# Jan 2008 #2 (QM)

electron starts in spin up in a uniform magnetic field $\vec{B}_0 = B_0 \hat{z}$

at $t=0$, $\vec{B}_1(t) = B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$ is turned on.

Probability of finding the electron with spin down for $t>0$?

Initially, $H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B}$ $\quad \gamma = \frac{-e}{m}$ for electron

$$H = \frac{e}{m} S_z B_0 = \frac{eB_0}{m}\frac{\hbar}{2}\sigma_z = \omega_1 \frac{\hbar}{2}\sigma_z$$

$$|\chi_+\rangle \Rightarrow \begin{pmatrix}1\\0\end{pmatrix} \qquad E_+ = \omega_1\frac{\hbar}{2}$$

$$|\chi_-\rangle \Rightarrow \begin{pmatrix}0\\1\end{pmatrix} \qquad E_- = -\omega_1\frac{\hbar}{2}$$

An exact solution exists; see Griffiths Example 10.1 for details

Here, I solve it with time dependent perturbation theory

$$H' = -\vec{\mu}\cdot\vec{B}_1 = -\gamma\vec{S}\cdot\vec{B}_1 = \frac{e}{m}B_1(S_x\cos\omega t + S_y\sin\omega t)$$

$$H' = \frac{eB_1}{m}\frac{\hbar}{2}\left(\sigma_x\cos\omega t + \sigma_y\sin\omega t\right)$$

Probability of transition $= |d|^2$ from $\chi_+$ to $\chi_-$

where $d^1_{i\to f} = \delta - \frac{i}{\hbar}\int_0^t dt' \langle f|H'|i\rangle e^{i\omega_0 t'}$ $\qquad \omega_0 = \frac{E_f - E_i}{\hbar}$

$$\omega_0 = \frac{E_f - E_i}{\hbar} = \frac{-\frac{\hbar\omega_1}{2} - \frac{\hbar\omega_1}{2}}{\hbar} = -\omega_1$$

$$\langle f|H'|i\rangle = \langle\chi_-|\frac{eB_1}{m}\frac{\hbar}{2}(\sigma_x\cos\omega t + \sigma_y\sin\omega t)|\chi_+\rangle$$

$$\langle\chi_-|\sigma_x|\chi_+\rangle = (0\ 1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1$$

$$\langle\chi_-|\sigma_y|\chi_+\rangle = (0\ 1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i$$

$$\Rightarrow \langle f|H'|i\rangle = \frac{eB_1}{m}\frac{\hbar}{2}(\cos\omega t + i\sin\omega t) = \frac{eB_1}{m}\frac{\hbar}{2}e^{i\omega t}$$

$$d = \frac{-i}{\hbar}\cdot\frac{eB_1}{m}\frac{\hbar}{2}\cdot\int_0^t dt'\, e^{i(\omega-\omega_1)t'}$$

$$= \frac{-i}{2}\frac{eB_1}{m}\cdot\frac{1}{i(\omega-\omega_1)}e^{i(\omega-\omega_1)t'}\Big|_0^t = \frac{-eB_1}{2m}\frac{1}{(\omega-\omega_1)}\cdot\left[e^{i(\omega-\omega_1)t} - 1\right]$$

$$P_{+\to -} = |d|^2 = \left(\frac{eB_1}{2m}\right)^2\cdot\frac{1}{(\omega-\omega_1)^2}\cdot\underbrace{\left[e^{i(\omega-\omega_1)t} - 1\right]\left[e^{-i(\omega-\omega_1)t} - 1\right]}_{1 - e^{i(\omega-\omega_1)t} - e^{-i(\omega-\omega_1)t} + 1}$$

$$1 - e^{i(\omega-\omega_1)t} - e^{-i(\omega-\omega_1)t} + 1 = 2 - 2\cos[(\omega-\omega_1)t]$$

$$= 4\cdot\sin^2\left(\frac{\omega-\omega_1}{2}t\right)$$

$$P = \left(\frac{eB_1}{m}\right)^2 \cdot \frac{1}{(\omega-\omega_1)^2} \cdot \sin^2\left(\frac{\omega-\omega_1}{2} t\right)$$