Generals Exam 2007, Part II, Problem 2

The Rayleigh-Taylor Instability and the Energy Principle (MHD)

Clayton Myers – 26 March 2009

The goal here is to analyze the stability of a slab-geometry plasma supported against a gravitational force by a magnetic field. In Cartesian coordinates, the gravitational force $\rho\mathbf{g}$ is defined to be x-directed such that

$$\rho\mathbf{g} = -\rho g \hat{\mathbf{e}}_x. \tag{1}$$

The magnetic field is perpendicular to the gravitational force. Thus, it is defined to be z-directed, and it is assumed to vary only along the direction of the gravitational force such that

$$\mathbf{B} = B(x)\hat{\mathbf{e}}_z. \tag{2}$$

Part (a)

Examine the momentum equation, which has been modified to include the gravitational force:

$$\rho\frac{d\mathbf{v}}{dt} = \mathbf{J}\times\mathbf{B} - \nabla p + \rho\mathbf{g}. \tag{3}$$

The pressure gradient is zero because the plasma is assumed to be cold ($p = 0$). In equilibrium the time derivative will vanish, leaving the simple equilibrium force balance equation

$$-\rho\mathbf{g} = \mathbf{J}\times\mathbf{B}. \tag{4}$$

Rewriting $\mathbf{J}$ using Ampère's Law ($\nabla\times\mathbf{B} = \mu_0\mathbf{J}$) and substituting for the gravitational force from Equation 1, the equilibrium force balance equation becomes

$$\begin{aligned} \rho\mathbf{g} = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} &= -\frac{1}{\mu_0}\mathbf{B}\times\left[\left(\frac{\partial}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial}{\partial y}\hat{\mathbf{e}}_y + \cancel{\frac{\partial}{\partial z}\hat{\mathbf{e}}_z}\right)\times B\hat{\mathbf{e}}_z\right] \\ &= -\frac{1}{\mu_0}(B\hat{\mathbf{e}}_z)\times\left[-\frac{\partial B}{\partial x}\hat{\mathbf{e}}_y + \cancel{\frac{\partial B}{\partial y}\hat{\mathbf{e}}_x}\right] = \frac{1}{\mu_0}BB'\hat{\mathbf{e}}_x, \end{aligned} \tag{5}$$

where the convention $\partial B/\partial x = B'$ has been adopted. Noting that this vector equation reduces to a scalar equation because $\mathbf{g} = -g\hat{\mathbf{e}}_x$, the final equilibrium force balance equation is simply

$$\rho g = -\frac{BB'}{\mu_0} \tag{6}$$

Part (b)

The energy principle integral modified to include the effect gravitational force is given in the problem to be

$$\delta W = \frac{1}{2}\int_V d^3\mathbf{r}\,\left[\gamma p|\nabla\cdot\boldsymbol{\xi}|^2 - \boldsymbol{\xi}^*\cdot\nabla(\boldsymbol{\xi}\cdot\nabla p) + \frac{1}{\mu_0}|Q|^2 - \boldsymbol{\xi}^*\cdot\mathbf{J}\times\mathbf{Q} + \boldsymbol{\xi}^*\cdot\mathbf{g}\nabla\cdot(\rho\boldsymbol{\xi})\right], \tag{7}$$

where

$$\mathbf{Q} \equiv \nabla\times(\boldsymbol{\xi}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\boldsymbol{\xi} - \mathbf{B}(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla)\mathbf{B}. \tag{8}$$

The necessary and sufficient condition for stability is $\delta W \geq 0$. The first two terms in Equation 7 vanish in the cold plasma limit because they both include a factor of p. This leaves the δW integral as

$$\delta W = \frac{1}{2}\int_V d^3\mathbf{r}\,\left[\frac{1}{\mu_0}|Q|^2 - \boldsymbol{\xi}^*\cdot\mathbf{J}\times\mathbf{Q} + \boldsymbol{\xi}^*\cdot\mathbf{g}\nabla\cdot(\rho\boldsymbol{\xi})\right]. \tag{9}$$

The form of the most destabilizing displacement $\boldsymbol{\xi}$ is given in the problem to be

$$\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}(x)e^{iky} \tag{10}$$

Note that this form of $\boldsymbol{\xi}$ is independent of z such that no energy from the displacement is expended in bending fieldlines. The mathematical equivalent of this no-fieldline-bending statement is that $(\mathbf{B}\cdot\nabla)\boldsymbol{\xi} = 0$. The expression for $\mathbf{Q}$ now reduces to

$$\mathbf{Q} = -\mathbf{B}(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla)\mathbf{B}. \tag{11}$$

To minimize δW, select the components of $\boldsymbol{\xi}$ such that $\mathbf{Q} = 0$. This ensures that the most possible free energy remains to drive the instability. Substituting Equations 2 and 10 into Equation 11 gives

$$\mathbf{Q} = -B\left[\frac{\partial\xi_x}{\partial x} + ik\xi_y\right]\hat{\mathbf{e}}_z - \xi_x\frac{\partial B}{\partial x}\hat{\mathbf{e}}_z = -\left[B(\xi_x' + ik\xi_y) + \xi_x B'\right]\hat{\mathbf{e}}_z = 0 \tag{12}$$

Solving for ξ_y gives

$$\xi_y = -\frac{1}{ikB}\left[B\xi_x' + \xi_x B'\right] \tag{13}$$

With the assertion that $\mathbf{Q} = 0$, the energy principle integral reduces to

$$\delta W = \frac{1}{2}\int_V d^3\mathbf{r}\,\left[\boldsymbol{\xi}^*\cdot\mathbf{g}\nabla\cdot(\rho\boldsymbol{\xi})\right]. \tag{14}$$

Defining the integrand as $\mathcal{I}$ and expanding gives

$$\mathcal{I} \equiv \boldsymbol{\xi}^*\cdot\mathbf{g}\nabla\cdot(\rho\boldsymbol{\xi}) = -\xi_x^* g\nabla\cdot(\rho\boldsymbol{\xi}) = -\xi_x^*\nabla\cdot(\rho g\boldsymbol{\xi}). \tag{15}$$

Substituting for ρg in the above expression from the equilibrium condition in Equation 6:

$$\mathcal{I} = -\xi_x^*\nabla\cdot(\rho g\boldsymbol{\xi}) = -\xi_x^*\nabla\cdot\left(-\frac{BB'}{\mu_0}\boldsymbol{\xi}\right) = \xi_x^*\left[\frac{\partial}{\partial x}\left(\frac{BB'}{\mu_0}\xi_x\right) + \frac{\partial}{\partial y}\left(\frac{BB'}{\mu_0}\xi_y\right)\right]. \tag{16}$$

Breaking out the above equation and evaluating the individual terms gives

$$\frac{\partial}{\partial x}\left(\frac{BB'}{\mu_0}\xi_x\right) = \frac{1}{\mu_0}\left[(B')^2\xi_x + BB''\xi_x + BB'\xi_x'\right] \tag{17}$$

$$\frac{\partial}{\partial y}\left(\frac{BB'}{\mu_0}\xi_y\right) = \frac{1}{\mu_0}ikBB'\xi_y. \tag{18}$$

Substituting ξ_y from Equation 13 into the latter expression above gives

$$\frac{\partial}{\partial y}\left(\frac{BB'}{\mu_0}\xi_y\right) = \frac{1}{\mu_0} ikBB'\xi_y = \frac{B'}{\mu_0}\cancel{ikB}\left(-\frac{1}{\cancel{ikB}}\Big[B\xi_x' + \xi_x B'\Big]\right) = -\frac{1}{\mu_0}\Big[BB'\xi_x' + \xi_x (B')^2\Big]. \quad (19)$$

Folding Equations 17 and 19 back into Equation 16 gives

$$\mathcal{I} = \xi_x^* \left\{\frac{1}{\mu_0}\left[\cancel{(B')^2\xi_x} + BB''\xi_x + \cancel{BB'\xi_x'}\right] - \frac{1}{\mu_0}\left[\cancel{BB'\xi_x'} + \cancel{\xi_x (B')^2}\right]\right\} = \frac{|\xi_x|^2}{\mu_0} BB''. \quad (20)$$

The condition that $\mathcal{I} \geq 0$ is equivalent to the energy principle condition for stability ($\delta W \geq 0$). Because $|\xi_x|^2 \geq 0$, the derived stability condition for the Rayleigh-Taylor instability is simply

$$BB'' \geq 0. \quad (21)$$

This condition can be placed in an alternate form using the equilibrium condition in Equation 6, which gives B' to be

$$B' = -\frac{\mu_0 \rho g}{B} \quad (22)$$

such that

$$B'' = \frac{\partial}{\partial x}\left(-\frac{\mu_0 \rho g}{B}\right) = -\frac{\mu_0 \rho g}{B}\left[\frac{\rho'}{\rho} - \frac{B'}{B}\right]. \quad (23)$$

Substituting back into the stability condition in Equation 21 gives

$$BB'' = \cancel{B}\left\{-\frac{\mu_0 \rho g}{\cancel{B}}\left[\frac{\rho'}{\rho} - \frac{B'}{B}\right]\right\} = -\mu_0 \rho g \left[\frac{\rho'}{\rho} - \frac{B'}{B}\right] \geq 0 \quad (24)$$

Noting that

$$\frac{\partial}{\partial x}\ln\left(\frac{\rho}{B}\right) = \frac{\rho'}{\rho} - \frac{B'}{B}, \quad (25)$$

and that

$$\mathbf{g}\cdot\nabla = -g\frac{\partial}{\partial x}, \quad (26)$$

the stability condition can be rewritten in its final form as

$$\mathbf{g}\cdot\nabla \ln\left(\frac{\rho}{B}\right) \geq 0. \quad (27)$$