1999 Part I Q4B Asymptotics

$$x\frac{d^2y}{dx^2} + (a-x)\frac{dy}{dx} + by = 0$$

(a). Integral representations

Try $y = \int e^{xt} f(t)dt \qquad y' = \int t e^{xt} f(t)dt \qquad y'' = \int t^2 e^{xt} f(t)dt$

$$\int \left(xt^2 + (a-x)t + b\right) e^{xt} f(t)\, dt = 0$$

$$x e^{xt} = \frac{d}{dt} e^{xt}$$

$$\Rightarrow \int f(t)\left(t^2\frac{d}{dt} + at - t\frac{d}{dt} + b\right) e^{xt} dt = 0$$

$$f(t^2-t)\frac{d}{dt}e^{xt} = f(t^2-t)e^{xt} - e^{xt}\left[f'(t^2-t) + f(2t-1)\right]$$

$$\Rightarrow f(t^2-t)e^{xt}\Big|_{t_1}^{t_2} = 0$$

$$\int\left[f(t)(at+b) - f'(t^2-t) - f(2t-1)\right]e^{xt} = 0$$

$$f(at+b) - f(2t-1) = f'(t^2-t)$$

$$f'(t^2-t) = f\left[(a-2)t + b + 1\right]$$

$$\frac{f'}{f} = \frac{(a-2)t + b+1}{t^2-t} = \frac{(a-2)t + b+1}{t(t-1)}$$

Partial fractions: $\dfrac{(a-2)t + b+1}{t(t-1)} = \dfrac{A}{t} + \dfrac{B}{t-1}$

$$(a-2)t + b+1 = A(t-1) + Bt$$
$$= (A+B)t - A$$

$\Rightarrow A = -(b+1)$

$a-2 = A+B \qquad B = a-2-A = a-2+b+1 = a+b-1$

$$\Rightarrow \frac{f'}{f} = \frac{-(b+1)}{t} + \frac{a+b-1}{t-1}$$

$$\ln f = -(b+1)\ln t + (a+b-1)\ln(t-1)$$

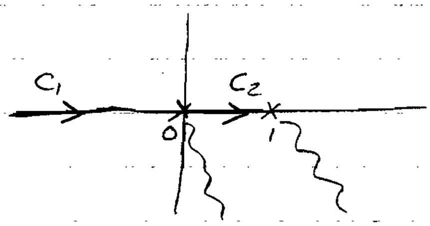
$$f = t^{-(b+1)} (t-1)^{(a+b-1)}$$

Endpoints: $f\,t(t-1)\,e^{xt}\Big|_{t_1}^{t_2} = 0$

$$t^{-b}(t-1)^{a+b} e^{xt}\Big|_{t_1}^{t_2} = 0$$

Choose $t = 0, 1, \frac{-\infty}{\arg(x)}$ (with certain conditions)

contour 1: $-\infty$ to 0   contour 2: 0 to 1



$$y_i = \int_{C_i} dt\, e^{xt}\, t^{-b-1} (t-1)^{a+b-1}$$

b. For these solutions to exist

$-b > 0$, $a+b > 0$

$$\Rightarrow \boxed{\begin{matrix} b < 0 \\ a > -b \end{matrix}}$$