Jan 2008 #1 (CM)

a. $L = T - U \qquad T = T_1 + T_2 \qquad T_1 = \frac{1}{2} M \dot{x}^2$

T_2: $x_2 = x + l\sin\theta \qquad y_2 = -l\cos\theta$

$T_2 = \frac{1}{2} m(\dot{x}_2^2 + \dot{y}_2^2) \qquad \dot{x}_2 = \dot{x} + l\cos\theta\,\dot{\theta}$

$\dot{y}_2 = l\sin\theta\dot{\theta}$

$\dot{x}_2^2 + \dot{y}_2^2 = \dot{x}^2 + 2l\cos\theta\,\dot{x}\dot{\theta} + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta$

$= \dot{x}^2 + 2l\cos\theta\,\dot{x}\dot{\theta} + l^2\dot{\theta}^2$

$U = mgy_2 = -mgl\cos\theta$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\cos\theta\,\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

b. $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\cos\theta\dot{\theta} \qquad \frac{\partial L}{\partial x} = 0$

$\frac{d}{dt}\left((M+m)\dot{x} + ml\cos\theta\dot{\theta}\right) = 0 \qquad (M+m)\dot{x} + ml\cos\theta\dot{\theta} = \text{constant}$

$[M+m]\ddot{x} + ml\cos\theta\ddot{\theta} - ml\sin\theta\dot{\theta}^2 = 0$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \qquad \frac{\partial L}{\partial \dot{\theta}} = ml\cos\theta\dot{x} + ml^2\dot{\theta} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml\cos\theta\ddot{x} - ml\sin\theta\dot{x}\dot{\theta} + ml^2\ddot{\theta}$

$\frac{\partial L}{\partial \theta} = -ml\sin\theta\dot{x}\dot{\theta} - mgl\sin\theta$

$ml\cos\theta\ddot{x} - ml\sin\theta\dot{x}\dot{\theta} + ml^2\ddot{\theta} = -ml\sin\theta\dot{x}\dot{\theta} - mgl\sin\theta$

$\cos\theta\ddot{x} + l\ddot{\theta} = -g\sin\theta$

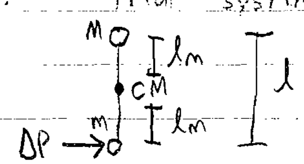
c. θ small; $\cos\theta \approx 1$, $\sin\theta \approx \theta$, $\dot{\theta}^2$ negligible

$[M+m]\ddot{x} + ml\ddot{\theta} = 0, \qquad \ddot{x} + l\ddot{\theta} = -g\theta$

$[M+m](-g\theta - l\ddot{\theta}) + ml\ddot{\theta} = 0 \qquad -g(M+m)\theta = Ml\ddot{\theta}$

$\ddot{\theta} = -\frac{g}{l}\frac{M+m}{M}\theta \qquad \omega = \left(\frac{g}{l}\frac{M+m}{M}\right)^{1/2}$

d. For a short-time after the tap, the constraint force on mass M won't matter: treat system as dumbbell in free space



distance to center of mass from M: $l_M = \frac{m}{m+M} l$

$$l_m = \frac{M}{m+M} l$$

• Net translational motion of center of mass: $V_{cm} = \frac{\Delta P}{M+m}$

• Angular Momentum about center of mass: $l_m \Delta P = L$

Moment of inertia about center of mass:

$$I = M l_M^2 + m l_m^2$$

$$= l^2\left(\frac{Mm^2}{(M+m)^2} + \frac{mM^2}{(M+m)^2}\right) = \frac{Mml^2}{M+m}$$

Angular velocity $L = I\omega$

$$\Rightarrow l_m \Delta p = \frac{Mml^2}{M+m}\omega \Rightarrow \frac{Ml}{M+m}\Delta p = \frac{Mml^2}{M+m}\omega$$

$$\Rightarrow \omega = \frac{\Delta p}{ml}$$

rotational velocity of M: $V_{rot-M} = -\omega l_M = \frac{-\Delta p}{ml} \cdot \frac{m}{M+m} l = \frac{-\Delta p}{m+M}$

$$\boxed{\dot{X} = V_{rot-M} + V_{cm} = 0}$$

rotational velocity of m: $V_{rot-m} = \omega l_m = \frac{\Delta p}{ml} \cdot \frac{M}{m+M} l = \frac{\Delta p\, M}{m(m+M)}$

Total translational velocity of m:

$$V_{rot-m} + V_{cm} = \frac{\Delta p}{M+m}\left(\frac{M}{m} + 1\right) = \frac{\Delta p}{m}$$

$$\boxed{\dot{\theta} = \frac{V_m}{l} = \frac{\Delta p}{ml}}$$