Jan 2004 #3 (QM)

In the S_z basis, $|\psi(0)\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$ magnetic moment μ

at $t=0$, $\vec{B} = B\hat{y}$ is turned on

a. $\langle \vec{S}(t)\rangle = \langle S_x\rangle\hat{x} + \langle S_y\rangle\hat{y} + \langle S_z\rangle\hat{z}$

method 1) Direct calculation of $|\psi(t)\rangle$ by using the S_y basis

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B}$$

$$H = -\gamma B S_y$$

$$S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

Diagonalize S_y: $\begin{bmatrix}-\lambda & -i\frac{\hbar}{2}\\ i\frac{\hbar}{2} & -\lambda\end{bmatrix}$: $\lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0$ $\lambda = \pm\frac{\hbar}{2}$

eigenvector with $\lambda = \frac{\hbar}{2}$: $-\frac{\hbar}{2}x_1 - i\frac{\hbar}{2}x_2 = 0$ $ix_2 = -x$ $x_2 = ix_1$ $|\psi_{y+}\rangle \to \begin{pmatrix}1\\ i\end{pmatrix}\frac{1}{\sqrt{2}}$

$|\psi_{y-}\rangle$: $\cancel{x_1} - ix_2 = 0$ $x_2 = -ix_1$ $|\psi_{y-}\rangle \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$

$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{y+}\rangle + |\psi_{y-}\rangle\right)$ energy of $|\psi_{y+}\rangle$: $-\frac{\gamma B\hbar}{2}$

$|\psi_{y-}\rangle$: $\frac{\gamma B\hbar}{2}$ $\omega \equiv \gamma B$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{y+}\rangle e^{i\omega t/2} + |\psi_{y-}\rangle e^{-i\omega t/2}\right)$$

$$\Rightarrow \frac{1}{2}\left[\begin{pmatrix}e^{i\omega t/2}\\ ie^{i\omega t/2}\end{pmatrix} + \begin{pmatrix}e^{-i\omega t/2}\\ -ie^{-i\omega t/2}\end{pmatrix}\right] = \begin{bmatrix}\cos\frac{\omega t}{2}\\ -\sin\frac{\omega t}{2}\end{bmatrix}$$

method 2) Calculate $|\psi(t)\rangle$ using the Propagator in the S_z basis

$$U(t) = e^{-iHt/\hbar} = e^{i\gamma\vec{S}\cdot\vec{B}t/\hbar}$$

- $\exp(-i\vec{\theta}\cdot\vec{S}/\hbar)$ is the operator that rotates by $\vec{\theta}$, so U rotates by angle $\vec{\theta} = -\gamma\vec{B}t$

$U(t) = \exp(i\gamma t S_y B/\hbar) = \exp(i\gamma t B\sigma_y/2)$ let $\omega = \gamma B$

$\sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$ $\sigma_y^2 = I$

$$U(t) = 1 + \frac{i\omega t}{2}\sigma_y - \frac{1}{2!}\left(\frac{\omega t}{2}\right)^2 - \frac{i}{3!}\left(\frac{\omega t}{2}\right)^3\sigma_y + \dots$$

$$= I\left(1 - \frac{1}{2!}\left(\frac{\omega t}{2}\right)^2 + \dots\right) + i\sigma_y\left(\frac{\omega t}{2} - \frac{1}{3!}\left(\frac{\omega t}{2}\right)^3 + \dots\right)$$

$$U(t) = I\cos\frac{\omega t}{2} + i\sigma_y \sin\frac{\omega t}{2} \xrightarrow{S_z \text{ basis}} \begin{bmatrix} \cos\frac{\omega t}{2} & \sin\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{bmatrix}$$

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

$$\longrightarrow \begin{bmatrix} \cos\frac{\omega t}{2} & \sin\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} \end{bmatrix}$$

Now with $|\psi(t)\rangle$ known, it is simple to calculate $\langle \vec{S} \rangle$

$$\langle S_x \rangle = \langle \psi | S_x | \psi \rangle = \frac{\hbar}{2}\left(\cos\frac{\omega t}{2} \;\; -\sin\frac{\omega t}{2}\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} \cos\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\frac{\omega t}{2} \;\; -\sin\frac{\omega t}{2}\right)\begin{pmatrix} -\sin\frac{\omega t}{2} \\ \cos\frac{\omega t}{2} \end{pmatrix} = -\hbar\cos\frac{\omega t}{2}\sin\frac{\omega t}{2} = -\frac{\hbar}{2}\sin\omega t$$

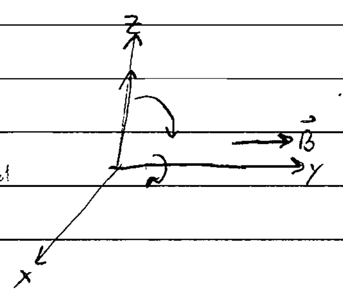
$$\langle S_y \rangle = \langle \psi | S_y | \psi \rangle = \frac{\hbar}{2}\left(\cos\frac{\omega t}{2} \;\; -\sin\frac{\omega t}{2}\right)\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{pmatrix} \cos\omega t/2 \\ -\sin\omega t/2 \end{pmatrix} = 0$$

$$\langle S_z \rangle = \langle \psi | S_z | \psi \rangle = \frac{\hbar}{2}\left(\cos\frac{\omega t}{2} \;\; -\sin\frac{\omega t}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\frac{\omega t}{2} \;\; -\sin\frac{\omega t}{2}\right)\begin{pmatrix} \cos\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} \end{pmatrix} = \frac{\hbar}{2}\left(\cos^2\frac{\omega t}{2} - \sin^2\frac{\omega t}{2}\right)$$

$$= \frac{\hbar}{2}\cos\omega t$$

$$\langle \vec{S} \rangle = \frac{\hbar}{2}\left(\cos\omega t\,\hat{z} - \sin\omega t\,\hat{x}\right)$$



same precession as classically predicted
$(\tau = \vec{\mu} \times \vec{B})$

b. $P(\text{spin down}) = |\langle \psi_{z-} | \psi(T) \rangle|^2$

$$\langle \psi_{z-} | \psi(T) \rangle = (0 \;\; 1)\begin{pmatrix} \cos\frac{\omega T}{2} \\ -\sin\frac{\omega T}{2} \end{pmatrix} = -\sin\frac{\omega T}{2}$$

$$P = \sin^2\frac{\omega T}{2}$$

c. at $t = \frac{T}{2}$, $P(\text{spin up}) = \cos^2\frac{\omega T}{4}$ $\quad P(\text{spin down}) = \sin^2\frac{\omega T}{4}$

Probability that at $t=T$, spin is down?

$P = P(\text{spin up at } \frac{T}{2}) \cdot P(\text{down at } T \mid \text{up at } \frac{T}{2}) + P(\text{down at } \frac{T}{2}) \cdot P(\text{down at } T \mid \text{down at } \frac{T}{2})$

$P(\text{down at } T \mid \text{up at } \frac{T}{2})$:

$|\psi(\frac{T}{2})\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\hookrightarrow |\psi(T)\rangle = U(T, \frac{T}{2}) |\psi(\frac{T}{2})\rangle$

and $U(T, \frac{T}{2}) = U(\frac{T}{2}, 0)$ [H not dependent on time]

$\Rightarrow |\psi(T)\rangle \rightarrow \begin{pmatrix} \cos \frac{\omega T}{4} \\ -\sin \frac{\omega T}{4} \end{pmatrix}$

$P(\text{down at } T \mid \text{up at } \frac{T}{2}) = \sin^2 \frac{\omega T}{4}$

$P(\text{down at } T \mid \text{down at } \frac{T}{2})$

$|\psi(\frac{T}{2})\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\hookrightarrow |\psi(T)\rangle = U(T, \frac{T}{2}) |\psi(\frac{T}{2})\rangle \longrightarrow \begin{pmatrix} \cos \frac{\omega T}{4} & \sin \frac{\omega T}{4} \\ -\sin \frac{\omega T}{4} & \cos \frac{\omega T}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \frac{\omega T}{4} \\ \cos \frac{\omega T}{4} \end{pmatrix}$

$P(\text{down at } T \mid \text{down at } \frac{T}{2}) = \cos^2 \frac{\omega T}{4}$

$P(\text{down at } T) = \cos^2 \frac{\omega T}{4} \cdot \sin^2 \frac{\omega T}{4} + \sin^2 \frac{\omega T}{4} \cos^2 \frac{\omega T}{4}$

$= 2 \left(\sin \frac{\omega T}{4} \cos \frac{\omega T}{4}\right)^2 = \frac{1}{2} \sin^2 \frac{\omega T}{2}$