boldsymbol{x}}^2\delta(\boldsymbol{x} - \boldsymbol{x}')$, where f is an arbitrary slowly varying function. It is not hard to show that through first order one has $\Omega_{\boldsymbol{k}}(\boldsymbol{X}) = f(\boldsymbol{X})k^2 + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\nabla}f(\boldsymbol{X})$; the $\boldsymbol{k}\cdot\boldsymbol{\nabla}f$ term is absent from equations like (F.2). Such terms arise when $\widehat{\Omega}$ is itself the product of two noncommuting operators $\widehat{A}$ and $\widehat{B}$. For reducing operator products like $\widehat{\Omega}\widehat{C}$, the proper generalization is the symmetrical expression

$$\widehat{A}\widehat{B}\widehat{C} \approx ABC + \tfrac{1}{2}\mathrm{i}(\{A, B\}C + \{A, C\}B + \{B, C\}A), \tag{F.8}$$

which can be written as

$$\widehat{\Omega}\widehat{C} \approx \Omega C + \tfrac{1}{2}\mathrm{i}(\{\Omega, C\} + \underline{\{A, B\}C}). \tag{F.9}$$

The underlined term, absent from the result of CM,[293] recovers the $\boldsymbol{k}\cdot\boldsymbol{\nabla}f$ term in the above example.

As an important practical example, consider the operator that arises in the HME to describe the advection of short scales by long ones (where the long-wavelength velocity is $\overline{\boldsymbol{V}}$, with $\boldsymbol{\nabla}\cdot\overline{\boldsymbol{V}} = 0$):

$$\mathrm{i}\widehat{\Omega} \doteq [(1 - \nabla^2)^{-1}][\overline{\boldsymbol{V}}(\boldsymbol{x}, t)\cdot\boldsymbol{\nabla}][-\nabla^2]. \tag{F.10}$$

This operator can be considered to be the product of the three operators delimited by brackets in Eq. (F.10). The necessary algebra was performed by KK; the result for $\gamma_{\boldsymbol{k}} = 0$ is

$$\partial_T C_{\boldsymbol{k}} - \sigma_W^{-1}\{\Omega_{\boldsymbol{k}}, W_{\boldsymbol{k}}\} = 0, \tag{F.11}$$

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[293] It seems to not be considered quite generally. McDonald (1988) reviewed the traditional derivation of WKE's, which takes into account a first-order dissipative correction $\widehat{\gamma}$ arising from the anti-Hermitian part of the dielectric operator. In contrast, the correction term in Eq. (F.9) arises from the Hermitian part of the dielectric and is nondissipative.

where $\sigma_W \doteq k^2(1+k^2)$ is the weight factor associated with enstrophy conservation: $W_{\boldsymbol{k}} = \sigma_W C_{\boldsymbol{k}}$. Thus upon integration over $\boldsymbol{X}$ and summation over $\boldsymbol{k}$, Eq. (F.11) correctly conserves enstrophy whereas the classical equation (F.5) violates enstrophy conservation and incorrectly conserves C.

Equation (F.11) agrees with a result of Smolyakov and Diamond (1999). Those authors proceeded by working with a Fourier representation of the $\boldsymbol{X}$ dependence. The present method isolates more cleanly the source of the difficulty in the analysis leading to Eq. (F.2) (the failure to take into account all first-order terms), and formulas such as (F.8) are arguably cleaner and easier to apply in general situations.

G DERIVATION OF WAVE KINETIC EQUATION FROM RENORMALIZED SPECTRAL BALANCE

In this appendix I sketch the reduction of the Vlasov DIA to the wave kinetic equation of Vlasov WTT. As an example, the GK WKE of Sagdeev and Galeev (1969) is derived.

G.1 General form of the wave kinetic equation

The calculation begins with the exact spectral balance equation (286) or the nonstationary version of Eq. (344). With $I(t,t') \doteq \langle \delta\varphi(t)\delta\varphi(t')\rangle$ and $\star$ denoting convolution in time, one has

$$I(t,t') = (\mathcal{D}^{-1} \star \widehat{\Phi} g) \star F \star (\mathcal{D}^{-1} \star \widehat{\Phi} g)^{\dagger} = \mathcal{D}^{-1} \star \langle \delta\widetilde{\varphi}^2 \rangle \star \mathcal{D}^{-1\dagger}. \tag{G1a,b}$$

Here the incoherent potential spectrum is $\langle \delta\widetilde{\varphi}^2 \rangle \doteq \widehat{\Phi} g \star F \star (\widehat{\Phi} g)^{\dagger}$; spatial and velocity-space variables are not written explicitly. Initially, one need not commit oneself to specific forms for F and the nonlinear terms that define the renormalized particle propagator g and the renormalized dielectric function $\mathcal{D}$.

To deduce a wave kinetic equation, the first step is to operate on Eq. (G1b) on the left with $\mathcal{D}(t,t')$, thereby obtaining

$$\mathcal{D} \star I = \langle \delta\widetilde{\varphi}^2 \rangle \star \mathcal{D}^{-1\dagger}. \tag{G.2}$$

This rearrangement introduces an apparent asymmetry. To better appreciate the structure of Eq. (G.2), first consider the stationary, symmetrical model correlation function $C(\tau) = \mathrm{e}^{-\nu|\tau|} C_0$. The (two-sided) Fourier transform of this function is $C(\omega) = [2\nu/(\omega^2 + \nu^2)] C_0$, which can be written as

$$C(\omega) = R(\omega)(2\nu C_0) R^*(\omega), \quad R(\omega) \doteq [-\mathrm{i}(\omega + \mathrm{i}\nu)]^{-1}, \tag{G3a,b}$$

$$C(\tau) = R(\tau) \star [2\nu C_0 \delta(\tau)] \star R(-\tau), \quad R(\tau) = H(\tau)\mathrm{e}^{-\nu\tau}. \tag{G4a,b}$$

A differential equation for (the two-sided) $C(\tau)$ follows by applying $R^{-1}(\tau) \doteq (\partial_\tau + \nu)\delta(\tau)$ to Eq. (G4a):

$$\partial_\tau C + \nu C = H(-\tau) 2\nu C_0 \mathrm{e}^{\nu\tau}. \tag{G.5}$$

The solution of this equation is indeed the given $C(\tau)$. The left-hand and the right-hand sides of Eq. (G.5) are each asymmetrical, but in just such a way that symmetry of the solution is maintained. The role of the right-hand side is to make the effective differential equation be $\partial_\tau C + \operatorname{sgn}(\tau)\nu C = 0$ with $C(0) = C_0$. Note that although R is causal (one-sided), R^{-1} is not.

In Eq. (G1b) $\mathcal{D}^{-1}$ is causal, analogous to the R in the above model. Consider, then, the time dependence of Eq. (G.2) in detail. In order to treat a possible weak nonstationarity due to slow growth or damping, I follow the discussion of Appendix F (p. 286), introduce $\tau \doteq t - t'$ and $T \doteq \frac{1}{2}(t + t')$, and write, for example, $I(t, t') \equiv I(\tau \mid T)$. With $\overline{\tau} \doteq \overline{t} - t'$, one finds

$$\mathcal{D} \star I = \int \mathrm{d}\overline{\tau}\, \mathcal{D}(t - \overline{\tau} \mid T + \tfrac{1}{2}\overline{\tau}) I\big(\overline{\tau} \mid T - \tfrac{1}{2}(\tau - \overline{\tau})\big) \tag{G6a}$$

$$\approx \mathcal{D}(\tau \mid T) \star I(\tau \mid T) + \frac{1}{2}\left(\frac{\partial \mathcal{D}(\tau \mid T)}{\partial T} \star [\tau I(\tau \mid T)] - [\tau \mathcal{D}(\tau \mid T)] \star \frac{\partial I(\tau \mid T)}{\partial T}\right), \tag{G6b}$$

where one expanded for mean time T longer than the characteristic autocorrelation time of the spectrum. Convolutions are now only in τ.

The Fourier transform of the resulting equation is

$$\mathcal{D}(\omega) I(\omega) + \frac{1}{2}\mathrm{i}\left[\frac{\partial}{\partial T}\left(\frac{\partial \mathcal{D}}{\partial \omega} I\right) - \frac{\partial}{\partial \omega}\left(\frac{\partial \mathcal{D}}{\partial T} I\right)\right] (\langle \delta\widetilde{\varphi}^2 \rangle / \mathcal{D}^*)(\omega), \tag{G.7}$$

where all quantities depend parametrically on T and wave-number labels have been suppressed. Recall the considerations about stability in the first paragraph of Sec. 6.5.4 (p. 176). In the vicinity of a normal-mode resonance at complex frequency[294] $\widehat{\Omega}_{\boldsymbol{k}}$, one can expand $\mathcal{D}(\omega) \approx (\omega - \widehat{\Omega}_{\boldsymbol{k}})\partial\mathcal{D}/\partial\widehat{\Omega}_{\boldsymbol{k}}$. Consideration of the real part of Eq. (G.7) and the reality of $I(\omega)$ shows that to lowest order $I_{\boldsymbol{k}}(\omega) \approx 2\pi I_{\boldsymbol{k}} \delta(\omega - \Omega_{\boldsymbol{k}})$. Then the wave kinetic equation for $I_{\boldsymbol{k}}$ emerges by integrating the imaginary part of Eq. (G.7) over ω. For a more symmetrical notation, I now use $\omega_{\boldsymbol{k}}$ instead of ω and write $k \equiv \{\boldsymbol{k}, \omega_{\boldsymbol{k}}\}$. Note that

$$\operatorname{Im}(1/\mathcal{D}_k^*) \approx \pi\delta(\omega_{\boldsymbol{k}} - \Omega_{\boldsymbol{k}})/[s_{\boldsymbol{k}}|\mathcal{D}'_{\boldsymbol{k}}|], \tag{G.8}$$

where $s_{\boldsymbol{k}} \doteq \operatorname{sgn}(\mathcal{D}'_{\boldsymbol{k}})$, $\mathcal{D}'_{\boldsymbol{k}} \doteq \operatorname{Re}\,\big(\partial\mathcal{D}(\boldsymbol{k}, \Omega)/\partial\Omega\big)\big|_{\Omega = \Omega_{\boldsymbol{k}}}$, and one recalled that necessarily $\gamma_{\boldsymbol{k}} < 0$.[295] Let us define the *action density* of the $\boldsymbol{k}$th mode by[296] $\mathcal{N}_{\boldsymbol{k}} \doteq |\mathcal{D}'_{\boldsymbol{k}}| k^2 I_{\boldsymbol{k}}$, where $I_{\boldsymbol{k}} \doteq \langle \delta\varphi^2 \rangle_{\boldsymbol{k}}$. One finally finds

$$\frac{\partial \mathcal{N}_{\boldsymbol{k}}}{\partial T} - 2\gamma_{\boldsymbol{k}} \mathcal{N}_{\boldsymbol{k}} = \frac{\operatorname{Im} k^2 \langle \delta\widetilde{\varphi}^2 \rangle_{\boldsymbol{k}}(\Omega_{\boldsymbol{k}})}{|\mathcal{D}'_{\boldsymbol{k}}|}, \tag{G.9}$$

where $\gamma_{\boldsymbol{k}}$ is given by formula (170b) evaluated with the full dielectric $\mathcal{D}$ (including nonlinear corrections).

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[294] Elsewhere in the article, $\omega_{\boldsymbol{k}}$ has been used for complex frequency. I use the caret notation here [not to be confused with the linear-operator notation in Appendix F (p. 286)] in order that $\omega_{\boldsymbol{k}}$ can be used for the real Fourier frequency associated with $\boldsymbol{k}$.

[295] Establishing the sign of Eq. (G.8) unambiguously is possible only by working with (or at least having firmly in mind the implications of) the fully renormalized theory. If the linear growth rate were used here, the sign of the resonance contribution would (incorrectly) depend on the sign of $\gamma_{\boldsymbol{k}}^{\mathrm{lin}}$.

[296] A conventional denominator of 8π is omitted since it merely clutters the subsequent formulas.

So far I have assumed spatially homogeneous statistics. When weak spatial inhomogeneities are allowed, $\partial/\partial T$ is conventionally replaced by the total time derivative along the *ray trajectories*:

$$\frac{\mathrm{d}}{\mathrm{d}T} \doteq \frac{\partial}{\partial T} + \boldsymbol{V_X} \cdot \frac{\partial}{\partial \boldsymbol{X}} + \boldsymbol{V_k} \cdot \frac{\partial}{\partial \boldsymbol{k}}, \tag{G.10}$$

where

$$\boldsymbol{V_X} \doteq \left(\frac{\partial \Omega}{\partial \boldsymbol{k}}\right)\bigg|_{\boldsymbol{X}} = -\frac{\partial \mathcal{D}/\partial \boldsymbol{k}}{\partial \mathcal{D}/\partial \Omega}, \qquad \boldsymbol{V_k} \doteq -\left(\frac{\partial \Omega}{\partial \boldsymbol{X}}\right)\bigg|_{\boldsymbol{k}} = \frac{\partial \mathcal{D}/\partial \boldsymbol{X}}{\partial \mathcal{D}/\partial \Omega}. \tag{G11a,b,c,d}$$

However, as discussed in Appendix F (p. 286), this assumes that the function $\mathcal{D}_{\boldsymbol{k},\omega}(\boldsymbol{X}, T)$ is of zeroth order in the long-scale gradient; otherwise extra correction terms arise. It is not clear that those correction terms have been dealt with consistently in most applications of the wave kinetic equation.

G.2 Wave kinetic equation through second order

I shall now indicate the reduction of Eq. (G.9) to weak-turbulence theory, first in general, then for the specific example of HM dynamics. In principle the procedure can be carried to any order; however, I shall use the DIA forms of Σ^{nl} and F^{nl}, so the results will be correct only through second order in the intensity. That includes three-mode decay processes as well as the wave–wave–particle interactions.

For definiteness I consider the Vlasov nonlinearity $\boldsymbol{E} \cdot \boldsymbol{\partial} f$; however, with simple transcriptions [see Eqs. (G.18)] the formulas will remain valid for the $\boldsymbol{E} \times \boldsymbol{B}$ nonlinearity $\boldsymbol{V}_E \cdot \boldsymbol{\nabla} f$. Consider first the nonlinear noise, given in the DIA by Eq. (351). Upon Fourier transformation for spatially homogeneous and temporally stationary statistics, this becomes

$$F_k^{\mathrm{nl}}(1,2) = \sum_{k+p+q=0} [I_p \boldsymbol{p} \cdot \boldsymbol{\partial}_1\, \boldsymbol{p} \cdot \boldsymbol{\partial}_2 C_q(1,2) + \boldsymbol{q} \cdot \boldsymbol{\partial}_2 \langle \delta\varphi\, \delta f(2) \rangle_p \boldsymbol{p} \cdot \boldsymbol{\partial}_1 \langle \delta f(1) \delta\varphi \rangle_q]^*. \tag{G.12}$$

Here $\sum_{k+p+q=0} \equiv \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}} (2\pi)^{-1} \int \mathrm{d}\omega_{\boldsymbol{p}}\, \mathrm{d}\omega_{\boldsymbol{q}}\, \delta(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}})$. The goal is to express all quantities in terms of spectral intensities. In Sec. 6.5.3 (p. 173) it was argued that in perturbation theory the coherent response dominates; from Eq. (355a),

$$C_q(1,2) \approx [g_{0,q} \boldsymbol{q} \cdot \boldsymbol{\partial} \langle f \rangle](1) I_q [g_{0,q} \boldsymbol{q} \cdot \boldsymbol{\partial} \langle f \rangle]^*(2). \tag{G.13}$$

The analogous expression for $\langle \delta\varphi\, \delta f \rangle_q$ is best calculated directly from Eq. (355b). If it is calculated from Eq. (G.13), the result involves the product $[\mathcal{D}(\boldsymbol{q}, \omega_q) - 1] I_q$; the $\mathcal{D}$ contribution is absent from Eq. (355b), an apparent inconsistency. However, in weak-turbulence theory $\mathcal{D}(\boldsymbol{q}, \omega_q) I_q \approx (\omega_{\boldsymbol{q}} - \Omega_{\boldsymbol{q}} - \mathrm{i}\gamma_{\boldsymbol{q}}) 2\pi \delta(\omega_{\boldsymbol{q}} - \Omega_{\boldsymbol{q}}) I_{\boldsymbol{q}} \propto -\mathrm{i}\gamma_{\boldsymbol{q}}$; such dissipative contributions will be of higher order in $\gamma_{\boldsymbol{q}}/\Omega_{\boldsymbol{q}}$ and can be neglected. The final result is that

$$\langle \delta\widetilde{\varphi}^2 \rangle_k = \frac{1}{2} \sum_{k+p+q=0} |\epsilon^{(2)}(k,p,q)|^2 I_p I_q. \tag{G.14}$$

It is now straightforward to perform all frequency integrations, thereby finding that the mode-coupling contribution to the wave kinetic equation is

$$\frac{1}{2}\left(\frac{\partial\widehat{\mathcal{N}}_{\boldsymbol{k}}}{\partial T}\right)^{\mathrm{mc}} = \frac{1}{2}\sum_{\Delta}\frac{|\epsilon^{(2)}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})|^2}{|\mathcal{D}'_{\boldsymbol{k}}||\mathcal{D}'_{\boldsymbol{p}}||\mathcal{D}'_{\boldsymbol{q}}|}\theta_{\boldsymbol{kpq}}\widehat{\mathcal{N}}_{\boldsymbol{p}}\widehat{\mathcal{N}}_{\boldsymbol{q}}, \tag{G.15}$$

where $\widehat{\mathcal{N}}_{\boldsymbol{k}} \doteq |\mathcal{D}'_{\boldsymbol{k}}|I_{\boldsymbol{k}}$ and the triad interaction time θ takes the weak-turbulence form (183c), $\theta_{\boldsymbol{kpq}} \approx \pi\delta(\Omega_{\boldsymbol{k}}+\Omega_{\boldsymbol{p}}+\Omega_{\boldsymbol{q}})$. The notation $\epsilon^{(2)}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ means $\epsilon^{(2)}(\boldsymbol{k},\Omega_{\boldsymbol{k}};\boldsymbol{p},\Omega_{\boldsymbol{p}};\boldsymbol{q},\Omega_{\boldsymbol{q}})$.

I now consider nonlinear contributions to the growth rate $\gamma_{\boldsymbol{k}} \approx -\operatorname{Im}[\mathcal{D}(\boldsymbol{k},\Omega_{\boldsymbol{k}})]/\mathcal{D}'_{\boldsymbol{k}}$. The result of tedious calculations leads to the weak-turbulence dielectric quoted in Eq. (190a). In order to further reduce the imaginary part of that formula, one must consider two possibilities: either the beat wave $\omega_{\boldsymbol{p}} = -(\Omega_{\boldsymbol{k}}+\Omega_{\boldsymbol{q}})$ is a normal mode $\Omega_{\boldsymbol{p}}$; or it is not. If it is, then the resonance approximation (G.8) can be employed in the second term of Eq. (190a) and one obtains the contribution

$$\gamma_{\boldsymbol{k}}^{\mathrm{mc}} = \sum_{\Delta} s_{\boldsymbol{k}}s_{\boldsymbol{p}}\frac{\operatorname{Re}\big(\epsilon^{(2)}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\epsilon^{(2)*}(\boldsymbol{p},\boldsymbol{q},\boldsymbol{k})\big)}{|\mathcal{D}'_{\boldsymbol{k}}||\mathcal{D}'_{\boldsymbol{p}}||\mathcal{D}'_{\boldsymbol{q}}|}\theta_{\boldsymbol{kpq}}\widehat{\mathcal{N}}_{\boldsymbol{q}}. \tag{G.16}$$

The term $\gamma_{\boldsymbol{k}}^{\mathrm{mc}}\widehat{\mathcal{N}}_{\boldsymbol{k}}$ obviously describes the process inverse to the three-wave decay term (G.15). If a modal energy is defined by $\mathcal{E} = \sum_{\boldsymbol{k}}\sigma_{\boldsymbol{k}}|\mathcal{D}'_{\boldsymbol{k}}|\widehat{\mathcal{N}}_{\boldsymbol{k}}$, it is easy to see that the mode-coupling terms conserve energy provided that $s_{\boldsymbol{k}}\sigma_{\boldsymbol{k}}\epsilon^{(2)}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) + \text{c.p.} = 0$. That will be true if only nonresonant contributions to $\epsilon^{(2)}$ are retained; see the GK example in the next section.

For the case in which the beat wave is a driven non-normal mode, imaginary contributions to Eq. (190a) stem dominantly from $\operatorname{Re} g_{0,\boldsymbol{p}} \approx \pi\delta(\omega_{\boldsymbol{p}} - \boldsymbol{p}\cdot\boldsymbol{v})$; this describes *induced scattering*. Thus one finds

$$\begin{aligned}\gamma_{\boldsymbol{k}}^{\mathrm{ind}} = -s_{\boldsymbol{k}}\operatorname{Im}\sum_{\Delta}\int\frac{\mathrm{d}\omega_{\boldsymbol{p}}}{2\pi}\,\delta(\Omega_{\boldsymbol{k}}+\omega_{\boldsymbol{p}}+\Omega_{\boldsymbol{q}})(|\mathcal{D}'_{\boldsymbol{k}}||\mathcal{D}'_{\boldsymbol{q}}|)^{-1}\\ \times[\epsilon^{(3)}(\boldsymbol{k}\mid\boldsymbol{q},-\boldsymbol{q},-\boldsymbol{k}) - \epsilon^{(2)}(\boldsymbol{k}\mid p,\boldsymbol{q})\operatorname{Re}(\mathcal{D}_p^*)^{-1}\epsilon^{(2)*}(p\mid\boldsymbol{q},\boldsymbol{k})]\widehat{\mathcal{N}}_{\boldsymbol{q}}.\end{aligned} \tag{G.17}$$

For some general manipulations of this form, see Horton and Choi (1979). A specific illustration is given in the next section.

Upon assembling all contributions, one is led to the WKE in the form (191). See Sec. 6.5.4 (p. 176) for important further discussion of its content.

G.3 Example: Wave kinetic equation for drift waves

As an example of the general formulas of the previous sections, consider the GKE for drift waves, in which the background PDF is assumed to be Maxwellian with a slowly varying density gradient. It is straightforward to find that one can use the previous Vlasov results provided one makes the transcriptions

$$\boldsymbol{k}\cdot\boldsymbol{\partial}\langle f\rangle \to -[k_\parallel v_\parallel - \omega_*(\boldsymbol{k})]J_0(\boldsymbol{k})(q/T)\langle f\rangle, \quad \boldsymbol{p}\cdot\boldsymbol{\partial}a_{\boldsymbol{q}} \to \mathrm{i}J_0(\boldsymbol{p})(c/B)\widehat{\boldsymbol{b}}\cdot\boldsymbol{p}\times\boldsymbol{q}a_{\boldsymbol{q}}, \tag{G18a,b}$$

where $a_{\boldsymbol{q}}$ is any fluctuation-related quantity and $J_0(\boldsymbol{k}) \equiv J_0(k_\perp v_\perp/\omega_c)$. One could proceed symmetrically for electrons and ions, but I shall consider here the common approximation of adiabatic electrons, for which only the ions are treated kinetically and the potential operator (for the normalized potential $\varphi = e\phi/T_e$) becomes $\widehat{\Phi}_{\boldsymbol{k}} = (1+\overline{k}^2)^{-1}\int d\boldsymbol{v}\, J_0(\boldsymbol{k})$, where $\overline{k}^2 \doteq \tau[1-\Gamma(\boldsymbol{k})]$, $\tau \doteq T_e/T_i$, and $\Gamma(\boldsymbol{k}) \doteq \langle J_0^2(\boldsymbol{k})\rangle_\perp = I_0(b)\mathrm{e}^{-b}$ (the $\perp$ average being over a perpendicular Maxwellian). Here the $(1+\overline{k}^2)^{-1}$ multiplier describes the effects of the ion polarization drift; in the limit $T_i \to 0$, $1+\tau(1-\Gamma) \to 1+k^2$ (independent of T_i). Thus the k^2 term is not a finite-Larmor-radius correction.

In the drift-wave problem one orders $\Omega_{\boldsymbol{k}} \gg k_\parallel v_\parallel$. Then

$$\mathcal{D}(\boldsymbol{k},\omega) \approx 1-(1+\overline{k}^2)^{-1}[\omega_*(\boldsymbol{k})\Gamma(\boldsymbol{k})/\omega], \quad \text{so} \quad \Omega_{\boldsymbol{k}} = \omega_*(\boldsymbol{k})\Gamma(\boldsymbol{k})/(1+\overline{k}^2). \tag{G19a,b}$$

It is straightforward to extract the nonresonant portion of $\epsilon^{(2)}(\boldsymbol{k}\mid\boldsymbol{p},\boldsymbol{q})$. From formula (188a) and $m_{\boldsymbol{k}} \doteq (c/B)\widehat{\boldsymbol{b}}\cdot\boldsymbol{p}\times\boldsymbol{q}$, one finds $\mathrm{Re}[\epsilon^{(2)}(\boldsymbol{k}\mid\boldsymbol{p},\boldsymbol{q})] = -\mathrm{i}M_{\boldsymbol{kpq}}/\Omega_{\boldsymbol{k}}$, where

$$M_{\boldsymbol{kpq}} \doteq \frac{m_{\boldsymbol{k}}\langle J_0(\boldsymbol{k})J_0(\boldsymbol{p})J_0(\boldsymbol{q})\rangle_\perp}{1+\overline{k}^2}\left(\frac{\omega_*(\boldsymbol{q})}{\Omega_{\boldsymbol{q}}}-\frac{\omega_*(\boldsymbol{p})}{\Omega_{\boldsymbol{q}}}\right); \tag{G.20}$$

as $T_i \to 0$, $M_{\boldsymbol{kpq}}$ properly reduces to the HM form $m_{\boldsymbol{k}}(q^2-p^2)/(1+k^2)$. From Eq. (G19b), note that $\Omega_{\boldsymbol{k}}\mathcal{D}'_{\boldsymbol{k}} = 1$. Then Eq. (G.15) becomes

$$\frac{1}{2}\left(\frac{\partial I_{\boldsymbol{k}}}{\partial T}\right)^{\mathrm{mc}} = \frac{1}{2}\sum_\Delta |M_{\boldsymbol{kpq}}|^2\theta_{\boldsymbol{kpq}}I_{\boldsymbol{p}}I_{\boldsymbol{q}}, \tag{G.21}$$

precisely the fluid form (183b) of the HM incoherent noise. Similarly, $-\gamma_{\boldsymbol{k}}^{\mathrm{mc}}$ reduces to the fluid expression (183a) for the nonlinear damping $\eta_{\boldsymbol{k}}^{\mathrm{nl}}$. One can verify that the GK mode-coupling terms conserve an energy $\mathcal{E} \propto \sum_{\boldsymbol{k}}(1+\overline{k}^2)\Gamma^{-1}(\boldsymbol{k})I_{\boldsymbol{k}}$ as well as an enstrophy (with an additional weighting of $\overline{k}^2$). If one introduces the GK action density [297] $\mathcal{N}_{\boldsymbol{k}} \doteq (1+\overline{k}^2)\Gamma^{-1}(\boldsymbol{k})\widehat{\mathcal{N}}_{\boldsymbol{k}}$, then one obtains the conventional quantum-mechanical form $\mathcal{E} = \sum_{\boldsymbol{k}}\Omega_{\boldsymbol{k}}\mathcal{N}_{\boldsymbol{k}}$.

Reduction of $\gamma_{\boldsymbol{k}}^{\mathrm{ind}}$ proceeds similarly although with somewhat more algebra. The final result, which should be compared with that of Sagdeev and Galeev (1969), is

$$\gamma_{\boldsymbol{k}}^{\mathrm{ind}} = -(1+\overline{k}^2)^{-1}\sum_\Delta \delta(\Omega_{\boldsymbol{k}}+\Omega_{\boldsymbol{q}})m_{\boldsymbol{k}}^2F(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\left(\frac{\omega_*(\boldsymbol{k})}{\Omega_{\boldsymbol{k}}}-\frac{\omega_*(\boldsymbol{q})}{\Omega_{\boldsymbol{q}}}\right)\frac{\Gamma(\boldsymbol{q})}{1+\overline{q}^2}\mathcal{N}_{\boldsymbol{q}}, \tag{G22a}$$

$$F(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) \doteq \langle J_0^2(\boldsymbol{k})J_0^2(\boldsymbol{q})\rangle_\perp - \Gamma(\boldsymbol{p})^{-1}\langle J_0(\boldsymbol{k})J_0(\boldsymbol{p})J_0(\boldsymbol{q})\rangle_\perp^2. \tag{G22b}$$

It can be seen from the antisymmetric form of Eq. (G22a) that the induced scattering properly conserves $\sum_{\boldsymbol{k}}\mathcal{N}_{\boldsymbol{k}}$. Also note that a Schwartz inequality guarantees $F \geq 0$. As $T_i \to 0$, $F \to 0$ as $k_\perp^2\rho_i^2$. This behavior represents the characteristic long-wavelength cancellation between Compton scattering from bare particles and nonlinear scattering from the shielding clouds.

[297] That this is the appropriate definition of GK action follows from considerations of GK energy conservation (Dubin et al., 1983). In the absence of the 1 term (adiabatic electron response), one has $\overline{k}^2\widehat{\mathcal{N}}_{\boldsymbol{k}} = (\overline{k}^2/k^2)k^2\widehat{\mathcal{N}}_{\boldsymbol{k}} = \epsilon_\perp(\boldsymbol{k})|\mathcal{D}'_{\boldsymbol{k}}|k^2I_{\boldsymbol{k}}$; see the discussion by Krommes (1993c) of $\epsilon_\perp$ as the permittivity of the GK vacuum.

H PROBABILITY DENSITY FUNCTIONALS; GAUSSIAN INTEGRATION; WHITE-NOISE ADVECTIVE NONLINEARITY

Here I collect a few miscellaneous results about functional integration and Gaussian statistics, including the interpretation of probability density functionals, Gaussian integration (Novikov's theorem), and the treatment of white-noise advective nonlinearity.

H.1 Probability density functions and functional integration

A general probability density functional of a field $\psi(t)$ is interpreted as $P[\psi] = \lim_{N\to\infty} P(\psi_{-N}, \ldots, \psi_{-1}, \psi_0, \psi_1, \ldots, \psi_N)$, where $\psi_n \doteq \psi(t_n)$ and $t_n \doteq n\,\Delta t$. The statistical average of an arbitrary functional $\mathcal{F}[\psi]$ is written $\langle \mathcal{F}[\psi]\rangle = \int \mathrm{D}[\psi]\,\mathcal{F}[\psi]P[\psi]$, where $D[\psi] \doteq \prod_n \mathrm{d}\psi_n$. For example, the PDF of a multivariate centered Gaussian field $\psi(t)$ has the form [298]

$$P[\psi] = [\det(2\pi C)]^{-1/2} \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}t'\,\psi(t)C^{-1}(t,t')\psi(t')\right), \tag{H.1}$$

where $C(t,t')$ is a specified two-time correlation function and C^{-1} obeys $\int_{-\infty}^{\infty}\mathrm{d}\bar{t}\,C(t,\bar{t})C^{-1}(\bar{t},t') = \delta(t-t')$. Under a discretization in which $C(t,t') \to \mathsf{C} \equiv C_{ij}$ and $\delta(t-t') \to \Delta t^{-1}\delta_{i,j}$, one has $C^{-1}(t,t') \to \Delta t^{-2}C^{-1}_{ij}$. Because C is symmetric and positive definite, it can be diagonalized with positive eigenvalues λ_i. Now $\det(2\pi\mathsf{C}) = \prod_i(2\pi\lambda_i)$ is formally infinite as $\Delta t \to 0$; however, that infinity will be canceled by the time integration over the infinite number of variables. For example, as a special case of Eq. (H.1), the PDF of Gaussian white noise with covariance $C(\tau) = 2D\delta(\tau)$ is $P[\psi] \propto \exp[-(4D)^{-1}\int_{-\infty}^{\infty}\mathrm{d}t\,\psi^2(t)]$; the argument of the exponential is clearly infinite, the size of the infinity scaling with the total integration time.

As a consistency check, one may compute $\langle\psi(t)\psi(t')\rangle_{ij} = \int \mathrm{D}[\psi]\,\psi(t)\psi(t')P[\psi]$. Upon introducing the square root S of C such that $\mathsf{C} = \mathsf{S}\cdot\mathsf{S}$, one may change variables to $\overline{\psi} \doteq \mathsf{S}^{-1}\cdot\psi$, whereupon

$$\langle\psi(t)\psi(t')\rangle = \int \mathrm{d}\overline{\psi}_1\ldots\mathrm{d}\overline{\psi}_N\,\det\mathsf{S}\,(S_{ik}\overline{\psi}_k)(S_{jl}\overline{\psi}_l)(2\pi)^{-N/2}(\det\mathsf{S})^{-1/2}\exp\left(-\frac{1}{2}\sum_{i=1}^{N}\overline{\psi}_i^2\right) \tag{H2a}$$

$$= S_{ik}S_{jl}\delta_{k,l} = C_{ij}. \tag{H2b}$$

For non-Gaussian functionals, explicit manipulations like this are rarely possible. Nevertheless, the mere existence of such functionals can be used to good advantage. Both Gaussian and non-Gaussian functionals have central roles in quantum field theory (Zinn-Justin, 1996). For further discussion, see the description of the MSR formalism in Sec. 6 (p. 146).

[298] An explicit reference to such a Gaussian functional was made by Taylor and McNamara (1971); see their Eq. (15).

H.2 Gaussian integration

For a centered Gaussian variable x with variance σ^2, and given an arbitrary function $F(x)$, one has

$$\langle x\, F(x)\rangle = \sigma^2 \left\langle \frac{\partial F}{\partial x} \right\rangle. \tag{H.3}$$

This result follows by an integration by parts upon noting that for a centered Gaussian distribution $P(x)$ one has $\partial_x \ln P = -x/\sigma^2$.[299] For a multivariate Gaussian PDF with correlation matrix C, the result (H.3) generalizes to

$$\langle x_i F(\boldsymbol{x})\rangle = \sum_j C_{ij} \left\langle \frac{\partial F}{\partial x_j} \right\rangle. \tag{H.4}$$

Finally, if F is a functional of a Gaussian time series $x(t)$, namely, $F = F[x]$, then

$$\langle x(t) F[x]\rangle = \int_{-\infty}^{\infty} \mathrm{d}\bar{t}\, C(t,\bar{t}) \left\langle \frac{\delta F}{\delta x(\bar{t})} \right\rangle. \tag{H.5}$$

This result is called *Novikov's theorem* (Novikov, 1964) and is frequently referred to as *Gaussian integration*. It can be used to good advantage in problems of passive advection with Gaussian coefficients; see the special case discussed in the next section and the more complicated applications treated by Schekochihin (2001).

H.3 White-noise advective nonlinearity

Consider an advective nonlinearity with a Gaussian coefficient that is white in time with zero mean—see, for example, the stochastic acceleration problem $\partial_t \psi + \widetilde{a}(t)\partial_v \psi = 0$, where $\langle \widetilde{a}(t)\widetilde{a}(t')\rangle = 2D_v \delta(t-t')$. For this limit it is known (Kraichnan, 1968b) that first-order perturbation theory is exact, namely, $\partial_t \langle \psi \rangle = \partial_v D_v \partial_v \langle \psi \rangle$. This can be demonstrated directly, and also follows from a straightforward application of Novikov's theorem (H.5); it can be viewed as a special case of Fokker–Planck theory. However, one can also employ the results for multiplicative passive statistics obtained in Sec. 6.4 (p. 166). This advective nonlinearity corresponds to the random vertex $U_2(1,\overline{1}) = \widetilde{a}(t_1)\partial_v(1-\overline{1})$, which after averaging over the Gaussian statistics enters the field equations as the spurious (symmetrical) vertex γ_4' with nonvanishing components

$$\gamma_4' \begin{pmatrix} 1 & 2 & \overline{1} & \overline{2} \\ - & - & + & + \end{pmatrix} = 2D_v(1,2)\delta(t_1-t_2)\frac{\partial}{\partial v_1}\delta(1-\overline{1})\frac{\partial}{\partial v_2}\delta(2-\overline{2}). \tag{H.6}$$

Of the three distinct terms in the contribution $\frac{1}{6}\gamma_4'(\langle\varphi\rangle\langle\varphi\rangle\langle\varphi\rangle + 3\langle\varphi\rangle\langle\langle\varphi\varphi\rangle\rangle + \langle\langle\varphi\varphi\varphi\rangle\rangle)$ to $-\mathrm{i}\sigma\dot{\varphi}$ [see Eq. (297); one must take the $-$ component to obtain $\dot{\psi}$], the first vanishes because $\langle\widehat{\psi}\rangle = 0$, and the last vanishes because $G_{3,-++}(t,t,t) = 0$. The second has a nonvanishing contribution from

[299] When F is a polynomial function of x, Eq. (H.3) can be verified directly by using the Gaussian result $\langle x^{2n}\rangle = (2n-1)!!$.

$\langle\varphi\rangle_+ G_{+-} = \langle\psi\rangle R$. That simplifies because the time locality reduces R to a delta function in the nontime coordinates at $t = t'$; one finally finds

$$\frac{\partial\langle\psi\rangle}{\partial t} + \cdots = \frac{\partial}{\partial v} D_v(v,v) \frac{\partial\langle\psi\rangle}{\partial v}. \tag{H.7}$$

This result is unambiguous when D_v is independent of v (or, in higher dimensions, when $\widetilde{\boldsymbol{a}}$ is divergence-free). Otherwise, one must worry about the Itô vs Stratonovich controversy (van Kampen, 1981), in which it is noted that the form of the Fokker–Planck equation that results from a Langevin equation of the form $\dot{v} = F(v)\delta a(t)$ is ambiguous unless one gives a definite prescription about how to finite-difference the time evolution.

I THE `DIA` CODE

"The rationale must be that studies of the DIA directly aid one's theoretical understanding of the various physical mechanisms. In view of the complexity of the [drift-wave] physics, ... such studies are absolutely essential." — *Krommes (1984b).*

Both the Eulerian DIA and the Markovian closures discussed in this review are causal initial-value problems, all of which can be accommodated by a common numerical superstructure. One such realization is the code `DIA` developed at Princeton University by the author and his collaborators for the numerical solution of various statistical closure problems. It has evolved from a FORTRAN version that was soon abandoned as being unmanageable, through a mixed C and RATFOR[300] implementation that has been used for most of the published results (Bowman et al., 1993; Bowman and Krommes, 1997; Hu et al., 1995, 1997), to its present realization in C++. An early collaborator was R. A. Smith. Bowman (1992) was principally responsible for the implementation of the Markovian closures; his careful and extensive work can justifiably be described as heroic.

In Fig. I.1 (p. 296) I show a slightly simplified C++ class hierarchy for the `DIA` code. The notation $A \leftarrow B$ means B is derived from A (A is a base class for B). The structure is relatively standard and emulates that of many simulation codes. The code employs a second-order predictor–corrector algorithm (Kraichnan, 1964b), which for the equation $\partial_t\psi = S(\psi(t), t)$ is

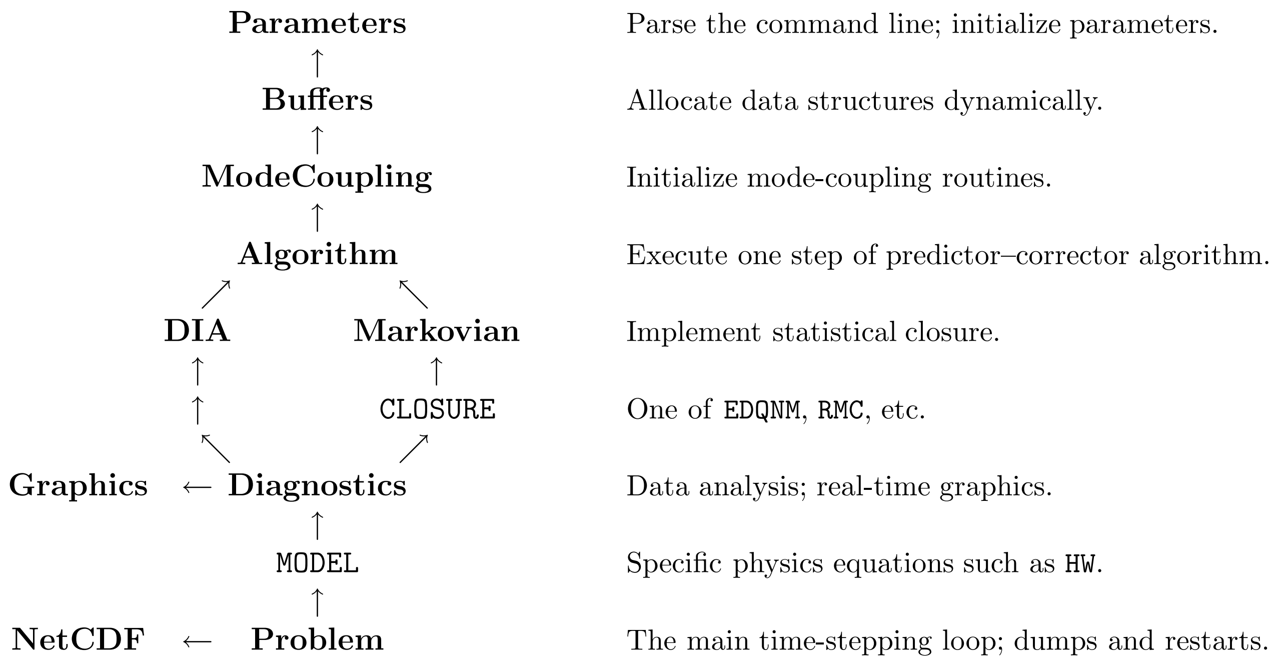
$$\widehat{\psi}(t+\Delta t) = \psi(t) + \Delta t\, S(\psi(t), t), \tag{I1a}$$

$$\psi(t+\Delta t) = \psi(t) + \tfrac{1}{2}\Delta t[S(\psi(t), t) + S(\widehat{\psi}(t+\Delta t), t+\Delta t)]. \tag{I1b}$$

Note that the DIA and Markovian closures are accommodated on equal footing in the inheritance tree; they are both advanced by the same time-stepping algorithm and, broadly speaking, can use the same data structures. (Additional data structures are required for the Markovian closures because there is no analog of the triad interaction time $\theta_{\boldsymbol{kpq}}$ in the DIA.) Information about a particular physics model is inserted in place of **class** `MODEL`. That includes both information about the equations as well as any model-specific diagnostic and/or graphics routines.

Both real-time as well as off-line graphics are used. The real-time graphics software manages the main integration loop, after each step of which on-screen graphics are updated. Data for postprocessing are written to files in the conventional manner.

[300] RATFOR (RATional FORtran) was introduced by Kernighan and Plauger (1976).



main( )
{
static Problem *problem*; // Realize the class hierarchy.
GraphicsStart( ); // Initialize real-time graphics and enter time-stepping loop.
}

Fig. I.1. Class hierarchy of the `DIA` code. Names in boldface type represent C++ classes; for example, **Parameters** ≡ **class Parameters**. The entirely uppercase class names in typewriter type are replaced by specific instances when the code is built—as examples, `CLOSURE` might be replaced by **RMC** (a particular Markovian closure), and `MODEL` might be replaced by **HW** (a particular set of equations, those of Hasegawa and Wakatani).

Integrated source code and documentation are managed by the `FWEB` utility (Krommes, 1993e), a nontrivial extension of Levy's `CWEB`,[301] which in turn was based on Knuth's Pascal-based `WEB` (Knuth, 1992) originally developed to maintain the computer typesetting language TeX. Some virtues of `WEB` utilities in general are book-quality documentation (including an automatically generated index) and the ability to modularize the code at a sub-function level without impacting performance. A feature unique to `FWEB` and extensively used in `DIA` is the built-in macro preprocessor (a generalization of the C preprocessor). Macros are used to build specialized versions of the code from a common general template. Thus for example, in addition to time t a general field ψ might depend on statistically homogeneous variables $\boldsymbol{x}$, inhomogeneous variables $\boldsymbol{v}$, and a discrete field index s; the two-point correlation function C would then be indexed by $C_{s,s'}(\boldsymbol{k}, \boldsymbol{v}, t; \boldsymbol{v}', t')$, where a Fourier transformation was performed with respect to $\boldsymbol{x} - \boldsymbol{x}'$. In general, loops over all of the indices must be done, which implies that performance can be noticeably improved when loops over unused variables are eliminated at the compile stage. Thus a single-field Hasegawa–Mima problem possesses a correlation function $C(k_x, k_y, t; t')$, and macro preprocessing can be used to generate loops that refer only to k_x and k_y.

[301] For references to `CWEB`, see the bibliography in Knuth (1992).

J THE EDQNM FOR THREE COUPLED MODES

For precisely three modes labeled by the distinct indices K, P, and Q, the steady-state solution of the EDQNM can be found explicitly, as shown by Ottaviani (1991) [see also Bowman (1992)]. The details of the calculation are presented here. The results provide nontrivial test cases for statistical closure codes [see, for example, Sec. I (p. 295)]. A number of useful insights about growth-rate scalings and parameter regimes also follow from generalizations of the solution.

Consider the generic amplitude equation (223) for three modes coupled by the *real* coefficients $M_{\boldsymbol{k}} \equiv M_{\boldsymbol{kpq}}$, etc. (A $\boldsymbol{k}$-independent phase can be absorbed into a redefinition of the ψ_k's.) One may also write $\theta \doteq \theta_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ since there is only one triad interaction time for just three modes, and similarly introduce the frequency mismatch $\Delta\Omega$ and the characteristic forcing $\Delta\gamma$; I define $\theta_r \equiv \operatorname{Re}\theta$ and $\theta_i \equiv \operatorname{Im}\theta$.

Upon taking account of the appropriate sums over modes, one finds from Eqs. (368) that

$$F_{\boldsymbol{k}} - (\operatorname{Re}\eta^{\mathrm{nl}}_{\boldsymbol{k}})C_{\boldsymbol{k}} = -M_{\boldsymbol{k}}A, \quad A \doteq -\theta_r(M_{\boldsymbol{k}}C_{\boldsymbol{p}}C_{\boldsymbol{q}} + \text{c.p.}). \tag{J1a,b}$$

The steady-state balance condition is then

$$-2\gamma_{\boldsymbol{k}}C_{\boldsymbol{k}} = -2M_{\boldsymbol{k}}A, \quad \text{or} \quad \frac{\gamma_{\boldsymbol{k}}C_{\boldsymbol{k}}}{M_{\boldsymbol{k}}} = \frac{\gamma_{\boldsymbol{p}}C_{\boldsymbol{p}}}{M_{\boldsymbol{p}}} = \frac{\gamma_{\boldsymbol{q}}C_{\boldsymbol{q}}}{M_{\boldsymbol{q}}} = A. \tag{J2a,b}$$

For nonzero γ's this allows one to eliminate $C_{\boldsymbol{p}}$ and $C_{\boldsymbol{q}}$ in terms of $C_{\boldsymbol{k}}$. (The Gibbsian thermal-equilibrium solutions $A = 0$ are lost at this point.) The results may be substituted into Eq. (J1b), then combined with Eq. (J2a), yielding the two solutions $C_{\boldsymbol{k}} = 0$ (which one assumes is unstable) and

$$C_{\boldsymbol{k}} = -\left(\frac{\gamma_{\boldsymbol{p}}\gamma_{\boldsymbol{q}}}{M_{\boldsymbol{p}}M_{\boldsymbol{q}}}\right)\left(\frac{1}{\theta_r\Delta\gamma}\right). \tag{J.3}$$

The solution is completed by determining θ from the steady-state solution (372):

$$\theta = (\mathrm{i}\Delta\Omega - \Delta\gamma + \Delta\eta^{\mathrm{nl}})^{-1} = \frac{-\mathrm{i}(\Delta\Omega + \Delta\eta^{\mathrm{nl}}_i) + (\Delta\eta^{\mathrm{nl}}_r - \Delta\gamma)}{(\Delta\Omega + \Delta\eta^{\mathrm{nl}}_i)^2 + (\Delta\eta^{\mathrm{nl}}_r - \Delta\gamma)^2}. \tag{J4a,b}$$

This result fixes the ratio

$$\iota \doteq \frac{\theta_i}{\theta_r} = -\left(\frac{\Delta\Omega + \Delta\eta^{\mathrm{nl}}_i}{\Delta\eta^{\mathrm{nl}}_r - \Delta\gamma}\right). \tag{J.5}$$

Upon inserting Eqs. (J2b) and (J.3) into the expression (368a) for $\eta^{\mathrm{nl}}_{\boldsymbol{k}}$, one finds

$$\Delta\eta^{\mathrm{nl}} = \alpha(1 - \mathrm{i}\iota), \quad \text{where} \quad \alpha \doteq 2(\gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{p}}\gamma_{\boldsymbol{q}})\left(\frac{\Delta(\gamma^{-1})}{\Delta\gamma}\right). \tag{J6a,b}$$

Upon inserting the real and imaginary parts of Eq. (J6a) into Eq. (J.5), one may solve for ι, finding

$$\iota = \Delta\Omega/\Delta\gamma. \tag{J.7}$$

All quantities can now be determined. One obtains[302]

$$\theta_r = -\left(\frac{(\Delta\gamma)^2}{\Delta(\gamma^2)}\right)\left(\frac{\Delta\gamma}{(\Delta\Omega)^2 + (\Delta\gamma)^2}\right), \quad C_{\boldsymbol{k}} = \left(\frac{\gamma_{\boldsymbol{p}}\gamma_{\boldsymbol{q}}}{M_{\boldsymbol{p}}M_{\boldsymbol{q}}}\right)\left[\left(\frac{\Delta(\gamma^2)}{(\Delta\gamma)^2}\right)\left(\frac{(\Delta\Omega)^2 + (\Delta\gamma)^2}{(\Delta\gamma)^2}\right)\right]. \tag{J8a,b}$$

One can see that the saturation level is controlled by the dimensionless parameter $\Delta\gamma/\Delta\Omega$ as well as by three independent combinations of the γ's such as $\Delta\gamma$, $\Delta(\gamma^2)$, and $\Delta(\gamma^{-1})$. Fundamentally, $\Delta\gamma$ is the forcing parameter.

The forms of these results demonstrate some constraints on the γ's. For example, if θ_r is to be positive, one must have $\Delta\gamma < 0$. [According to the theory of entropy evolution described in Sec. 7.2.3 (p. 188), this is equivalent to the requirement that a nonequilibrium steady state exists.] One must also have $\gamma_{\boldsymbol{p}}\gamma_{\boldsymbol{q}}/M_{\boldsymbol{p}}M_{\boldsymbol{q}} > 0$.

The saturation level summed over all modes is $\mathcal{E} \doteq \frac{1}{2}\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}$, where $\sigma_{\boldsymbol{k}}$ is a positive weighting coefficient that is determined by the conservation properties of the original nonlinear equation; it is assumed that the mode-coupling coefficients obey[303] $\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} M_{\boldsymbol{k}} = 0$. The symmetries inherent in the closure lead to the spectral balance

$$\frac{\partial \mathcal{E}}{\partial t} = \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} \gamma_{\boldsymbol{k}} C_{\boldsymbol{k}}; \tag{J.9}$$

this also follows from the primitive equation. It is readily verified that the result (J8b) consistently obeys $\sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} \gamma_{\boldsymbol{k}} C_{\boldsymbol{k}} = 0$. As another independent consistency check, one may verify that the right-hand side of the entropy evolution equation (383) vanishes when the formulas (J8a) and (J8b) are inserted.[304]

If one defines a characteristic mode-coupling coefficient M and a characteristic frequency W by

$$\frac{1}{M^2} \doteq \sum_{\text{c.p.}} \frac{1}{M_{\boldsymbol{p}}M_{\boldsymbol{q}}}, \quad W^2 \doteq M^2 \sum_{\text{c.p.}} \sigma_{\boldsymbol{k}} \left(\frac{\gamma_{\boldsymbol{p}}\gamma_{\boldsymbol{q}}}{M_{\boldsymbol{p}}M_{\boldsymbol{q}}}\right), \tag{J10a,b}$$

then one has

$$\mathcal{E} = \left(\frac{W^2}{M^2}\right)\left(\frac{\Delta(\gamma^2)}{(\Delta\gamma)^2}\right)\left(\frac{(\Delta\Omega)^2 + (\Delta\gamma)^2}{(\Delta\gamma)^2}\right). \tag{J.11}$$

One may define the strong-turbulence limit by $\Delta\gamma/\Delta\Omega \gg 1$. For strong turbulence,

$$\mathcal{E} \to \left(\frac{\Delta(\gamma^2)}{(\Delta\gamma)^2}\right)\left(\frac{W^2}{M^2}\right). \tag{J.12}$$

[302] Exact steady-state solutions of the three-wave model can also be found (Johnston, 1989; Bowman, 1992). The result is identical to Eq. (J8b) without the factor $(\Delta\gamma)^2/\Delta(\gamma^2)$. For further discussion, see Bowman (1992).

[303] For appropriate M's one can satisfy this for two independent sets of σ's, corresponding to the conservation of both energy and enstrophy by the nonlinear terms.

[304] The proof uses the result that $\sum_k \sum_{q\neq k} \sum_p \delta_{k+p+q=0} = 3 \times 2 \times 1 = 6$.

If one lets all growth rates scale with a common parameter γ, then $W \sim \gamma$ and in the strong-turbulence limit

$$\mathcal{E} \sim \gamma^2/M^2. \tag{J.13}$$

This quadratic scaling is in accord with the argument given in Sec. 3.8.2 (p. 73) that equates the rate of forcing with the eddy turnover rate for the energy-containing modes:

$$\gamma \sim k\overline{V}_E \sim M\psi \tag{J.14}$$

($M \propto k_\perp^2$).

For $\Delta\gamma/\Delta\Omega \ll 1$ one has

$$\mathcal{E} \to \mathcal{E}^{\mathrm{ml}} \doteq \left(\frac{(\Delta\Omega)^2}{(\Delta\gamma)^2}\right)\left(\frac{W^2}{M^2}\right) \sim \frac{(\Delta\Omega)^2}{M^2}. \tag{J.15}$$

This is the mixing-length level for this problem. Note that this quantity formally remains finite as $\gamma \to 0$. If this problem were describing drift waves, the mixing-length level would correspond to an rms potential

$$\psi \doteq \sqrt{\mathcal{E}^{\mathrm{ml}}} \sim \frac{\Delta\Omega}{M} \sim \frac{\omega_*}{k_\perp^2} \sim \frac{\kappa}{k_\perp} = \frac{1}{k_\perp L_n}, \tag{J.16}$$

which is the usual result (Kadomtsev, 1965).

This model has difficulty capturing the appropriate weak-turbulence scaling. As we have seen in Sec. 4.2.3 (p. 100), in the weak-turbulence theory of many interacting waves the quantity $\delta(\Delta\Omega)$ arises in a description of the resonant three-wave interaction. The effects of that term are then evaluated by appropriate integrals over (approximately continuous) $\boldsymbol{p}$ and $\boldsymbol{q}$. In the present problem one has a very discrete and essentially trivial $\boldsymbol{k}$ sum, not an integral, so one cannot work with the delta function in the usual way. However, if one treats these equations as structural prototypes for the many-wave problem, one may assume that the effect of θ_r is to introduce an appropriate autocorrelation time τ_{ac} for the wave–wave interactions. In the limit $\Delta\gamma/\Delta\Omega \ll 1$ one can then write

$$\theta_r \to \pi\delta(\Delta\Omega + \Delta\eta_i^{\mathrm{nl}}) = \pi\delta\big(\Delta\Omega[\Delta(\gamma^2)/(\Delta\gamma)^2]\big). \tag{J.17}$$

[Notice from Eqs. (J6a) and (J.7) that in this three-wave model $\Delta\eta_i^{\mathrm{nl}} \sim \Delta\Omega$, so one obtains an extra factor of order unity over the usual $\delta(\Delta\Omega)$.] One now finds

$$\mathcal{E} \sim \left(\frac{W^2}{M^2}\right)\left(\frac{1}{|\Delta\gamma|\tau_{\mathrm{ac}}}\right). \tag{J.18}$$

If one assumes that $\Delta\Omega\,\tau_{\mathrm{ac}} = O(1)$, then one finds

$$\mathcal{E} \sim \left(\frac{\Delta\gamma}{\Delta\Omega}\right)\mathcal{E}^{\mathrm{ml}}, \tag{J.19}$$

which is the conventional weak-turbulence scaling.

The result (J.19) comes from a simple modification of the strong-turbulence balance. Instead of writing $\gamma^2 \sim k^2\overline{V}_E^2 \sim M^2\mathcal{E}$, one estimates

$$\gamma^2 \sim \left(\frac{\gamma}{\omega}\right) k^2\overline{V}_E^2 = \left(\frac{\gamma}{\omega}\right) M^2\mathcal{E}, \tag{J.20}$$

which leads readily to Eq. (J.19). The extra factor of γ/ω describes the fraction of the energy that can be transferred during one eddy turnover time when the motion is predominantly oscillatory.

The results for the scaling of the saturation level with $\Delta\gamma/\Delta\Omega$ are summarized in Fig. J.1. The strict model predicts the dashed lines. However, if one extrapolates to a weak-turbulence regime (in which all modes can couple by three-wave resonance), then $\mathcal{E}/\mathcal{E}^{\mathrm{ml}} \sim \Delta\gamma/\Delta\Omega$. Furthermore, if one allows the possibility of ion stochasticity, then in real life the fluctuations can never grow above the mixing-length level. In all models that level is achieved at the crossover point $\Delta\gamma/\Delta\Omega \sim 1$. The more realistic scalings are indicated by the solid lines.

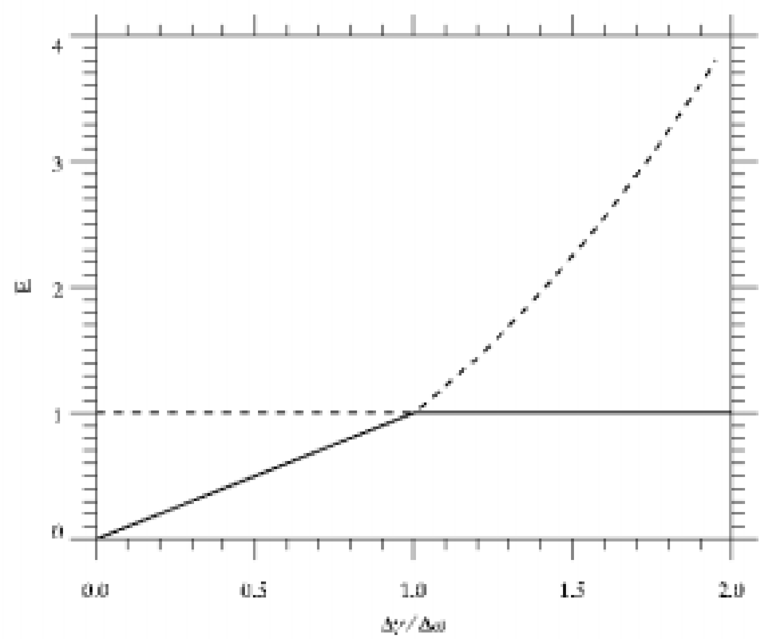


Fig. J.1. Saturation levels $\mathcal{E}$ vs the driving parameter $\Delta\gamma/\Delta\Omega$ for generic systems of forced, dissipative, coupled, dispersive waves. The three-wave model treated in this appendix strictly predicts the dashed lines. The solid line is the more realistic scaling for plasma systems with distributed forcing.

It should be emphasized that these general scaling laws apply to situations in which the forcing and dissipation are distributed. They must be modified when the forcing is localized, as in a dual cascade (Ottaviani and Krommes, 1992).

The structure of Eq. (J.11), $\mathcal{E} \propto M^{-2}$, provides an explicit demonstration that saturation level for a general system of coupled modes does not depend on the Taylor microscale Sec. 3.6.3, p. 66 or, equivalently, the rms shear; see the discussion in Sec. 4.4.4 (p. 123). Such a result would appear as $\mathcal{E} \propto M^{+2}$, since if Eq. (223) were derived for an advective nonlinearity, M would represent the gradient operator. The present results can be interpreted as a description of generic energy-containing modes. If the model were supplemented by additional small-scale modes, the fluctuation level would not sensibly change since it is determined by the eddy turnover rate for the energy-containing modes; however, the total rms shear would increase.

The saturation levels of the three-mode model can be used to illustrate some of the general discussion about zonal flows given in Sec. 12.7 (p. 248). A consistent interpretation of the three modes is given in Table J.1.

Table J.1
A possible interpretation of the three-mode system as comprising two drift waves and one zonal-flow mode.

mode index	interpretation	sgn γ	sgn M
K	drift wave (unstable)	$+$	$+$
P	drift wave (damped)	$-$	$-$
Q	zonal flow (damped)	$-$	$-$

For simplicity it is assumed that $\sigma_{\boldsymbol{k}} = 1$, so $M_K + M_P + M_Q = 0$. The steady-state solution is

$$C_K = a\left(\frac{\gamma_P\gamma_Q}{M_P M_Q}\right), \quad C_P = a\left(\frac{\gamma_Q\gamma_K}{M_Q M_K}\right), \quad C_Q = a\left(\frac{\gamma_K\gamma_P}{M_K M_P}\right), \tag{J21a,b,c}$$

where a is the square-bracketed quantity in Eq. (J8b). The total energy $\mathcal{E}$, the drift-wave energy $\mathcal{E}_*$, and the zonal-flow energy $\overline{\mathcal{E}}$ are easily shown to be

$$\mathcal{E} = b\left(\frac{M_K}{\gamma_K} + \frac{M_P}{\gamma_P} + \frac{M_Q}{\gamma_Q}\right), \quad \mathcal{E}_* = b\left(\frac{M_K}{\gamma_K} + \frac{M_P}{\gamma_P}\right), \quad \overline{\mathcal{E}} = b\left(\frac{M_Q}{\gamma_Q}\right), \tag{J22a,b,c}$$

where $b \doteq a(\gamma_K\gamma_P\gamma_Q/M_K M_P M_Q)$. One also finds

$$\gamma_*\mathcal{E}_* = b(M_K + M_P), \quad -\overline{\gamma}\overline{\mathcal{E}} = bM_Q; \tag{J23a,b}$$

thus $\gamma\mathcal{E} = \gamma_*\mathcal{E}_* - \overline{\gamma}\overline{\mathcal{E}} = b(M_K + M_P + M_Q) = 0$. As predicted by Eqs. (461), it is readily seen that $\mathcal{E}_* \propto \overline{\gamma} = -\gamma_Q$; the result $\overline{\mathcal{E}} \propto \gamma_*$ can be verified as well. A condensation into a pure zonal-flow state is found by considering the limit $\overline{\gamma} \to 0$, in which case

$$C_K \to 0, \quad C_P \to 0, \quad C_Q \to a\left(\frac{\gamma_K\gamma_P}{M_K M_P}\right), \tag{J24a,b,c}$$

or $\mathcal{E}_* \to 0$, $\overline{\mathcal{E}} > 0$. Nevertheless, as noted in Sec. 12.7 (p. 248), such condensation need not happen in a realistic drift-wave–zonal-flow system because such a three-mode model does not properly capture cascades of drift-wave invariants.

K NOTATION

K.1 Abbreviations

An appended E means Equation, as in gyrokinetic equation (GKE).

BBGKY — Bogoliubov–Born–Green–Kirkwood–Yvon
BGK — Bernstein–Greene–Kruskal
CA — clump algorithm
CDIA — cubic direct-interaction approximation
CTR — controlled thermonuclear fusion research
CUDIA — cumulant-update direct-interaction approximation
c.p. — cyclic permutations
DIA — direct-interaction approximation
DNS — direct numerical simulation
EDQNM — eddy-damped quasinormal Markovian
FDA — fluctuation–dissipation Ansatz
FDT — fluctuation–dissipation theorem
FFT — fast Fourier transform
FLR — finite-Larmor-radius
FNS — Forster–Nelson–Stephen
GK — gyrokinetic
gBL — generalized Balescu–Lenard
HM — Hasegawa–Mima
ITG — ion temperature gradient
KAM — Kolmogorov–Arnold–Moser
LES — large-eddy simulation
LHDIA — Lagrangian-history direct-interaction approximation
MHD — magnetohydrodynamic
MSR — Martin–Siggia–Rose
NEMD — nonequilibrium molecular dynamics
NS — Navier–Stokes
PDE — partial differential equation
PDF — probability density function
PDIA — particle direct-interaction approximation
QED — quantum electrodynamics
QFT — quantum field theory
QLT — quasilinear theory
RBT — resonance-broadening theory
RCM — random-coupling model
RG — renormalization group
RMC — Realizable Markovian Closure
SO — stochastic oscillator
SOC — self-organized criticality
TFM — test-field model
TFTR — Tokamak Fusion Test Reactor
TH — Terry–Horton
TTM — turbulent trapping model
WKE — wave kinetic equation
WTT — weak-turbulence theory

K.2 Basic physics symbols

— A, a, α —

$\boldsymbol{A}(\boldsymbol{x}, t)$ — vector potential ($\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$)
a — interparticle spacing [$a \doteq (3/4\pi n)^{1/3}$]
$\widetilde{\boldsymbol{a}}$ — random acceleration
α — inverse temperature in Gibbs distribution; fine-structure constant ($\alpha \doteq e^2/\hbar c \approx \frac{1}{137}$); in MHD, helical contribution to the kinematic dynamo
$\widehat{\alpha}$ — parallel coupling operator in HW equations ($\widehat{\alpha} \doteq -\omega_{ci}^{-1} D_\parallel \nabla_\parallel^2$); ratio of inverse temperatures in two-parameter Gibbs distribution ($\widehat{\alpha} \doteq \alpha/\beta$)

— B, b, β —

$\boldsymbol{B}(\boldsymbol{x}, t)$ — magnetic field ($\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$)
b — $k_\perp^2 \rho_i^2$; impact parameter
b_0 — impact parameter for 90° scattering ($b_0 \doteq q_1 q_2/T$)
$\widehat{\boldsymbol{b}}$ — unit vector in direction of magnetic field ($\widehat{\boldsymbol{b}} \doteq \boldsymbol{B}/|\boldsymbol{B}|$)
β — inverse temperature in a two-parameter Gibbs distribution; rms frequency of stochastic oscillator ($\beta \doteq \langle \delta\widetilde{\omega}^2 \rangle^{1/2}$)
$\widehat{\beta}$ — in HM equation, projection operator into $k_\parallel \neq 0$ subspace

— C, c, Γ, γ —

$C_{\boldsymbol{k}}$ — Fourier transform of homogeneous correlation function
C_n — nth cumulant [$C_n(1, \dots, n) \equiv \langle\langle \psi(1) \dots \psi(n) \rangle\rangle$]
$C_s[f]$ — nonlinear collision operator for species s ($\partial_t f + \cdots = -C_s[f]$, where $C_s \doteq \sum_{s'} C_{s,s'}$)
C_E, C_W — Kolmogorov constants for energy and enstrophy cascades
$\widehat{C}$ — linearized collision operator [$\partial_t \chi + \cdots = -\widehat{C}\chi$, where $f \doteq f_M(1+\chi)$]
$C(\tau)$ — Lagrangian correlation function
$\mathsf{C}(1, 1')$ — Eulerian correlation matrix

c — speed of light
c_A — Alfvén velocity ($c_A^2 \doteq B^2/4\pi\overline{n}M$)
c_s — sound speed [$c_\mathrm{s} \doteq (ZT_e/M)^{1/2}$]
Γ — generic notation for a point in Γ space ($\Gamma \doteq \{X_i \mid i = 1, \ldots, N\}$); strong-coupling parameter ($\Gamma \doteq e^2/aT$)
$\Gamma_s(\boldsymbol{x}, t)$ — flux of some quantity of species s
$\Gamma_n(1, 2, \ldots, n)$ — renormalized vertex of order n
$\Gamma(\boldsymbol{k})$ — $\langle J_0^2(k_\perp v_\perp/\omega_{ci})\rangle_M = I_0(b)\mathrm{e}^{-b}$
γ — Poincaré–Cartan one-form
$\gamma(\boldsymbol{k}), \gamma_{\boldsymbol{k}}$ — growth rate [$\omega(\boldsymbol{k}) = \Omega(\boldsymbol{k}) + \mathrm{i}\gamma(\boldsymbol{k})$]
$\gamma_{\boldsymbol{k}}^{(1)}$ — rate of wave-number evolution in weakly inhomogeneous medium ($\gamma_{\boldsymbol{k}}^{(1)} \doteq \boldsymbol{k} \cdot \boldsymbol{\nabla}\Omega_{\boldsymbol{k}}$)
$\gamma_{\boldsymbol{q}}^{\mathrm{nl}}$ — long-wavelength nonlinear growth rate
$\gamma_n(1, 2, \ldots, n)$ — bare vertex (spurious when primed)

— D, d, Δ, δ —

D — diffusion coefficient
D_B — Bohm diffusion coefficient ($D_B \doteq cT_e/eB$)
D_m — magnetic diffusion coefficient
D — diffusion tensor
$D_\parallel$ — diffusion coefficient in the parallel direction (along the magnetic field lines)
$D_\perp$ — diffusion coefficient in the perpendicular direction (across the magnetic field lines)
$D(1, 2; 3, 4)$ — doubly connected graphs in field theory
$\boldsymbol{D}$ — electric displacement
$\mathcal{D}$ — dissipation term in turbulent energy balance
$\mathcal{D}(\boldsymbol{k}, \omega)$, $\mathsf{D}(\boldsymbol{k}, \omega)$ — dielectric function and tensor
$\mathcal{D}^{\mathrm{lin}}$, $\mathcal{D}^{\mathrm{nl}}$ — linear and nonlinear contributions to $\mathcal{D}(\boldsymbol{k}, \omega)$
$\mathrm{D}[q]$ — volume element in functional integration
d — number of spatial dimensions
$\Delta(x; L)$ — periodic delta function with period L
Δk — spectral width of wave packet
Δt, Δv, Δx — elementary steps in random-walk process
$\Delta\Omega$ — frequency mismatch ($\Delta\Omega \doteq \Omega_{\boldsymbol{k}} + \Omega_{\boldsymbol{p}} + \Omega_{\boldsymbol{q}}$, with $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}$)
δf_c, $\delta\widetilde{f}$ — coherent and incoherent fluctuations
δk — Fourier mode spacing in a box of size L ($\delta k \doteq 2\pi/L$)
$\delta_{\boldsymbol{k}}$ — nonadiabaticity factor in the iδ model [$\delta n_{\boldsymbol{k}}/n = (1 - \mathrm{i}\delta_{\boldsymbol{k}})\delta\varphi_{\boldsymbol{k}}$]
$\delta_{k,k'}$ — Kronecker delta function [$\delta_{k,k'} \doteq 1 \ \ (k' = k)$ or 0 otherwise; $\delta_k \equiv \delta_{k,0}$]
$\delta(x - y)$ — Dirac delta function
$\delta f/\delta\eta(1)$ — functional derivative of $f[\eta]$ with respect to η
$\boldsymbol{\partial}$ — $(q/m)\partial/\partial\boldsymbol{v}$

— E, e, ϵ, ε —

E — total energy of Hamiltonian system ($E = K + W$)
$E(k)$ — omnidirectional energy spectrum, normalized such that the total fluctuation energy $E = \int_0^\infty \mathrm{d}k\, E(k)$
$\boldsymbol{E}(\boldsymbol{x}, t)$ — electric field
$\overline{E}_\parallel$ — effective, gyro-averaged $E_\parallel$ [$\overline{E}_\parallel \doteq J_0(k_\perp\rho)E_\parallel$]
$\mathcal{E}$ — wave or fluctuation energy
$\widehat{\boldsymbol{\mathcal{E}}}$ — electrostatic electric-field operator. The solution of Poisson's equation $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho[f]$ is $\boldsymbol{E} = \widehat{\boldsymbol{\mathcal{E}}}f$. In Fourier space, its kernel is $\widehat{\boldsymbol{\mathcal{E}}}_{\boldsymbol{k}}(\boldsymbol{v}, s; \boldsymbol{v}', s') = \boldsymbol{\epsilon}_{\boldsymbol{k}}(\overline{n}q)_{s'}$. $\widehat{\boldsymbol{\mathcal{E}}} = -\boldsymbol{\nabla}\widehat{\Phi}$.
e — base of the natural logarithms, approximately equal to 2.7 (Candlestickmaker, 1972)
e — electronic charge ($e = |e|$). Sometimes (it should be clear from the context), e is used like q, a generic charge whose species is not specified.
e_e, e_i — $e_e \doteq -e$, $e_i \doteq Ze$
ϵ — a positive infinitesimal; the wave-number ratio $q/k \ll 1$
$\epsilon_\perp$ — dielectric constant of the gyrokinetic vacuum ($\epsilon_\perp \approx \omega_{pi}^2/\omega_{ci}^2$)
$\epsilon^{(2)}, \epsilon^{(3)}$ — coupling coefficients in Vlasov WTT, defined by Eqs. (188)
ϵ_{ijk} — Levi–Civita fully antisymmetric permutation symbol
$\boldsymbol{\epsilon}(\boldsymbol{x})$, $\boldsymbol{\epsilon}_{\boldsymbol{k}}$ — Coulomb interaction field, *sans* charge [$-\boldsymbol{\nabla}\overline{\varphi} \equiv -\boldsymbol{\nabla}(1/|\boldsymbol{x}|) = \boldsymbol{x}/|\boldsymbol{x}|^3$; the Fourier transform is $\boldsymbol{\epsilon}_{\boldsymbol{k}} = -4\pi\mathrm{i}\boldsymbol{k}/k^2$]
ϵ_p — plasma discreteness parameter ($\epsilon_p \doteq 1/\overline{n}\lambda_D^3$)
ε — constant rate of energy flow through k space in steady-state homogeneous, isotropic turbulence

— F, f —

F — flatness statistic ($F \doteq \langle\delta\psi^4\rangle/\langle\delta\psi^2\rangle^2$)
$F(\boldsymbol{x}, \mu, v_\parallel, t)$ — gyrokinetic PDF
F^{nl} — covariance of incoherent noise (in MSR formalism, $F^{\mathrm{nl}} \doteq \Sigma_{--}$)
$f^{\mathrm{ext}}(\boldsymbol{x}, t)$ — external random forcing in primitive amplitude equation

$\widetilde{f}_{\boldsymbol{k}}(\boldsymbol{x},t)$ — incoherent noise in generalized Langevin equation

$f(X,t)$ — one-particle PDF $[f(X,t)=\langle\widetilde{N}(X,t)\rangle]$

$f_M(\boldsymbol{v})$ — Maxwellian $[f_M(\boldsymbol{v}) \doteq (2\pi v_t^2)^{-3/2}\times \exp(-v^2/2v_t^2)]$

— G, g, η —

G, G_0 — Green's functions (renormalized and bare)

$G(1,1')$ — correlation matrix in MSR formalism $\left[G(1,1')\doteq\langle\langle\Phi(1)\Phi^T(1')\rangle\rangle=\begin{pmatrix}C(1,1') & R(1;1')\\ R(1';1) & 0\end{pmatrix}\right]$

$g(1;1')$, $g_0(1;1')$ — particle propagators (renormalized and bare)

$g(1,2)$ — pair correlation function

$g_{\alpha\beta}$ — metric tensor

η — resistivity; rate of enstrophy injection

$\eta_{\boldsymbol{k}}$ — total damping (linear plus nonlinear) in Markovian closures ($\eta_{\boldsymbol{k}}=\eta^{\text{lin}}_{\boldsymbol{k}}+\eta^{\text{nl}}_{\boldsymbol{k}}$)

$\eta^{\text{nl}}_{\boldsymbol{k}}$ — nonlinear damping in Markovian closures

$\eta(r)$ — two-particle eddy diffusivity

$\eta(1)$, $\widehat{\eta}(1)$ — additive sources in field theory

η_i — $\mathrm{d}\ln T_i/\mathrm{d}\ln n_i$

— H, h, Θ, θ —

$H(\Gamma)$ — N-particle Hamiltonian

$H(\tau)$ — Heaviside unit step function [$H(\tau)=0$ $(\tau<0)$, $\frac{1}{2}$ $(\tau=0)$, or 1 $(\tau>0)$]

$\mathcal{H}$ — fluid helicity ($\mathcal{H}\doteq\langle\boldsymbol{u}\cdot\boldsymbol{\omega}\rangle$)

$\mathcal{H}_c$ — cross helicity ($\mathcal{H}_c\doteq\langle\boldsymbol{v}\cdot\boldsymbol{B}\rangle$)

$\mathcal{H}_m$ — magnetic helicity ($\mathcal{H}_m\doteq\langle\boldsymbol{A}\cdot\boldsymbol{B}\rangle$)

$h(1,2,3)$ — triplet correlation

Θ — canonical angle (new coordinates)

$\Theta_{\boldsymbol{kpq}}$ — modified triad interaction time in RMC

θ — canonical angle (old coordinates)

$\theta_{\boldsymbol{kpq}}$ — triad interaction time in EDQNM

— I, i —

I — identity matrix

$I_{\boldsymbol{k}}$ — fluctuation intensity (usually $I_{\boldsymbol{k}}\doteq\langle\delta\varphi^2_{\boldsymbol{k}}\rangle$)

I_ν — modified Bessel function (first kind)

$I(1,2;1',2')$ — interaction kernel in BSE

$\mathcal{I}$ — nonlinearly conserved invariant

i — $\sqrt{-1}$

— J, j —

J, $\mathcal{J}$ — action variable; the Poisson tensor $\widehat{\omega}^{-1}$

$J_\nu(z)$ — Bessel function (first kind)

$\boldsymbol{j}(\boldsymbol{x},t)$ — current density

— K, k, κ —

K — kurtosis statistic ($K\doteq F-3$)

K_ν — modified Bessel function (second kind)

$K(1,2;1',2')$ — two-body scattering matrix in MSR formalism

$\mathcal{K}$ — Kubo number ($\mathcal{K}\doteq\beta\tau_{\text{ac}}$)

k — usually, a one-dimensional Fourier transform variable; occasionally, the set $\{\boldsymbol{k},\omega_{\boldsymbol{k}}\}$

$\overline{k}$ — typical k {in gyrokinetics, $\overline{k}^2\doteq\tau[1-\Gamma(\boldsymbol{k})]$}

$\boldsymbol{k}$ — Fourier-transform variable conjugate to $\boldsymbol{x}$ $[f(\boldsymbol{x})\sim\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})]$

k_d — Kolmogorov dissipation wave number ($k_d\doteq\lambda_d^{-1}$)

k_D — Debye wave number [$k_D^2\doteq\sum_s k^2_{Ds}$, where $k^2_{Ds}\doteq4\pi(nq^2/T)_s$]

k_T — Taylor wave number ($k_T\doteq\lambda_T^{-1}$)

κ — thermal conductivity

κ_n, κ_T — inverses of density and temperature scale lengths ($\kappa_n\doteq L_n^{-1}$, $\kappa_T\doteq L_T^{-1}$)

— L, l, Λ, λ —

L — box length for discrete Fourier transform; macroscopic system size

$L[F,G]$ — Legendre transform in MSR formalism

L_c — correlation length

L_n, L_T — density and temperature scale lengths ($L_n^{-1}\doteq-\nabla\ln\langle n\rangle$, $L_T^{-1}\doteq-\nabla\ln\langle T\rangle$)

L_s — magnetic shear length

Lu — Lundquist number ($\mathrm{Lu}\doteq v_AL/\mu_m$)

ℓ — mixing length in turbulence; occasionally, arc length along $\boldsymbol{B}$

ln, log — $\log_{\mathrm{e}}$, $\log_{10}$

Λ — in $\ln\Lambda$, the Spitzer factor ($\Lambda\doteq\lambda_D/b_0$ for $B=0$)

L — Onsager matrix ($\mathsf{L}\doteq\mathsf{M}\cdot\mathsf{C}$)

$\mathcal{L}$ — Lagrangian

$\widehat{\mathcal{L}}$, $\widehat{\mathsf{L}}$ — linear operator ($\partial_t\psi=-\mathrm{i}\widehat{\mathcal{L}}\psi$)

λ, λ_c — general parameter and critical parameter in bifurcation theory

λ_D — Debye length ($\lambda_D\doteq k_D^{-1}$)

λ_T — Taylor microscale $\{\lambda_T\doteq[-C''(0)/2C(0)]^{-1/2}\}$

λ_d — Kolmogorov dissipation microscale

λ_{mfp} — kinetic mean free path ($\lambda_{\text{mfp}}\doteq v_t/\nu$)

— M, m, μ —

$M_{\boldsymbol{kpq}}$ — symmetrized mode-coupling coefficient (symmetrical in $\boldsymbol{p}$ and $\boldsymbol{q}$)
$M^U_{\boldsymbol{kpq}}$ — unsymmetrized mode-coupling coefficient ($M_{\boldsymbol{kpq}} = M^U_{\boldsymbol{kpq}} + M^U_{\boldsymbol{kqp}}$)
m_e, m — electron mass
m_i, M — ion mass
μ — kinematic viscosity; magnetic moment ($\mu \approx v_\perp^2/2B$)

— N, n, ν —

N, N_s — total number of particles
$\widetilde{N}(X,t)$ — Klimontovich microdensity [$\widetilde{N}(X,t) \doteq \overline{n}_s^{-1} \sum_{i=1}^{N_s} \delta\big(X - \widetilde{X}_i(t)\big)$]
$\mathcal{N}$ — wave action [$\mathcal{N}_k \doteq (\partial \operatorname{Re} \mathcal{D}/\partial\omega_k) \times (\langle|\delta E_k|^2\rangle/8\pi)$]
$\mathcal{N}_e$ — Hasegawa–Wakatani invariant ($\mathcal{N}_e \doteq \frac{1}{2}\langle n_e^2\rangle$)
$\mathcal{N}_i$ — Hasegawa–Wakatani invariant ($\mathcal{N}_i \doteq \frac{1}{2}\langle n_i^2\rangle$)
$\overline{n}$, $\overline{n}_e$, $\overline{n}_i$ — mean number density ($\overline{n} \doteq N/V$)
$n_s(\boldsymbol{x},t)$, $n_s^{\mathrm{G}}(\boldsymbol{x},t)$ — fluid density of particles (n) or gyrocenters (n^{G})
ν — collision frequency
ν_d — resonance-broadening or diffusion frequency ($\nu_d \doteq \tau_d^{-1}$)

— P, p, Π, π, ϖ —

$P(\boldsymbol{x},t)$ — scalar pressure
$P(x)$ — probability density function
P, $\mathsf{\Pi}$ — projection operators ($\mathsf{P}^2 = \mathsf{P}$)
Pr — Prandtl number ($\mathrm{Pr} \doteq \mu/\kappa$)
$\mathcal{P}$ — production term in turbulent energy balance; total wave momentum
p — canonical momentum
$\boldsymbol{p}$ — Fourier wave vector, as in $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$
Π — dimensionless variable
π — approximately 3.14 (Sagan, 1985)
ϖ — general probability measure

— Q, q —

Q — orthogonal projection operator ($\mathsf{Q} \doteq \mathsf{I} - \mathsf{P}$)
q — generic charge, of unspecified species; canonical coordinate
$\boldsymbol{q}$ — Fourier wave vector, as in $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q} = \boldsymbol{0}$

— R, r, ρ —

$R(1;1')$, $\widetilde{R}(1;1')$ — mean and random infinitesimal response functions. If $\tilde{f}_\eta$ is a random variable depending on an external source function η, then $R(1;1') \doteq \langle\widetilde{R}(1;1')\rangle_{\eta=0}$, where $\widetilde{R}(1;1') \doteq \delta\tilde{f}(1)/\delta\eta(1')$. At linear order, where there are no fluctuations, the averaging is irrelevant and R is the usual Green's function, called R_0 to emphasize the linear approximation.
Re, $\mathcal{R}$ — Reynolds number ($\mathrm{Re} \doteq UL/\mu$)
$\boldsymbol{\rho}$, ρ — relative separation ($\rho = |\boldsymbol{\rho}|$)
$\rho(\boldsymbol{x},t)$ — charge density
ρ_e, ρ_i — gyroradii [$\rho_s \doteq v_{ts}/\omega_{cs}$ (s is in italics)]
ρ_{s} — sound radius [$\rho_{\mathrm{s}} \doteq c_{\mathrm{s}}/\omega_{ci}$ (s is in Roman)]
ρ^{pol} — polarization charge density

— S, s, Σ, σ —

S — entropy; stochasticity parameter; action; skewness statistic ($S \doteq \langle\delta\psi^3\rangle/\langle\delta\psi^2\rangle^{3/2}$)
$S(1,1')$ — structure function $\{S(1,1') = 2[C(1,1) - C(1,1')]\}$
$\mathcal{S}$ — information-theoretic entropy
s — species label ($s = e, i$)
s — rate-of-strain tensor $\{\mathsf{s} \doteq \frac{1}{2}[(\boldsymbol{\nabla u}) + (\boldsymbol{\nabla u})^T] - \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathsf{I}\}$
Σ — MSR mass-operator matrix
Σ^{nl} — turbulent collision operator [$\Sigma^{\mathrm{nl}} \doteq -\Sigma_{-+}$ (the sign convention is consistent with dissipation)]
σ — Pauli matrix $\left[\mathrm{i}\sigma \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$
$\sigma_{\boldsymbol{k}}$ — weighting factor that defines nonlinear invariant ($\mathcal{I} \doteq \sum_{\boldsymbol{k}} \sigma_{\boldsymbol{k}} C_{\boldsymbol{k}}$)

— T, t, τ —

T — periodicity length or total integration time in temporal Fourier transform; mean time [$T \doteq \frac{1}{2}(t+t')$]
T_s — temperature of species s
$T_s(\boldsymbol{x},t)$ — fluid temperature field
$T_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}$ — triplet correlation function
$T(z)$ — Lie transformation operator
t, $\bar{t}$, t' — time
τ — time difference ($\tau \doteq t - t'$)
$\boldsymbol{\tau}$ — Reynolds stress ($\boldsymbol{\tau} \doteq -\rho_m\langle\delta\boldsymbol{u}\,\delta\boldsymbol{u}\rangle$)
τ_{ac} — autocorrelation time $\{\tau_{\mathrm{ac}} \doteq [C(0)]^{-1} \times \int_0^\infty \mathrm{d}\tau\, C(\tau)\}$
τ_c — collision time ($\tau_c \doteq \nu^{-1}$)
τ_{cl} — clump lifetime

τ_d — turbulent diffusion time ($\tau_d \doteq \nu_d^{-1}$)
τ_r — recurrence time in QLT
τ_K — Kolmogorov time for exponential separation

— U, u, Υ —

$\widetilde{U}, U$ — random and mean particle propagator in RBT
$\boldsymbol{U}$ — mean velocity
U — ubiquitous tensor in the theory of the Landau operator [$\mathsf{U}(\boldsymbol{u}) \doteq (1 - \widehat{\boldsymbol{u}}\,\widehat{\boldsymbol{u}})/|\boldsymbol{u}|$]
$\overline{u}$ — rms velocity fluctuation
$\boldsymbol{u}_s(\boldsymbol{x}, t)$ — fluid velocity
$\Upsilon(\tau)$ — covariance of random coefficient for passive statistics

— V, v —

V — total volume of system (often taken to infinity in such a way that $\overline{n} \doteq N/V$ remains finite)
$\overline{V}$ — RMS velocity
$\boldsymbol{V}_E$ — $\boldsymbol{E} \times \boldsymbol{B}$ velocity ($\boldsymbol{V}_E \doteq c\boldsymbol{E} \times \widehat{\boldsymbol{b}}$)
V_{*s} — diamagnetic velocity ($V_{*s} \doteq -cT_s/q_s B L_n$)
$\boldsymbol{V}_s^{\mathrm{pol}}$ — polarization drift velocity [$\boldsymbol{V}_s^{\mathrm{pol}} \doteq \omega_{cs}^{-1}\partial_t(c\boldsymbol{E}_\perp/B)$]
$\mathcal{V}$ — Hasegawa–Wakatani invariant ($\mathcal{V} \doteq \frac{1}{2}\langle|\nabla\varphi|^2\rangle$)
$\boldsymbol{v}$ — particle velocity
$v_{\mathrm{gr}}(\boldsymbol{k})$ — group velocity [$v_{\mathrm{gr}} \doteq \partial\Omega(\boldsymbol{k})/\partial\boldsymbol{k}$]
$v_{\mathrm{ph}}(\boldsymbol{k})$ — phase velocity [$v_{\mathrm{ph}} \doteq [\Omega(\boldsymbol{k})/k]\widehat{\boldsymbol{k}}$]
v_t, v_{te}, v_{ti} — thermal velocity [$v_{ts} \doteq (T_s/m_s)^{1/2}$]
$v_{\mathrm{tr}}, V_{\mathrm{tr}}$ — microscopic (single-resonance) and macroscopic (all resonances) trapping velocity [$v_{\mathrm{tr},\boldsymbol{k}} \doteq (2q|\varphi_{\boldsymbol{k}}|/m)^{1/2}$]

— W, w —

$W(k)$, $W[\eta]$ — cumulant generating function and functional ($W \doteq \ln Z$)
$\mathcal{W}$ — enstrophy ($\mathcal{W} \doteq \frac{1}{2}\langle\omega^2\rangle$)
$\widetilde{w}(t)$ — Gaussian white noise

— X, x, ξ —

X — phase-space point ($X \doteq \{\boldsymbol{x}, \boldsymbol{v}, s\}$)
$\widetilde{X}(t)$ — trajectory in μ space
$\mathcal{X}$ — Hasegawa–Wakatani invariant ($\mathcal{X} \doteq \langle\omega\, n\rangle$)
$\boldsymbol{x}$ — vector position in configuration space [$\boldsymbol{x} = (x, y, z)$]
x — Cartesian component of $\boldsymbol{x}$
$\widetilde{\xi}$ — external random variable in Langevin representations for DIA and EDQNM

— y —

$\boldsymbol{y}$ — vector position in configuration space
y — poloidal direction (perpendicular to both density gradient and magnetic field); Cartesian component of $\boldsymbol{x}$

— Z, z —

Z — atomic number ($\overline{n}_e = Z\overline{n}_i$)
$Z(k)$, $Z[\eta]$ — characteristic (moment-generating) function and functional
$Z(z)$ — plasma dispersion function [$Z(z) \doteq \pi^{-1/2}\int_{-\infty}^{\infty}\mathrm{d}t\,(t - z)^{-1}\mathrm{e}^{-t^2}\quad(\mathrm{Im}\, z > 0)$]
$\mathcal{Z}$ — Terry–Horton invariant
z — toroidal direction; Cartesian component of $\boldsymbol{x}$

— Φ, φ, ϕ —

Φ — dimensionless function
$\Phi(1)$ — extended vector in MSR formalism [$\Phi \doteq (\psi, \widehat{\psi})^T$]
$\widehat{\Phi}$ — electrostatic potential operator ($\widehat{\boldsymbol{\mathcal{E}}} \doteq -\boldsymbol{\nabla}\widehat{\Phi}$)
$\varphi(\boldsymbol{x}, t)$ — electrostatic potential
$\overline{\varphi}$ — effective (gyro-averaged) potential [$\overline{\varphi}_{\boldsymbol{k}} \doteq J_0(k_\perp\rho)\varphi_{\boldsymbol{k}}$]
$\phi_n(x)$ — orthonormal eigenfunction

— χ —

$\chi(\boldsymbol{x}, \boldsymbol{v}, t)$ — deviation of the one-particle distribution from a Maxwellian [$f = (1 + \chi)f_M$]
$\chi(\boldsymbol{k}, \omega)$ — susceptibility ($\mathcal{D} = 1 + \sum_s \chi_s$)

— ψ —

ψ — generic field
$\widehat{\psi}$ — adjoint field or operator in MSR formalism
$\psi_n(t)$ — amplitude of orthonormal eigenfunction $\phi_n(x)$

— Ω, ω —

Ω — solid angle.
$\Omega(\boldsymbol{k})$ — real normal-mode frequency
$\widehat{\Omega}(\boldsymbol{k})$, $\omega(\boldsymbol{k})$ — complex mode frequency ($\widehat{\Omega} = \Omega + \mathrm{i}\gamma$)
ω — Fourier variable conjugate to time t [$f(t) \sim \exp(-\mathrm{i}\omega t)$]; the fundamental two-form of differential geometry, i.e., the exterior derivative of the one-form γ
$\widehat{\omega}$ — Lagrange tensor
$\widetilde{\omega}(t)$ — random coefficient in stochastic oscillator ($\langle\widetilde{\omega}^2\rangle \doteq \beta^2$)

$\boldsymbol{\omega}(\boldsymbol{x}, t)$ — vorticity ($\boldsymbol{\omega} \doteq \boldsymbol{\nabla} \times \boldsymbol{u}$)
$\omega(\boldsymbol{x}, t)$ — z component of vorticity ($\omega \doteq \nabla_\perp^2 \varphi$)
ω_c, ω_{ce}, ω_{ci} — gyrofrequencies ($\omega_{cs} \doteq q_s B/m_s c$)
ω_d — toroidal precession frequency
ω_p, ω_{pe}, ω_{pi} — plasma frequency [$\omega_p^2 \doteq \sum_s \omega_{ps}^2$, where $\omega_{ps}^2 \doteq 4\pi(\overline{n}q^2/T)_s$]
ω_s — shearing rate ($\omega_s \doteq k\Delta v$, where Δv is the velocity difference across a structure of size k^{-1})
$\omega_*(k_y)$ — drift (diamagnetic) frequency ($\omega_* \doteq k_y V_*$)
ω_t — transit frequency ($\omega_t \doteq kv_t$)
$\omega_{\rm tr}$, $\Omega_{\rm tr}$ — microscopic (single-resonance) and macroscopic (all resonances) trapping frequencies ($\omega_{\rm tr} \doteq kv_{\rm tr}$)

K.3 Miscellaneous notation

$\widetilde{A}$ — Tilde signifies a random variable.
A^* — complex conjugate of A
A_k, $A(k)$ — The subscript and parentheses distinguish the discrete Fourier amplitude A_k from the continuum transform $A(k)$ when both are in use simultaneously.
$A_{ij}^\dagger$ — Hermitian conjugate ($A_{ij}^\dagger \doteq A_{ji}^*$)
$\overline{A(t)}$ — time average of A
$\overline{A}$ — space and ensemble average of A [$\overline{A} \doteq L^{-1}\int_0^L \mathrm{d}x\, \langle A\rangle(x)$]
$\langle A\rangle$, δA — ensemble average and fluctuation of $\widetilde{A}$ ($\widetilde{A} = \langle A\rangle + \delta A$; the tilde is frequently omitted)
$\langle A \mid B\rangle$ — scalar product of A and B
$\langle\, A\,|, \,|\, B\,\rangle$ — Dirac bra and ket
$\langle\langle x\,y\,z\rangle\rangle$ — cumulant of x, y, and z
$[\![v_1 v_2 \ldots]\!]$ — fluid velocity cumulant
$A[f]$ — A depends functionally on f.
$(f[\mathsf{A}])_+$ — time-ordered operator product expansion, with later times to the left. See footnote 121 (p. 88) for discussion of the time-ordered exponential. By convention, $\langle f[\mathsf{A}]\rangle_+ \equiv \langle (f[\mathsf{A}])_+\rangle$.
$\{A, B\}$ — Poisson bracket of A and B
$[A, B]$ — commutator of A and B ($[A, B] \doteq AB - BA$)
$(a \star b)$ — convolution of a and b [$(a\star b)(x) \doteq \int_{-\infty}^{\infty}\mathrm{d}y\, a(x-y)b(y)$]
$A_\pm(t)$ — one-sided functions [$A_\pm(t) \doteq H(\pm t)A(t)$]
A^T — transpose [$A_{ij}^T \doteq A_{ji}$]
$|A|$ — absolute value of A
$\|\mathsf{A}\|$ — norm of A
$[A]$ — dimensions of A
$\widehat{\boldsymbol{k}}$ — unit vector ($\widehat{\boldsymbol{k}} \doteq \boldsymbol{k}/|\boldsymbol{k}|$)
$O(\epsilon)$ — asymptotically of order ϵ
$\mathrm{P}\int$ — principal value $\left[\mathrm{P}\int \doteq \lim_{\epsilon\to 0}\left(\int_{-\infty}^{\epsilon} + \int_{\epsilon}^{\infty}\right)\right]$
Re, Im — real and imaginary parts
$\doteq$ — definition
$\equiv$ — equivalent to $\left(\doteq \equiv \stackrel{\rm def}{=}\right)$
$\mathrm{sgn}(x)$ — sign of x
$\sum_\Delta$ — $\sum_{\substack{\boldsymbol{p},\boldsymbol{q}\\ \boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}}}$

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