2004 Part II Q4 Asymptotics

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

a) local series solution about $x=0$. Ordinary point $\to$ Taylor series

$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad y'' = \sum n(n-1) a_n x^{n-2}$

$$\sum_{n=2}^{\infty}\left(n(n-1)a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (n+1) a_n x^n = 0$$

$$= \sum_{n=2} (n-1) a_{n-2} x^{n-2}$$

$$\Rightarrow \sum_{n=2}^{\infty}\left[(n-1)n a_n + (n-1)a_{n-2}\right]x^{n-2} = 0$$

$$\Rightarrow (n-1)\left[n a_n + a_{n-2}\right] = 0 \qquad n \geq 2$$

$$\Rightarrow a_n = \frac{-a_{n-2}}{n} \qquad n \geq 2$$

e.g. $a_2 = -\frac{a_0}{2}$ $\quad a_3 = -\frac{a_1}{3}$, etc. $\quad a_0, a_1$ are arbitrary

General terms: for n even, let $n = 2k$

$$a_{2k} = \frac{-a_{2k-2}}{2k} \qquad a_{2k} = \frac{(-1)\cdots(-1)}{2^k k!} a_0 = \frac{(-1)^k}{2^k k!} a_0$$

For n odd, let $n = 2k+1$

$$a_{2k+1} = \frac{-a_{2k-1}}{2k+1} = \frac{-a_{2k-1}}{2(k+\frac{1}{2})} \qquad a_{2k+1} = \frac{(-1)^k}{2^k (k+\frac{1}{2})(k-\frac{1}{2})\cdots(\frac{3}{2})} a_1$$

$$a_{2k+1} = \frac{(-1)^k\, \Gamma(\frac{3}{2})}{2^k\, \Gamma(k+\frac{3}{2})} a_1$$

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} = 1 - \frac{1}{2}x^2 + \ldots$$

$$y_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}\, \Gamma(\frac{3}{2})}{2^k\, \Gamma(k+\frac{3}{2})} = x - \frac{1}{3}x^3 + \ldots$$

b. Asymptotic behaviors at $x \to +\infty$

$$S'' + \underbrace{(S')^2 + xS'}\; \underbrace{+\, 1} = 0$$

$$(S')^2 + xS' = 0 \Rightarrow \quad S' = -x$$

$S' = -x$, but this ends up being an exact solution, because $S'' = -1$

$S = -\frac{1}{2}x^2$ $\quad \boxed{y = e^{-\frac{1}{2}x^2}}$ (exact)

$xS' = -1 \qquad S' = -\frac{1}{x} \qquad S = -\ln x$

$$\boxed{y \sim \frac{1}{x}}$$

c. Try integral solution $y(x) = \int_C e^{ixt} f(t)\,dt$

$$\Rightarrow \int f\left(-t^2 + ixt + 1\right) e^{ixt} dt = 0$$

$$ixe^{ixt} = \frac{d}{dt}e^{ixt}$$

$$\Rightarrow \int f(1 - t^2 + t\tfrac{d}{dt}) e^{ixt} dt = 0$$

$$ft\frac{d}{dt}e^{ixt} = fte^{ixt} - e^{ixt}[f't + f]$$

$$\Rightarrow fte^{ixt}\Big|_a^b = 0 \qquad \int \left[f(1-t^2) - f - f't\right] e^{ixt} dt = 0$$

$$\Rightarrow -ft^2 - f't = 0 \qquad f't = -ft^2$$

$$\frac{f'}{f} = -t \qquad \ln f = -\tfrac{1}{2}t^2 \qquad f = e^{-\frac{1}{2}t^2}$$

Endpoints: $\quad t\, e^{-\frac{1}{2}t^2 + ixt}\Big|_a^b = 0$ $\qquad$ Endpoints of $0, \pm\infty$ work

Take $a = 0$, $b = \infty$

$y = \int_0^\infty dt\, e^{-\frac{1}{2}t^2} e^{ixt}$ $\qquad$ real and imaginary parts are each solutions

$$y_A = \int_0^\infty dt\, e^{-\frac{1}{2}t^2} \cos xt \qquad y_B = \int_0^\infty dt\, e^{-\frac{1}{2}t^2} \sin xt$$

d. $y_A = \int_0^\infty dt\, e^{-\frac{1}{2}t^2} \cos xt$

Want to find asymptotic behavior as $x \to 0$. Taylor expand in x

$$y_A = \int_0^\infty dt\, e^{-\frac{1}{2}t^2}\left[1 - \frac{x^2t^2}{2} + \ldots\right]$$

First term: $\int_0^\infty dt\, e^{-\frac{1}{2}t^2} = \frac{1}{2}\int_{-\infty}^\infty dt\, e^{-\frac{1}{2}t^2} = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$

$$\Rightarrow \boxed{y_A = \sqrt{\tfrac{\pi}{2}} \cdot y_1(x)}$$

$y_B = \int_0^\infty dt\, e^{-\frac{1}{2}t^2}\left[xt - \ldots\right]$ $\qquad u = \frac{1}{2}t^2 \quad du = t\,dt$

$\Rightarrow \quad x\int_0^\infty du\, e^{-u} = x$

$$\boxed{y_B = y_2(x)}$$

e. Find asymptotic behavior of integral solutions for $x \to \infty$.

Method of steepest descent.

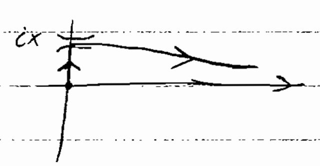
$$y_A = \int_0^\infty dt\, e^{-\frac{1}{2}t^2} \cos xt = \mathrm{Re}\int_0^\infty dt\, e^{-\frac{1}{2}t^2 + ixt}$$

$\phi = -\frac{1}{2}t^2 + ixt \qquad \phi \to -\infty$ for $t \to \pm\infty$

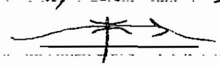
$\phi' = -t + ix \qquad \phi' = 0$ at $t = ix$

$\phi'' = -1$

Endpoint: at $t=0$, $\phi = 0$, $\phi' = ix$ $\qquad$ direction: $ixt = -1 \qquad it \sim -1 \qquad t \sim +i$



This path goes straight up the imaginary axis, and then gets only half the saddle contribution.

Alternately, we could say our integral is $\frac{1}{2}$ the integral from $-\infty$ to ∞ then have  entire saddle

Even though you usually seem to get the correct answer without doing this to be rigorous you should change variables so that $|\phi| \to \infty$, $|\phi''| \to \infty$

$\to$ i.e. let $t = xs$

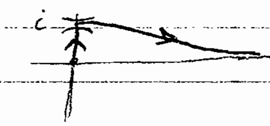
$$y_A = \mathrm{Re}\int_0^\infty x\,ds\ e^{-\frac{1}{2}x^2s^2 + ix^2s} = x\ \mathrm{Re}\int_0^\infty ds\ e^{x^2(-\frac{1}{2}s^2 + is)}$$

$\phi = x^2(-\frac{1}{2}s^2 + is)$

$\phi' = x^2(-s+i) \longrightarrow$ $s = i$ is a saddle

$\phi'' = x^2(-1)$

• Now, $|\phi''| \to \infty$, so method of steepest descent actually works and this also has the pleasing effect of making the saddle point location not depend on x.



$\phi'(0) = ix^2 \to$ steepest descent is up

$\phi(0) = 0$ $\qquad \phi(i) = x^2(\frac{1}{2} - 1) = -\frac{1}{2}x^2$

saddle contribution: $e^{\phi_0} \cdot \displaystyle\int_i^{i+\infty} ds\ e^{-\frac{1}{2}x^2(s-i)^2} = e^{\phi_0} \cdot \frac{1}{2}\int_{i-\infty}^{i+\infty} ds\ e^{-\frac{1}{2}x^2(s-i)^2}$

$$= \frac{e^{\phi_0}}{2} \cdot \sqrt{\frac{2\pi}{x^2}} = \frac{1}{x}\sqrt{\frac{\pi}{2}}\, e^{-\frac{1}{2}x^2}$$

Endpoint contribution: $\displaystyle\int_0^{i\infty} ds\ e^{ix^2s}$ $\qquad s = iu \to i\displaystyle\int_0^\infty du\ e^{-x^2s}$

$$= \frac{i}{x^2}$$

$$y_A = x \cdot \mathrm{Re}\int = \mathrm{Re}\left[\sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}x^2} + \frac{i}{x}\right]$$

The endpoint doesn't give any real part (to any order), so we just have $\sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}x^2}$ (which is the exact solution anyway)

$\longrightarrow$ $y_1(x) \sim e^{-\frac{1}{2}x^2}$

To get y_B, we just take the imaginary part of the same integral.

Thus, $y_B \sim \frac{1}{x}$

Thus $y_2(x) \sim \frac{1}{x}$ as $x \to \infty$