May 2008 #1 (EM)

$B_x(0,0,z) = B_0 \cos kz$
$B_y(0,0,z) = B_0 \sin kz$
$B_z(0,0,z) = 0$

Field $\vec{B}$ in the vacuum region?

$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{B} = 0$

$\Rightarrow \vec{B} = \nabla \phi$ magnetic scalar potential, satisfies $\nabla \times \vec{B} = 0$ automatically

$\Rightarrow \nabla \cdot \vec{B} = \nabla^2 \phi = 0$ with "boundary conditions" $B_x(0,0,z)$, $B_y(0,0,z)$, $B_z(0,0,z)$

$B_x = \frac{\partial \phi}{\partial x} \quad B_y = \frac{\partial \phi}{\partial y} \quad B_z = \frac{\partial \phi}{\partial z}$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Take the x-component of the field first:

$B_x(0,0,z) = B_0 \cos kz = \frac{\partial \phi}{\partial x}\Big|_{x,y=0}$

Suppose that the full z-dependence of B_x was $\cos kz$

Then $\phi \propto \cos kz$. $\quad \phi = g(x) \cos kz$

$$\nabla^2 \phi = \frac{\partial^2 g}{\partial x^2} \cos kz - k^2 g \cos kz = 0 \Rightarrow \frac{d^2 g}{dx^2} = k^2 g$$

Solutions to Laplace's Eqn have one term oscillating, where $\frac{\partial^2}{\partial z^2}$ gives the function back with a $-k^2$; another term with $\frac{\partial^2}{\partial x^2}$ must then give the function back multiplied by k^2; this is an exponential

$$g(x) = Ae^{kx} + Be^{-kx} \quad \text{or} \quad A \sinh kx + B \cosh kx$$

Boundary Condition: $\frac{\partial \phi}{\partial z}\Big|_{x,y=0} = 0$ $\quad B\cosh kx$ term must be 0

$\Rightarrow \phi \sim \sinh kx \cos kz$ for the x-field

$$\phi = \frac{B_0}{k} \sinh kx \cos kz$$

The y-component of the field is exactly the same, except with $\sin kz$:

$$\phi = \frac{B_0}{k}(\sinh kx \cos kz + \sinh ky \sin kz)$$

$$\boxed{\begin{aligned} B_x &= B_0 \cosh kx \cos kz \\ B_y &= B_0 \cosh ky \sin kz \\ B_z &= B_0(-\sinh kx \sin kz + \sinh ky \cos kz) \end{aligned}}$$