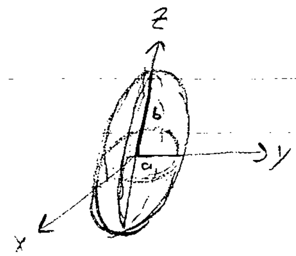
May 2007 #2 (CM)

mass m, ellipsoid, no gravity



a. $x = a\sin\theta\cos\phi \qquad y = a\sin\theta\sin\phi \qquad z = b\cos\theta$

$\dot{x} = a(\cos\theta\cos\phi\,\dot{\theta} - \sin\theta\sin\phi\,\dot{\phi})$

$\dot{y} = a(\cos\theta\sin\phi\,\dot{\theta} + \sin\theta\cos\phi\,\dot{\phi})$

$\dot{z} = -b\sin\theta\,\dot{\theta}$

$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$\dot{x}^2 = a^2(\cos^2\theta\cos^2\phi\,\dot{\theta}^2 + \sin^2\theta\sin^2\phi\,\dot{\phi}^2 - 2\sin\theta\cos\theta\sin\phi\cos\phi\,\dot{\theta}\dot{\phi})$

$\dot{y}^2 = a^2(\cos^2\theta\sin^2\phi\,\dot{\theta}^2 + \sin^2\theta\cos^2\phi\,\dot{\phi}^2 + 2\sin\theta\cos\theta\sin\phi\cos\phi\,\dot{\theta}\dot{\phi})$

$\dot{z}^2 = b^2\sin^2\theta\,\dot{\theta}^2$

$$T = \frac{1}{2}m\left((\cos^2\theta\,\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)a^2 + b^2\sin^2\theta\,\dot{\theta}^2\right)$$

$L = T - U = T \qquad (U = 0)$

$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = \frac{\partial L}{\partial\phi} = 0 \qquad \frac{\partial L}{\partial\dot{\phi}} = ma^2\dot{\phi}\sin^2\theta$ = constant in time

$\Rightarrow \quad \dot{\phi}\sin^2\theta \equiv A$ constant in time

$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \qquad \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta}\cos^2\theta + mb^2\dot{\theta}\sin^2\theta$

$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = m(a^2\ddot{\theta}\cos^2\theta - 2a^2\dot{\theta}^2\sin\theta\cos\theta + b^2\ddot{\theta}\sin^2\theta + 2b^2\dot{\theta}^2\sin\theta\cos\theta)$

$\frac{\partial L}{\partial\theta} = \frac{1}{2}m(-2a^2\dot{\theta}^2\sin\theta\cos\theta + 2a^2\dot{\phi}^2\sin\theta\cos\theta + 2b^2\dot{\theta}^2\sin\theta\cos\theta)$

$\Rightarrow a^2\ddot{\theta}\cos^2\theta - a^2\dot{\theta}^2\sin\theta\cos\theta + b^2\ddot{\theta}\sin^2\theta + b^2\dot{\theta}^2\sin\theta\cos\theta = a^2\dot{\phi}^2\sin\theta\cos\theta$

Eqn of motion for θ, using $\dot{\phi} = \frac{A}{\sin^2\theta}$:

$$\ddot{\theta}(a^2\cos^2\theta + b^2\sin^2\theta) + (b^2 - a^2)\dot{\theta}^2\sin\theta\cos\theta = \frac{a^2A^2\cos\theta}{\sin^3\theta}$$

b. $E = T = \frac{1}{2}m\left[a^2(\cos^2\theta\dot\theta^2 + \sin^2\theta\dot\phi^2) + b^2\sin^2\theta\dot\theta^2\right]$

$\sin^2\theta\,\dot\phi^2 = \frac{A^2}{\sin^2\theta}$

$$2E - \frac{ma^2A^2}{\sin^2\theta} = m(a^2\cos^2\theta + b^2\sin^2\theta)\,\dot\theta^2$$

$$\Rightarrow \quad \dot\theta^2 = \frac{2E\sin^2\theta - ma^2A^2}{m\sin^2\theta\,(a^2\cos^2\theta + b^2\sin^2\theta)} = -V_{E,A}(\theta)$$

$$\dot\theta = \sqrt{-V_{E,A}(\theta)} = \frac{d\theta}{dt} \quad \Rightarrow \quad dt = \frac{d\theta}{\sqrt{-V_{E,A}}}$$

Turning points occur when $\dot\theta = 0$, or when $V_{E,A}(\theta) = 0$, at θ_-, θ_+

(Half period between θ_-, θ_+)

$$T = 2\int_{\theta_-}^{\theta_+} \frac{d\theta}{\sqrt{-V_{E,A}(\theta)}}$$