Generals Study Guide

December 6, 2004

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1 Definitions

Basic Plasma

λ_D **- Debye Length** Section 2.1 on page 8

$$\lambda_D = \sqrt{\frac{kT_e\epsilon_0}{e^2 n_0}} \tag{1.1}$$

Ω_s **- Cyclotron Frequency**

$$\Omega_i = \frac{q_s B}{m_s c} \tag{1.2}$$

ω_{ps} **- Plasma Frequency** - Section 2.4 on page 10
CGS:

$$\omega_{ps}^2 = \frac{4\pi n_0 q_s^2}{m_s} \tag{1.3}$$

MKS:

$$\omega_{ps}^2 = \frac{n_0 q_s^2}{m_s \epsilon_0} \tag{1.4}$$

ρ_s **- Larmour Radius** - Section 3.1 on page 12

$$\rho_s = \frac{v_\perp}{\Omega_s} = \frac{m_s v_\perp}{q_s B} \tag{1.5}$$

v_{th} **- Thermal Velocity**

$$v_{th,s} = \sqrt{\frac{kT_s}{m_s}} \tag{1.6}$$

Mirror Machines - Section 4.1 on page 16

μ **- Magnetic Moment**

$$\mu = \frac{1}{2}\frac{mv_\perp^2}{B} \tag{1.7}$$

Tokamaks - Section 4.2.2 on page 18

ϵ **- Inverse Aspect Ratio**

$$\epsilon = \frac{r}{R_0} \tag{1.8}$$

q **- Safety Factor** q is the average pitch of the magnetic field line.

$$q = \left\langle \frac{d\phi}{d\theta} \right\rangle = \frac{rB_\phi}{R_0 B_\theta} = \epsilon \frac{B_T}{B_P} = \frac{\text{winding \# long way}}{\text{winding \# short way}} \tag{1.9}$$

ω_v **- Bounce Frequency** - Section 7.3.2 on page 34

$$\omega_b = \epsilon^{\frac{1}{2}} \frac{v_{th}}{qR} \tag{1.10}$$

$(v_D)_r$ **- Radial Drift Velocity** - Section 7.3.3 on page 35

$$(v_D)_r = \frac{\rho v_{th}}{R} \tag{1.11}$$

Waves

v_g **&** v_{ph} **- Group and Phase Velocity** - Section 9.3.1 on page 47

$$v_g = \frac{\partial \omega}{\partial k} \tag{1.12}$$

$$v_{ph} = \frac{\omega}{k} \tag{1.13}$$

V_A **- Alfvèn Velocity** - Section 9.6 on page 48

$$V_A^2 = \frac{B_0^2}{4\pi n_i m_i} \tag{1.14}$$

$$\frac{c^2}{V_A^2} = \frac{\omega_{pi}^2}{\Omega_i^2} \tag{1.15}$$

γ - Section 9.7 on page 52

$$\gamma = \frac{4\pi n_i m_i c^2}{B_0^2} \tag{1.16}$$

2 Simple Plasma Derivations

2.1 Plasma Shielding

Starting with a Gibbs Distribution for both particle types[1]:

$$n_s = n_0 e^{\frac{q\phi}{kT_s}} \approx n_0 \left(1 + \frac{a\phi}{kT_s}\right) \tag{2.1}$$

Using Poisson's Equation and substituting in our particle distributions:

$$\begin{aligned}\nabla^2\phi &= \frac{e}{\epsilon_0}\left(Z_i n_i - n_e\right)\\ &= \frac{e}{\epsilon_0} n_0 \left(e^{\frac{Z_i e\phi}{kT_i}} - e^{-\frac{e\phi}{kT_e}}\right)\end{aligned}$$

We then expand using the approximation $e^x \approx 1 + x$:

$$\begin{aligned}\nabla^2\phi &= \frac{e}{\epsilon_0}\left(1 + \frac{Z_i q\phi}{kT_i} - 1 + \frac{q\phi}{kT_e}\right)\\ &= \frac{e^2 n_0}{kT_e\epsilon_0}\phi\left(\frac{Z_i T_e}{T_i} + 1\right)\\ &= \frac{e^2 n_0}{k\epsilon_0}\phi\left(\frac{Z_i}{T_i} + \frac{1}{T_e}\right)\\ &= \frac{1}{\lambda_D^2}\phi\end{aligned}$$

Giving us the definition for the Debye Length:

$$\boxed{\lambda_D = \sqrt{\frac{kT_e\epsilon_0}{e^2 n_0}}} \tag{2.2}$$

2.2 Distance of Closest Approach

At the distance of closest approach, the potential energy of the system is at a maximum, while the kinetic energy is zero. Before the particles interact, they are moving at the thermal velocity V_t. This is show in the following equation:

[1] With shorter time scales, the ions can be seen as stationary. If this is the case, replace $n_i = n_0 e^{\frac{Z_i q\phi}{kT_i}}$ with $n_i = n_0$.

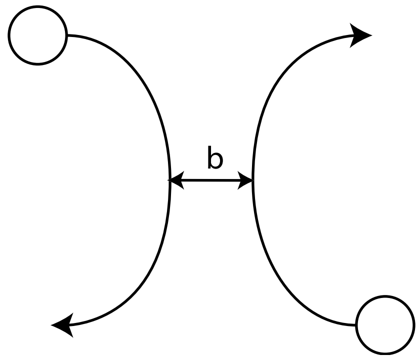

Figure 1: Distance of Closest Approach

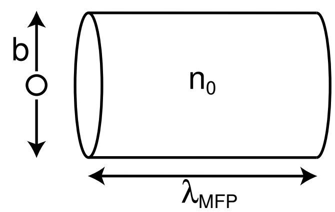

Figure 2: Mean Free Path

$$\frac{e^2}{4\pi\epsilon_0 b} = \frac{mV_T^2}{2} + \frac{mV_T^2}{2} = mV_T^2$$

Rearranging for b, we get:

$$b = \frac{e^2}{4\pi\epsilon_0 mV_T^2}$$

Using the fact that $\frac{1}{2}mV_T^2 = \frac{1}{2}kT$, we finally arrive at:

$$\boxed{b = \frac{e^2}{4\pi\epsilon_0 kT}} \tag{2.3}$$

2.3 Mean Free Path

The number of particles in a cylinder of radius b is the area of the volume times the density n_0. If we define the *mean free path* as the average length travelled by a particle before it undergoes one collision, then we see that:

$$\pi b^2 n_0 \lambda_{\rm mfp} = 1$$

Rearranging, we find that

$$\boxed{\lambda_{\rm mfp} = \frac{1}{\pi b^2 n_0} = \frac{V_{T_e}}{\nu_{es}}} \tag{2.4}$$

2.4 Plasma Oscillations

Starting off with the following three equations:

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot n\underline{\boldsymbol{v}} = 0 \qquad \text{(continuity eq.)} \tag{2.5}$$

$$\boldsymbol{\nabla} \cdot \epsilon_0 \underline{\boldsymbol{E}} = e(n_i - n_e) \qquad \text{(Poisson's eq.)} \tag{2.6}$$

$$\frac{\partial}{\partial t}(mn\underline{\boldsymbol{v}}) + \boldsymbol{\nabla} \cdot (mn\underline{\boldsymbol{v}\boldsymbol{v}}) = en\underline{\boldsymbol{E}} \qquad \text{(force eq.)} \tag{2.7}$$

We then linearize these equations by making the following substitutions and neglecting 2nd order terms [2]:

$$n = n_0 + n_1$$
$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_1$$
$$\underline{\boldsymbol{v}} = \underline{\boldsymbol{v}}_1$$

arriving at:

$$\frac{\partial n_1}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \underline{\boldsymbol{v}}_1 = 0 \tag{2.8}$$

$$\boldsymbol{\nabla} \cdot \epsilon_0 \underline{\boldsymbol{E}}_1 = -en_1 \tag{2.9}$$

$$mn_0 \frac{\partial \underline{\boldsymbol{v}}_1}{\partial t} = -en_0 \underline{\boldsymbol{E}}_1 \tag{2.10}$$

Taking $\frac{\partial}{\partial t}$ of (2.8), and combining the result with (2.10) gives us the following:

$$\frac{\partial^2 n_1}{\partial t^2} = -n_0 \boldsymbol{\nabla} \cdot \left(\frac{-e\underline{\boldsymbol{E}}_1}{m} \right)$$

[2] Any quantity with a 1 subscript is small.

Then, subbing in (2.9), we finally arrive at:

$$\frac{\partial^2 n_1}{\partial t^2} + \frac{e^2 n_0}{m\epsilon_0} n_1 = 0 \tag{2.11}$$

Obviously, this is the equation of a simple harmonic oscillator, with a frequency ω_{ps}, called the plasma frequency.

$$\ddot{n}_1 + \omega_{ps}^2 n = 0$$

$$\boxed{\omega_{ps}^2 = \frac{q_s^2 n_0}{m_s \epsilon_0}} \tag{2.12}$$

3 Single Particle Motion

3.1 Cyclotron Frequency

Starting with the equations of motion for a charged particle in a magnetic field $\underline{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}}$ and no electric field, and breaking it into components

$$\frac{d\boldsymbol{v}}{dt} = \frac{q}{m}\left(\underline{\boldsymbol{E}} + \underline{\boldsymbol{V}} \times \underline{\boldsymbol{B}}\right) \tag{3.1}$$

$$\frac{dv_z}{dt} = 0 \tag{3.2}$$

$$\frac{dv_x}{dt} = \frac{q}{m}v_y B_z = \Omega_i v_y \tag{3.3}$$

$$\frac{dv_y}{dt} = -\frac{q}{m}v_x B_z = -\Omega_i v_x \tag{3.4}$$

where we have defined the cyclotron frequency as $\Omega_i = \frac{qB_z}{m_i}$. We can then proceed in finding the equations of motion by taking the time derivative of Eq.'s (3.3) and (3.4), and subbing in Eq.'s (3.3) and (3.4) into the results.

$$\dot{y} = -v_\perp \sin\left(\Omega_i t + \alpha\right) \tag{3.5}$$

$$\dot{x} = v_\perp \cos\left(\Omega_i t + \alpha\right) \tag{3.6}$$

To finish, we merely integrate the above equations with respect to time.

The gyro radius (Larmour radius) of the particle is defined as the radius of the circular motion of the particles.

$$\rho_s = \frac{v_\perp}{\Omega_s} = \frac{m_s v_\perp}{q_s B} \tag{3.7}$$

3.2 $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}$ Drift

Now we will look at the motion of a charged particle in a magnetic field $\underline{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}}$ and an electric field of $\underline{\boldsymbol{E}} = E_\perp\hat{\boldsymbol{y}} + E_\parallel\hat{\boldsymbol{z}}$. We break down Eq. (3.1) into its components:

$$\ddot{x} = \Omega_i \dot{y} \tag{3.8}$$

$$\ddot{y} = -\Omega_i \dot{x} + \frac{q}{m}E_\perp \tag{3.9}$$

$$\ddot{z} = \frac{q}{m}E_\parallel \tag{3.10}$$

The $\hat{\boldsymbol{z}}$ equation can be directly integrated. The other two equations are solved by the same time derivative procedure as used in the previous section, arriving at:

$$\dddot{y} = -\Omega_i^2 \dot{y} \tag{3.11}$$

$$\dddot{x} = -\Omega_i^2 \dot{x} + \frac{q}{m} E_\perp \Omega_i \tag{3.12}$$

For solutions to this set of equations, we plug in a slightly modified version of Eq.'s (3.5) and (3.6).

$$\dot{y} = -v_\perp \sin(\Omega t + \alpha) \tag{3.13}$$

$$\dot{x} = v_\perp \cos(\Omega t + \alpha) + V_E \tag{3.14}$$

Plugging in the $\hat{\boldsymbol{x}}$ component of the above equation into Eq. (3.12), we find that

$$V_E = \frac{q}{m} \frac{E_\perp}{\Omega_i} = \frac{E_\perp}{B_z} \tag{3.15}$$

Generally speaking, this velocity, the $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}$ drift velocity, is written as:

$$\underline{\boldsymbol{V}}_E = \frac{\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}}{B^2} \tag{3.16}$$

3.3 Gravity Drift

With the gravity drift, we merely have a force in the $\hat{\boldsymbol{y}}$ direction:

$$F_y = mg$$

We can see from the $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}$ drift that if we set $qE_y = F_y$ in Eq. (3.9), we can then plug in our new value for F_y and proceed, arriving at a generalized drift for forces:

$$\boxed{\underline{\boldsymbol{V}}_F = \frac{\underline{\boldsymbol{F}} \times \underline{\boldsymbol{B}}}{qB^2}} \tag{3.17}$$

In the gravity drift case, our drift is merely:

$$\underline{\boldsymbol{V}}_g = \frac{m\underline{\boldsymbol{g}} \times \underline{\boldsymbol{B}}}{qB^2} \tag{3.18}$$

3.4 $\nabla \underline{\boldsymbol{B}}$ Drift

The $\nabla \underline{\boldsymbol{B}}$ drift can also be written in terms of Eq. (3.17). The relevant force is:

$$\underline{\boldsymbol{F}} = -\mu \nabla B$$

Since $\mu = \frac{mV_\perp^2}{2B}$ this drift can be rewritten as:

$$\underline{\boldsymbol{V}}_{\nabla B} = W_\perp \frac{\underline{\boldsymbol{B}} \times \nabla B}{qB^3} \tag{3.19}$$

3.5 Curvature Drift

If we define a centrifugal "pseudo-force":

$$\underline{\boldsymbol{F}}_{cf} = mv_\parallel^2 \frac{\underline{\boldsymbol{R}}_c}{R_c^2}$$

where

$$\frac{\underline{\boldsymbol{R}}_c}{R_c^2} = -(\hat{\boldsymbol{b}} \cdot \nabla)\hat{\boldsymbol{b}}$$

We can then utilize Eq. (3.17) for the curvature drift:

$$\underline{\boldsymbol{V}}_{curv} = 2W_\parallel \frac{\underline{\boldsymbol{B}} \times \nabla B}{qB^3} \tag{3.20}$$

3.6 Polarization Drift

Polarization drift arises from a time dependent electric field. The equations of motion can be written as the following:

$$\ddot{x} = \Omega_i \dot{y}$$
$$\ddot{y} = -\Omega_i \dot{x} + \frac{q}{m} E_y(t)$$

Proceeding much as we did in Section 3.2, we find that:

$$\dddot{y} + \Omega_i^2 \dot{y} = \frac{q}{m} \dot{E}_y$$
$$\dddot{x} + \Omega_i^2 \dot{x} = \frac{q}{m} \Omega_i E$$

Solving these equations lead to the normal $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}$ term in the $\hat{\boldsymbol{x}}$ equation, while in the $\hat{\boldsymbol{y}}$ equation, we get the following:

$$\dot{y} = v_\perp \sin(\Omega_i t + \alpha) + V_{pol} \tag{3.21}$$

where

$$\underline{\boldsymbol{V}}_{pol} = \frac{1}{\Omega_i^2}\frac{q}{m}\dot{\underline{\boldsymbol{E}}} \tag{3.22}$$

3.7 $\nabla B_\parallel$ Drift

The $\nabla B_\parallel$ is not a perpendicular drift like the previous drifts. This comes about from the flux through a gyro loop being constant. The force felt by the particle is:

$$F_\parallel = -\mu \frac{\partial B_z}{\partial z} = q v_\parallel B_r$$

The "drift" associated with this force is, by using Eq. (3.17):

$$V_d = v_\parallel \frac{B_r^2}{B^2} \approx \pm v_\parallel \frac{B_r}{B}$$

This "drift" of the particle motion leads to the bouncing seen in a mirror machine, i.e. μ conservation. More in mirror machines can be found in Section 4.1.

4 Confinement Schemes

4.1 Mirror Machines

4.1.1 Magnetic Invariance

μ is the magnetic moment, and is defined as follows:

$$\mu = \frac{1}{2}\frac{mv_\perp^2}{B} \tag{4.1}$$

This definition of μ is derived from the standard definition of $\mu = IA$, where $I = \frac{e\omega_c}{2\pi}$ and the area $A = \pi r_L^2 = \frac{\pi v_\perp^2}{\omega_c^2}$. By writing down the force associated with μ invariance,

$$\underline{\boldsymbol{F}}_\parallel = m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial \underline{\boldsymbol{s}}} = -\mu\boldsymbol{\nabla}_\parallel B \tag{4.2}$$

where $d\underline{\boldsymbol{s}}$ is a line element along $\underline{\boldsymbol{B}}$ and multiplying the second and third terms of the above equation by $v_\parallel = \frac{ds}{dt}$, one gets

$$mv_\parallel\frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{dB}{dt}$$

The particle's total energy must be conserved, so

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \mu B\right) = 0$$

Combining the two preceding equations gives us

$$-\mu\frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0 \tag{4.3}$$

$$\frac{d\mu}{dt} = 0 \tag{4.4}$$

4.1.2 Adiabatic Invariants

μ is the first adiabatic invariant in a mirror configuration. Adiabatic invariants (J) are defined by

$$2\pi J = \oint \mathrm{p}\mathrm{d}q = constant \tag{4.5}$$

where p and q are the generalized momentum and coordinate that the motion repeats itself over.

For μ, the calculation is as follows

$$\begin{aligned}
p_\phi &= mv_\phi + \frac{eA_\phi}{c} \\
J_1 &= \frac{1}{2\pi}\oint p_\phi \mathrm{d}q \\
&= \frac{1}{2\pi}\left[\oint mr_L\omega_c r_L \, d\theta + \frac{e}{c}\oint \underline{\boldsymbol{B}} \cdot \hat{\boldsymbol{n}}\mathrm{d}a\right] \\
&= \frac{1}{2\pi}\left(mr_L^2\omega_c 2\pi - \frac{e}{c}B\pi r_L^2\right) \\
&= \frac{m}{2}r_L^2\omega_c = \mu\left(\frac{mc}{q}\right)
\end{aligned}$$

The second adiabatic invariant, J is the bounce time between mirrors, with $p = \frac{v_\parallel}{L_{machine}}$. The third adiabitic invariant is the drift time around the device Φ with $p = \frac{v_D}{2\pi r_{machine}}$.

$$\Phi = \oint \underline{\boldsymbol{A}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \approx constant \tag{4.6}$$

See pg. 43 of Chen.

4.1.3 Trapping Condition

The trapping condition on particles in a mirror machine is based on μ invariance. As can be seen from Eq. (4.4), μ is constant in time. As a particle moving in a mirror machine sees an increasing B, its $v_\perp$ must also increase in order to keep μ constant. Total energy of the particle also remains constant, so as $v_\perp$ increases, $v_\parallel$ decreases. At some point, if the magnetic field is high enough, $v_\parallel = 0$ and the particle turns around.

One can easily find the condition on the particles initial velocities that determines whether or not the particle is trapped. By taking a particle with an initial velocity of $v_{\parallel 0}$ at the midplane ($B = B_0$) as having too just enough energy to escape

$$\frac{1}{2}mv_{\parallel 0}^2 + \mu B_0 \leq \mu B_{max} \tag{4.7}$$

Rearranging

$$\begin{aligned}
\frac{\frac{1}{2}mv_{\parallel 0}^2}{\mu B_0} + 1 &\leq \frac{B_{max}}{B_0} \\
\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} + 1 &\leq \frac{B_{max}}{B_0}
\end{aligned}$$

Finally, defining the mirror ration $R = \frac{B_{max}}{B_0}$ and using $v^2 = v_\perp^2 + v_\parallel^2$, one gets

$$\frac{v^2}{v_\perp^2} \leq R \tag{4.8}$$

Noticing that $\sin\theta = \frac{v_\perp}{v}$, one can rearrange Eq. (4.8) to the following

$$\sin\theta \geq \left(\frac{1}{R}\right)^{\frac{1}{2}} = \left(\frac{B_0}{B_{max}}\right)^{\frac{1}{2}} \tag{4.9}$$

Thus, the trapping condition in Eq. (4.9) states that the any θ of a confined particle that is smaller than defined in the condition is not trapped by a magnetic mirror.

4.2 Other Confinement Schemes

4.2.1 Common Machine Parameters

Device	$\frac{R}{a}$	$\frac{B_T}{B_\theta}$	β
Tokamak	3 - 5	10 - 20	5%
Spherical Torus	1.2 - 1.5	5	25%
RFP	3 - 5	1	25%
Spheromak	1 - 1.5	.5	50%
FRC	1	0	1

4.2.2 Simple Tokamak Definitions

For other Tokamak related information/parameters, please refer to sections 7.3 on page 34 and 7 on page 33.

ϵ - Inverse Aspect Ratio

$$\epsilon = \frac{r}{R_0} \tag{4.10}$$

q - Safety Factor q is the average pitch of the magnetic field line.

$$q = \left\langle \frac{d\phi}{d\theta} \right\rangle = \frac{rB_\phi}{R_0 B_\theta} = \epsilon \frac{B_T}{B_P} = \frac{\text{winding \# long way}}{\text{winding \# short way}} \tag{4.11}$$

ω_v - Bounce Frequency

$$\omega_b = \epsilon^{\frac{1}{2}} \frac{v_{th}}{qR} \tag{4.12}$$

$(v_D)_r$ **- Radial Drift Velocity**

$$(v_D)_r = \frac{\rho v_{th}}{R} \tag{4.13}$$

5 GPP2 Miscellaneous

5.1 Flux Functions

$$\nabla\psi \times \nabla\phi = B(R, z) \tag{5.1}$$

$$\underline{\boldsymbol{B}} = \nabla\psi \times \nabla\phi + B_\phi \hat{\boldsymbol{\phi}} \tag{5.2}$$

Flux surfaces have constant ψ (ψ is the flux function). The $\nabla\psi \times \nabla\phi$ part represents the poloidal field, while the $B_\phi \hat{\boldsymbol{\phi}}$ is the toroidal field. Other relevant equations:

$$\underline{\boldsymbol{B}} \cdot \nabla\psi = 0 \tag{5.3}$$

$$\psi = \int_0^R R' B_0(R', z) dR' \tag{5.4}$$

Most places have B_0 in the definition of ψ equal to B_P, B_z, or B_θ. This can be gotten from from the fact that

$$\underline{\boldsymbol{B}} = \nabla \times \underline{\boldsymbol{A}} \tag{5.5}$$

$$B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) \tag{5.6}$$

$$B_R = -\frac{\partial A_\phi}{\partial z} \tag{5.7}$$

Since we know that $\underline{\boldsymbol{B}}_p = \nabla\psi \times \nabla\phi$, or

$$\underline{\boldsymbol{B}}_p = -\frac{1}{R}\frac{\partial\psi}{\partial z}\hat{\boldsymbol{R}} + \frac{1}{R}\frac{\partial\psi}{\partial R}\hat{\boldsymbol{z}} \tag{5.8}$$

we can equate the two expressions for $\underline{\boldsymbol{B}}$, arriving at the above integral definition of ψ from the integral form of $\nabla \times \underline{\boldsymbol{A}} = \underline{\boldsymbol{B}}$ and

$$\psi = RA_\phi \tag{5.9}$$

5.1.1 Slab Geometry

$$\psi(x, y) = \psi_0(x) - \epsilon\psi_1(x) \cos ky \tag{5.10}$$

$$\underline{\boldsymbol{B}} = \nabla\psi \times \hat{\boldsymbol{z}} + B_z \hat{\boldsymbol{z}} \tag{5.11}$$

Expand $x = x_0 + \epsilon x_1$ for flux surfaces. $y = 0 = 2\pi r$ is periodic, as is $z = 0 = 2\pi R_0$.

5.1.2 Cylindrical Geometry

$$\psi(r,\theta,z) = \psi_0(r) - \epsilon\psi_1(r)\cos(m\theta - kz) \tag{5.12}$$

$$\underline{\boldsymbol{B}} = \boldsymbol{\nabla}\psi \times \hat{\boldsymbol{z}} + B_z\left(\hat{\boldsymbol{z}} + \frac{kr}{m}\hat{\boldsymbol{\theta}}\right) \tag{5.13}$$

Expand $r = r_0 + \epsilon r_1$ for flux surfaces. $z = 0 = 2\pi R_0$ is periodic.

5.1.3 Toroidal Geometry

$$\psi = \psi(R,z) = RA_\phi \tag{5.14}$$

$$\underline{\boldsymbol{B}} = \boldsymbol{\nabla}\psi \times \frac{1}{R}\hat{\boldsymbol{\phi}} + B_\phi\hat{\boldsymbol{\phi}} \tag{5.15}$$

5.2 Field Line Curvature

We start with Ampere's law.[3]

$$\underline{\boldsymbol{J}} = \frac{1}{\mu}\boldsymbol{\nabla} \times \underline{\boldsymbol{B}} \tag{5.16}$$

$$\underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} = \frac{1}{\mu}\left(\boldsymbol{\nabla} \times \underline{\boldsymbol{B}}\right) \times \underline{\boldsymbol{B}} \tag{5.17}$$

$$\tag{5.18}$$

If we rewrite the vector identity

$$\boldsymbol{\nabla}(\underline{\boldsymbol{A}} \cdot \underline{\boldsymbol{B}}) = \underline{\boldsymbol{A}} \times (\boldsymbol{\nabla} \times \underline{\boldsymbol{B}}) + \underline{\boldsymbol{B}} \times (\boldsymbol{\nabla} \times \underline{\boldsymbol{A}}) + (\underline{\boldsymbol{A}} \cdot \boldsymbol{\nabla})\underline{\boldsymbol{B}} + (\underline{\boldsymbol{B}} \cdot \boldsymbol{\nabla})\underline{\boldsymbol{A}} \tag{5.19}$$

setting $\underline{\boldsymbol{A}} = \underline{\boldsymbol{B}}$, and using it on the RHS of the $\underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}}$ equation, we get

$$(\boldsymbol{\nabla} \times \underline{\boldsymbol{B}}) \times \underline{\boldsymbol{B}} = \underline{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\underline{\boldsymbol{B}} - \frac{1}{2}\boldsymbol{\nabla}B^2 \tag{5.20}$$

Now we have to examine the $\underline{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\underline{\boldsymbol{B}}$ term.

$$\underline{\boldsymbol{B}} \cdot \boldsymbol{\nabla}\underline{\boldsymbol{B}} = B\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}(B\hat{\boldsymbol{b}}) = B^2\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}\hat{\boldsymbol{b}} + \underbrace{B\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla}B}_{=\frac{1}{2}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}(B^2)} \tag{5.21}$$

[3] If we choose, we can add a $-\boldsymbol{\nabla}P$ to each side at this point. If we don't want a scalar P, we use $\boldsymbol{\nabla}\cdot(\overline{\overline{\mathbb{1}}}P)$. Of course, this complicates things, but if we carry out the derivation we would get $\underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} - \boldsymbol{\nabla}P = \boldsymbol{\nabla}\cdot\left[\frac{\underline{\boldsymbol{B}}\,\underline{\boldsymbol{B}}}{\mu} - \overline{\overline{\mathbb{1}}}\left(P + \frac{B^2}{2\mu}\right)\right]$. See Section 6.4 on page 27

Simplify using

$$\underline{\boldsymbol{\kappa}} = \hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \hat{\boldsymbol{b}} \tag{5.22}$$

$$\boldsymbol{\nabla}_{\perp} = \boldsymbol{\nabla} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \cdot \boldsymbol{\nabla} \tag{5.23}$$

we arrive at

$$\mu \underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} = \underline{\boldsymbol{\kappa}} B^2 - \boldsymbol{\nabla}_{\perp} \frac{B^2}{2} \tag{5.24}$$

The first term on the RHS of equation 5.24 represents the tension in the field lines, while the second term represents magnetic pressure.

5.3 Conservative Forms

Conservative forms are useful when working with mass, energy, and density. When one puts equations representing these quantities in the following form, it is known as the conservative form:

$$\frac{\partial}{\partial t}(\;) + \boldsymbol{\nabla} \cdot (\;) = 0 \tag{5.25}$$

Three equations for mass, energy, and momentum in their conservative forms follow below.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \underline{\boldsymbol{v}}) = 0 \qquad \text{Mass (density)} \tag{5.26}$$

$$\frac{\partial W}{\partial t} + \boldsymbol{\nabla} \cdot \underline{\boldsymbol{S}} = 0 \qquad \text{Energy} \tag{5.27}$$

where W is energy and S is defined below

$$W = \frac{1}{2}\rho v^2 + \frac{B^2}{2\mu} + \frac{P}{\gamma - 1}$$

$$\underline{\boldsymbol{S}} = \underline{\boldsymbol{v}} \left(\rho \frac{v^2}{2} + \frac{\gamma}{\gamma - 2} P \right) + \frac{1}{\mu} \underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}$$

$$\frac{\partial}{\partial t}(\rho \underline{\boldsymbol{v}}) + \boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{T}}} \qquad \text{Momentum} \tag{5.28}$$

where T is the stress tensor, defined below

$$\overline{\overline{\boldsymbol{T}}} = \rho \underline{\boldsymbol{v}\boldsymbol{v}} + \left(P + \frac{B^2}{2\mu} \right) \overline{\overline{\boldsymbol{I}}} - \frac{1}{\mu} \underline{\boldsymbol{B}\boldsymbol{B}}$$

5.4 δW

To look for stability in a system, one tries to make δW as small as possible. If δW is still > 0 after one does this, then it is stable.

$$\delta W = \frac{1}{2}\int_v \Bigg[\underbrace{\frac{|Q_\perp|^2}{\mu}}_{1} + \underbrace{\frac{B^2}{\mu}\left|\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\zeta}}_\perp + 2\underline{\boldsymbol{\zeta}}_\perp\cdot\underline{\boldsymbol{\kappa}}\right|^2}_{2} - \underbrace{2\left(\underline{\boldsymbol{\zeta}}_\perp\cdot\boldsymbol{\nabla}P\right)\left(\underline{\boldsymbol{\zeta}}^*_\perp\cdot\underline{\boldsymbol{\kappa}}\right)}_{3} - \underbrace{j_\parallel\underline{\boldsymbol{\zeta}}^*_\perp\cdot\hat{\boldsymbol{b}}\times\underline{\boldsymbol{Q}}_\perp}_{4} + \underbrace{\gamma P\left|\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\zeta}}\right|^2}_{5}\Bigg]d^3x \quad (5.29)$$

Terms 1, 2, and 5 in equation (5.29) are all stabilizing terms. 1 corresponds to shear Alfvén waves and the energy required to bend field lines. 2 matches up with compressional Alfvén waves and the energy required to compress field lines. 5 represents sound waves and the energy necessary to compress the plasma. The last two terms, 3 and 4 are destabilizing terms, and are pressure and current driven terms.

6 MHD

6.1 MHD Equations

To begin, we start out with Boltzmann's Equation.

$$\frac{\partial f}{\partial t} + \underline{\boldsymbol{v}} \cdot \boldsymbol{\nabla} f + \frac{q}{m}\left(\underline{\boldsymbol{E}} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}\right) \cdot \boldsymbol{\nabla}_v f = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{6.1}$$

The collision term is 0 for ideal MHD. If we take moments of this, we can get the continuity equation and force balance equation. With Ampere's Law, Faraday's Law, Ohm's Law and an equation of state, we have the MHD equations.

Conservation of Momentum:

$$\rho\frac{\partial \underline{\boldsymbol{v}}}{\partial t} + \rho\underline{\boldsymbol{v}} \cdot \boldsymbol{\nabla}\underline{\boldsymbol{v}} = -\boldsymbol{\nabla} p + \underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} + \rho g \tag{6.2}$$

Continuity Equation:

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho\underline{\boldsymbol{v}}) = -\rho\boldsymbol{\nabla} \cdot \underline{\boldsymbol{v}} - \underline{\boldsymbol{v}} \cdot \boldsymbol{\nabla}\rho \tag{6.3}$$

Ampere's Law:

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{B}} = \mu_0\underline{\boldsymbol{J}} + \mu_0\epsilon_0\frac{\partial \underline{\boldsymbol{E}}}{\partial t} \tag{6.4}$$

Faraday's Law:

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{E}} = -\frac{\partial B}{\partial t} \tag{6.5}$$

Ohm's Law:

$$\underline{\boldsymbol{E}} - \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}} + \left(- \frac{\underline{\boldsymbol{J}} \times \underline{\boldsymbol{V}}}{ne\epsilon_0} + \frac{\boldsymbol{\nabla} \cdot \Pi_e}{ne\epsilon_0} + \frac{\rho g}{ne\epsilon_0}\right) = \eta\underline{\boldsymbol{J}} \tag{6.6}$$

Equation of State

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{6.7}$$

or

$$\frac{dp}{dt} + \gamma p\boldsymbol{\nabla} \cdot \underline{\boldsymbol{v}} = 0 \tag{6.8}$$

For the derivation of the alternate form of the adiabatic equation, use Eq. (6.3). Also, see Section 9.11.2 on page 61.

For Ohm's Law, the terms in the parenthesis are often ignored for MHD. η is the resistivity

$$\eta = \frac{m_e \nu_{ei}}{ne^2} \tag{6.9}$$

For Ideal MHD, $\eta = 0$.

Alternately, for the Equation of State, one can use either of the following two equations, the first being for incompressible fluids, while the second two are derived from the first.

$$\boldsymbol{\nabla} \cdot \underline{\boldsymbol{v}} = 0 \tag{6.10}$$

$$\frac{d\rho}{dt} = 0 \tag{6.11}$$

$$\frac{dp}{dt} = 0 \tag{6.12}$$

Lastly the Force Balance equation is simply Eq. (6.2) with $\rho \frac{\partial v}{\partial t} = 0$, so

$$\boldsymbol{\nabla} p = \underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} \tag{6.13}$$

6.2 Assumptions for MHD

Assumptions made for MHD to be valid are:

1. small gyroradius
2. high collisionality (scalar p)
3. small resistivity, with $\eta = 0$ in Ideal MHD.

6.3 Magnetic Islands

In slab symmetry

$$\underline{\boldsymbol{B}}(x, y) = \boldsymbol{\nabla}\psi(x, y) \times \hat{\boldsymbol{z}} + B_z(x)\hat{\boldsymbol{z}} \tag{6.14}$$

$$\psi(x, y) = \psi_0(x) - \epsilon\psi_1(x)\cos(ky) \tag{6.15}$$

$$x = x_0 + \epsilon x_1 \tag{6.16}$$

First we sub in the definition of x into our equation for ψ and Taylor expand around x_0. Finally we solve for x_1.

$$\psi_0(x_0) \rightarrow$$

$$\psi(x_0+\epsilon x_1, y) = \psi_0(x_0+\epsilon x_1) - \epsilon\psi_1(x_0+\epsilon x_1)\cos(ky) = \text{const.} \tag{6.17}$$

$$\psi = \psi_0(x_0) - \epsilon x_1\psi_0'(x_0) - \epsilon\big[\psi_1(x_0) - \underbrace{\epsilon x_1\psi_1'(x_0)}_{O(\epsilon^2)\rightarrow 0}\big]\cos(ky) = \underbrace{\psi_0(x_0)}_{\text{const.}} \tag{6.18}$$

$$x_1 = \frac{\psi_1(x_0)}{\psi_0'(x_0)}\cos(ky) \tag{6.19}$$

All this leads us to the perturbed flux surfaces.[4]

$$x = x_0 + \epsilon\frac{\psi_1(x_0)}{\psi_0'(x_0)}\cos(ky) \tag{6.20}$$

When $\psi_0' \rightarrow 0$, then $x \rightarrow \infty$. Thus, this description breaks down at these surfaces, called rational surfaces. Magnetic islands occur in a plasma at rational surfaces, or when $q(r_0) = \frac{m}{n}$.

To continue the calculation to see what occurs at these rational surfaces, one must go the 2nd order. In order to do this, we set

$$\psi_0'\big|_{x=x_0} = \psi_0(x_0) = 0 \tag{6.21}$$

We then Taylor expand ψ again to the second order.

$$\psi = \delta\psi = \frac{x_1^2}{2}\psi_0''(x_0) - \epsilon\psi_1(x_0)\cos(ky) \tag{6.22}$$

After solving for x_1, we get

$$x_1 = \pm 2\sqrt{\frac{\epsilon\psi_1(x_0)}{\psi_0''(x_0)}}\cos\left(\frac{ky}{2}\right) \tag{6.23}$$

To normalize this equation, one notes that

$$\cos(ky) = 2\cos^2\left(\frac{ky}{2}\right) - 1$$
$$\cos^2 = 1 - \sin^2$$

[4]If we wish to do the same for cylindrical geometry, simple replace the x's with r's, and the argument of the cos with $m\theta - kz$.

Also, one defines the following terms

$$w = 4\sqrt{\frac{\epsilon\psi_1}{\psi_0''}} \tag{6.24}$$

$$\rho^2 = \frac{\delta\psi + \epsilon\psi_1}{2\epsilon\psi_1} \tag{6.25}$$

where w is the island width and ρ is the flux coordinate. Putting all of this together, one gets

$$x_1 = \pm\frac{w}{2}\sqrt{\rho^2 - \sin^2\left(\frac{ky}{2}\right)} \tag{6.26}$$

For more on tearing modes, see Section 6.6 on page 31.

6.4 MHD Equilibrium

To derive the static MHD equilibrium equations, take $\frac{\partial}{\partial t} = 0$ and $\underline{\boldsymbol{v}} = 0$ in the MHD equations to get the following.

$$\boldsymbol{\nabla} p = \underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} \tag{6.27}$$

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{B}} = \mu_0 \underline{\boldsymbol{J}} \tag{6.28}$$

$$\boldsymbol{\nabla} \cdot \underline{\boldsymbol{B}} = 0 \tag{6.29}$$

For non scalar pressures, the first equation above can be written as

$$\underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} = -\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{T}}}_m \tag{6.30}$$

$$\overline{\overline{\boldsymbol{T}}} = \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} T_\parallel + (\overline{\overline{\boldsymbol{I}}} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}) T_\perp \tag{6.31}$$

$$\overline{\overline{\boldsymbol{T}}} = \begin{pmatrix} T_\perp & 0 & 0 \\ 0 & T_\perp & 0 \\ 0 & 0 & T_\parallel \end{pmatrix} \tag{6.32}$$

where

$$T_\perp = p_{cm\perp} + \frac{B^2}{2\mu_0} \tag{6.33}$$

$$T_\parallel = p_{cm\parallel} - \frac{B^2}{2\mu_0} \tag{6.34}$$

In the steady state (setting V_{cm} in Hazeltine to 0), we can write

$$\boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{T}}} = 0 \tag{6.35}$$

and by multiplying by $\underline{\boldsymbol{x}}$, integrating, and applying Gauss's law, we can calculate the virial.

$$\underline{\boldsymbol{x}} \cdot \boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{T}}} = \boldsymbol{\nabla} \cdot (\underline{\boldsymbol{x}} \cdot \overline{\overline{\boldsymbol{T}}}) - Tr(\overline{\overline{\boldsymbol{T}}}) \tag{6.36}$$

$$\int_V \boldsymbol{\nabla} \cdot (\underline{\boldsymbol{x}} \cdot \overline{\overline{\boldsymbol{T}}}) d^3x = \int_S (\underline{\boldsymbol{x}} \cdot \overline{\overline{\boldsymbol{T}}}) \cdot d\underline{\boldsymbol{S}} = \int_V \left(3p_c m + \frac{B^2}{2\mu_0} \right) d^3x \tag{6.37}$$

In a confined plasma, one can set volume V large enough so that the surface S lies outside the plasma. p_{cm} and T_m vanish here and the magnetic field is due purely to plasma currents. Thus the virial become arbitrarily small. In this same large-V limit, the right hand side of the above equation becomes independent of V and remains finite. Thus, *a plasma cannot be confined by its self-generated electromagnetic field alone.*

See pages 77-79 in Hazeltine.

6.4.1 Screw Pinch Equilibrium

To derive the screw pinch equilibrium, one starts off with the force balance equation, Eq. (6.38), and the equation for field line curvature derived above, Eq. (5.24).

$$\boldsymbol{\nabla} p = \underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}} \tag{6.38}$$

Combining the two by setting both equations equal to $\underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}}$, one notes that

$$R_c = \frac{1}{\kappa} = r \left(1 + \frac{B_z^2}{B_\theta^2} \right)$$

$$\kappa = -\frac{B_\theta^2}{rB^2} \hat{\boldsymbol{r}}$$

Putting this all together takes us to the screw pinch equilibrium equation.

$$\frac{\partial}{\partial r} \left(p + \frac{B^2}{2\mu} \right) - \frac{1}{\mu} \frac{B_\theta^2}{r} = 0 \tag{6.39}$$

$$\frac{\partial}{\partial r} \left(\text{pressure} + \begin{matrix}\text{magnetic} \\ \text{pressure}\end{matrix} \right) + \left(\begin{matrix}\text{field line} \\ \text{tension}\end{matrix} \right) = 0$$

6.5 Grad-Shafranov Equation - Axisymmetric Equilibrium

To derive the Grad-Shafranov equation, we essentially take the Force Balance equation (Eq. (6.13)) and Ampere's Law. Starting with an equation for $\underline{\boldsymbol{B}}$

$$\underline{\boldsymbol{B}} = \underbrace{\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi}_{\underline{\boldsymbol{B}}_p} + \underbrace{B_\phi \hat{\boldsymbol{\phi}}}_{\underline{\boldsymbol{B}}_T} \tag{6.40}$$

We also know

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{\nabla} \times \underline{\boldsymbol{B}}_p = 0 \tag{6.41}$$

$$\hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \underline{\boldsymbol{B}}_p = 0 \tag{6.42}$$

We then take the curl of $\underline{\boldsymbol{B}}_p$ to get the Grad-Shafranov operator, defined below.

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{B}}_p = \hat{\boldsymbol{\phi}} \left(\frac{\partial B_{pr}}{\partial z} - \frac{\partial B_{pz}}{\partial r} \right) = \hat{\boldsymbol{\phi}} \left[-\frac{1}{R} \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) \right] = -\frac{\hat{\boldsymbol{\phi}}}{R} \Delta^* \psi \tag{6.43}$$

where

$$\Delta^* \psi = -R\hat{\boldsymbol{\phi}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\psi \boldsymbol{\nabla}\phi) = \frac{\partial^2 \phi}{\partial z^2} + R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \phi}{\partial R} \right) \tag{6.44}$$

Since

1. Take $\boldsymbol{\nabla} \times \underline{\boldsymbol{B}} = \mu \underline{\boldsymbol{j}}$.

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{B}} = \mu \underline{\boldsymbol{j}} \tag{6.45}$$

or

$$\mu \underline{\boldsymbol{j}} = (\boldsymbol{\nabla} \times \underline{\boldsymbol{B}})_\phi + (\boldsymbol{\nabla} \times \underline{\boldsymbol{B}})_p \tag{6.46}$$

which, using Eq. (6.43) and

$$\boldsymbol{\nabla} \times B_\phi \hat{\boldsymbol{\phi}} = \boldsymbol{\nabla} \times (RB_\phi \boldsymbol{\nabla}\phi) = \boldsymbol{\nabla}(RB_\phi) \times \boldsymbol{\nabla}\phi + \underbrace{(RB_\phi)\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi}_{\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi \to 0} \tag{6.47}$$

we can rewrite as

$$\mu \underline{\boldsymbol{j}} = -\Delta^* \psi \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}(RB_\phi) \times \boldsymbol{\nabla}\phi \tag{6.48}$$

We then take a minute to look at the following. Since

2. Operate on the Force Balance Eq. with $\underline{\boldsymbol{B}} \cdot (\quad)$.

$$\underline{\boldsymbol{j}} \times \underline{\boldsymbol{B}} = \boldsymbol{\nabla} P \tag{6.49}$$

then

$$\underline{\boldsymbol{B}} \cdot (\underline{\boldsymbol{j}} \times \underline{\boldsymbol{B}}) = \underline{\boldsymbol{B}} \cdot \boldsymbol{\nabla} P = 0 \tag{6.50}$$

The above statement is true since $\underline{\boldsymbol{B}} \perp (\underline{\boldsymbol{j}} \times \underline{\boldsymbol{B}})$. Since this means that P is constant along $\underline{\boldsymbol{B}}$ lines, then $P(\psi)$. Finally,

$$\boldsymbol{\nabla} P = \frac{dP}{d\psi} \boldsymbol{\nabla} \psi \tag{6.51}$$

Using Eq. (6.49) again, we find that if we dot $\underline{\boldsymbol{j}}$ into it, we get

3. Operate on the Force Balance Eq. with $\underline{\boldsymbol{j}} \cdot (\quad)$.

$$\underline{\boldsymbol{j}} \cdot \underline{\boldsymbol{j}} \times \underline{\boldsymbol{B}} = 0 = \underline{\boldsymbol{j}} \cdot \boldsymbol{\nabla} P \tag{6.52}$$

which, using the result from Eq. (6.51), becomes

$$\underline{\boldsymbol{j}} \cdot \boldsymbol{\nabla} \psi = 0 \tag{6.53}$$

Returning back to Eq. (6.48), dotting it with $\boldsymbol{\nabla}\psi$, and using the preceding result to set the whole thing to zero,

4. Dot $\boldsymbol{\nabla} \times \underline{\boldsymbol{B}} = \mu \underline{\boldsymbol{j}}$ with $\boldsymbol{\nabla}\psi$.

$$[-\Delta^* \psi \boldsymbol{\nabla} \phi + \boldsymbol{\nabla}(R B_\phi) \times \boldsymbol{\nabla} \phi] \cdot \boldsymbol{\nabla} \psi = 0 \tag{6.54}$$

The first term in the brackets goes away since ψ is independent of $\hat{\boldsymbol{\phi}}$. The rest of the equation becomes

$$\boldsymbol{\nabla}(R B_\phi) \times \boldsymbol{\nabla} \phi \cdot \boldsymbol{\nabla} \psi = \boldsymbol{\nabla}(R B_\phi) \cdot \underbrace{\boldsymbol{\nabla} \phi \times \boldsymbol{\nabla} \psi}_{-\underline{\boldsymbol{B}}_p} = 0 \tag{6.55}$$

$$\underline{\boldsymbol{B}}_p \cdot \boldsymbol{\nabla}(R B_\phi) = 0 \tag{6.56}$$

$$\Downarrow$$

$$R B_\phi = F(\psi) \tag{6.57}$$

Plugging all of this into equations (6.40) and (6.48)

$$\underline{\boldsymbol{B}} = \boldsymbol{\nabla} \psi \times \boldsymbol{\nabla} \phi + F(\psi) \boldsymbol{\nabla} \phi \tag{6.58}$$

$$\mu \underline{\boldsymbol{j}} = -\Delta^* \psi \boldsymbol{\nabla} \phi + F'(\psi) \boldsymbol{\nabla} \psi \times \boldsymbol{\nabla} \phi \tag{6.59}$$

Crossing these two equations together gives us

5. Take $\mu \underline{\boldsymbol{J}} \times \underline{\boldsymbol{B}}$.

$$\mu \underline{\boldsymbol{j}} \times \underline{\boldsymbol{B}} = \underbrace{\boldsymbol{\nabla} \phi \times (\boldsymbol{\nabla} \psi \times \boldsymbol{\nabla} \phi)}_{\Downarrow} \left[-\Delta^* \psi - F F'\right] = \mu \boldsymbol{\nabla} P = \mu P'(\psi) \boldsymbol{\nabla} \psi \tag{6.60}$$

$$= \underbrace{\boldsymbol{\nabla} \psi (\boldsymbol{\nabla} \phi)^2}_{=\frac{1}{R^2} \boldsymbol{\nabla} \psi} - \boldsymbol{\nabla} \phi \underbrace{(\boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \phi)}_{=0} \tag{6.61}$$

Which leads us to the final form of the Grad-Shafranov equation:

$$\frac{1}{R^2}\nabla\psi(-\Delta^*\psi - FF') = \mu P'(\psi)\nabla\psi$$
$$\Downarrow$$
$$\boxed{\Delta^*\psi = -FF' - \mu R^2 P'(\psi)} \tag{6.62}$$

6.6 Tearing Modes

Consider a system with a magnetic field in slab geometry

$$\underline{\boldsymbol{B}} = \nabla\psi \times \hat{\boldsymbol{z}} + B_z\hat{\boldsymbol{z}} \tag{6.63}$$

$B_z \gg B_x, B_y \Rightarrow \nabla \cdot \underline{\boldsymbol{v}} = 0$. We then look at the Grad-Shafranov equation with $P = 0$

$$\nabla^2\psi = -\mu j_z \tag{6.64}$$

The next step is to expand ψ and j_z about the equilibrium (Fourier decompose?).

$$\psi = \psi_0(x) - \epsilon\psi_1(x)\cos(ky) + \epsilon^2(\psi_2(x)\cos(2ky) + \delta\psi_0) \tag{6.65}$$
$$j_z = j_0(x) + \epsilon j_1(x)\cos(ky) \tag{6.66}$$

The zeroth order Grad-Shafranov eq. is

$$\frac{\partial^2\psi_0}{\partial x^2} = -\mu j_0 \tag{6.67}$$

while to the first order is

$$\frac{\partial^2\psi_1}{\partial x^2} - k^2\psi_1 = \mu j_1 \tag{6.68}$$

We can also decompose the current j_z to an equilibrium component ($j_z^0(\psi)$) and an island component, or perturbation ($\delta j_z(\psi)$). After we do that, we can Taylor expand j_z about ψ_0.

$$j_z(\psi) = j_{z0} + \delta j_z(\psi) = j_z(\psi_0 - \epsilon\psi_1\cos(ky)) \approx j_z(\psi_0) - \epsilon\psi_1\cos(ky)\left.\frac{\partial j_z}{\partial\psi}\right|_{\psi_0}$$
$$= j_{z0}(\psi_0) + \delta j_z(\psi_0) - \epsilon\psi_1\cos(ky)\left.\frac{\partial j_{z0}}{\partial\psi}\right|_{\psi_0} \tag{6.69}$$

So, by comparing Eq. (6.69) to Eqs. (6.64), (6.65), and (6.66), we can identify the following. Far from the island, the perturbation component of j_z goes to zero ($\delta j_z \to 0$).

$$\epsilon^2 \nabla^2 \delta\psi_0 = -\mu \delta j_z \tag{6.70}$$

$$j_1 = -\psi_1 \frac{\partial j_{z0}}{\partial \psi_0}\bigg|_{\psi_0} = -\psi_1 \frac{dj_{z0}/dx}{d\psi_0/dx} \tag{6.71}$$

Plugging Eq. (6.71) back into the first order Grad-Shafranov equation, Eq. (6.68), we get the exterior tearing mode equation.

$$\boxed{\psi_1'' = k^2\psi_1 - \mu \frac{dj_{z0}/dx}{d\psi_0/dx}\psi_1} \tag{6.72}$$

$\psi_1 \to 0$ at the boundaries. The tearing mode equation breaks down where $\epsilon\psi_1 \sim \psi_0 \sim \psi_0'' \frac{x^2}{2}$. This happens near $x = 0$.

If we take Eq. (6.68) and integrate across layers, we can get the jump condition.

$$\psi_1'(\delta) - \psi_1'(-\delta) - \underbrace{k^2\psi_1 2\delta}_{\text{small}} = \mu \underbrace{\int_{-\delta}^{\delta} j_1(x)dx}_{\equiv \kappa_1} \tag{6.73}$$

κ_1 is the surface current. We can define Δ' as

$$\Delta' \equiv \frac{\psi_1'(\delta) - \psi_1'(-\delta)}{\psi_1(x=0)} = \mu \frac{\kappa_1}{\psi_1(x=0)} \tag{6.74}$$

The surface current can be expressed as

$$\kappa_1 = .41w\frac{\dot{\psi}_1}{\eta} \tag{6.75}$$

where w is the width of the island. One can use this to calculate the growth rate.

$$\frac{dw}{dt} = 1.2\frac{\eta}{\mu}\Delta' \quad \text{where if} \quad \begin{cases} \Delta' > 0 & \Rightarrow \text{unstable} \\ \Delta' < 0 & \Rightarrow \text{stable} \end{cases} \tag{6.76}$$

7 Transport Coefficients

To begin our heuristic estimates of the transport coefficients in a tokamak, we first present the classic for of a diffusion coefficient.

$$D = \frac{(\Delta x)^2}{\Delta t} \tag{7.1}$$

where Δx is the characteristic length over which the diffusion takes place over a time Δt.

7.1 Classical Diffusion

Classical diffusion comes from the particles' collisions as they travel along the magnetic field lines. As a result, the length associated with this diffusion is the Larmor radius ρ_e, while the time scale is the reciprocal of the collision time ν_{ei}. Related to $\eta_\perp$.

$$D_{class} = \frac{(\Delta x)^2}{\Delta t} = \frac{\rho_e^2}{1/\nu_{ei}} = \boxed{\rho_e^2 \nu_{ei}} \tag{7.2}$$

7.2 Neoclassical Diffusion

Neoclassical diffusion comes from the transition from a cylindrical system to a toroidal system. Thus it takes into account $\underline{\boldsymbol{B}} \times \boldsymbol{\nabla} \underline{\boldsymbol{B}}$ drifts. It is also identified with the Pfirsh-Schlüter regime. Related to $\eta_\parallel$.

$$D_{ps} = (v_D)_r^2 \Delta t_\parallel \tag{7.3}$$

$$D_\parallel = \frac{(\Delta x_\parallel)^2}{\Delta t_\parallel} = \frac{v_{th}^2}{\nu_{ei}} \tag{7.4}$$

where $\Delta x_\parallel = R_q$ is the connection length.

$$\Delta t_\parallel = \frac{R^2 q^2 \nu_{ei}}{v_{th}^2} \tag{7.5}$$

Plugging this all into the forumula for D_{ps} gives us

$$D_{ps} = \frac{R^2 q^2 \nu_{ei}}{v_{th}^2} \cdot \frac{v_{th}^2 \rho_e^2}{R^2} = \boxed{\rho_e^2 q^2 \nu_{ei}} \tag{7.6}$$

For a derivation of $(v_D)_r$, look at Eqs. (7.19)-(7.21) in Section 7.3.3.

7.3 Tokamak Parameters

Before we go any further with derivations, we have to get a couple of tokamak parameters out of the way.

7.3.1 Tokamak Trapping

$$B = B_0(1 - \epsilon\cos\theta) \tag{7.7}$$

$$\frac{1}{2}v_\parallel^2 + \mu B = E = \text{constant} \tag{7.8}$$

Reflection is at $v_\parallel = 0$.

$$B_{refl} = \frac{E}{\mu} \tag{7.9}$$

Trapped Particle: $B_{min} \leq \frac{E}{\mu} \leq B_{max}$

$$\underbrace{\frac{1}{2}v_\parallel^2\Big|_{B_{min}} + \frac{1}{2}v_\perp^2\Big|_{B_{min}}}_{E} \leq \underbrace{B_{max}}_{\frac{1}{2}v_\perp^2\big|_{B_{min}}\frac{B_{max}}{B_{min}}} \tag{7.10}$$

$$\Downarrow$$

$$\frac{v_\parallel^2}{v_\perp^2} \leq \frac{B_{max}}{B_{min}} - 1 \sim \frac{1+\epsilon}{1-\epsilon} - 1 \simeq 2\epsilon \tag{7.11}$$

$$\boxed{\left|\frac{v_\parallel}{v_\perp}\right| \leq (2\epsilon)^{\frac{1}{2}}} \tag{7.12}$$

If $\frac{E}{\mu} > B$, then the particle is untrapped.

7.3.2 Bounce Frequency

$$\tau_b = \frac{2\pi}{\omega_b} = 2\int_{-\theta_0}^{\theta_0} d\theta \frac{R_0 q}{\left|v_\parallel(\theta)\right|} \tag{7.13}$$

The 2 in front of the integral comes from the fact that we have to take into account the particle moving up and then back. The following definitions apply.

$$v_\parallel = \sqrt{2(E - \mu B)} \tag{7.14}$$

$$\lambda = \frac{\mu B_0}{E} \quad \text{pitch angle} \tag{7.15}$$

$$v_\parallel(\theta) = \sqrt{2E(1 - \lambda(1 - \epsilon\cos\theta))} \tag{7.16}$$

$$\omega_b = \frac{\epsilon^{\frac{1}{2}} v_{th}}{qR_0} \tag{7.17}$$

7.3.3 Banana Width or Excursion

$$\Lambda_j \simeq \frac{(v_D)_{rad}}{\omega_b} \simeq \frac{\rho_j v_{th}}{R_0}\frac{qR_0}{\epsilon^{\frac{1}{2}} v_{th}} \Rightarrow \boxed{\Lambda_j = \frac{\rho_j q}{\epsilon^{\frac{1}{2}}}} \tag{7.18}$$

where $\epsilon = \frac{r}{R}$. To get $(v_D)_{rad}$, one must calculate the radial component of the drift velocity v_D.

$$v_D = \underbrace{\frac{c}{B^2}\boldsymbol{E}\times\boldsymbol{B}}_{\boldsymbol{E}\times\boldsymbol{B}} + \frac{1}{\Omega}\hat{\boldsymbol{n}} \times \big(\underbrace{\mu\boldsymbol{\nabla}B}_{\boldsymbol{\nabla}B} + \underbrace{\hat{\boldsymbol{n}}v_\parallel^2 \cdot \boldsymbol{\nabla}\hat{\boldsymbol{n}}}_{\text{curvature}}\big) \tag{7.19}$$

$$(v_D)_{rad} \propto (\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}B)_{rad} = -\frac{1}{r}\frac{\partial}{\partial\theta}B = -\frac{B_0}{R_0}\sin\theta \tag{7.20}$$

$$\Rightarrow (v_D)_{rad} \sim \frac{v_{th}^2}{\Omega R_0} = \frac{\rho v_{th}}{R_0} \tag{7.21}$$

Another way to calculate banana width is using conservation of toroidal angular momentum.

$$P_\zeta = mRV_\zeta - \underbrace{eRA_\zeta}_{e\psi} = C \tag{7.22}$$

take

$$A_\zeta = -\int_0^r dr' B_p(r') \approx -\int_0^{r_0} dr' B_p(r') - \underbrace{(r - r_0)}_{\Lambda_j = r - r_0} B_p(r_0) \tag{7.23}$$

$$P_\zeta = R\left[mV_\zeta + \frac{e}{c}\int_0^{r_0} dr' B_p(r') + \frac{e}{c}\Lambda B_p\right] \tag{7.24}$$

we then choose r_0 so that at the turning points, $V_\zeta = 0$ and $\Lambda = 0$, so

$$P_\zeta = \frac{Re}{c}\int_0^{r_0} dr' B_p(r') \tag{7.25}$$

To ensure constancy, those two terms must cancel each other out.

$$\Lambda_j = \frac{m_j c}{e_j B_p} v_\zeta = \frac{v_\zeta}{\Omega\big|_{B_p}} \tag{7.26}$$

where

$$\begin{aligned} v_\zeta \approx v_\parallel &\approx \epsilon^{\frac{1}{2}} v_{th} \\ q &= \epsilon \frac{B_T}{B_p} \\ B_T &\sim B_p \end{aligned} \tag{7.27}$$

so

$$\epsilon^{\frac{1}{2}} \sim \frac{q}{\epsilon^{\frac{1}{2}}}$$

To wrap it all up, we get

$$\boxed{\Lambda_j = \frac{\epsilon^{\frac{1}{2}} v_{th}}{\frac{eB_p}{mc}} = \frac{\rho_j q}{\epsilon^{\frac{1}{2}}}} \tag{7.28}$$

7.4 Banana Diffusion

For banana diffusion, we start with a modified version of the standard diffusion coefficient.

$$D_{ban} = f_{tr} \left(\frac{\Delta x^2}{\Delta t} \right) \tag{7.29}$$

where $f_{tr} = \epsilon^{\frac{1}{2}}$ is the fraction of trapped particles and $\Delta x^2 = \Lambda^2$. We also use $\Delta t = \frac{1}{\nu_{eff}}$. We define ν_{eff} by the following.

$$\nu_{eff} = \frac{\nu_{ei}}{\Delta\theta^2} = \frac{\nu_{ei}}{\epsilon} \tag{7.30}$$

$\Delta\theta = \epsilon^{\frac{1}{2}}$ is the angle that the trapped particles need to be deflected to become untrapped. Putting all of this together, we get the following for D_{ban}.

$$D_{ban} = \left(\epsilon^{\frac{1}{2}}\right) \left(\frac{\rho q}{\epsilon^{\frac{1}{2}}}\right)^2 \left(\frac{\nu_{ei}}{\epsilon}\right) = \boxed{\frac{\rho^2 q^2 \nu_{ei}}{\epsilon^{\frac{3}{2}}}} \tag{7.31}$$

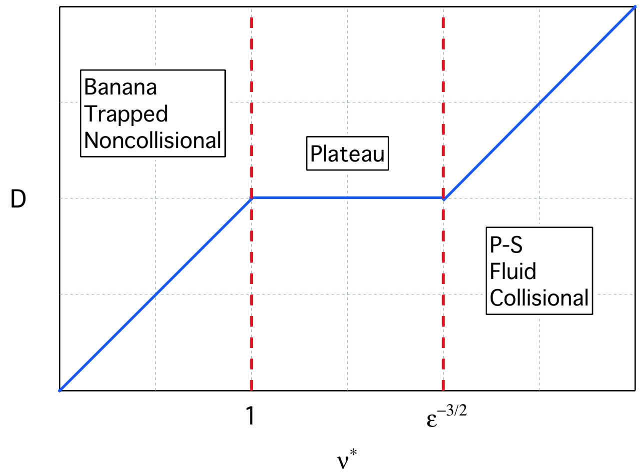

Figure 3: Three regimes in a tokamak plasma.

7.5 Plateau Regime Diffusion

For the plateau regime between the banana regime and the Pfirsh-Schlüter regime (see Fig. 3), we start with the collision frequency.

$$\nu^* = \frac{\nu_{eff}}{\omega_b} = \frac{\text{collision freq}}{\text{bounce freq}} = \frac{\nu_{ei} q R_0}{\epsilon^{\frac{3}{2}} v_{th}} \tag{7.32}$$

In the banana regime, particles must bounce more than they collide to be trapped. In the P-S regime, particles must collide more than they bounce or pass ($\frac{v_{th}}{Rq}$ - the stronger condition). So

$$\frac{\nu_{ei}}{v_{th}/Rq} > 1 \Rightarrow \nu^* \epsilon^{\frac{3}{2}} \Rightarrow \nu^* > \epsilon^{-\frac{3}{2}} \tag{7.33}$$

We then plug this back into our equation for ν^*.

$$\nu^* = \frac{\nu_{ei} q R_0}{\epsilon^{\frac{3}{2}} v_{th}} \quad \text{for } \nu_{ei}, \text{ using } 1 \;\&\; \epsilon^{-\frac{3}{2}} \text{ for } \nu^* \tag{7.34}$$

$$\boxed{D_{plat} = \rho_e^2 \frac{q v_{th,e}}{R_0}} \tag{7.35}$$

One uses the calculated ν_{ei} with $\nu^* = \epsilon^{-\frac{3}{2}}$ in D_{neo} or with $\nu^* = 1$ in D_{ban}.

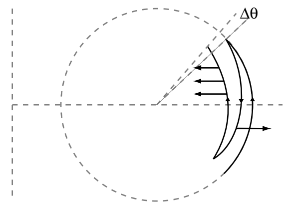

Figure 4: As a result of the E_ζ, particles in banana orbits tend to spend more time moving in the direction parallel to E_ζ, which also corresponds to spending more time above the midplane. As a result, the particles are subjected to an inward drift for a longer time then they are subjected to an outward drift. The cumulative effect of all this is the Ware drift.

7.6 Ware Pinch

Tokamaks have a constant $E_\parallel = E_\zeta$. As a result, as particles bounce around in their banana orbits, they are accelerated by E_ζ while moving on one half of their orbit, and slowed down on the other half. At the end of this slower half, they bounce and again speed up; however, the length of this side is smaller and smaller each bounce, and thus slightly closer to the magnetic axis. The result of all this is a slow drift of particles in banana orbit towards the center of the plasm over the period of many banana bounces. See Fig. 4.

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{E}} = -\frac{\partial \underline{\boldsymbol{B}}}{\partial t} = -\boldsymbol{\nabla} \times \frac{\partial \underline{\boldsymbol{A}}}{\partial t} \Rightarrow E_\zeta = -\frac{\partial A_\zeta}{\partial t} \tag{7.36}$$

This is the quantity that causes the asymmetry. The acceleration caused by this is

$$\text{acceleration} = \frac{\Delta v_\parallel}{\Delta t} = \frac{eE_\parallel}{m} \tag{7.37}$$

where

$$\Delta t \sim \frac{1}{\omega_b} = \frac{Rq}{\epsilon^{\frac{1}{2}} v_{th}} \tag{7.38}$$

Solving for $\Delta v_\parallel$ gives us

$$\Delta v_\parallel = \frac{eE_\zeta}{m\omega_b} \tag{7.39}$$

The net inward drift due to the Ware pinch effect is simply

$$(v_D)_{ware} = (v_D)_r \Delta\theta \tag{7.40}$$

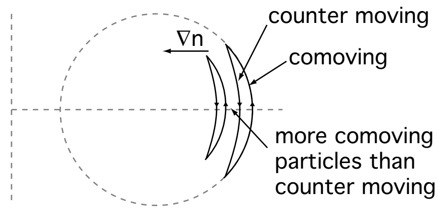

Figure 5: Boot strap current.

where $\Delta\theta$ is the amount the endpoints of the banana orbit shift with each orbit. Again, see Fig. 4. The value for $\Delta\theta$ is

$$\Delta\theta = \frac{\Delta v_\parallel}{(v_\parallel)_{TP}} = \frac{\Delta v_\parallel}{\epsilon^{\frac{1}{2}} v_{th}} \tag{7.41}$$

Putting this all together lets us solve for the value of $(v_D)_{ware}$.

$$\begin{aligned}
(v_D)_{ware} &= -\left|(v_D)_r\right| \Delta\theta = -\frac{v_{th}\rho}{R v_{th}} \frac{eE_\zeta}{\epsilon^{\frac{1}{2}} m\omega_b} \\
&= -\frac{v_{th}\rho}{R} \frac{eE_\zeta}{\epsilon^{\frac{1}{2}} m} \frac{Rq}{\epsilon^{\frac{1}{2}} v_{th}^2} = -\frac{v_{th}^2}{\Omega R} \frac{eE_\zeta}{\epsilon m} \frac{Rq}{v_{th}^2} \\
&= -\frac{mc}{eB_T} \frac{eE_\zeta}{\epsilon m} q = -\frac{qc}{\epsilon} \frac{E_\zeta}{B_T} = -\frac{r}{R} \frac{R}{r} \frac{B_T}{B_p} \frac{cE_\zeta}{B_T} \\
&= \boxed{-c\frac{E_\zeta}{B_p}}
\end{aligned} \tag{7.42}$$

7.7 Boot Strap Current

Boot strap current is a pressure gradient driven toroidal current produced by the collisional trapping and detrapping of particles in the banana regime. For boot strap current, we will first go through a heuristic calculation of it before we outline the more in depth derivation.

Boot strap current arises from the fact that the density gradient $\boldsymbol{\nabla} n$ increases towards the inside of the plasma. As a result, at a given location, the number of particles in a comoving trajectory (outside of an interior banana orbit) is more than the number of particles counter moving (on the inside of an exterior banana orbit). See Fig. 5.

From Wesson, pg. 166:

A heuristic explanation of the bootstrap current can be given. For an inverse aspect-ration $\epsilon = r/R$ there is a fraction $\epsilon^{\frac{1}{2}}$ of trapped particles and they typically have a parallel velocity $\epsilon^{\frac{1}{2}} v_{th}$. They execute a banana orbit of width $w_b \sim \epsilon^{-\frac{1}{2}} q\rho$, where ρ is the Larmor radius. Thus in the presence of a density gradient they carry a current analogous to the diamagnetic current of untrapped particles but here parallel to the magnetic field. This current is

$$j_T \sim -e\epsilon^{\frac{1}{2}}(\epsilon^{\frac{1}{2}} v_{th}) w_b \frac{dn}{dr} \sim -q\frac{\epsilon^{\frac{1}{2}}}{B} T \frac{dn}{dr} \tag{4.9.1}$$

Both trapped ions and trapped electrons carry such a current and there is a transfer of momentum to the passing particles of both species which adjust their velocities accordingly. The dominant current turns out to the that arising from the difference in velocity between passing ions and the passing electrons and this is the bootstrap current j_b.

The heuristic argument from class goes as follows. The boot current is simply

$$J_{BS} = -|e|\, v_{th}(\#\text{ of passing particles}) \tag{7.43}$$

The number of passing particles can be expressed as $\Lambda \frac{dn}{dr} = \frac{\rho q}{\epsilon^{\frac{1}{2}}} \frac{dn}{dr}$. Plugging in for $\rho = \frac{v_{th}}{eB/mc}$ and $q = \frac{\epsilon B_T}{B_p}$, we get

$$\boxed{J_{BS} = -\epsilon^{\frac{1}{2}} \frac{c}{B_p} T \frac{dn}{dr}} \tag{7.44}$$

The more detailed derivation starts with the Drift-Kinetic equation.

$$(\hat{\boldsymbol{n}} v_\parallel + \underline{\boldsymbol{v}}_D) \cdot \boldsymbol{\nabla} f + \frac{e}{m} E_\parallel \underbrace{v_\parallel \frac{\partial f}{\partial E}}_{\frac{\partial}{\partial v_\parallel}} = C_L[f] \tag{7.45}$$

We first have to separate $f = f_\parallel + f_1$ where $f_1 = f_s + \hat{f}$. Skipping some steps, we proceed till we get the following expression.

$$J_\parallel = -|e| \underbrace{\int d^3v v_\parallel (f_s}_{\text{spitzer} = \sigma_\parallel E_\parallel} + \hat{f}) = \sigma_\parallel E_\parallel - \underbrace{|e| \int d^3 v \hat{f} v_\parallel}_{\text{boot strap}} \tag{7.46}$$

To continue, other highlights in the derivation follow.

$$\Gamma_{neo} = \frac{c}{|e|\, B_0} \left\langle \int d^3 v m_e v_\parallel C_{ei}(\hat{f}_e) \right\rangle \tag{7.47}$$

$$\left\langle j_\parallel - j_s \right\rangle = -\,|e| \left\langle \int d^3 v v_\parallel \hat{f}_e \right\rangle \Rightarrow$$

$$\left\langle j_\parallel - j_s \right\rangle \sim -\epsilon^{\frac{1}{2}} \Bigg(\underbrace{\frac{1}{B_p}\frac{dp_e}{dr}}_{\text{boot strap}} + \underbrace{\sigma_\parallel E_\parallel}_{\substack{\text{neoclassical} \\ \text{reduction}}} \Bigg) \tag{7.48}$$

$$J_\parallel^{total} = \sigma_\parallel E_\parallel (1 - \epsilon^{\frac{1}{2}}) + \epsilon^{\frac{1}{2}} \left(\frac{1}{B_p}\frac{dp_e}{dr} \right) \tag{7.49}$$

$$\Gamma_{neo} = -D_{ban} \left(\frac{dn_0}{dr} - \frac{n_0}{2}\frac{d}{dr}\ln T_0 \right) - n\epsilon^{\frac{1}{2}} \frac{cE_\parallel}{B_p} \tag{7.50}$$

See GPPII notes for more on this. This probably won't come up.

More imporantly, we should stress the neoclassical reduction on parallel current due to the fact that trapped particles can't carry current.

$$\boxed{\text{neoclassical reduction} = -\epsilon^{\frac{1}{2}} \sigma_\parallel E_\parallel} \tag{7.51}$$

7.8 Transport Coefficients

To sum it all up, we write down the transport coefficients matrix. There is a 3×3 version of this, with the extra terms coming from heat diffusion and related cross terms. Here we will deal only with density and current terms.

$$\begin{bmatrix} \Gamma_e \\ \left\langle j_\parallel - j_s \right\rangle \end{bmatrix} = \begin{bmatrix} \frac{D_{ban}}{T_e} & \epsilon^{\frac{1}{2}} \frac{c}{B_p} \\ \epsilon^{\frac{1}{2}} \frac{c}{B_p} & \frac{1}{n_0} \sigma_\parallel \epsilon^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \frac{dp_e}{dr} - \frac{5}{2} n_0 \frac{dT_e}{dr} \\ -n_0 E_\parallel \end{bmatrix} \tag{7.52}$$

The term in the upper right hand corner of the middle matrix is the ware pinch term, while the one in the lower left hand corner is for boot strap current. Onsager symmetry holds here, as coefficients are mirrored about the diagonal.

8 Drift Waves

The different types of diffusion described in the previous section don't account for all the diffusion observed in physical systems. The unaccounted for component was named anomalous diffusion. Drift waves are one source of anomalous diffusion.

8.1 Diamagnetic Drift Velocity

Start with the force balance equation and the definition of $\underline{\boldsymbol{j}}$. $\underline{\boldsymbol{B}} = B_z\hat{\boldsymbol{z}}$ and $\boldsymbol{\nabla} n = \frac{\partial n}{\partial x}\hat{\boldsymbol{x}}$.

$$\boldsymbol{\nabla} p_j = \frac{1}{c}\underline{\boldsymbol{j}} \times \underline{\boldsymbol{B}} \tag{8.1}$$

$$\underline{\boldsymbol{j}} = e_j n_0 \underline{\boldsymbol{v}}_j \tag{8.2}$$

We note that $\boldsymbol{\nabla} p = T\boldsymbol{\nabla} n$. Putting this together and solving for the $\hat{\boldsymbol{y}}$-component of v_j, we get the following.

$$T_j\boldsymbol{\nabla} n_0 = \frac{1}{c}e_j n v_j \hat{\boldsymbol{y}} \times B_z\hat{\boldsymbol{z}} \tag{8.3}$$

$$T_j\frac{\partial n_0}{\partial x}\hat{\boldsymbol{x}} = \frac{1}{c}e_j n v_j B_z\hat{\boldsymbol{x}} \tag{8.4}$$

$$\underline{\boldsymbol{v}}_j = \underbrace{\left(\frac{cT_j}{e_j B_z n_j}\right)}_{v_{*j}}\hat{\boldsymbol{y}} \tag{8.5}$$

8.2 Drift Waves

$$n_e = n_0\left(1 + \frac{e\phi}{kT}\right) \tag{8.6}$$

where the last term in parentheses is equivalent to $\frac{n_{1e}}{n_0}$.

$$n_{1i} \Rightarrow \text{continuity \& force}$$

where $v_0 = 0$, $v = v_1$, and $n = n_0 + n_1$.

1. Force $\rightarrow$ components
2. comps into continuity
3. solve for $\frac{n_1}{n_0}$
4. Plug into charge neutrality: $\sum\limits_s n_{s1}e_s = 0$

9 Cold Plasma Waves

9.1 The Wave Equation

To derive the wave equations and various other goodies for cold plasma waves, we first start out with Ampere's Law and Fourier transform it.

$$\begin{aligned}\nabla \times \underline{\boldsymbol{B}} &= \frac{4\pi}{c}\underline{\boldsymbol{J}} + \frac{1}{c}\frac{\partial \underline{\boldsymbol{E}}}{\partial t} = \frac{1}{c}\frac{\partial \underline{\boldsymbol{D}}}{\partial t} \\ \Rightarrow \text{F.T.} \Rightarrow \underline{\boldsymbol{D}}(\omega, k) &= \underline{\boldsymbol{E}} - \frac{4\pi}{i\omega}\underline{\boldsymbol{J}} = \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}} \end{aligned} \tag{9.1}$$

We also note that the plasma dielectric tensor is defined as follows.

$$\boxed{\overline{\overline{\boldsymbol{\epsilon}}} = \overline{\overline{\mathbb{1}}} + \sum_s \overline{\overline{\boldsymbol{\chi}}}_s} \tag{9.2}$$

Plugging this stuff together and solving for $\underline{\boldsymbol{J}}$ gives us

$$\boxed{\underline{\boldsymbol{J}} = -\frac{i\omega}{4\pi}\sum_s \overline{\overline{\boldsymbol{\chi}}}_s \cdot \underline{\boldsymbol{E}} = \overline{\overline{\boldsymbol{\sigma}}}_s \cdot \underline{\boldsymbol{E}} = \sum_s n_s q_s \underline{\boldsymbol{v}}_s} \tag{9.3}$$

For the next step of the derivation, one must bring in the equation of motion.

$$n_s m_s \frac{\partial \underline{\boldsymbol{v}}_s}{\partial t} = n_s q_s \left(\underline{\boldsymbol{E}} + \frac{\underline{\boldsymbol{v}}_s}{c} \times \underline{\boldsymbol{B}}\right) - \nabla \cdot \overline{\overline{\boldsymbol{\Phi}}} \tag{9.4}$$

The last term is the fluid stress tensor. For cold plasmas, this is equal to zero. We then linearize the equation of motion using $\underline{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}}$ and $n = n_0$, setting all other terms to 0 in the 0th order.

$$m_s \frac{\partial v_{1s}}{\partial t} = q_s(\underline{\boldsymbol{E}}_1 + \underline{\boldsymbol{v}}_1 \times \underline{\boldsymbol{B}}_0) \tag{9.5}$$

After we linearize, we Fourier transform and use the following three equations

$$E^{\pm} = \frac{1}{2}(E_x \pm iE_y) \tag{9.6}$$

$$v^{\pm} = \frac{1}{2}(v_x \pm iv_y) \tag{9.7}$$

$$n_s q_s v_s^{\pm} = -\frac{i\omega}{4\pi}\overline{\overline{\boldsymbol{\chi}}}_s^{\pm} \cdot \underline{\boldsymbol{E}}_s^{\pm} \tag{9.8}$$

to arrive at

$$\chi_s^{\pm} = -\frac{\omega_{ps}^2}{\omega(\omega \mp \Omega_s)} \tag{9.9}$$

$$\chi_{zz,s} = -\frac{\omega_{ps}^2}{\omega^2} \tag{9.10}$$

$$\chi_{xx} = \frac{1}{2}(\chi^+ + \chi^-) \tag{9.11}$$

$$\chi_{xy} = \frac{i}{2}(\chi^+ - \chi^-) \tag{9.12}$$

One of the nice things about writing this in this form is that it is additive. Thus we can calculate the susceptibilities for just the changed species and replace their contribution to the total susceptibility, species by species. Putting this all together and writing it in terms of the dielectric using Eq. (9.2), we get

$$\boxed{\overline{\overline{\epsilon}} \cdot \underline{\boldsymbol{E}} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}} \tag{9.13}$$

where S is the sum of R and L, D is the difference, and P stands for plasma.

$$\boxed{R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \Omega_s)}} \tag{9.14a}$$

$$\boxed{L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \Omega_s)}} \tag{9.14b}$$

$$\boxed{S = \frac{1}{2}(R + L)} \tag{9.14c}$$

$$\boxed{D = \frac{1}{2}(R - L)} \tag{9.14d}$$

$$\boxed{P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}} \tag{9.14e}$$

Other relations to note are

$$S^2 - D^2 = RL \tag{9.15a}$$

Now we use Faraday's law and Ampere's law to get the wave equation. We take the curl of Faraday's law, Fourier transform these two equations (see Eq. (9.1) for the FT of Ampere's law), and then plug the FT'd Ampere's law into our modified Faraday's law.

$$i\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = \frac{i\omega}{c}\underline{\boldsymbol{B}} \tag{9.16}$$

$$i\underline{\boldsymbol{k}} \times \underline{\boldsymbol{B}} = \frac{4\pi}{c}\underline{\boldsymbol{J}} - \frac{i\omega}{c}\underline{\boldsymbol{E}} = -\frac{i\omega}{c}\underline{\boldsymbol{D}} = -\frac{i\omega}{c}\overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}} \tag{9.17}$$

$$\Downarrow$$

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = \frac{\omega}{c}\underline{\boldsymbol{k}} \times \underline{\boldsymbol{B}} = -\frac{\omega^2}{c^2}\overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}} \tag{9.18}$$

This is the wave equation. Putting it all on the same side will get it in the standard form. One note to make is that the coefficient to the $\overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}}$ term can be thought of as the free space wavelength (k_0^2), since $\omega_0 = ck_0$. We can also write it in terms of index of refraction, where $\underline{\boldsymbol{n}} = \frac{c\underline{\boldsymbol{k}}}{\omega}$.

$$\boxed{\underline{\boldsymbol{k}} \times \underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} + \frac{\omega^2}{c^2}\overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}} = 0}$$

$$\boxed{\underline{\boldsymbol{n}} \times \underline{\boldsymbol{n}} \times \underline{\boldsymbol{E}} + \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}} = 0} \tag{9.19}$$

Another way to write it out is in matrix form, as follows.

$$\boxed{\begin{bmatrix} S - n_\parallel^2 & -iD & n_\parallel n_\perp \\ iD & S - n^2 & 0 \\ n_\parallel n_\perp & 0 & P - n_\perp^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0} \tag{9.20}$$

where $n_\parallel = n\cos\theta$, $n_\perp = n\sin\theta$, and $\cos\theta = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{b}}$.

9.2 Dispersion Relation

Now that we have the wave equation, we can take the determinant of it, set it to 0, and solve. We find that we get an equation of the form of

$$An^4 - Bn^2 + C = 0 \tag{9.21}$$

where

$$A = P\cos^2\theta + S\sin^2\theta \tag{9.22a}$$

$$B = RL\sin^2\theta + PS(1 + \cos^2\theta) \tag{9.22b}$$

$$C = PRL \tag{9.22c}$$

The solution to this equation follows. This is the dispersion relation.

$$n^2 = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm F}{2A} \tag{9.23}$$

where

$$F^2 = (RL - PS)^2 \sin^4\theta + 4P^2D^2\cos^2\theta \tag{9.24}$$

If $n^2 > 0$, then the wave is purely propagating. If $n^2 < 0$, then it is evanescent (it has the form of e^{-kx}). Something else to note is that if $F = 0$, then the wave can mode convert. More on this in Section 9.5.

There are different classifications of waves depending on there attributes. Note that θ is defined as the angle between $\underline{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{n}}$ or $\hat{\boldsymbol{k}}$.

1. Longitudinal waves $\Rightarrow \underline{\boldsymbol{E}} \parallel \underline{\boldsymbol{k}} \Rightarrow$ electrostatic
   Transverse waves $\Rightarrow \underline{\boldsymbol{E}} \perp \underline{\boldsymbol{k}} \Rightarrow$ electromagnetic
2. Right (electrons) and Left (ions) polarized waves are for when $\theta = 0$, the wave is propagating along the $\underline{\boldsymbol{B}}$ field.
3. X ($\underline{\boldsymbol{E}} \perp \underline{\boldsymbol{B}}_0$) and O ($\underline{\boldsymbol{E}} \parallel \underline{\boldsymbol{B}}_0$ - independent of magnetic field) waves occur when $\theta = \frac{\pi}{2}$, the wave is propagating perpendicular to the $\underline{\boldsymbol{B}}$ field.
4. Fast waves - higher v_{ph} and lower n^2
   Slow waves - lower v_{ph} and higher n^2

Another form for the dispersion relation is

$$\boxed{\tan^2\theta = \frac{-P(n^2 - R)(n^2 - L)}{(S^2n^2 - RL)(n^2 - P)}} \tag{9.25}$$

We can see from this form of the dispersion relation that when the wave is propagating parallel to $\underline{\boldsymbol{B}}$ ($\theta = 0$), then

$$P = 0 \qquad n^2 = R \qquad n^2 = L \tag{9.26}$$

Likewise, if the wave is propagating perpendicular to $\underline{\boldsymbol{B}}$ ($\theta = \frac{\pi}{2}$), then

$$\underbrace{n^2 = \frac{RL}{S}}_{X} \qquad \underbrace{n^2 = P}_{O} \tag{9.27}$$

9.3 Cutoffs and Resonances

A cutoff is simply when the wave no longer propagates and $n^2 \to 0$ (or $n_\perp^2 \to 0$), where $v_{ph} = \frac{\omega}{k} \to \infty$. Going back to Eq. (9.23), since $n = 0$

$$C = PRL = 0 \tag{9.28}$$

A resonance occurs when $n^2 \to \infty$ or where $v_{ph} \to 0$. Here, $A \sim 0$, so we get

$$\boxed{\tan^2\theta = -\frac{P}{S}} \tag{9.29}$$

One can also look at the alternate dispersion relation Eq. (9.25) and rewrite it.

$$\tan^2\theta = -\frac{P}{S}\frac{\left(1-\frac{R}{n^2}\right)\left(1-\frac{L}{n^2}\right)}{\left(1-\frac{RL}{Sn^2}\right)\left(1-\frac{P}{Sn^2}\right)} \approx -\frac{P}{S} \tag{9.30}$$

We can rewrite it as such because the second fraction goes to zero. One must be careful in this region because at resonance, cold plasma theory really doesn't hold. Also, this resonance equation is the same as the electrostatic approximation, *though the two effects are not the same.*

Principle resonances occur at

$$\theta = 0 \qquad S \to \infty \qquad S = \frac{R+L}{2} \quad \text{so if } R \text{ or } L \to \infty \tag{9.31}$$

$$\theta = \frac{\pi}{2} \qquad S \to 0 \qquad \text{hybrid type resonances} \tag{9.32}$$

9.3.1 Group and Phase Velocity

Two quick definitions. Group velocity is the velocity at which the wave packets are traveling, or at which the wave itself is moving. Phase velocity is the velocity of the phase.

$$\boxed{v_g = \frac{\partial\omega}{\partial k}} \tag{9.33}$$

$$\boxed{v_{ph} = \frac{\omega}{k}} \tag{9.34}$$

9.4 Polarization

Before we get into specific waves and propagation regimes, one last thing we should look at is wave polarization. One can read the wave polarization from the middle line of Eq. (9.20).

$$\boxed{\frac{iE_x}{E_y} = \frac{n^2 - S}{D} = \begin{cases} 1 & \text{if RH}, n^2 = R \\ -1 & \text{if LH}, n^2 = L \end{cases}} \tag{9.35}$$

It is important to note that $\theta = 0$ for these cases. Another way to examine polarization is with

$$\frac{E^+}{E^-} = \frac{n^2 - R}{n^2 - L} \tag{9.36}$$

9.5 Mode Conversion

As stated earlier, in a slowly varying plasma, the discriminant in Eq. (9.23) can go to zero and $B^2 \gg 4AC$, then two waves in the plasma can exist at the same conditions and therefore they can couple to each other and mode convert. When this happens, there is usually a large n^2 wave and a small n^2 wave. Since $n_\parallel^2$ is usually constant, the two roots are large and small $n_\perp^2$.

For the two roots of the dispersion relation, one balances the appropriate terms to get the asymptotic expressions for the waves.

$$\underbrace{An_\perp^4 - B}_{large}\underbrace{n_\perp^2 + C}_{small} = 0 \tag{9.37}$$

Doing so gives us the following two dispersion relations.

$$n_\perp^2 \sim \frac{B}{A} \qquad \text{large } n_\perp^2 \tag{9.38}$$

$$n_\perp^2 \sim \frac{C}{B} \qquad \text{small } n_\perp^2 \tag{9.39}$$

The large $n_\perp^2$ wave is often an electrostatic wave (since when $P = 0$, $C \to 0$.), while the small root is often an electromagnetic fast wave. Also, one wave usually comes from cold plasma theory while the other usually comes from warm/kinetic effects. The point of mode conversion happens at $B^2 = 4AC$ and it typically happens near a cutoff or resonance where the wave numbers vary quickly.

For more on this derivation, look at Section 9.8 on page 53.

9.6 Parallel and Perpendicular Propagation

9.6.1 Parallel Propogation

For parallel propagation, $\theta = 0$, we can see from Eqs. Eq. (9.25) and Eq. (9.26) that we have one of three forms. For the $P = 0$ wave, we can solve to see that (dropping $\frac{m_e}{m_i}$ terms)

$$\omega^2 = \omega_{ps}^2 \approx \omega_{pe}^2 \tag{9.40}$$

These are plasma oscillations. For the $n^2 = n_\parallel^2 = R, L$ waves, we can solve that to find

$$\frac{k_\parallel^2 c^2}{\omega^2} = \frac{\omega^2 \pm \omega\Omega_e + \Omega_e\Omega_i - \omega_{pe}^2}{(\omega \pm \Omega_e)(\omega \pm \Omega_i)} \tag{9.41}$$

For both the $n^2 = R$ and $n^2 = L$ waves, at low frequencies ($\omega \to 0$),

$$n^2 \approx 1 + \frac{c^2}{V_A^2} \tag{9.42}$$

where

$$V_A^2 = \frac{B_0^2}{4\pi n_i m_i} \tag{9.43}$$

and

$$\frac{c^2}{V_A^2} = \frac{\omega_{pi}^2}{\Omega_i^2} \tag{9.44}$$

At high frequencies ($\omega^2 \to> \omega_{ps}^2, \Omega_s^2$), $n^2 \approx 1$. Both waves are transverse waves, $\underline{\boldsymbol{k}} \perp \underline{\boldsymbol{E}}$.

The resonances are at $\omega = \Omega_e$ for the R wave and $\omega = \Omega_i$ for the L wave. The L wave can be used for magnetic beach heating. The cutoff frequencies for these waves is as follows. For $n^2 = R$

$$\omega_{cr} = \begin{cases} \omega_{pe} - \frac{\Omega_e}{2} & \text{high density - } \frac{\omega_{pe}^2}{\Omega_e^2} \gg 1 \\ -\Omega_e - \frac{\omega_{pe}^2}{\Omega_e} & \text{low density - } \frac{\omega_{pe}^2}{\Omega_e^2} \ll 1 \end{cases} \tag{9.45}$$

For $n^2 = L$

$$\omega_{cr} = \begin{cases} \Omega_i + \frac{\omega_{pi}}{\Omega_i} & \text{high density - } \frac{\omega_{pe}^2}{\Omega_e^2} \gg 1 \\ \omega_{pe} + \frac{1}{2}\Omega_e & \text{low density - } \frac{\omega_{pe}^2}{\Omega_e^2} \ll 1 \end{cases} \tag{9.46}$$

9.6.2 Perpendicular Propagation

For perpendicular propagation, $\theta = \frac{\pi}{2}$, it can be seen from Eqs. Eq. (9.25) and Eq. (9.27) that there are 2 waves, an $n^2 = n_\perp^2 = P$ wave and an $n^2 = n_\perp^2 = \frac{RL}{S}$ wave. For the $n^2 = n_\perp^2 = P$ wave (O-wave)

$$\omega^2 = \omega_{pe}^2 + k_\perp^2 c^2 \tag{9.47}$$

This is the typical/generic dispersion relation of perpendicular propagation of a wave, for space, etc. The cutoff for this wave is at $\omega^2 = \omega_{ps}^2$. The polarization of the wave is arbitrary (see Wave notes, Lecture 3, page 4).

For the $n^2 = n_\perp^2 = \frac{RL}{S}$ wave (X-wave)

$$\frac{k_\perp^2 c^2}{\omega^2} = \frac{(\omega^2 + \omega\Omega_e + \Omega_e\Omega_i - \omega_{pe}^2)(\omega^2 - \omega\Omega_e + \Omega_e\Omega_i - \omega_{pe}^2)}{(\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2)} \tag{9.48}$$

where

$$\omega_{LH} = \left[\frac{1}{\Omega_i^2 + \omega_{pi}^2} + \frac{1}{|\Omega_i\Omega_e|}\right]^{-1} \qquad \text{(Lower Hybrid)} \tag{9.49}$$

$$\omega_{UH} = \Omega_e^2 + \omega_{pe}^2 \qquad \text{(Upper Hybrid)} \tag{9.50}$$

For the X-wave, there are cutoffs where $R = 0$ and $L = 0$. These are the same as for parallel propagation above. There is a resonance at $S = 0$, which occurs at the two hybrid frequencies. The high and low frequency limits of this wave are the same as the parallel propagation case. The polarization of this wave is semitransverse, since as $\omega \to 0$, $E_x \to 0$, $\underline{\boldsymbol{E}} \to E_y\hat{\boldsymbol{y}}$ and $\underline{\boldsymbol{k}} = k_\perp\hat{\boldsymbol{x}}$.

$$\frac{iE_x}{E_y} = -\frac{R - L}{R + L} \tag{9.51}$$

The high and low density limits for the hybrid resonances follow.

$$\omega_{LH}^2 \approx \begin{cases} |\Omega_i\Omega_e| & \text{high density} \\ \omega_{pi}^2 + \Omega_i^2 & \text{low density} \end{cases} \tag{9.52}$$

$$\omega_{UH}^2 \approx \begin{cases} \omega_{pe}^2\left(1 + \frac{\Omega_e^2}{\omega_{pe}^2}\right) & \text{high density} \\ \Omega_e^2\left(1 + \frac{\omega_{pe}^2}{\Omega_e^2}\right) & \text{low density} \end{cases} \tag{9.53}$$

The lower hybrid resonance can be visualized as the following (see Fig. 6). Quoting Stix, pg 36:

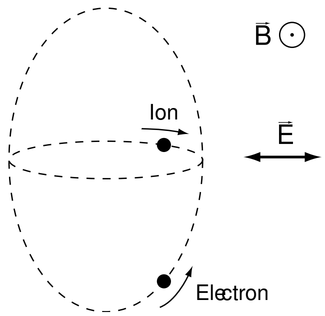

Figure 6: The lower hybrid resonance is when the minor diameter of the electron orbit is equal to the major diameter of the ion orbit. The ratio of the large axis to the small axis for both ellipses is $(\frac{m_i}{Zm_e})^{\frac{1}{2}}$.

> If we take the wave electric field in the x direction, the ion motion will be principally in the x direction, oscillating back and forth in almost a straight line unaffected by the magnetic field. The electrons will move predominantly in the y direction with an $\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}_0$ drift, but the deviation of the electron motion from a straight line plays an important role. The x displacement of the electrons, which is the movement along the minor diameter of their elliptic trajectory, is in phase with and equal to the x displacement of the ions at the root-mean gyrofrequency $\omega = |\Omega_e \Omega_i|^{\frac{1}{2}}$. The ion space charge is thereby neutralized at high plasma densities, and the hybrid oscillation can take place.

Also, see problem 2.6.

The upper hybrid resonance can be thought of in the following manner, also from Stix, pg. 36.

> An elementary picture will describe the *upper-hybrid* resonance. Let us consider a cylinder of uniform plasma, and inside this cylinder draw an imaginary cylindrical surface of radius r. The cylindrical axis is in the $z(B_0)$ direction. We anticipate that the oscillation frequency will be high enough so that ion motion may be neglected, and consider a collective mode of motion for the electrons. if electrons on the surface r move outward by an amount Δr, the radial electric field will be, by Gauss' theorem,

$E_r = 4\pi n_e e \Delta r$. The equation of collective motion is then

$$m\frac{d^2}{dt^2}\Delta\underline{\boldsymbol{r}} = -e\left[4\pi n_e e(\hat{\boldsymbol{r}}\cdot\Delta\underline{\boldsymbol{r}})\hat{\boldsymbol{r}} + \frac{1}{c}\frac{d(\Delta\underline{\boldsymbol{r}})}{dt}\times\underline{\boldsymbol{B}}_0\right] \qquad (30)$$

and the characteristic frequency for transverse oscillations will be just the upper hybrid resonant frequency.

9.7 Alfvèn Waves

9.7.1 V_A and γ

$$V_A^2 = \frac{B_0^2}{4\pi n_i m_i} \qquad (9.54)$$

$$\gamma = \frac{4\pi n_i m_i c^2}{B_0^2} \qquad (9.55)$$

One can easily see that these two quantities are easily related.

$$V_A^2 = \frac{B_0^2}{4\pi n_i m_i} = \frac{c^2}{\gamma} = \frac{c^2\Omega_i^2}{\omega_{pi}^2} \qquad (9.56)$$

9.7.2 Shear and Compressional Alfvèn Waves

For $\omega \ll \Omega_e$, we find that $P \gg 1$, and $P > S, D$. For these waves, $E_z \sim 0$.

Alfvèn waves are low frequency ($\omega \ll \Omega_i$) that occur in a magnetized plasma. In this regime, $R \simeq L \simeq S \simeq 1+\gamma$. As a result of this, $D = 0$. There are two dispersion relations here.

$$S - n_\parallel^2 = 0 \quad \Rightarrow \quad 1+\gamma = n_\parallel^2 \quad \text{(slow)} \qquad (9.57)$$

$$S - n^2 = 0 \quad \Rightarrow \quad 1+\gamma = n^2 \quad \text{(fast)} \qquad (9.58)$$

For the slow wave, $E_x \neq 0$ and $E_y = E_z = 0$. Thus $\underline{\boldsymbol{E}}\hat{\boldsymbol{x}} + \frac{\boldsymbol{v}}{c}\times\underline{\boldsymbol{B}}_0\hat{\boldsymbol{z}} = 0$ leading to the fact that $\underline{\boldsymbol{V}} = v\hat{\boldsymbol{y}} = v\cos(k_\perp x - \omega t + k_\parallel z)\hat{\boldsymbol{y}}$. This wave is also known as the (torsional) shear wave since $\underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{v}} = 0$. It is incompressible. There is a cutoff at $\omega = |\Omega_i|$. It is a LH polarized wave near $\theta = 0$, and when $\frac{V_A^2}{c^2} \ll 1$, $\omega^2 \approx k_\parallel^2 V_A^2$. Also $\underline{\boldsymbol{k}}\times\underline{\boldsymbol{E}} \sim \underline{\boldsymbol{B}}$, leading to $k_\parallel E\hat{\boldsymbol{y}} \sim B(k,\omega)\hat{\boldsymbol{y}}$. The magnetic field lines in this wave are like a string, with $V = \sqrt{\frac{T}{\rho}} \Rightarrow V_A = \sqrt{\frac{B^2}{nm}}$. See Fig. 7a.

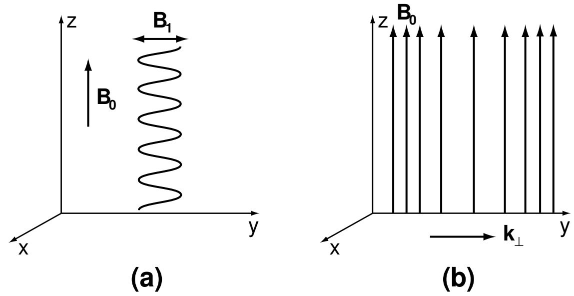

Figure 7: Here we see the two types of Alfvèn waves, the a) shear wave and the b) compressional wave.

The fast wave, also known as the compressional wave, is RH polarized near $\theta = 0$. $E_x = E_z = 0$, while $E_y \neq 0$. Since $\underline{\boldsymbol{k}} = k_\perp \hat{\boldsymbol{x}} + k_\parallel \hat{\boldsymbol{z}}$, this is a transverse wave. Since the force equation looks like $E\hat{\boldsymbol{y}} + \frac{\boldsymbol{v}}{c} \times \underline{\boldsymbol{B}}_0 \hat{\boldsymbol{z}} = 0$, one can easily see that $\underline{\boldsymbol{V}} \sim v\hat{\boldsymbol{x}}$. Thus $\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{v}} \neq 0$, leading to the identification of this wave as the compressional wave. See Fig. 7b. If one lets $\theta = \frac{\pi}{2}$, then $\underline{\boldsymbol{k}} = k_\perp$ and $n_\perp^2 = \frac{RL}{S} \sim S$. Thus this wave is the low frequency limit of the X-mode wave.

9.8 Ion cyclotron Waves and $\omega \sim \Omega_i$

In this regime, $\omega \ll \Omega_e$ and $\omega \sim O(\Omega_i)$. The following relations for S and D hold true.

$$S \approx \gamma \frac{\Omega_i^2}{\Omega_i^2 - \omega^2} \tag{9.59}$$

$$D \approx -\frac{\omega}{\Omega_i} S \tag{9.60}$$

Also note that $|P| \gg |S|, |D|$ and $S^2 - D^2 = \gamma S$.

The Alfvèn resonance becomes evident when examining the following dispersion relation.

$$n_\perp^2 = \frac{(R - n_\parallel^2)(L - n_\parallel^2)}{S - n_\parallel^2} \tag{9.61}$$

The resonance is occurs at $n_\parallel^2 = S$ when $\theta = \frac{\pi}{2}$.

A quick look at waves in this in region shows the dispersion relations for the fast and slow waves as

$$\omega^2 = k^2 V_A^2 (1 + \cos^2\theta) \qquad \text{(fast)} \tag{9.62}$$

$$\omega^2 = \Omega_i^2 \left[1 + \frac{\omega_{pi}^2}{k_\parallel^2 c^2} + \frac{\omega_{pi}^2}{k_\parallel^2 c^2 + k_\perp^2 c^2} \right]^{-1} \qquad \text{(slow)} \tag{9.63}$$

The shear/slow Alfvèn wave in this regime is LH circularly polarized at $\theta = 0$. It has a resonance at $\theta = 0$ and $\omega \to \Omega_i$. The fast wave in the regime $\omega > \Omega_i$ is known as the magnetosonic wave.

	$\theta = 0$	$\theta = \frac{\pi}{2}$
fast	$n_\parallel^2 = R$	$n_\perp^2 = \frac{RL}{S}$
slow	$n_\parallel^2 = L$	evanescant ($n_\perp^2 = P$)

Another way to look at waves in this region is to examine Eq. (9.20). Since $E_z = 0$, we can deal with just the 2×2 form of it, getting

$$(S - n_\parallel^2)(S - n^2) - D^2 = 0 \tag{9.64}$$

Using the above forms for S and D and rearranging, we get

$$n^4 \cos^2\theta - n^2 S(1 + \cos^2\theta) + \gamma S = 0 \tag{9.65}$$

We then apply the quadratic formula and solve for n^2, and then expand for $\frac{4ac}{b^2} \ll 1$.

$$n^2 \approx -\frac{b}{2a}\left[1 \mp \left(1 - \frac{2ac}{b^2}\right)\right] \tag{9.66}$$

From this relation, we get two asymptotic forms for solutions (this is similar to the mode conversion stuff). We find that

$$n^2 \approx \begin{cases} -\frac{c}{b} & \text{“low n root” - fast} \\ -\frac{b}{a} & \text{“high n root” - slow} \end{cases} \tag{9.67}$$

Looking at the fast solution first and plugging in for b and c, we find that

$$n^2 \approx -\frac{c}{b} \approx \frac{-\gamma S}{-S(1 + \cos^2\theta)} = \frac{\gamma}{1 + \cos^2\theta} \tag{9.68}$$

We then examine the polarization, plugging values for S and D.

$$i\frac{E_x}{E_y} = \frac{n^2 - S}{D} \xrightarrow{\omega \to \Omega_i} -\frac{S}{D} \to 1 \tag{9.69}$$

So, this wave is RH at $\omega \to \Omega_i$, since $S, D \to \infty$. This is with single ions. As a result, even though there might be a resonance with ions at this frequency, *since the polarization is in the wrong direction, there is no resonance.*

Now we move onto the slow wave solution. This is the ion cyclotron wave. *It does not propagate above* Ω_i ($\omega^2 < \Omega_i^2$ for propagation).

$$n^2 \approx -\frac{b}{a} = \frac{S(1+\cos^2\theta)}{\cos^2\theta}$$
$$n_\parallel^2 \approx S(1+\cos^2\theta) \approx \gamma\left(\frac{\Omega_i^2}{\Omega_i^2-\omega^2}\right)\left(1+\cos^2\theta\right) \tag{9.70}$$

Looking at the polarization for this wave gives us

$$i\frac{E_x}{E_y} = \frac{n^2-S}{D} = \frac{\frac{S(1+\cos^2\theta)}{\cos^2\theta}-S}{-\frac{\omega}{\Omega_i}S} = -\frac{\Omega_i}{\omega}\frac{1}{\cos^2\theta} \tag{9.71}$$

If we look at $\theta = 0$ and $\omega \to \Omega_i$, then we see that the above equation goes to -1 and the polarization is LH (resonates with ions). In general however, this wave is elliptically polarized.

9.9 High Frequency Waves

9.9.1 Whistler Wave

We now move on to look at higher frequency cold plasma waves. We will assume that $\Omega_i \ll \omega \ll \Omega_e \sim \omega_{pe}$ is true for ω. In this region, $|P| \gg |D| \gg |S|$. *There are no slow waves here, only fast wave.*

We will start out with the Whistler wave, also known as the electron cyclotron wave. Assume $P \gg 1$ and $E_z = 0$. Using the 2×2 version of Eq. (9.20) and $S \ll D$, we find that

$$n_\parallel^2 n^2 \approx D$$
$$n^4\cos^2\theta \approx D^2$$
$$n^2 \approx \left|\frac{D}{\cos\theta}\right| \Rightarrow \frac{c^2k^2}{\omega^2} = \frac{\omega_{pe}^2}{\omega\Omega_e\cos\theta} \tag{9.72}$$

This is the Whistler wave.

A couple quick things about the Whistler wave. Assume that $\theta = \frac{\pi}{2}$ (don't remember why right now). Solving for v_g shows us something interesting.

$$
\begin{aligned}
\frac{c^2k^2}{\omega^2} &= \frac{\omega_{pe}^2}{\omega\Omega_e\cos\theta} \\
k &= \left(\frac{\omega_{pe}^2\omega^2}{c^2\omega\Omega_e\cos\theta}\right)^{\frac{1}{2}} \\
v_g^{-1} &= \frac{dk}{d\omega} \propto \omega^{-\frac{1}{2}} \\
v_g &\propto \sqrt{\omega}
\end{aligned} \tag{9.73}
$$

Thus, the higher frequencies in the Whistler wave travel faster than the lower frequencies.

The other interesting thing about Whistler's is found by looking at the angle of propagation. Start off by looking at the $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{z}}$ components of v_g and take their ratio to find the angle of propagation α.

$$
\begin{aligned}
v_{gx} &= \frac{\partial\omega}{\partial k_x} = \frac{c^2\left|\Omega_e\right|}{\omega_{pe}^2}\left(\frac{k_\parallel k_x}{k}\right) \\
v_{gz} &= \frac{\partial\omega}{\partial k_z} = \frac{c^2\left|\Omega_e\right|}{\omega_{pe}^2}\left(k + \frac{k_\parallel^2}{k}\right) \\
\frac{v_{gx}}{v_{gz}} = \tan\alpha &= \frac{k_\parallel k_x}{k^2 + k_\parallel^2} = \frac{\cos\theta\sin\theta}{2\cos^2\theta + \sin^2\theta} = \frac{\zeta}{2+\zeta^2} \\
\zeta_{max} &= \sqrt{2} \\
\tan\alpha_{max} &= \frac{1}{\sqrt{8}} \Rightarrow \alpha_{max} = 19^\circ 28'
\end{aligned} \tag{9.74}
$$

So the Whistler wave travels mostly in the direction of $\underline{\boldsymbol{B}}$, which is why it was theorized that they have the ability to be generated in one hemisphere and travel to the other. However, now it is believed that this phenomena is the result of wave guide physics.

9.9.2 Quasi Longitudinal/Transverse Waves

To look at these waves, we assume that $\omega \sim O(\Omega_e)$. We can't ignore E_z here. We also use the fact that $n^2 \sim O(1) + \ldots$. Normally, we start with

$$
An^4 - Bn^2 + C = 0 \tag{9.75}
$$

and use the quadratic formula to solve for n^2. We do that here, getting

$$\begin{aligned}
n^2 &= \frac{B \pm \sqrt{B^2 - 4AC}}{2A} + 1 - 1 && \text{(add/subtract 1)} \\
&= 1 + \frac{B \pm \sqrt{B^2 - 4AC} - 2A}{2A} && \text{(rearrange)} \\
&= 1 - \frac{\left((2A - B) \mp \sqrt{(\ )}\right)\left((2A - B) \pm \sqrt{(\ )}\right)}{2A\left((2A - B) \pm \sqrt{(\ )}\right)} && \text{(crank)} \\
&= 1 - \frac{2[A - B + C]}{2A - B \pm \sqrt{B^2 - 4AC}} && \text{(plug in } A, B, C\text{)} \\
n^2 &= 1 - \frac{2\omega_{pe}^2(\omega^2 - \omega_{pe}^2)/\omega^2}{2(\omega^2 - \omega_{pe}^2) - \Omega_e^2 \sin^2\theta \pm \Omega_e \Delta} && (9.76)
\end{aligned}$$

where

$$\Delta = \left[\Omega_e^2 \sin^4\theta + \frac{(\omega^2 - \omega_{pe}^2)\cos^2\theta}{4\omega^2}\right]^{\frac{1}{2}} \tag{9.77}$$

In the steps above when we add and subtract 1, remember that since $n^2 \sim 1$, the rest is a correction. These waves are quasi longitudinal/transverse with respect to $\underline{\boldsymbol{B}}_1$, not $\underline{\boldsymbol{k}}$.

Some QT and QL waves of interest:

QT-O $\qquad n^2 \simeq \dfrac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{pe}^2 \cos^2\theta}$ $\qquad$ (9.78)

QT-X $\qquad$ ECE

QL-R/L $\qquad n^2 = 1 - \dfrac{\omega_{pe}^2}{\omega(\omega \pm \Omega_e \cos\theta)}$ $\qquad$ (9.79)

The QT-O wave is interesting at $\theta = \frac{\pi}{2}$. The dispersion relation becomes

$$n^2 = P = 1 - \frac{\omega_{pe}^2}{\omega} \Rightarrow \omega^2 = \omega_{pe}^2 + k_\perp^2 c^2 \tag{9.80}$$

This is the electromagnetic plasma wave. It is an O-mode wave.

The + solution (RH) of the QL-R/L resonates at $\omega = |\Omega_e|$. This is the electron cyclotron wave or the Whistler wave from Section 9.9.1.

9.9.3 Even Higher Frequencies

At frequencies of $\omega^2 \gg \omega_{pe}^2 \gg \Omega_e^2$, $P \to 1$, $S \to 1$, and $D \to 0$. At these frequencies, the dispersion relation is

$$n^2 \simeq 1 \tag{9.81}$$

At frequencies of $\Omega_e \ll \omega \ll \omega_{pe}$, $P < 0$, $S < 0$, and $D < 0$. There is no propagation here.

9.10 Electrostatic Waves

Electrostatic waves have $\underline{\boldsymbol{E}}_1 \parallel \underline{\boldsymbol{k}}$. $\underline{\boldsymbol{B}}_1 \approx 0$ in most cases. The electrostatic approximation relies on the fact that

$$\underline{\boldsymbol{E}}_1 = -\boldsymbol{\nabla}\phi \tag{9.82}$$

If we use the fact that there is no free charge

$$\boldsymbol{\nabla} \cdot \underline{\boldsymbol{D}} = \boldsymbol{\nabla} \cdot (\overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}}) = 0 \tag{9.83}$$

and plug in the approximation and FT the result, we get

$$i\underline{\boldsymbol{k}} \cdot \overline{\overline{\boldsymbol{\epsilon}}} \cdot (-i\underline{\boldsymbol{k}}\phi) = 0 \tag{9.84}$$

This can be simplified to get the electrostatic wave equation.

$$\boxed{\underline{\boldsymbol{k}} \cdot \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{k}} = \underline{\boldsymbol{n}} \cdot \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{n}} = 0} \tag{9.85}$$

One can further break the $\underline{\boldsymbol{E}}$ field into longitudinal ($\|$) and transverse ($\perp$) components ($\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_L + \underline{\boldsymbol{E}}_T$). Plugging this into the wave equation and dotting everything into $\underline{\boldsymbol{n}}$ gives us

$$\underline{\boldsymbol{n}} \cdot \left[\underline{\boldsymbol{n}} \times \underline{\boldsymbol{n}} \times \underline{\boldsymbol{E}} + \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}} = 0\right] \Rightarrow \underline{\boldsymbol{n}} \cdot \overline{\overline{\epsilon}} \left[\underline{\boldsymbol{E}}_L + \underline{\boldsymbol{E}}_T\right] = 0 \tag{9.86}$$

Thus, if $\underline{\boldsymbol{E}}_L \gg \underline{\boldsymbol{E}}_T$, then the electrostatic approximation holds. By looking at the wave equation again

$$\underline{\boldsymbol{n}} \times (\underline{\boldsymbol{n}} \times \underline{\boldsymbol{E}}) + \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}} = 0 \tag{9.87}$$

$$(n^2\overline{\overline{\mathbb{1}}} - \overline{\overline{\boldsymbol{\epsilon}}}) \cdot \underline{\boldsymbol{E}}_\perp = \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{E}}_\| \tag{9.88}$$

we can see that $\underline{\boldsymbol{E}}_L \gg \underline{\boldsymbol{E}}_T$ is true when

$$\boxed{n^2 \gg |\epsilon_{ij}|} \tag{9.89}$$

Since $n^2 \gg \epsilon_{ij}$ or as we see $n^2 \gg 1$, electrostatic waves tend to be short wavelength waves ($n^2 \sim \frac{1}{\lambda^2}$). We can see by examining Eq. (9.23)

$$An^4 - Bn^2 + C = 0 \tag{9.90}$$

that if we have $n^2 \gg 1$, then $A \to 0$ for it to be true. So we set $A = 0$.

$$A = S\sin^2\theta + P\cos^2\theta = 0 \tag{9.91}$$

$$\Downarrow$$

$$\boxed{\tan^2\theta = -\frac{P}{S}} \tag{9.92}$$

or

$$\boxed{n_\perp^2 = -\frac{P}{S}n_\parallel^2} \tag{9.93}$$

or

$$\boxed{k_\perp^2 S + k_\parallel^2 P = 0} \tag{9.94}$$

Notice that this form of the electrostatic dispersion relation for when $n^2 \gg |\epsilon_{ij}|, 1$ is exactly the same as a resonance for when $n^2 \to \infty$. Sometimes they are both true, but sometimes they are not. The cyclotron resonance ($S \to \infty$ at $\theta = 0$) is not an electrostatic wave since $n^2 \gg |\epsilon_{ij}|$ does not hold true.

This form of the dispersion relation can be taken to the next order by including B in the derivation

$$An^2 + B = 0$$
$$\left[S\sin^2\theta + P\cos^2\theta\right] n^2 - PS \simeq 0$$
$$n_\perp^2 S + n_\parallel^2 P - PS = 0$$
$$\boxed{n_\perp^2 = -\frac{P}{S}\left(n_\parallel^2 - S\right)} \tag{9.95}$$

where

$$B = RL\sin^2\theta + PS(1+\cos^2\theta) \tag{9.96}$$
$$P \gg |R|, |L|, |S| \tag{9.97}$$

One thing to note about ES waves is that as a wave moves toward cutoff, it starts to pick up an EM part.

At large ω, $P \sim 1$ and $S \sim 1$, so ES waves don't propagate. Thus at high frequencies, all waves are EM waves.

The electromagnetic approximation is

$$\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} \neq 0 \tag{9.98}$$

9.10.1 Examples of ES Waves

Some quick ES waves. The Langmuir-Tonks wave is a QL wave.

$$\omega^2 = \omega_{pe}^2 \tag{9.99}$$

The CESICW (cold electrostatic ion cyclotron wave) and the LH wave are also ES waves.

9.11 Finite Temperature Effects

9.11.1 Cold Plasma Approximation

The cold plasma approximation is simply that v_{th} is much less than the phase velocity of the wave, and that the Larmor radius is small (FLR).

$$\frac{\omega}{k_\parallel} \gg v_{th} \tag{9.100}$$

$$k_\perp \rho_s \ll 1 \tag{9.101}$$

If we examine Eq. (9.100), we see that it breaks down for electrons first.

$$\begin{gathered} \frac{\omega}{k_\parallel} \gg v_{th,s} \sim \sqrt{\frac{T_s}{m_s}} \\ \Downarrow \\ v_{th,e} > v_{th,i} \end{gathered} \tag{9.102}$$

On the other hand, FLR breaks down for ions first.

$$\begin{gathered} v_{th,s} = \rho_s \Omega_s \\ \sqrt{\frac{T_s}{m_s}} \sim \rho_s \frac{q_s B}{m_s} \\ \rho_s \sim \sqrt{T_s m_s} \\ \Downarrow \\ \rho_i > \rho_e \end{gathered} \tag{9.103}$$

9.11.2 Finite $T_{\parallel,e}$

The first step to take in including some temperature effects is to take into account the parallel thermal motions of the electrons. In order to do this, we use either an adiabatic or isothermal equation of state to close the fluid equations.

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad \text{(adiabatic)} \qquad (9.104)$$

$$p_{e1} = n_{e1}\kappa T_{e0} \qquad \text{(isothermal)} \qquad (9.105)$$

κ is Boltzmann's constant, and is ignored when using eV of T. $\gamma = \frac{n+2}{n}$ where n is the number of degrees of freedom. The adiabatic equation of state is used for warm electrons (or ions) where the density and T move with the fluid element, while the isothermal one is used for hot particles, where electrons (T) can move along the $\underline{\boldsymbol{B}}$ field very quickly, but the density moves with the fluid element. A couple alternate ways to write the adiabatic equation follow.

$$0 = \frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) \qquad \text{(adiabatic eq.)} \qquad (9.106)$$

$$= \frac{1}{\rho^\gamma}\frac{dp}{dt} - \frac{p\gamma}{\rho^{\gamma+1}}\frac{d\rho}{dt} \qquad \text{(chain rule)} \qquad (9.107)$$

$$= \frac{dp}{dt} - \frac{p\gamma}{\rho}\left[\frac{\partial\rho}{\partial t} + \underline{\boldsymbol{v}}\cdot\boldsymbol{\nabla}\rho\right] \qquad \text{(convective deriv)} \qquad (9.108)$$

$$= \frac{dp}{dt} - \frac{p\gamma}{\rho}\left[-\rho\boldsymbol{\nabla}\cdot\underline{\boldsymbol{v}} - \underline{\boldsymbol{v}}\cdot\boldsymbol{\nabla}\rho + \underline{\boldsymbol{v}}\cdot\boldsymbol{\nabla}\rho\right] \qquad \text{(Eq. (6.3))} \qquad (9.109)$$

$$\boxed{\frac{dp}{dt} + \gamma p\boldsymbol{\nabla}\cdot\underline{\boldsymbol{v}} = 0} \qquad (9.110)$$

In general, this form of the equation is a lot easier to work with. If we take Eq. (9.107), write it in a nice (linearized) form, and then manipulate it some, we find the following.

$$\frac{1}{p_{e0}}\frac{dp_{e1}}{dt} = \frac{\gamma}{n_{e0}}\frac{dn_{e1}}{dt} \qquad (9.111)$$

$$\frac{1}{p_{e0}}\left[\frac{\partial p_{e1}}{\partial t} + \underbrace{\underline{\boldsymbol{v}}_1\cdot\boldsymbol{\nabla}p_0}_{=0}\right] = \frac{\gamma}{n_{e0}}\left[\frac{\partial n_{e1}}{\partial t} + \underbrace{\underline{\boldsymbol{v}}_1\cdot\boldsymbol{\nabla}n_0}_{=0}\right] \qquad (9.112)$$

we know

$$p_{e0} = n_{e0} T_{e0} \tag{9.113}$$

from this we get

$$p_{e1} = \gamma n_{e1} T_{e0} \tag{9.114}$$

Since we are using the susceptibility formulation of plasma waves, we can easily use what we have already calculated and just plug in the new value of the warm/hot electrons' susceptibility χ^e_{zz}. To do this we simply linearize and FT the force and continuity equations. We assume low frequency ES ($\omega < |\Omega_e|$).

$$-i\omega n_{e0} m_e v_z = -n_{e0} e E_z - ik_\parallel \gamma n_{e1} T_{e0} \tag{9.115}$$

$$-i\omega n_{e1} + i\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{v}} n_{e0} = 0 \tag{9.116}$$

Continuing to work with this, the second equation above simplifies once we see that

$$\underline{\boldsymbol{E}} = -\boldsymbol{\nabla}\phi = \ i\underline{\boldsymbol{k}}\phi \tag{9.117}$$

$$\underline{\boldsymbol{v}}_{\perp,e} = \frac{e\underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}_0}{B_0^2} \tag{9.118}$$

$$\Downarrow$$

$$\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{v}}_{\perp,e} = 0 \tag{9.119}$$

Using this on that equation gives us

$$-i\omega n_{e1} + ik_\parallel v_z n_{e0} = 0 \Rightarrow n_{e1} = \frac{k_\parallel v_z}{\omega} n_{e0} \tag{9.120}$$

Now, we take this equation and Eq. (9.115) and stick them together.

$$-i\omega n_{e0} m_e v_z = -n_{e0} e E_z - ik_\parallel \gamma T_{e0} \left(\frac{k_\parallel v_z}{\omega} n_{e0} \right) \tag{9.121}$$

$$\left(\omega^2 - \frac{k_\parallel^2 \gamma T_{e0}}{m_e} \right) v_z = -\frac{ie\omega}{m_e} E_z \tag{9.122}$$

We then use Eq. (9.3)

$$n_{0e} q_e v_{ze} = j_{1e} = \frac{-i\omega}{4\pi} \chi_{e,zz} E_z \tag{9.123}$$

to get

$$\boxed{(\chi_{zz})_e = \frac{-\omega_{pe}^2}{\omega^2 - \frac{k_\parallel^2 \gamma T_{e0}}{m_e}}} \tag{9.124}$$

where $\gamma = 1$ for an isothermal system.

9.11.3 Plasma Oscillations

There are a couple applications of this kind of wave. The first is for plasma oscillations (Vlasov-Bohm or $P = 0$ waves). In the cold plasma, we simply use $\omega^2 = \omega_{ps}^2$. If we recall the definition of P,

$$P = 1 + \sum_s \chi_{zz} \tag{9.125}$$

we can see that with the correction for warm electrons (and cold ions), P would be

$$P = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2 - \frac{k_\parallel^2 \gamma T_{e0}}{m_e}} = 0 \tag{9.126}$$

This can be rewritten using the binomial expansion (since $\frac{\omega}{k_\parallel} \gg v_{th,e}$) as

$$0 = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}\left[1 + \frac{k_\parallel^2 \gamma T_{e0}}{m_e \omega^2}\right] \tag{9.127}$$

The final form of the dispersion relation is

$$\omega^2 = \omega_{pi}^2 + \omega_{pe}^2 \left[1 + \frac{k_\parallel^2 \gamma T_{e0}}{m_e \omega^2}\right] \tag{9.128}$$

For the hot plasmas seen later, when P is written, Eq. (9.126) is expanded for $\frac{\omega}{k_\parallel} \ll v_{th,e}$ and the electron term goes like the following:

$$-\frac{\omega_{pe}^2}{k_\parallel^2 v_{th,e}^2}\left[\frac{1}{1 - \frac{\omega^2}{k_\parallel^2 v_{th,e}^2}}\right] \approx \frac{\omega_{pe}^2}{k_\parallel^2 v_{th,e}^2} \tag{9.129}$$

9.11.4 Ion Acoustic Wave

Another wave in which we include finite temperature effects is the ion acoustic wave. In this kind of wave, we treat the electrons as hot (isothermal $\rightarrow \gamma = 1$) and the ions as warm (adiabatic $\rightarrow \gamma = 3$). Thus $v_{th,i} < \frac{\omega}{k} < v_{th,e}$. We start with the $P = 0$ wave, and we throw out the displacement term (the 1) because it is small.

$$0 = \frac{\omega_{pi}^2}{\omega^2 - \frac{3k_\parallel^2 T_{i0}}{m_i}} - \frac{\omega_{pe}^2}{\omega^2 - \frac{k_\parallel^2 T_{e0}}{m_e}} \tag{9.130}$$

$$\omega_{pi}^2\left(\underbrace{\omega^2}_{=0} - \frac{k_\parallel^2 T_{e0}}{m_e}\right) = -\omega_{pe}^2\left(\omega^2 - \frac{3k_\parallel^2 T_{i0}}{m_i}\right) \tag{9.131}$$

$$\omega^2 - \frac{3k_\parallel^2 T_{i0}}{m_i} = \underbrace{\frac{\omega_{pi}^2}{\omega_{pe}^2}}_{\frac{Z_i m_e}{m_i}} \frac{k_\parallel^2 T_{e0}}{m_e} \tag{9.132}$$

$$\omega^2 = k_\parallel^2\left[\frac{Z_i m_e}{m_i}\frac{T_{e0}}{m_e} + \frac{3T_{i0}}{m_i}\right] \tag{9.133}$$

$$\boxed{\omega^2 = k_\parallel^2\left[\frac{Z_i T_{e0} + 3T_{i0}}{m_i}\right]} \tag{9.134}$$

or

$$\boxed{\omega^2 = k_\parallel^2 C_s^2} \tag{9.135}$$

where $C_s = \frac{T_e}{m_i}$ is the sound speed. What is going on here is that the electrons can move faster and tend to overshoot their position. This sets up and $\underline{\boldsymbol{E}}$ field which drags the ions along. This sets up a pressure driven wave, much like a sound wave.

9.11.5 Electrostatic ICW

One of the other waves that we will look at is the electrostatic ion cyclotron wave. This is a ES version of the ion cyclotron wave. We begin with $\omega \sim \Omega_i$ and $v_{th,i} \ll \frac{\omega}{k_\parallel} < v_{th,e}$ or cold ions and hot electrons. We use the ES dispersion relation

$$n_\perp^2 S + n_\parallel^2 P = 0 \tag{9.136}$$

We simply plug in for values of S and P, using our temperature modified form for P.

$$n_\perp^2\left[1-\underbrace{\frac{\omega_{pi}^2}{\omega^2-\Omega_i}}_{\text{dominant}}{}^2+\frac{\omega_{pe}^2}{\Omega_e^2}\right]=-n_\parallel^2\left[1-\frac{\omega_{pi}^2}{\omega^2}-\underbrace{\frac{\omega_{pe}^2}{\omega^2-\frac{k_\parallel^2T_e}{m_e}}}_{\text{dominant}}\right] \tag{9.137}$$

If one keeps these dominant terms, the result is

$$\omega^2=\Omega_i^2+\frac{k_\perp^2 ZT_{e0}}{m_i} \tag{9.138}$$

9.11.6 Drift Waves

For drift waves, there is a y dependence for p for electrons, so $p_{e0}=p_{e0}(y)$ ($\boldsymbol{\nabla} n\sim\hat{\boldsymbol{y}}$). Ions are the same as before. We are still assuming ES waves, so $\underline{\boldsymbol{E}}=-\boldsymbol{\nabla}\phi=\imath\underline{\boldsymbol{k}}\phi$. As before,

$$J_z=-\frac{i\omega}{4\pi}\chi_{zz}E_z \tag{9.139}$$

We then linearize the force equation.

$$n_{e0}m_e\frac{d\underline{\boldsymbol{v_e}}}{dt}=-n_{e0}e\underline{\boldsymbol{E}}-\frac{n_{e0}e}{c}\underline{\boldsymbol{v}}\times\underline{\boldsymbol{B}}_0-\frac{\partial p_{e1}}{\partial z}\hat{\boldsymbol{z}} \tag{9.140}$$

We then take $\underline{\boldsymbol{B}}_0\times(\quad)$ this equation and solve for $\underline{\boldsymbol{v}}_\perp$.

$$\underline{\boldsymbol{v}}_\perp=-c\frac{\boldsymbol{\nabla}\phi\times\underline{\boldsymbol{B}}_0}{B_0^2}\Rightarrow\hat{\boldsymbol{y}} \tag{9.141}$$

This is the same for ions and electrons in the first order in the $\hat{\boldsymbol{y}}$ direction.

$$\boldsymbol{\nabla}\cdot\underline{\boldsymbol{v}}_\perp\sim i\underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{v}}_\perp\sim 0 \tag{9.142}$$

$$\boldsymbol{\nabla}\phi\sim(\hat{\boldsymbol{x}}+\hat{\boldsymbol{z}})\times B\hat{\boldsymbol{z}}\Rightarrow\hat{\boldsymbol{y}} \tag{9.143}$$

So we have

$$\underline{\boldsymbol{v}}_\perp=\frac{ik_xc\phi}{B_0}\hat{\boldsymbol{y}} \tag{9.144}$$

$$\underline{\boldsymbol{k}}=k_\perp\hat{\boldsymbol{x}}+k_\parallel\hat{\boldsymbol{z}} \tag{9.145}$$

We then proceed to linearize and FT the continuity equation.

$$-i\omega n_1 + \underline{\boldsymbol{v}}_1 \cdot \boldsymbol{\nabla} n_0 + n_0 \boldsymbol{\nabla} \cdot \underline{\boldsymbol{v}}_{\parallel} = 0$$
$$-i\omega n_1 + \frac{ik_x c\phi}{B_0}\frac{\partial n_0}{\partial y} + ik_{\parallel} v_z n_0 = 0$$

We then take this equation and plug it into the linearized FT'd equation of state and solve for p_1.

$$-i\omega p_1 + V_{Dy}\frac{\partial p_0}{\partial y} = \frac{\gamma p_0}{n_0}\left[-i\omega n_1 + \frac{ik_x c\phi}{B_0}\frac{\partial n_0}{\partial y}\right]$$
$$p_1 = \frac{k_x c\phi}{\omega B_0}\frac{\partial p_0}{\partial y} + \frac{k_{\parallel} v_z}{\omega}\gamma B_0$$

Now we go to the equation of motion,

$$mn\frac{dv_z}{dt} = qn_0 E_z - ik_{\parallel} p_1$$

and plug in for E_z

$$= -ik_{\parallel} q n_0 \phi - ik_{\parallel} p_1$$

At this point we plug in our expression for p_1 and rearrange.

$$v_z = \frac{\left[\frac{k_{\parallel} q\phi\omega}{m} + \frac{k_{\parallel} k_{\perp} c\phi}{qn_0 B_0}\frac{\partial p_0}{\partial y}\right]}{\left(\omega^2 - \frac{k_{\parallel}^2 \gamma T_{e0}}{m_e}\right)} \tag{9.146}$$

Then we use Eq. (9.139).

$$n_0 q v_z = -\frac{\omega}{4\pi}\chi_{zz} k_{\parallel}\phi$$
$$\chi_{zz} = -\frac{4\pi}{\omega}\frac{n_0 q}{k_{\parallel}\phi} v_z$$
$$\chi_{zz} = \frac{-\omega_{ps}^2\left[\omega - \frac{k_{\perp} c}{n_0 B_0}\frac{\partial p_0}{\partial y}\right]}{\omega^2 - k_{\parallel}^2\frac{\gamma T_s}{m_s}} \tag{9.147}$$

If we set χ_{zz} to zero, we can solve for ω^*.

$$\omega^* = \frac{k_{\perp} c}{qn_0 B_0}\frac{\partial p_0}{\partial y} \tag{9.148}$$

9.12 Energy Transfer

Poynting's theorem expresses the conservation of energy in an electromagnetic wave as follows.

$$\nabla \cdot \underline{\boldsymbol{P}} + \frac{\partial W}{\partial t} = -2\underline{\boldsymbol{k}}_i \cdot (\underline{\boldsymbol{P}} + \underline{\boldsymbol{T}}) + 2\omega_i W + \left.\frac{\partial W}{\partial t}\right|_{lossy} \tag{9.149}$$

The different parts of this equation represent different bits of energy. $\underline{\boldsymbol{P}}$ is the Poynting vector and it represents the flux of electromagnetic energy.

$$\underline{\boldsymbol{P}} = \frac{c}{16\pi}\left[\underline{\boldsymbol{E}}^* \times \underline{\boldsymbol{B}} + \underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}^*\right] e^{2\phi_i} \tag{9.150}$$

$\underline{\boldsymbol{T}}$ is the kinetic flux, or the amount of energy transported by acoustic means or particle flux.

$$\underline{\boldsymbol{T}} = -\frac{\omega_r}{16\pi}\underline{\boldsymbol{E}}^* \cdot \frac{\partial \overline{\overline{\boldsymbol{\epsilon}}}_h}{\partial \underline{\boldsymbol{k}}} \cdot \underline{\boldsymbol{E}} e^{2\phi_i} \tag{9.151}$$

where $\overline{\overline{\boldsymbol{\epsilon}}}_{a/h} = \frac{1}{2}\left[\overline{\overline{\boldsymbol{\epsilon}}} \pm \overline{\overline{\boldsymbol{\epsilon}}}^\dagger\right]$. W and $\left.\frac{\partial W}{\partial t}\right|_{lossy}$ are defined as

$$\frac{1}{16\pi}\left[|B|^2 + \underline{\boldsymbol{E}}^* \cdot \frac{\partial}{\partial \omega}(\omega \overline{\overline{\boldsymbol{\epsilon}}}_h) \cdot \underline{\boldsymbol{E}} e^{2\phi_i}\right] \tag{9.152}$$

$$\left.\frac{\partial W}{\partial t}\right|_{lossy} = \frac{\omega_r}{8\pi}\underline{\boldsymbol{E}}^* \cdot \overline{\overline{\boldsymbol{\epsilon}}}_a \cdot \underline{\boldsymbol{E}} e^{2\phi_i} \tag{9.153}$$

If there is no dissipation in the system, then

$$\delta \underline{\boldsymbol{k}} \cdot (\underline{\boldsymbol{P}} + \underline{\boldsymbol{T}}) = -\delta\omega W \tag{9.154}$$

$$\frac{\delta\omega}{\delta \underline{\boldsymbol{k}}} = \frac{\partial \omega}{\partial \underline{\boldsymbol{k}}} = \underline{\boldsymbol{v}}_g = \frac{\underline{\boldsymbol{P}} + \underline{\boldsymbol{T}}}{W} \tag{9.155}$$

When $D(\underline{\boldsymbol{k}}, \omega) = 0$, then

$$D(\underline{\boldsymbol{k}}, \omega) = 0 = D(\underline{\boldsymbol{k}} + \Delta\underline{\boldsymbol{k}}, \omega + \Delta\omega) = D(\underline{\boldsymbol{k}}, \omega) + \delta\omega\frac{\partial D}{\partial \omega} + \delta\underline{\boldsymbol{k}}\frac{\partial D}{\partial \underline{\boldsymbol{k}}} \Rightarrow$$

$$\frac{\partial \omega}{\partial \underline{\boldsymbol{k}}} = \frac{-\partial D/\partial \underline{\boldsymbol{k}}}{\partial D/\partial \omega} = \underline{\boldsymbol{v}}_g \tag{9.156}$$

9.12.1 Ray Tracing

These are the ray tracing equations. τ is a dummy variable (actually its the distance along the ray).

$$\frac{\partial \underline{\boldsymbol{r}}}{\partial \tau} = \frac{\partial g}{\partial \underline{\boldsymbol{k}}} \tag{9.157}$$

$$\frac{\partial t}{\partial \tau} = -\frac{\partial g}{\partial \omega} \tag{9.158}$$

$$\frac{\partial \underline{\boldsymbol{k}}}{\partial \tau} = -\frac{\partial g}{\partial \underline{\boldsymbol{r}}} \tag{9.159}$$

$$\frac{\partial \omega}{\partial \tau} = \frac{\partial g}{\partial t} \tag{9.160}$$

If we fiddle with this a bit, we get the following.

$$\frac{\frac{\partial \underline{\boldsymbol{r}}}{\partial \tau}}{\frac{\partial t}{\partial \tau}} = \frac{\partial \underline{\boldsymbol{r}}}{\partial t} = -\frac{\frac{\partial g}{\partial \underline{\boldsymbol{k}}}}{\frac{\partial g}{\partial \omega}} = \underline{\boldsymbol{v}}_g \tag{9.161}$$

If $\underline{\boldsymbol{v}}_g \cdot \underline{\boldsymbol{v}}_{ph} < 0$, then the wave is a backwards wave. Random fact: α is the angle between $\underline{\boldsymbol{v}}_g$ and $\underline{\boldsymbol{v}}_{ph}$.

$$\tan \alpha = -\frac{1}{n}\frac{\partial n}{\partial \theta} \tag{9.162}$$

10 Hot Plasma Waves

Several of the approximations used for cold plasmas are no longer valid for hot plasmas. $\frac{\omega}{k} \sim v_{th}$ for hot plasmas (the parallel component of v), while $k_\perp \rho \gtrsim O(1)$. Usually, $k_\perp \rho_i \gg k_\perp \rho_e$ for the same temperatures ($T_i \sim T_e$), so if $k_\perp \rho_i \ll 1$, then $k_\perp \rho_e$. Two handy parameters for summing all of this up are λ and ζ_n. The details of these two will be explained later, but for now we will list their values for hot and cold plasmas.

$$\underline{\textbf{Cold}} \qquad \lambda \ll 1 \rightarrow \text{gyroradius} \ll \text{wavelength} \tag{10.1}$$

$$\zeta_n \gg 1 \rightarrow v_{ph} \gg v_{th} \tag{10.2}$$

$$\underline{\textbf{Hot}} \qquad \lambda \gg 1 \tag{10.3}$$

$$\zeta_n \ll 1 \tag{10.4}$$

where

$$\lambda = \frac{1}{2}\rho_s^2 k_\perp^2 \tag{10.5}$$

$$\zeta_n = \frac{\omega - n\Omega - k_\parallel v_\parallel}{k_\parallel W_\parallel} \tag{10.6}$$

10.1 Plasma Kinetic Equations

The general procedure for hot waves is to use the kinetic equations, which consist of Vlasov's equation and Maxwell's equations. Vlasov's equation is simply Boltzmann's equation with the $\frac{df_s}{dt}\big|_{coll} = 0$.

$$\frac{\partial f_s}{\partial t} + \underline{\boldsymbol{v}} \cdot \frac{\partial f_s}{\partial \underline{\boldsymbol{r}}} + \frac{q_s}{m_s}\left[\underline{\boldsymbol{E}} + \frac{\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}}{c}\right] \cdot \frac{\partial f_s}{\partial \underline{\boldsymbol{v}}} = \underbrace{\frac{df_s}{dt}\bigg|_{coll}}_{=0} \tag{10.7}$$

We will also be using Maxwell's equations and the definition of $\underline{\boldsymbol{J}}$.

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{B}} = \frac{4\pi}{c}\underline{\boldsymbol{J}} + \frac{1}{c}\frac{\partial \underline{\boldsymbol{E}}}{\partial t} \tag{10.8}$$

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial \underline{\boldsymbol{B}}}{\partial t} \tag{10.9}$$

$$\underline{\boldsymbol{J}} = n_0 q \underline{\boldsymbol{v}}_1 \tag{10.10}$$

For the density $n(\underline{\boldsymbol{r}}, t)$, we will use the following equation.

$$n(\underline{\boldsymbol{r}}, t) = n_0 \underbrace{\int d^3 v f_s(\underline{\boldsymbol{r}}, \underline{\boldsymbol{v}}, t)}_{=1} \tag{10.11}$$

Depending on whether we are dealing with electrostatic or electromagnetic waves, we will use the one of following equations.

$$\sigma = \sum_s q_s \int \mathrm{d}^3 v f_{s1}(\underline{\boldsymbol{r}}, \underline{\boldsymbol{v}}, t) \qquad \text{(electrostatic)} \tag{10.12}$$

$$\underline{\boldsymbol{J}} = \sum_s q_s \int \mathrm{d}^3 v \underline{\boldsymbol{v}} f_{s1}(\underline{\boldsymbol{r}}, \underline{\boldsymbol{v}}, t) \qquad \text{(electromagnetic)} \tag{10.13}$$

Finally, again depending on whether or not the waves are ES or EM, one of the following equations is used.

$$\nabla^2 \phi = -4\pi\sigma \qquad \text{(electrostatic)} \tag{10.14}$$

$$\underline{\boldsymbol{J}} = -\frac{i\omega}{4\pi} \sum_s \overline{\overline{\boldsymbol{\chi}}}_s \cdot \underline{\boldsymbol{E}} \qquad \text{(electromagnetic)} \tag{10.15}$$

10.2 Electrostatic Oscillations

Now that we have our basic set of equations that we are going to work with, we can outline the basic procedure. Assume 1-D, ES limit ($\underline{\boldsymbol{B}}_1 = 0$, $\underline{\boldsymbol{k}} \parallel \underline{\boldsymbol{E}}$) in an unmagnetized plasma. The first thing we do is to break up f_s.

$$f_s = f_0 + f_1 \tag{10.16}$$

where $f_1 \ll f_0$. We then linearize the Vlasov equation by assuming small amplitude waves ($\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_1$), getting a zeroth order and a first order equation.

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial z} = 0 \tag{10.17}$$

$$\underbrace{\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial z}}_{\frac{df_1}{dt}} = -\frac{q}{m} E \frac{\partial f_0}{\partial v} \tag{10.18}$$

At this point, there is some trickery involving Landau contours which we will go into in a bit. However, the next step is to FT in space and Laplace transform in time the first order equation and rearrange to get the following.

$$\tilde{f}_{s1}(k, \omega, v) = \frac{-i\frac{q_s}{m_s}\tilde{E}(k,\omega)\frac{\partial f_{s0}}{\partial v} + i g_s(k,v)}{\omega - kv} \tag{10.19}$$

$g_s(k, v) = f_{s1}(k, v, t = 0)$ is the spatial transform of the initial disturbance. From this point on, $\tilde{F}$ is short for $\tilde{F}(\omega, k)$ as opposed to $F = F(r, t)$. Also, a similar transform of Poisson's equation gives us

$$ik\tilde{E}(k,\omega) = 4\pi \sum_s q_s \int_{-\infty}^{\infty} \mathrm{d}v \tilde{f}_s(k,\omega,v) \tag{10.20}$$

Combining the first order Vlasov equation and Poisson's equation gives us

$$\tilde{E}(\omega,k) = \frac{\frac{4\pi}{k}\sum_s q_s \int_{-\infty}^{\infty} \mathrm{d}v \frac{g_s(v,k)}{\omega - kv}}{1 + \frac{4\pi}{k}\sum_s \frac{q_s^2}{m_s}\int_{-\infty}^{\infty} \mathrm{d}v \frac{df_{s0}(v)/dv}{\omega - kv}} \tag{10.21}$$

The integrals in the previous equation are along the real axis. However, there is a pole at $\omega = kv$, so this must be accounted for. Only the poles of this equation contribute to the solution $E(z,t)$, and those poles only arise when the denominator is equal to zero. So the dispersion relation for electrostatic oscillations in an unmagnetized plasma is simply

$$\boxed{H(\omega,k) = D(\omega,k) = 1 + \frac{\omega_{po}^2}{k}\int_{-\infty}^{\infty} \mathrm{d}v \frac{df_{s0}(v)/dv}{\omega - kv} = 0} \tag{10.22}$$

where

$$\omega_{po}^2 f_o(v) = \sum_s \frac{4\pi q_s^2}{m_s} f_{s0}(v) \tag{10.23}$$

where $f_o(v)$ is a single "reduced" velocity distribution consisting of the combined zero-order velocity distributions for ions and electrons. $f_o(v)$ is normalized to 1 and not n_0. The integral in the dispersion relation is evaluated along the Landau contour.

10.3 The Landau Contour

To back out an expression for $E(z,t)$, one must take the inverse Laplace transform and the inverse FT of Eq. (10.22). Likewise for $f_{s1}(v,z,t)$. This inverse transform is evaluated along a contour that goes along the real axis and closes above in the v plane ($v_r vs v_i$). If one extends the top part of the contour to ∞, then the contribution of it goes to 0 since v gets big. In this plane, there is one pole at $v = \frac{\omega}{k} = \frac{\omega_r}{k} + i\frac{\sigma}{k}$. The contour should always contain this pole.

In the ω plane, we drop the contour down so that the contribution goes to 0. This makes the integral easier to perform since the contributions come only from the poles. Also of note is that as t gets large, the uppermost pole(s) dominate. We must make sure as we do that our contours stay above the poles. Thus the value of the integral comes from the sum of the residues. As we do this, (for $k > 0$), the pole in the v plane drops. We must deform the contour to make sure that we still contain it.

the result of all this is the following dispersion relation, the form of which depends on the value of ω_i. This form is for $k > 0$.

$$H(\omega,k) = \begin{cases} 1 + \frac{\omega_{po}^2}{k}\int_{-\infty}^{\infty} \mathrm{d}v \frac{\frac{\partial f_o}{\partial v}}{\omega - kv} & \omega_i > 0,\ k > 0 \\ 1 + \frac{\omega_{po}^2}{k} P \int_{-\infty}^{\infty} \mathrm{d}v \frac{\frac{\partial f_o}{\partial v}}{\omega - kv} - \frac{i\pi\omega_{po}^2}{k^2}\frac{\partial f_o}{\partial v}\Big|_{\frac{\omega}{k}=0} & \omega_i = 0,\ k > 0 \\ 1 + \frac{\omega_{po}^2}{k}\int_{-\infty}^{\infty} \mathrm{d}v \frac{\frac{\partial f_o}{\partial v}}{\omega - kv} - \frac{2i\pi\omega_{po}^2}{k^2}\frac{\partial f_o}{\partial v}\Big|_{\frac{\omega}{k}} & \omega_i < 0,\ k > 0 \end{cases} \tag{10.24}$$

In these equations, the $P\int$ is similar to the principal part (which is along the real axis) but instead is on a line parallel to the real axis that runs through the pole.

We can rewrite this as the following for all ω_i by using this principal part notation. It ends up being similar to an average of the contour just above and just below the pole (see page 190 of Stix).

$$H(\omega,k) = 1 + \sum_s \frac{4\pi q_s^2}{m_s} n_s \int\limits_L \mathrm{d}v \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} =$$
$$1 + \frac{\omega_{po}^2}{k} P \int_{-\infty}^{\infty} \mathrm{d}v \frac{\frac{\partial f_0}{\partial v}}{\omega - kv} - \frac{i\pi\omega_{po}^2}{k\,|k|}\frac{\partial f_0}{\partial v}\Big|_{v=\frac{\omega}{k}} = 0 \tag{10.25}$$

10.4 The Plasma Dispersion Function

If we let the distribution function of each species be a Maxwellian to the zeroth order

$$f_0 = \frac{1}{\sqrt{\pi}}\frac{1}{v_{th}} e^{-\frac{v^2}{v_{th}^2}} \tag{10.26}$$

and plug this into our dispersion relation, using the $\omega_i > 0$ for H, we start to see where the PDF comes from. The $\frac{1}{v - v_{ph}}$ in the equation above comes from pulling out a $-\frac{1}{k}$. The $v_{th,s}^2 = \frac{2T}{m_s}$.

$$
\begin{aligned}
0 &= 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} \mathrm{d}v \frac{\frac{\partial f_0}{\partial v}}{v - v_{ph}} \\
&= 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} \mathrm{d}v \; \underbrace{\frac{d}{dv}\left[\frac{f_0(v)}{v - v_{ph}}\right]}_{\to 0 \text{ since } \frac{f_0(\infty)}{\pm\infty} \to 0} - \frac{f_0}{(v - v_{ph})^2}
\end{aligned}
$$

since

$$
\frac{d}{dv}\left[\frac{f_0(v)}{v - v_{ph}}\right] = \frac{1}{v - v_{ph}} \frac{\partial f_0}{\partial v} - \frac{f_0}{(v - v_{ph})^2}
$$

Also

$$
\int_{-\infty}^{\infty} \mathrm{d}v \frac{\frac{\partial f_0}{\partial v}}{v - v_{ph}} = \int_{-\infty}^{\infty} \mathrm{d}v \frac{f_0}{(v - v_{ph})^2} = \frac{d}{dv_{ph}} \int_{-\infty}^{\infty} \mathrm{d}v \frac{f_0(v)}{v - v_{ph}}
$$

Here we used $\frac{d}{dv_{ph}}\left(\frac{f_0(v)}{v - v_{ph}}\right) = \frac{f_0(v)}{(v - v_{ph})^2}$. Plugging all this into H give us

$$
H(\omega, k) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \frac{1}{v_{th}\sqrt{\pi}} \frac{d}{dv_{ph}} \int_{-\infty}^{\infty} \mathrm{d}v \frac{1}{v_{th}} \frac{e^{-v^2/v_{th}^2}}{(v - v_{ph})/v_{th}} \tag{10.27}
$$

At this point, we make a change of variables.

$$
u = \frac{v}{v_{th}} \qquad \zeta = \frac{v_{ph}}{v_{th}} = \frac{\omega}{k v_{th}} \tag{10.28}
$$

$$
\mathrm{d}u = \frac{\mathrm{d}v}{v_{th}} \qquad \mathrm{d}\zeta = \frac{1}{v_{th}} \mathrm{d}v_{ph} \tag{10.29}
$$

Making the switch gives us

$$
\begin{aligned}
H(\omega, k) &= 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \frac{1}{v_{th}\sqrt{\pi}} \frac{v_{th}}{v_{th}^2} \frac{d}{d\zeta} \int_{-\infty}^{\infty} \mathrm{d}u \frac{e^{-u^2}}{u - \zeta} \\
&= 1 - \sum_s \frac{\omega_{ps}^2}{k^2 v_{th}^2} \frac{d}{d\zeta} \boxed{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \frac{e^{-u^2}}{u - \zeta}}
\end{aligned} \tag{10.30}
$$

The expression in the box is the plasma dispersion fucnction (PDF) $Z(\zeta)$.

$$
\boxed{Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \frac{e^{-u^2}}{u - \zeta}} \tag{10.31}
$$

with

$$\boxed{\zeta = \frac{\omega}{kv_{th}}} \tag{10.32}$$

For cold plasmas, use

$$\zeta \gg 1 \qquad \frac{\omega}{k} \gg v_{th} \tag{10.33}$$

For hot plasmas, use

$$\zeta \ll 1 \qquad \frac{\omega}{k} \ll v_{th} \tag{10.34}$$

For a drifting Maxwellian

$$f_0 = \frac{1}{\sqrt{\pi}}\frac{1}{v_{th}}e^{-\frac{(v-v_D)^2}{v_{th}^2}} \tag{10.35}$$

we would use for our change of variables

$$u = \frac{v - v_D}{v_{th}} \qquad \zeta = \frac{v_{ph} - v_D}{v_{th}} = \frac{\omega - k_\parallel v_D}{k_\parallel v_{th}} \tag{10.36}$$

$$\mathrm{d}u = \frac{\mathrm{d}v}{v_{th}} \qquad \mathrm{d}\zeta = \frac{1}{v_{th}}\mathrm{d}v_{ph} \tag{10.37}$$

The derivative of the PDF can be rewritten.

$$\frac{dZ(\zeta)}{d\zeta} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}u\frac{e^{-u^2}}{(u-\zeta)^2} \tag{10.38}$$

Also

$$\frac{d}{du}\left[\frac{e^{-u^2}}{u-\zeta}\right] = \frac{1}{u-\zeta}\left(-2ue^{-u^2}\right) - \frac{1}{(u-\zeta)^2}e^{-u^2} \tag{10.39}$$

so

$$\begin{aligned}
\int_{-\infty}^{\infty}\mathrm{d}u\frac{e^{-u^2}}{(u-\zeta)^2} &= \int_{-\infty}^{\infty}\mathrm{d}u\frac{-2ue^{-u^2}}{u-\zeta} - \underbrace{\int_{-\infty}^{\infty}\mathrm{d}u\frac{d}{du}\frac{e^{-u^2}}{u-\zeta}}_{=0} \\
&= -2\int_{-\infty}^{\infty}\mathrm{d}u\frac{\left[(u-\zeta)+\zeta\right]e^{-u^2}}{u-\zeta} \\
&= -2\underbrace{\int_{-\infty}^{\infty}\mathrm{d}ue^{-u^2}}_{=\sqrt{\pi}} - 2\zeta\underbrace{\int_{-\infty}^{\infty}\mathrm{d}u\frac{e^{-u^2}}{u-\zeta}}_{=\sqrt{\pi}Z(\zeta)}
\end{aligned} \tag{10.40}$$

As a result, one can see that

$$\boxed{\frac{dZ(\zeta)}{d\zeta} = -2\left[1 + \zeta Z(\zeta)\right]} \tag{10.41}$$

This lets us write the dispersion relation for longitudinal waves in an unmagnetized Maxwellian plasma (Eq. (10.22)) as the following.

$$\boxed{H(\omega, k) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2 v_{th}^2} \frac{d}{d\zeta_s} Z(\zeta_s)} \tag{10.42}$$

$$\boxed{H(\omega, k) = 1 + \sum_s 2\frac{\omega_{ps}^2}{k^2 v_{th}^2}\left[1 + \zeta_s Z(\zeta_s)\right]} \tag{10.43}$$

One final way to write this is

$$\boxed{k^2 = \frac{1}{2}\sum_s \frac{1}{\lambda_{ds}^2} Z_0'(\zeta_s)} \tag{10.44}$$

where

$$\boxed{\frac{1}{\lambda_{ds}^2} = \frac{4\pi n_s q_s^2}{\kappa T_s}} \tag{10.45}$$

The hot expansion for the PDF is

$$Z(\zeta \ll 1) = -2\zeta + \frac{4}{3}\zeta^3 + \cdots + i\sqrt{\pi}\,\mathrm{sgn}(k_\parallel) e^{-\zeta^2} \tag{10.46}$$

The cold expansion is

$$Z(\zeta \gg 1) = -\frac{1}{\zeta}\left[1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \ldots\right] + i\sqrt{\pi}\sigma\,\mathrm{sgn}(k_\parallel) e^{-\zeta^2} \tag{10.47}$$

where

$$\sigma = \begin{cases} 0 & \text{for } \mathrm{sgn}(k_\parallel)\mathsf{Im}\zeta = \mathsf{Im}(\omega) > 0 \\ 2 & \text{for } \mathrm{sgn}(k_\parallel)\mathsf{Im}\zeta = \mathsf{Im}(\omega) < 0 \\ 1 & \text{for } |\mathsf{Re}\zeta| \gg 1 \text{ and } |\mathsf{Re}\zeta|\,|\mathsf{Im}\zeta| \lesssim \frac{\pi}{4} \end{cases} \tag{10.48}$$

The term $\propto ie^{-\zeta^2}$ in the above expansions is the damping term coming from Landau damping.

In the hot plasma case, the dispersion relation for electrostatic waves with Maxwellian ions and electrons reduces to the Debye shielding condition, while in the cold limit, it reduces to Eq. (3.54) in Stix, which was derived when setting $\epsilon_{zz} = 0$ with the adiabatic correction.

10.5 Nyquist Stability Criterion

One can count the number of unstable roots for a given zero-order velocity distribution without finding the roots explicitly by using the Nyquist criterion. The Nyquist criterion states that the total number of unstable modes of a plasma will be equal to the number of zeros of $H(\frac{\omega}{k}, k)$ within the upper half-plane. So the number of zeros in the upper half-plane is given by

$$N_0 = \frac{1}{2\pi i} \int\limits_C \mathrm{d}\omega \frac{H'(\omega)}{H(\omega)} \tag{10.49}$$

This can also be mapped onto the complex H plane.

$$N_0 = \frac{1}{2\pi i} \int\limits_D \frac{\mathrm{d}H}{H} \tag{10.50}$$

If $f_0(v)$ is singly peaked, then it is stable. If it is doubly peaked, the stability is determined by the separation of the peaks (two stream instability).

10.6 Miscellaneous

10.6.1 Van Kampen Modes

The Van Kampen modes are the steady state solution for the linearized Boltzmann equation Eq. (10.18).

$$f_{s1} = -\frac{iq_s E}{m_s} \frac{1}{\omega - kv} \frac{df_{s0}}{dv} \tag{10.51}$$

This solution represents the poles in the *numerator* of Eq. (10.21) as opposed to the poles in the *denominator*. Van Kampen came up with a meaningful way of dealing with the singularity at $v = \frac{\omega}{k}$ and the form of this equation was

$$f_{s1} = -\frac{iq_s E}{m_s} \frac{df_{s0}}{dv} \left[P\left(\frac{1}{\omega - kv}\right) + \lambda\delta\left(v - \frac{\omega}{k}\right)\right] \tag{10.52}$$

δ here is a Dirac delta function and λ is chosen to satisfy Poisson's equation.

$$\boldsymbol{\nabla} \cdot \underline{\boldsymbol{E}} = ikE = 4\pi \sum_s q_s \int_{-\infty}^{\infty} f_{s1} \mathrm{d}v \tag{10.53}$$

10.6.2 Growth Rate of Instabilities

To calculate the growth rate γ, one must expand H into real and imaginary components.

$$H_r(\omega_r - i\gamma) + iH_i(\omega_r - i\gamma) = 0 \tag{10.54}$$

$$\underbrace{H_r(\omega_r)}_{\to 0} - i\gamma\frac{\partial H_r}{\partial \omega_r} + iH_i(\omega_r) \simeq 0 \tag{10.55}$$

$$\Downarrow$$

$$\boxed{\gamma = \frac{H_i}{H_r'}} \tag{10.56}$$

10.7 Hot Plasma Waves in a Magnetized Plasma

For magnetized plasmas, we will start out with uniformly magnetized, homogeneous (locally) plasmas where $\underline{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}} + \underline{\boldsymbol{B}}_1$, while $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_1$. The Vlasov equation for this kind of system is

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial t} + \underline{\boldsymbol{v}}_1\cdot\boldsymbol{\nabla} f_1 + \frac{q}{m}\underline{\boldsymbol{v}}\times\underline{\boldsymbol{B}}_0\cdot\frac{\partial f_1}{\partial \underline{\boldsymbol{v}}} = -\frac{q}{m}\left[\underline{\boldsymbol{E}}_1 + \frac{1}{c}\underline{\boldsymbol{v}}\times\underline{\boldsymbol{B}}_1\right]\cdot\frac{\partial f_0}{\partial \underline{\boldsymbol{v}}} \tag{10.57}$$

10.7.1 Electrostatic Waves

For ES waves in a magnetized plasma we know that $\underline{\boldsymbol{B}}_1 = 0$ and $\underline{\boldsymbol{E}} = -\boldsymbol{\nabla}\phi = -i\underline{\boldsymbol{k}}\Phi = -i(k_\perp\hat{\boldsymbol{x}} + k_\parallel\hat{\boldsymbol{z}})\Phi$. See Waves lecture (L12.2). The Vlasov equation reduces to

$$\frac{\partial}{\partial t}f_1(\underline{\boldsymbol{r}},\underline{\boldsymbol{v}},t) = -\frac{q}{m}(-i)(k_\perp\hat{\boldsymbol{x}} - k_\parallel\hat{\boldsymbol{z}})\Phi\cdot\frac{\partial f_0}{\partial \underline{\boldsymbol{v}}} = -\frac{iq}{m}\Phi\left[k_\perp\frac{\partial f_0}{\partial v_x} + k_\parallel\frac{\partial f_0}{\partial v_z}\right] \tag{10.58}$$

where

$$f_1 = \frac{iq}{m}\int_{-\infty}^{t} \mathrm{d}t'\Phi(\underline{\boldsymbol{r}},t')\left[k_\perp\frac{\partial f_0}{\partial v_x} + k_\parallel\frac{\partial f_0}{\partial v_z}\right] \tag{10.59}$$

Also, $\Phi \sim e^{i(\underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{r}}-\omega t)}$. A quick derivation shows us that $\frac{\partial f_0}{\partial v_x} = \frac{\partial f_0}{\partial v_\perp}\frac{v_x}{v_\perp}$:

$$\begin{aligned} v_\perp^2 &= v_x^2 + v_y^2 \\ \frac{\partial f_0}{\partial v_x} &= \frac{\partial f_0}{\partial v_\perp}\frac{\partial v_\perp}{\partial v_x} \\ 2v_\perp\frac{\partial v_\perp}{\partial v_x} &= 2v_x \\ \frac{\partial v_\perp}{\partial v_x} &= \frac{v_x}{v_\perp} \\ \therefore \frac{\partial f_0}{\partial v_x} &= \frac{\partial f_0}{\partial v_\perp}\frac{v_x}{v_\perp} \end{aligned} \tag{10.60}$$

Furthermore $\frac{dz'}{dt'} = v_\parallel$ since $z'(t') = z + v_\parallel(t'-t)$. Likewise, $\frac{d\phi'}{dt} = -\Omega$, since $\phi'(t'=t) = \phi(t)$ and $\phi'(t') = -\Omega(t'-t) + \phi(t)$. We also have $v_x'(t') = v_\perp\cos(\phi'(t'))$. Plugging in our expression for $\phi'(t')$ gives us

$$v_\perp\cos\left[\phi(t) + \Omega(t-t')\right] \tag{10.61}$$

Following from this

$$x'(t') - x'(t) = -\frac{v_\perp}{\Omega}\left[\sin\left[\phi + \Omega(t-t')\right] - \sin\phi(t)\right] \tag{10.62}$$

Returning to our expression for f_1 and substituting these values in gives u

$$f_1 = i\frac{q}{m}\int_{-\infty}^{t}\mathrm{d}t'\left\{k_\perp\frac{v_x'(t')}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + k_\parallel\frac{\partial f_0}{\partial v_\parallel}\right\}\underbrace{\tilde{\Phi}e^{i\left[k_\perp x'(t') + k_\parallel z'(t') - \omega t'\right]}}_{=\Phi\to\text{comes out of}\int\mathrm{d}t'}e^{i\omega t}e^{-i\omega t} \tag{10.63}$$

We also make a change of variables.

$$\tau = t - t' \qquad \mathrm{d}t' = -\mathrm{d}\tau \tag{10.64}$$

$$\int_{-\infty}^{t}\mathrm{d}t' \to -\int_{\infty}^{0}\mathrm{d}\tau \to \int_0^\infty \mathrm{d}\tau \tag{10.65}$$

Now, f_1 is

$$\begin{aligned} f_1 = i\tilde{\Phi}\frac{q}{m}\int_0^\infty \mathrm{d}\tau\left\{k_\perp\cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial v_\perp} + k_\parallel\frac{\partial f_0}{\partial v_\parallel}\right\}\exp\Bigg[k_\perp x(t) \\ -\frac{k_\perp v_\perp}{\Omega}\big(\sin(\phi+\Omega\tau) - \sin\phi\big) + k_\parallel x(t) - k_\parallel v_\parallel\tau - \omega t + \omega\tau\Bigg] \end{aligned} \tag{10.66}$$

In this equation, $e^{i\underline{\boldsymbol{k}}\cdot r} = e^{ik_\perp x'(t')}e^{ik_\parallel x'(t')}$. We will also make a substitution to make the exponential term a little more manageable.

$$\exp\{i\tilde{\beta}(\tau)\} = \exp(i\beta)\exp(i\beta') \tag{10.67}$$

$$f_1 = i\tilde{\Phi}\frac{q}{m}e^{i\beta}\int_0^\infty \mathrm{d}\tau\left[k_\perp\cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial v_\perp} + k_\parallel\frac{\partial f_0}{\partial v_\parallel}\right]e^{i\beta'(\tau)} \tag{10.68}$$

We then turn to Poisson's equation.

$$\begin{aligned}\boldsymbol{\nabla}\cdot\underline{\boldsymbol{E}} = \underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{E}} = k^2\tilde{\Phi}e^{i\beta} &= 4\pi\sum_s q_s\int_L f_1\mathrm{d}^3v \\ &= 4\pi\sum_s q_s\int_L \mathrm{d}^3v\, i\tilde{\Phi}\frac{q}{m}e^{i\beta}\int_0^\infty \mathrm{d}\tau\big(\quad\big)_s e^{i\beta'(\tau)}\end{aligned} \tag{10.69}$$

From this we continue on in the derivation.

$$1 = i\sum_s\frac{\omega_{ps}^2}{k^2}\int_L \mathrm{d}^3v\int_0^\infty \mathrm{d}\tau\left[k_\perp\cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial v_\perp} + k_\parallel\frac{\partial f_0}{\partial v_\parallel}\right]e^{i\beta'(\tau)} \tag{10.70}$$

where the integral over the Landau contour can be written as

$$\int_L \mathrm{d}^3v = \int \mathrm{d}\phi\int v_\perp \mathrm{d}v_\perp\int \mathrm{d}v_\parallel \tag{10.71}$$

Proceeding

$$\begin{aligned}1 = i\sum_s\frac{\omega_{ps}^2}{k^2}&\int_0^\infty \mathrm{d}\tau\int_{-\infty}^\infty \mathrm{d}v_\parallel\int_0^\infty \mathrm{d}v_\perp v_\perp e^{i(\omega-k_\parallel v_\parallel)\tau} \\ &\underbrace{\int_0^{2\pi}\mathrm{d}\phi\left[k_\perp\cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial v_\perp} + k_\parallel\frac{\partial f_0}{\partial v_\parallel}\right]e^{-iz\left(\sin(\phi+\Omega\tau)-\sin\phi\right)}}\end{aligned} \tag{10.72}$$

The term with the underbrace can be rewritten using

$$\frac{\partial}{\partial\tau}\left[e^{-iz\left(\sin(\phi+\Omega\tau)-\sin\phi\right)}\right] = -iz\Omega\cos(\phi+\Omega\tau)e^{-iz\left(\sin(\phi+\Omega\tau)-\sin\phi\right)} \tag{10.73}$$

Also, we use the following relations.

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty}e^{in\phi}J_n(z) \tag{10.74}$$

$$e^{-izsin(\phi+\Omega\tau)} = \sum_{m=-\infty}^{\infty}e^{-im(\phi+\Omega\tau)}J_m(z) \tag{10.75}$$

Continuing

$$1 = \int_0^{2\pi} \mathrm{d}\phi \left[\frac{k_\perp}{-iz\Omega} \frac{\partial}{\partial \tau} \frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel} \right] \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(z) \sum_{m=-\infty}^{\infty} e^{-im(\phi+\Omega\tau)} J_m(z)$$
$$= \left[\frac{k_\perp}{-iz\Omega} \frac{\partial}{\partial \tau} \frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel} \right] \sum_{n=-\infty}^{\infty} J_n(z) \sum_{m=-\infty}^{\infty} J_m(z) \int_0^{2\pi} \mathrm{d}\phi e^{in\phi} e^{-im(\phi+\Omega\tau)} \tag{10.76}$$

where

$$\int_0^{2\pi} \mathrm{d}\phi e^{in\phi} e^{-im(\phi+\Omega\tau)} = \int_0^{2\pi} \mathrm{d}\phi e^{-i(n-m)\phi} e^{-im\Omega\tau} = e^{-im\Omega} 2\pi\delta(n-m) \tag{10.77}$$

Obviously, taking $\frac{\partial}{\partial \tau}$ of the above equation will bring down a $-im\Omega$. We now look at the other integrals.

$$\int \mathrm{d}\phi \rightarrow 2\pi \left[\frac{nk_\perp}{z} \frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel} \right] \sum_n J_n^2(z) e^{-in\Omega\tau} \tag{10.78}$$

$$\int \mathrm{d}\tau \sum_n e^{i(\omega - k_\parallel v_\parallel - n\Omega)\tau} = \sum_n \frac{1}{i(\omega - k_\parallel v_\parallel - n\Omega)} e^{i(\quad)} \Big|_0^\infty \tag{10.79}$$

where $\lim_{\nu\rightarrow 0^+}$ for causality purposes. The exponential term goes away when evaluated, since $\omega = \omega + i\nu$, so $e^{i\omega\tau} \sim e^{i\omega\tau} e^{-\nu\tau} \rightarrow 0$. Thus we end up with

$$\boxed{1 = -2\pi \sum_s \frac{\omega_{ps}^2}{k^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \int_0^\infty \mathrm{d}v_\perp v_\perp \frac{\left[\frac{nk_\perp}{z} \frac{\partial f_0}{\partial v_\perp} + k_\parallel \frac{\partial f_0}{\partial v_\parallel} \right]_s}{\omega - k_\parallel v_\parallel - n\Omega_s} J_n^2(z_s)} \tag{10.80}$$

Both integrals are along the Landau contour.

To continue this derivation, we now stick in the Maxwellians for $f_0(v_\perp, v_\parallel)$, with $\underline{\boldsymbol{v}}_{Ds} = 0$.

$$f_0(v_\perp, v_\parallel) = \underbrace{\left[\frac{1}{\pi w_{\perp s}^2} e^{-\frac{v_\perp^2}{w_{\perp s}^2}} \right]}_{f_M(v_\perp)} \underbrace{\left[\frac{1}{\pi w_{\parallel s}^2} e^{-\frac{v_\parallel^2}{w_{\parallel s}^2}} \right]}_{f_M(v_\parallel)} \tag{10.81}$$

Taking the derivatives of f_0 with respect to $v_\perp$ and $v_\parallel$

$$\frac{\partial f_0}{\partial v_\perp} = f_M(v_\parallel)\left[-\frac{2v_\perp^2}{w_{\perp s}^2}\right] f_M(v_\perp) \tag{10.82}$$

$$\frac{\partial f_0}{\partial v_\parallel} = f_M(v_\perp)\left[-\frac{2v_\parallel^2}{w_{\parallel s}^2}\right] f_M(v_\parallel) \tag{10.83}$$

and subbing in the results in Eq. (10.80) gives us

$$1 = -2\pi \sum_s \frac{\omega_{ps}^2}{k^2} \sum_{n=-\infty}^{\infty} \int_0^\infty v_\perp \mathrm{d}v_\perp J_n^2(z) f_M(v_\perp)\times \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \frac{\left[-\frac{2n\Omega_s}{w_{\perp s}^2} + k_\parallel \frac{\partial}{\partial v_\parallel}\right] f_M(v_\parallel)}{\omega - k_\parallel v_\parallel - n\Omega_s} \tag{10.84}$$

We use the following integral to help us evaluate the above equation.

$$\int_0^\infty t\mathrm{d}t J_\nu(at) J_n(bt) e^{-p^2t^2} = \frac{1}{2p^2}\exp\left[-\frac{a^2+b^2}{4p^2}\right] I_\nu\left(\frac{ab}{2p^2}\right) \tag{10.85}$$

where I_n is the modified Bessel function.

$$I_\nu(x) = \frac{1}{\nu!}\frac{x}{2}^\nu\left[1 + \frac{(x/2)^2}{1(\nu+1)} + \frac{(x/2)^4}{1\cdot 2(\nu+1)(\nu+2)} + \cdots\right] \tag{10.86}$$

Subbing in our values for this integral gives us the following.

$$\int_0^\infty v_\perp \mathrm{d}v_\perp J_n\left(\frac{k_\perp v_\perp}{\Omega}\right) J_n\left(\frac{k_\perp v_\perp}{\Omega}\right) e^{-\frac{v_\perp^2}{w_\perp^2}} = \frac{1}{2}w_\perp^2 \exp\left[-\frac{k_\perp^2 w_\perp^2}{2\Omega^2}\right] I_n\left(\frac{k_\perp^2 w_\perp^2}{2\Omega^2}\right) \tag{10.87}$$

where $a = b = \frac{k_\perp}{\Omega}$, $p = \frac{1}{w_\perp^2}$, and $\nu = n$. Since $\lambda = \frac{1}{2}\frac{k_\perp^2 w_\perp^2}{\Omega_s^2} \sim \frac{1}{2}k_\perp^2\rho_s^2 \rightarrow \frac{\rho_s}{\lambda_\perp}$, the above integral $\sim e^{-\lambda} I_\nu(\lambda)$. Using this evaluation for the integration and changing variables to correspond to the PDF ($t = \frac{v_\parallel}{w_{\parallel s}}$), we get a final form for the dispersion relation for uniformly magnetized, locally homogeneous ES waves in hot plasma. See Eq. (11.85) in Stix.

$$\boxed{k^2 = -\sum_s \frac{2\omega_{ps}^2}{w_{\parallel s}^2} \sum_{n=-\infty}^{\infty} e^{-\lambda_s} I_n(\lambda_s)\left\{1 + \left[\frac{n\Omega_s T_{\parallel s} + (\omega - n\Omega_s)T_{\perp s}}{k_\parallel w_{\parallel s} T_{\perp s}}\right] Z(\zeta_{ns})\right\}} \tag{10.88}$$

where

$$\sum_n I_n(\lambda_s) = e^{\lambda_s} \Rightarrow \boxed{\sum_n e^{-\lambda_s} I_n(\lambda_s) = 1} \tag{10.89}$$

$$\lambda_s = \frac{1}{2}\frac{k_\perp^2 w_{\perp s}^2}{\Omega_s^2} \tag{10.90}$$

$$\zeta_{ns} = \frac{\omega - n\Omega_s}{k_\parallel w_{\parallel s}} \tag{10.91}$$

$$w_{\parallel s}^2 = \frac{2T_s}{m_s} \tag{10.92}$$

In the cold plasma limit, this does indeed reduce to $k_\parallel^2 P + k_\perp^2 S = 0$.

10.7.2 Susceptibilities in a Magnetized, Hot Plasma

Now we examine the case where $\underline{\boldsymbol{B}} = B_0\hat{\boldsymbol{z}} + \underline{\boldsymbol{B}}_1$ and $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_1$. Again, we will linearize the equations to find our solution. Note that we are switching to relativistic notation, with

$$\underline{\boldsymbol{r}}, \underline{\boldsymbol{v}}, t \rightarrow \underline{\boldsymbol{r}}, \underline{\boldsymbol{p}}, t \tag{10.93}$$

$$\underline{\boldsymbol{p}} = m\underline{\boldsymbol{v}} \tag{10.94}$$

$$p^2 = p_x^2 + p_y^2 + p_z^2 = p_\perp^2 + p_\parallel^2 \tag{10.95}$$

$$m = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \gamma m_0 \tag{10.96}$$

$$\Omega_s = \frac{q_s B_0}{m_s c} = \frac{q_s B_0}{\gamma m_{s0} c} = \frac{\Omega_{s0}}{\gamma} \tag{10.97}$$

$$f_0(p_\perp, p_\parallel) \qquad f_1(\underline{\boldsymbol{r}}, \underline{\boldsymbol{p}}, t) \tag{10.98}$$

Using this notation, our first order distribution function is

$$f_1(\underline{\boldsymbol{r}}, \underline{\boldsymbol{p}}, t) = -q\int_{-\infty}^{t} \mathrm{d}t' \left[\underline{\boldsymbol{E}}_1\big(\underline{\boldsymbol{r}}'(t'), t'\big) + \frac{1}{c}\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}_1\big(\underline{\boldsymbol{r}}'(t'), t'\big)\right] \cdot \frac{\partial f_0(p_\perp, p_\parallel)}{\partial p} \tag{10.99}$$

Here is a quick rundown of some of the derivatives involved.

$$\frac{\partial f_0}{\partial p} = \frac{\partial f_0}{\partial p_x}\hat{\boldsymbol{x}} + \frac{\partial f_0}{\partial p_y}\hat{\boldsymbol{y}} + \frac{\partial f_0}{\partial p_z}\hat{\boldsymbol{z}} = \underbrace{\frac{\partial f_0}{\partial p_\perp}\frac{\partial p_\perp}{\partial p_x}}_{\frac{v_x}{v_\perp}\frac{\partial f_0}{\partial p_\perp}}\hat{\boldsymbol{x}} + \underbrace{\frac{\partial f_0}{\partial p_\perp}\frac{\partial p_\perp}{\partial p_y}}_{\frac{v_y}{v_\perp}\frac{\partial f_0}{\partial p_\perp}}\hat{\boldsymbol{y}} + \frac{\partial f_0}{\partial p_z}\hat{\boldsymbol{z}} \tag{10.100}$$

Again

$$\underline{\boldsymbol{J}} = q \int \mathrm{d}^3 p \underline{\boldsymbol{v}} f_1 \rightarrow \underline{\boldsymbol{J}}_1 = -\frac{i\omega}{4\pi} \overline{\overline{\boldsymbol{\chi}}} \cdot \underline{\boldsymbol{E}}_1 \tag{10.101}$$

Starting off with Faraday's equation, FT'ing it, and then taking $\frac{q}{c}\underline{\boldsymbol{v}} \times (\quad)$, we get

$$\boldsymbol{\nabla} \times \underline{\boldsymbol{E}}_1 = -\frac{1}{c}\frac{\partial \underline{\boldsymbol{B}}_1}{\partial t} \rightarrow \underline{\boldsymbol{B}}_1 = \frac{c}{\omega}\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}}_1 \tag{10.102}$$

$$\frac{q}{c}\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}_1 = \frac{q}{\omega}\underline{\boldsymbol{v}} \times \underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}}_1 = \frac{q}{\omega}\big[(\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{E}}_1)\underline{\boldsymbol{k}} - (\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{k}})\underline{\boldsymbol{E}}_1\big]$$

$$q\underline{\boldsymbol{E}}_1 + \frac{q}{c}\underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}_1 = q\underline{\boldsymbol{E}}_1 \cdot \left[\overline{\overline{\mathbb{1}}} + \frac{\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{k}}}{\omega} - \frac{\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{k}}}{\omega}\overline{\overline{\mathbb{1}}}\right]$$

$$= q\underline{\boldsymbol{E}}_1 \cdot \left[\overline{\overline{\mathbb{1}}}\left(-\frac{\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{k}}}{\omega}\right) + \frac{\underline{\boldsymbol{v}}\underline{\boldsymbol{k}}}{\omega}\right]$$

We now look at the integral over t' in the distribution function. $\underline{\boldsymbol{E}}_1$ behaves $\sim e^{i(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}} - \omega t)}$. The exponential term can be broken up like so.

$$e^{i(\underline{\boldsymbol{k}} \cdot \underline{\boldsymbol{r}}'(t') - \omega t')} \rightarrow e^{ik_\perp x'(t')} e^{ik_\parallel z'(t')} e^{0\omega t'} e^{i\omega t} e^{-i\omega t} \tag{10.103}$$

where

$$x'(t') = x(t) - \frac{v_\perp}{\Omega}\big[\sin\big(\phi - \Omega(t - t')\big) - \sin\phi\big] \tag{10.104}$$

$$z'(t') = z(t) - v_\parallel (t - t') \tag{10.105}$$

$$\tau = t - t' \tag{10.106}$$

Thus, in the integral

$$e^{ik_\perp x(t)} e^{ik_\parallel z(t)} e^{-i\omega t} \int_0^\infty \mathrm{d}\tau (\quad) \tag{10.107}$$

Putting it together, our distribution function looks like

$$f_1(\underline{\boldsymbol{k}}, \underline{\boldsymbol{p}}, \omega) = -q \int_0^\infty \mathrm{d}\tau \underline{\boldsymbol{E}}_1(\underline{\boldsymbol{k}}, \omega) \cdot \left[\left(1 - \frac{\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{k}}}{\omega}\right)\overline{\overline{\mathbb{1}}} + \frac{\underline{\boldsymbol{v}}\underline{\boldsymbol{k}}}{\omega}\right] \cdot \left[\cos(\phi + \Omega\tau)\frac{\partial f_0}{\partial p_\perp}\hat{\boldsymbol{x}} + \sin(\phi + \Omega\tau)\frac{\partial f_0}{\partial p_\perp}\hat{\boldsymbol{y}} + \frac{\partial f_0}{\partial p_\parallel}\hat{\boldsymbol{z}}\right] e^{i\beta} \tag{10.108}$$

where

$$\beta = -\frac{k_\perp v_\perp}{\omega}\left[\sin(\phi+\Omega\tau) - \sin\phi\right] + (\omega - k_\parallel v_\parallel)\tau \tag{10.109}$$

$$\underline{\boldsymbol{k}} = k_\perp \hat{\boldsymbol{x}} + k_\parallel \hat{\boldsymbol{z}} \tag{10.110}$$

$$\underline{\boldsymbol{v}} \cdot \underline{\boldsymbol{k}} = \underbrace{k_\perp v_\perp \cos(\phi+\Omega\tau)}_{v_x} + k_\parallel v_\parallel \tag{10.111}$$

since

$$k_\perp \hat{\boldsymbol{x}} \cdot \hat{\perp} \sim k_\perp \cos(\phi+\Omega\tau) \tag{10.112}$$

Looking at some of the dot products with $\underline{\boldsymbol{E}}$, we see that

$$\underline{\boldsymbol{E}} \cdot \underline{\boldsymbol{v}} = E_x v_\perp \cos(\phi+\Omega\tau) + E_y v_\perp \sin(\phi+\Omega\tau) + E_z v_\parallel \tag{10.113}$$

$$\begin{aligned}\left(1 - \frac{\underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{v}}}{\omega}\right)\underline{\boldsymbol{E}} \cdot \frac{\partial f_0}{\partial \underline{\boldsymbol{p}}} = a\underline{\boldsymbol{E}} \cdot \frac{\partial f_0}{\partial \underline{\boldsymbol{p}}} &= aE_x \cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial p_\perp} + \\ &aE_y \sin(\phi+\Omega\tau)\frac{\partial f_0}{\partial p_\perp} + aE_z \frac{\partial f_0}{\partial p_\parallel}\end{aligned} \tag{10.114}$$

$$(\underline{\boldsymbol{E}}\cdot\underline{\boldsymbol{v}})\frac{\underline{\boldsymbol{k}}}{\omega} \cdot \frac{\partial f_0}{\partial \underline{\boldsymbol{p}}} = b\underline{\boldsymbol{k}} \cdot \frac{\partial f_0}{\partial \underline{\boldsymbol{p}}} = vk_\perp \cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial p_\perp} + bk_\parallel \frac{\partial f_0}{\partial p_\parallel} \tag{10.115}$$

where

$$a = 1 - \frac{k_\perp v_\perp}{\omega}\cos(\phi+\Omega\tau) - k_\parallel v_\parallel \tag{10.116}$$

$$b = \underline{\boldsymbol{E}} \cdot \underline{\boldsymbol{v}} \tag{10.117}$$

Plowing through this gives us the following expression for the integrand in Eq. (10.108)

$$\begin{aligned}&E_x \cos(\phi+\Omega\tau)\left[a\frac{\partial f_0}{\partial p_\perp} + \frac{k_\perp v_\perp}{\omega}\cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel v_\perp}{\omega}\frac{\partial f_0}{\partial p_\parallel}\right] + \\ &\quad E_y \sin(\phi+\Omega\tau)\left[a\frac{\partial f_0}{\partial p_\perp} + \frac{k_\perp v_\perp}{\omega}\cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel v_\perp}{\omega}\frac{\partial f_0}{\partial p_\parallel}\right] + \\ &\qquad E_z\left[a\frac{\partial f_0}{\partial p_\parallel} + \frac{k_\perp v_\parallel}{\omega}\cos(\phi+\Omega\tau)\frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel v_\parallel}{\omega}\frac{\partial f_0}{\partial p_\parallel}\right]\end{aligned} \tag{10.118}$$

Simplifying this expression and introducing the expressions U and V

$$U = \frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel}{\omega} \underbrace{\left[v_\perp \frac{\partial f_0}{\partial p_\parallel} - p_\parallel \frac{\partial f_0}{\partial p_\perp} \right]}_{\to 0 \text{ for Maxwellian}} \tag{10.119}$$

$$V = \frac{k_\perp}{\omega} \left[v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp} \right] \tag{10.120}$$

our distribution function finally reduces to the following form.

$$f_{s1} = -q_s \int_0^\infty \mathrm{d}\tau e^{i\beta} \Bigg[U \cos(\phi + \Omega\tau) E_x + U \sin(\phi + \Omega\tau) E_y + \left(\frac{\partial f_0}{\partial p_\parallel} - V \cos(\phi + \Omega\tau) \right) E_z \Bigg] \tag{10.121}$$

Now we use our distribution function in Eq. (10.101), the equation for $\underline{\boldsymbol{J}}$, remembering that

$$\int \mathrm{d}^3 p = \int_0^\infty p_\perp \mathrm{d}p_\perp \int_0^\infty \mathrm{d}p_\parallel \int_0^{2\pi} \mathrm{d}\phi \tag{10.122}$$

We will be using the identities in Equations (10.74) and (10.75) again to simplify $\underline{\boldsymbol{J}}$ and solving for $\overline{\overline{\boldsymbol{\chi}}}$, we get

$$\overline{\overline{\boldsymbol{\chi}}}_s = \sum_s \frac{\omega_{ps,0}^2}{\omega \Omega_{s,0}} \int_0^\infty 2\pi p_\perp \mathrm{d}p_\perp \int_{-\infty}^\infty \mathrm{d}p_\parallel \sum_{n=-\infty}^\infty \left(\frac{\Omega}{\omega - k_\parallel v_\parallel - n\Omega} \right) \overline{\overline{\boldsymbol{S}}}_n \tag{10.123}$$

$$\overline{\overline{\boldsymbol{S}}}_n = \begin{bmatrix} p_\perp U \frac{n^2}{z^2} N_n^2 & p_\perp U \frac{in}{z} J_n J_n' & p_\perp \frac{n}{z} J_n^2 W \\ -p_\perp U \frac{in}{z} J_n J_n' & p_\perp U \frac{n^2}{z^2} (J_n')^2 & -i J_n J_n' p_\perp W \\ p_\parallel U \frac{n}{z} J_n^2 & i p_\parallel U J_n J_n' & p_\parallel J_n^2 W \end{bmatrix}$$

where

$$W = \frac{\partial f_0}{\partial p_\parallel} 0 \frac{n}{z} V = \frac{\partial f_0}{\partial p_\parallel} \left[1 - \frac{n\Omega}{\omega} \right] + \frac{n\Omega}{\omega} \frac{p_\parallel}{p_\perp} \frac{\partial f_0}{\partial p_\perp} \tag{10.124}$$

To get here, see Problem (10.4) in Stix. Now we use the following identities

$$\sum_{n=-\infty}^{\infty} nJ_n^2 = 0 \tag{10.125}$$

$$\sum_{n=-\infty}^{\infty} J_n J_n' = 0 \tag{10.126}$$

$$\sum_{n=-\infty}^{\infty} J_n^2 = 1 \tag{10.127}$$

to change $W \to U$ inside the sum. This lets us rewrite the susceptibility as (Eq. (10.48) in Stix)

$$\overline{\overline{\boldsymbol{\chi}}} = \frac{\omega_{ps,0}^2}{\omega\Omega_{s,0}} \int_0^\infty 2\pi p_\perp \mathrm{d}p_\perp \int_{-\infty}^{\infty} \mathrm{d}p_\parallel \left[\frac{\Omega}{\omega} \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} p_\parallel^2 \left(\frac{1}{p_\parallel}\frac{\partial f_0}{\partial p_\parallel} - \frac{1}{p_\perp}\frac{\partial f_0}{\partial p_\perp} \right) + \sum_{n=-\infty}^{\infty} \frac{\Omega p_\perp U}{\omega - k_\parallel v_\parallel - n\Omega} \overline{\overline{\boldsymbol{T}}}_n \right] \tag{10.128}$$

$$\overline{\overline{\boldsymbol{T}}}_n = \begin{bmatrix} \frac{n^2 J_n^2}{x^2} & \frac{inJ_nJ_n'}{z} & \frac{nJ_n^2 p_\parallel}{p_\perp z} \\ \frac{-inJ_nJ_n'}{z} & \frac{n^2 (J_n')^2}{z^2} & \frac{-inJ_nJ_n' p_\parallel}{p_\perp} \\ \frac{nJ_n^2 p_\parallel}{p_\perp z} & \frac{inJ_nJ_n' p_\parallel}{p_\perp} & \frac{J_n^2 p_\parallel^2}{p_\perp^2} \end{bmatrix}$$

10.7.3 Susceptibilities for a Perpendicular Maxwellian f_0

Now we set our zero-order distribution to a Maxwellian, but only in the perpendicular direction. We are also examining non-relativistic distributions. We use the following for our distribution function.

$$f_0(v_\perp, v_\parallel) = \frac{1}{\pi w_\perp^2} e^{-\frac{v_\perp^2}{w_\perp^2}} h(v_\parallel) \tag{10.129}$$

For the susceptibility tensor, we find

$$\overline{\overline{\boldsymbol{\chi}}}_s = \hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \frac{2\omega_{ps}^2}{\omega k_\parallel w_\perp^2} \langle v_\parallel \rangle + \frac{\omega_{ps}^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda} \overline{\overline{\boldsymbol{W}}}_n \tag{10.130}$$

$$\overline{\overline{\boldsymbol{W}}}_n = \begin{bmatrix} \frac{n^2 I_n}{\lambda} A_n & -in(I_n - I_n')A_n & \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n \\ in(I_n - I_n')A_n & \left(\frac{n^2}{\lambda} I_n + 2\lambda I_n - 2\lambda I_n' \right) & \frac{ik_\perp}{\Omega}(I_n - I_n')B_n \\ \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n & -\frac{ik_\perp}{\Omega}(I_n - I_n')B_n & \frac{2(\omega - n\Omega)}{k_\parallel w_\perp^2} I_n B_n \end{bmatrix}$$

where

$$A_n = \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \frac{H(v_\parallel)}{\omega - k_\parallel v_\parallel - n\Omega}$$
$$B_n = \int_{-\infty}^{\infty} \mathrm{d}v_\parallel \frac{v_\parallel H(v_\parallel)}{\omega - k_\parallel v_\parallel - n\Omega}$$
$$H(v_\parallel) = -\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right) h(v_\parallel) + \frac{k_\parallel w_\perp^2}{2\omega} h'(v_\parallel)$$
$$B_n = \frac{1}{\omega k_\parallel}(\omega - k_\parallel \langle v_\parallel \rangle) + \frac{\omega - n\Omega}{k_\parallel} A_n \tag{10.131}$$

See problem (10.5) in Stix. We can often evaluate $\overline{\overline{\boldsymbol{\chi}}}_s$ to the lowest order (FLR). $n = 0, \pm 1, \pm 2$ is often sufficient. See Equations (10.61)-(10.63) for these approximations.

A_n an B_n simplify some when the parallel velocity distribtion $h(v_\parallel)$ is also a Maxwellian.

$$h_s(v_\parallel) = \frac{1}{\sqrt{\pi} w_\parallel} \exp\left[-\frac{(v_\parallel - v_D)^2}{w_\parallel^2}\right] \tag{10.132}$$
$$A_n = \frac{1}{\omega}\frac{T_\perp - T_\parallel}{T_\parallel} + \frac{1}{k_\parallel w_\parallel}\frac{(\omega - k_\parallel v_D - n\Omega)T_\perp + n\Omega T_\parallel}{\omega T_\parallel} Z_0 \tag{10.133}$$

$$B_n = \frac{1}{k_\parallel}\frac{(\omega - n\Omega)T_\perp - (k_\parallel v_D - n\Omega)T_\parallel}{\omega T_\parallel} + \frac{1}{k_\parallel}\frac{\omega - n\Omega}{k_\parallel w_\parallel}\frac{(\omega - k_\parallel v_D - n\Omega)T_\perp + n\Omega T_\parallel}{\omega T_\parallel} Z_0 \tag{10.134}$$

For derivation, see Problem (10.6) in Stix.

When $T_\parallel = T_\perp$ and $v_D = 0$ (non drifting Maxwellian), the above form of the susceptibility simplifies greatly.

$$\overline{\overline{\boldsymbol{\chi}}}_s = \frac{\omega_{ps}^2}{\omega} \sum_{n=-\infty}^{\infty} e^{-\lambda} \overline{\overline{\boldsymbol{W}}}_n \tag{10.135}$$

$$\overline{\overline{\boldsymbol{W}}}_n = \begin{bmatrix} \frac{n^2 I_n}{\lambda k_\parallel v_{th}} Z(\zeta_n) & -\frac{in}{k_\parallel v_{th}}(I_n - I_n')Z_n & -\frac{k_\perp}{2k_\parallel \Omega}\frac{nI_n}{\lambda} Z_n' \\ \frac{in}{k_\parallel v_{th}}(I_n - I_n')Z_n & \left[\frac{n^2 I_n}{\lambda} + 2\lambda(I_n - I_n')\right]\frac{1}{k_\parallel v_{th}} Z_n & \frac{ik_\perp}{2k_\parallel \Omega}(I_n - I_n')Z_n' \\ -\frac{k_\perp}{2k_\parallel \Omega}\frac{nI_n}{\lambda} Z_n' & \frac{ik_\perp}{2k_\parallel \Omega}(I_n - I_n')Z_n' & -\frac{\zeta_n}{k_\parallel v_{th}} I_n Z_n' \end{bmatrix}_s$$

10.7.4 Bernstein Waves

Bernstein waves are electrostatic waves. For Bernstein waves, we will be using Eq. (10.88) or Eq. (11.85) in Stix. We will be using a Maxwellian distribution function $f_0 = f_M(v_\perp)f_M(v_\parallel)$. We will also be setting $T_\perp = T_\parallel$, $k_\parallel = 0$, and $\theta = 0$. Even though for hot waves, where one typically has $\zeta_n \ll 1$, since $k_\parallel \sim 0$, we will be using the asymptotic expansion for $\zeta_n \gg 1$, where $\zeta = \frac{\omega+n\Omega}{k_\parallel v_{th}}$. So, our dispersion relation becomes

$$k_\perp^2 = -\sum_s \frac{2\omega_{ps}^2}{v_{th}^2}\left[1 + \sum_{n=-\infty}^{\infty} \frac{\omega}{k_\parallel v_{th}} e^{-\lambda_s} I_n(\lambda_s) Z_0(\zeta_n)\right] \tag{10.136}$$

The bracketed term is, remembering that $I_n(\lambda) = I_{-n}(\lambda)$,

1. Expand sum and use $Z_n \to -\frac{1}{\zeta_n}$.

$$\left[\quad\right] \Rightarrow 1 + \underbrace{\frac{\omega}{k_\parallel v_{th}} e^{-\lambda_s} I_0(\lambda_s) \frac{-k_\parallel v_{th}}{\omega}}_{n=0 \text{ term}} + \sum_{n=1}^{\infty} \frac{\omega}{k_\parallel v_{th}} e^{-\lambda_s} I_n(\lambda_s) \times \left[\frac{-k_\parallel v_{th}}{\omega - n\Omega} - \frac{k_\parallel v_{th}}{\omega + n\Omega}\right] \tag{10.137}$$

The 1 term is expanded using $1 = \sum I_n(\lambda)e^{-\lambda}$ when $\theta = 0$, giving us

2. Expand 1 term using $1 = \sum I_n e^{-\lambda}$.

$$1 = \sum_{n=-\infty}^{\infty} I_n(\lambda)e^{-\lambda} = e^{-\lambda} I_0(\lambda_s) + \sum_{n=1}^{\infty} \left[I_n(\lambda_s) + I_{-n}(\lambda_s)\right] \tag{10.138}$$

The first term on the RHS of the equation cancels with the $n = 0$ term from equation Eq. (10.137). Combining this with the rest of Eq. (10.137), we get

3. Combine terms.

$$\begin{aligned} -\sum_{n=1}^{\infty} e^{-\lambda_s} I_n(\lambda_s) &\left[\frac{\omega}{\omega - n\Omega} + \frac{\omega}{\omega + n\Omega} - 2\right] \\ &= -\sum_{n=1}^{\infty} e^{-\lambda_s} I_n(\lambda_s) \left[\frac{2\omega^2}{\omega^2 - n^2\Omega^2} - \frac{2\omega^2 - 2n^2\Omega^2}{\omega^2 - n^2\Omega^2}\right] \\ &= -2\sum_{n=1}^{\infty} e^{-\lambda_s} I_n(\lambda_s) \left[\frac{n^2}{\nu_s^2 - n^2}\right] \end{aligned} \tag{10.139}$$

where $\nu_s = \frac{\omega}{\Omega_s}$.

4. Set $\nu_s = \frac{\omega}{\Omega_s}$.

So, Eq. (10.137) becomes

$$-k_\perp^2 = -\sum_s \frac{4\omega_{ps}^2}{v_{th}^2} \sum_{n=1}^{\infty} e^{-\lambda_s} I_n(\lambda_s) \frac{n^2}{\nu_s^2 - n^2} \tag{10.140}$$

One term can be rewritten.

$$\frac{4\omega_{ps}^2}{v_{th}^2}\frac{\lambda_s}{\lambda_s} = \frac{4\pi n_s m_s c^2}{B_0^2}\frac{k_\perp^2}{\lambda_s} \tag{10.141}$$

Thus the dispersion relation becomes

$$1 = \sum_s \frac{4\pi n_s m_s c^2}{B_0^2} \underbrace{\left[2\sum_{n=1}^{\infty}\frac{e^{-\lambda_s}I_n(\lambda_s)}{\lambda_s}\frac{n^2}{\nu_s^2 - n^2}\right]}_{\frac{\alpha(\nu_s,\lambda_s)}{\lambda_s}} \tag{10.142}$$

One must have $\alpha > 1$ for the wave to propagate. So, since $k_\parallel = 0$ and $\underline{\boldsymbol{k}} \cdot \overline{\overline{\boldsymbol{\epsilon}}} \cdot \underline{\boldsymbol{k}} = 0$ for ES waves, we get $k_\perp^2 \epsilon_{xx} = 0$. For the simplest Bernstein waves, $\epsilon_{xx} = 0$ in a hot plasma.

In a cold plasma ($v_\perp^2 \ll 1 \rightarrow \lambda \ll 1$), we can expand α.

$$\alpha(\nu,\lambda) = \frac{\lambda}{\nu^2 - 1} + \frac{3\lambda^2}{(\nu^2-1)(\nu^2-4)} + \frac{15\lambda^3}{(\nu^2-1)(\nu^2-4)(\nu^3-9)} + \dots \tag{10.143}$$

To get the nth harmonic, keeps terms to $O(\lambda^{n-1})$ in $\frac{\alpha}{\lambda}$.

10.8 Damping

For weak damping, $\omega = \omega_r + i\omega_i$ and $\underline{\boldsymbol{k}} = \underline{\boldsymbol{k}}_r + i\underline{\boldsymbol{k}}_i$ where the imaginary components are much smaller than the real components. Power absorbed per unit volume is (see Eq. (11.68) in Stix)

$$P \sim \underline{\boldsymbol{E}} \cdot \underline{\boldsymbol{J}} = \sum_s n_s q_s \langle \underline{\boldsymbol{v}}_s \rangle \cdot \underline{\boldsymbol{E}} \tag{10.144}$$

$$\langle \underline{\boldsymbol{v}}_s \rangle = \mathsf{Re}\left\{ \langle \underline{\boldsymbol{v}}_s \rangle e^{i(\underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{r}} - \omega t)} \right\} \tag{10.145}$$

$$P_{abs} = -\frac{i\omega}{4\pi}\sum_s \underline{\boldsymbol{E}}^\dagger \cdot \left(\overline{\overline{\boldsymbol{\chi}}}_s - \overline{\overline{\boldsymbol{\chi}}}_s^\dagger\right) \cdot \underline{\boldsymbol{E}} \tag{10.146}$$

So for power absorption, we need to calculate $(\overline{\overline{\boldsymbol{\chi}}}_s - \overline{\overline{\boldsymbol{\chi}}}_s^\dagger)$. Using the hot plasma tensors such as Eq. (10.135), we see we need to calculate $Z_n - Z_n^*$

and $Z'_n - Z'^*_n$ where Z is the PDF.

$$Z_n(\zeta_n) = \underbrace{i\sqrt{\pi}e^{-\zeta_n^2}}_{\text{Im}} - \underbrace{Ze^{-\zeta_n^2}\int_0^\zeta \mathrm{d}te^{t^2}}_{\text{Re}} \tag{10.147}$$

$$Z_n - Z_n^* = 2i\sqrt{\pi}e^{-\zeta_n^2} \tag{10.148}$$

$$Z'_n - Z'^*_n = (i\sqrt{\pi})(-2\zeta)e^{-\zeta_n^2} - \text{c.c.} = -2\zeta\left(2i\sqrt{\pi}e^{-\zeta_n^2}\right) \tag{10.149}$$

Thus,

$$\overline{\overline{\chi}}_s - \overline{\overline{\chi}}_s^\dagger = \frac{\omega_{ps}^2}{\omega}\sum_n e^{-\lambda}\left(2i\sqrt{\pi}e^{-\zeta_n^2}\right)\begin{bmatrix} \frac{n^2 I_n}{\lambda k_\parallel v_{th}} & -\frac{in(I_n - I'_n)}{k_\parallel v_{th}} & \frac{k_\perp \zeta}{k_\parallel \Omega}\frac{nI_n}{\lambda} \\ \frac{in(I_n - I'_n)}{k_\parallel v_{th}} & \frac{\frac{n^2 I_n}{\lambda} + 2\lambda(I_n - I'_n)}{k_\parallel v_{th}} & \frac{ik_\perp \zeta(I_n - I'_n)}{k_\parallel \Omega} \\ \frac{k_\perp \zeta}{k_\parallel \Omega}\frac{nI_n}{\lambda} & -\frac{ik_\perp \zeta(I_n - I'_n)}{k_\parallel \Omega} & \frac{2\zeta(\omega - n\Omega)I_n}{k_\parallel^2 v_{th}^2} \end{bmatrix} \tag{10.150}$$

As will be stated below, cyclotron damping is in the xx, xy, yx, and yy terms. TTMP is in the yy term while Landau damping is in the zz term. The yz and zy terms are cross terms for TTMP and Landau damping.

10.8.1 Landau Damping

Landau damping is when a particle and a wave resonant due to the fact the that particle's thermal velocity is on the order of the phase velocity of the wave ($v_{th} \sim \frac{\omega}{k}$).

If we plug in $\omega = \omega_r + i\omega_i$ into Eq. (10.25) and do some expanding and rearranging, we find that we get an expression for ω_i.

$$\omega_i = \frac{\pi}{2}\frac{(\omega_r - k\langle v\rangle)\omega_{po}^2}{k\,|k|}f'_o\left(\frac{\omega}{k}\right) \tag{10.151}$$

This agrees with the value calculated from particle trajectories in Section 8.2 of Stix. More importantly, we will look at the valid collisionality regimes for Landau damping.

For Landau damping to occur there must be at least a certain number of collisions, but not to many. This is analogous to the banana regime in tokamaks, except on a smaller scale. To find these limits, let us define the oscillation frequency of a particle in a potential well as

$$\frac{1}{\tau_{osc}} = \left(\frac{qkE_1}{m}\right)^{\frac{1}{2}} \tag{10.152}$$

The upper limit on collisionality must be low enough to allow for interaction between the wave and the particle. Other wise the particle doesn't have a chance to resonate with the wave.

$$\lambda_{\text{mfp}} k_{\parallel} = \frac{\lambda_{\text{mfp}}}{\lambda} \leq 1 \tag{10.153}$$

At the other end of the spectrum, the particle can't see the other side of the potential well. Otherwise it will begin slow down, etc. as nonlinear effects begin to take over as the particle becomes "trapped". For Landau damping, we assume $v_{\parallel} \sim$ constant.

$$\tau_{osc}\nu_{coll} \geq 1 \tag{10.154}$$

Another way of stating that is

$$\nu_{coll} \geq \omega_{osc} \tag{10.155}$$

Since Landau damping depends on the $\underline{\boldsymbol{E}}$ field, and we know the force equation for this interaction is

$$F = -qe = m\frac{dv_{\parallel}}{dt} \tag{10.156}$$

we can easily see that $\underline{\boldsymbol{E}} = E_{\parallel}\hat{\boldsymbol{z}}$. As a result, the component of the dielectric tensor that plays a role in this kind of damping is $\epsilon_{\parallel}$.

Using Equations (10.146) and (10.150), we find that the power absorbed is

$$P_{abs} = \sum_s \frac{4\pi n_s q_s^2}{k_{\parallel}^2 T_s} e^{-\lambda} I_0(\lambda) \frac{\left|E_{\parallel}\right|^2}{8\pi} \frac{\sqrt{\pi}\omega^2}{k_{\parallel} v_{th}} e^{-\frac{\omega^2}{k_{\parallel}^2 v_{th}^2}} \tag{10.157}$$

As we can see, since the power absorbed is $P_{abs} \propto e^{-\zeta_n^2}$, for cold plasmas where $\zeta_n \rightarrow 0$, $P_{abs} \rightarrow 0$.

10.8.2 Cyclotron Damping

Cyclotron damping is similar to Landau damping except that the resonating particle interacts with the field at its cyclotron frequency as opposed to at zero frequency. To quote Stix, page 239:

> Cyclotron damping occurs for oscillations that are periodic both in time and in axial distance and in which there exists a component of $\underline{\boldsymbol{E}}$ that is perpendicular to $\underline{\boldsymbol{B}}_0$. Ions or electrons moving along lines of force will see thee oscillations of the perpendicular electric field at a frequency that differs from the laboratory-frame

> frequency by the Doppler shift. Some charged particles will see the oscillations at their own cyclotron frequency, and they will absorb energy from the field. If the electromagnetic field is produced by a plasma wave, this absorption of energy will cause the wave to damp out with time or with distance.

Cyclotron damping occurs when

$$\omega - k_{\parallel} v_{\parallel} - n\Omega_s = 0 \tag{10.158}$$

The $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{xy}$, and ϵ_{yx} terms all contribute to cyclotron damping.

For cyclotron damping, we will set $E_{\parallel} = 0$ and assume small λ. From the equation below, we can see that absorption goes down for large n (from $\frac{n}{(n-1)!}$). Also, hotter particles absorb better for $n \geq 2$ (from $\left(\frac{\lambda}{2}\right)^{n-1}$).

$$P_{abs} = \sum_s \frac{\omega_{ps}^2}{8\pi} \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \left(\frac{\lambda}{2}\right)^{n-1} \times \left\{ \frac{|E_x + iE_y|^2\, 2i\sqrt{\pi} e^{-\zeta_n^2}}{k_{\parallel} v_{th}} + \frac{|E_x - iE_y|^2\, 2i\sqrt{\pi} E^{-\zeta_n^2}}{k_{\parallel} v_{th}} \right\} \tag{10.159}$$

It is worth noting that the left term in the brackets is the $|E_+|$ term and represents ion cyclotron damping. The right term is the $|E_-|$ or electron damping term. The fast magnetosonic wave can't heat at $\omega = \Omega_i$ since it is RH and ions are LH. If there is a minority ion species, then its contribution to S is small and it can be heated.

10.8.3 Transit Time Magnetic Pumping

Transit Time Magnetic Pumping (TTMP) is similar to Landau damping, but instead of an interaction with the $\underline{\boldsymbol{E}}$ field, there is a $\mu \boldsymbol{\nabla} B$ interaction with the $\underline{\boldsymbol{B}}$ field. To follow this analogy through, we change the $E_{\parallel} \rightarrow B_{\parallel}$ where $B_{\parallel} = \frac{c}{\omega} k_{\perp} E_y$. In this sense, the perpendicular component of the dielectric tensor $\epsilon_{\perp}$ contributes. This corresponds to the χ_{yy} component of the susceptibility tensor. The force equation is

$$F = -\mu \boldsymbol{\nabla} B = m\frac{dv_x}{dt} = -\mu \frac{\partial B_1}{\partial z} \sim -\mu k_{\parallel} B_1 \tag{10.160}$$

As an example, suppose $\omega \sim n\Omega$. We will let $n = 0$ and $E_{\parallel} = 0$ too, so we will just examine the yy term.

$$P_{abs} = -\frac{i\omega}{16\pi} \sum_s \frac{\omega_{ps}^2}{\omega} e^{-\lambda} 2i\sqrt{\pi} e^{-\zeta^2} \frac{2\lambda(I_0 - I_0')}{k_{\parallel} v_{th}} \left|E_y^2\right| \tag{10.161}$$

We know from Faraday's law

$$\nabla \times \underline{\boldsymbol{E}} = -\frac{1}{c}\frac{\partial \underline{\boldsymbol{B}}}{\partial t}$$

$$i\underline{\boldsymbol{k}} \times \underline{\boldsymbol{E}} = i(k_\perp \hat{\boldsymbol{x}} + k_\parallel \hat{\boldsymbol{z}}) \times (E_y \hat{\boldsymbol{y}} + E_x \hat{\boldsymbol{x}}) = \frac{i\omega}{c}\underline{\boldsymbol{B}}_1$$

$$k_\perp E_y \hat{\boldsymbol{z}} = \frac{\omega}{c} B_\parallel \hat{\boldsymbol{z}}$$

$$|E_y|^2 = \frac{\omega^2}{c^2}\frac{1}{k_\perp^2}|B_{1z}|^2$$

$$\beta = \frac{8\pi n_s T_s}{B_0^2}$$

Thus, for TTMP,

$$P_{abs} = \sum_s \beta_s \frac{\omega^2}{k_\parallel v_{th}} \sqrt{\pi} e^{-\lambda}(I_0 - I_0') \frac{|B_{1z}|^2}{8\pi} e^{-\zeta^2} \tag{10.162}$$

We can see as $\beta \uparrow$, TTMP becomes more important.

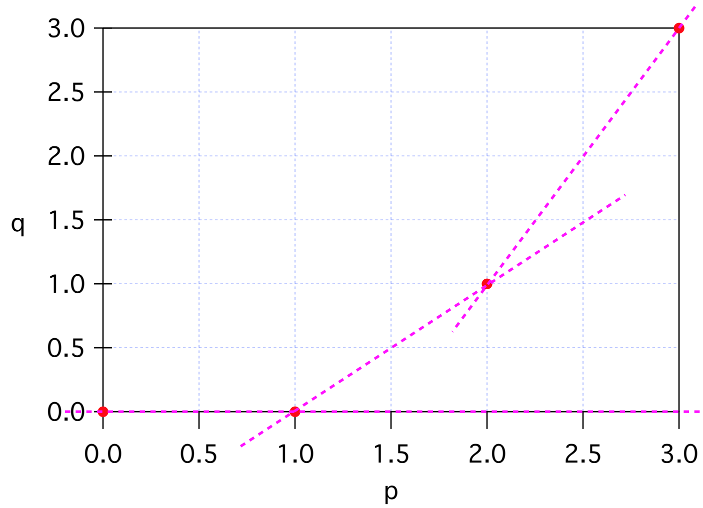

Figure 8: Kruskal Graph. The equation that is plotted is $1 + x + \epsilon x^2 + 2\epsilon^3 x^3 = 0$. The lines drawn between the pairs of points are dotted. All lines need to be examined more closely following the process in the text.

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11 Asymptotics

11.1 Kruskal Graphs

In using a Kruskal graph, one must have a small parameter, such as $\epsilon \ll 1$. The equations in question are algebraic and have the form of

$$\boxed{\sum \epsilon^q x^p} \tag{11.1}$$

One then simply plots each term on a p vs. q graph. Lines are then drawn between pairs of points. Lines that have nothing under them or to the left of them are examined more closely. Others are ignored.

One then sets the 2 quantities on the lines as dominant, making up the LHS of an equation. The rest of the terms go on the RHS of the equation, and get set as small when $x \sim x_0$. The LHS is then reduced to just x. At this point, we have a function $x = f(x)$. The iteration converges if $|f'(x_0)| < 1$. The zero-order solution is x_0 = non small stuff. The 1st order solution is found by subbing in x_0 into the expression for x. Higher order solutions are found by iterating.

Example: For an example, we will examine

$$1 + x + \epsilon x^2 + 2\epsilon^3 x^3 = 0 \tag{11.2}$$

See Fig. 8.

For case 1, $1 + x$ is dominant, so

$$\begin{aligned} x + 1 &= -\underbrace{(\epsilon x^2 + 2\epsilon^3 x^3)}_{\text{small}} \\ x &= -1 - (\epsilon x^2 + 2\epsilon^3 x^3) \\ x_0 &= 1 \\ x_1 &= -1 - (\epsilon x_0^2 + 2\epsilon^3 x_0^3) \\ x_2 &= \text{iterate...} \end{aligned} \tag{11.3}$$

For case 2, $x + \epsilon x^2$ is dominant, so

$$\begin{aligned} x &= -\frac{1}{\epsilon} - \underbrace{\frac{(1 + 2\epsilon^3 x^3)}{x\epsilon}}_{\text{small}} \\ x_0 &= -\frac{1}{\epsilon} \\ &\text{iterate...} \end{aligned} \tag{11.4}$$

Case 3 has $\epsilon x^3 + 2\epsilon^3 x^3$ dominant. It follows a similar route as the previous two cases.

11.2 Nonlinear Equations

When solving nonlinear equations, the method usually involves making the given equation look like a nonlinear equation that has already been solved. Here are several nonlinear equations that have solutions. A common substitution that seems to work a lot is

$$u = \frac{x}{y} \tag{11.5}$$

Bernoulli The Bernoulli equation has the form of

$$y' = a(x)y + b(x)y^p \tag{11.6}$$

To solve the Bernoulli equation, one makes the substitution of

$$u = y^{1-p} \tag{11.7}$$

This makes the equation linear in u. One can then find u' and sub in for all the y's. Once this is done, one can find the solution to the equation.

Riccati The Riccati equation is in the form of

$$y' = a(x)y^2(x) + b(x)y(x) + c(x) \tag{11.8}$$

There is no general method for solving this equation. However, if you can find $y_1(x)$, you can sub $y(x) = y_1(x)+u(x)$ and get to the Bernoulli equation.

Exact These equations can be written in the form of

$$M\big(x, y(x)\big) + N\big(x, y(x)\big)y'(x) = 0 \tag{11.9}$$

One can reduce this to

$$\frac{df}{dx}(x, y) = 0 \tag{11.10}$$

Using this lets us rewrite the equation as

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' = 0 \tag{11.11}$$

where

$$M = \frac{\partial f}{\partial x} \qquad N = \frac{\partial f}{\partial y} \tag{11.12}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{11.13}$$

11.3 Complex Integration

Complex integration is often performed using the theory of residues. The theory of residues states that if

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(u)}{(u-z)}\mathrm{d}u \tag{11.14}$$

then

$$f(z) = \frac{1}{2\pi i}\sum\left(\text{residues of } \frac{1}{(u-z)}\right) \tag{11.15}$$

The residues are

$$\frac{1}{(u-z)} \rightarrow \frac{1}{(z-z_1)}\frac{1}{(z-z_2)}\cdots \tag{11.16}$$

where the residue of the nth order zero z_1 is

$$\mathsf{Residue}(z_1) = \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z^{n-1}}\left[\frac{f(u)}{(z-z_2)(z-z_3)\ldots}\right|_{z=z_1} \tag{11.17}$$

So

$$\boxed{\oint \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z = 2\pi i\frac{1}{n!}f^{(n)}(z_0)} \tag{11.18}$$

11.4 Misc. Series

The Taylor series

$$f(z) = \sum_k \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{11.19}$$

converges in a radius given by the closest singularity in f.

The radius of convergence for series is given by

$$R = \lim \left| \frac{a_{2k-2}}{a_{2k}} \right| \tag{11.20}$$

11.5 Homogeneous Linear Differential Equations

Homogeneous linear differential equations have the form of

$$y^{(n)}(x) + P_{n-1}(x)y^{(n-1)} + \ldots P_0(x)y = 0 \tag{11.21}$$

There are three types points in these equations.

1. Ordinary Points
2. Regular Singular Points
3. Irregular Singular Points

Ordinary Points x_0 is an ordinary point if all $P_k(x)$ are analytic at x_0. In other words, its not singular. In this case, one can use a Taylor series for $y(x)$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \rightarrow \text{calculate } y', y'' \text{ and plug in} \tag{11.22}$$

Regular Singular Points The differential equation has regular singular points if no term is more singular than the highest derivative $y^{(n)}$. Think of $y^{(n)}(x) \sim \frac{y}{x^n}$. In this case, the first solution is in the form of a series with an indicial exponent α.

$$y = \sum a_n x^{n+\alpha} \tag{11.23}$$

Again, we find the derivatives of y and sub them into the equation. The next step is to break out terms of the series and change summing indices until there are a couple leading terms and just one summation whose terms are of the same nth order (x^n). Then, the indicial index is set from the leading term in front of a_0 and one solves normally from here.

The second solution is the following.

$$y_2 = \left.\frac{dy_1}{d\alpha}\right|_{\alpha=0} = y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^n \tag{11.24}$$

where the recursion relation is

$$b_n = \left.\frac{\partial a_n}{\partial \alpha}\right|_{\alpha=0} \tag{11.25}$$

$$b_0 = 0 \tag{11.26}$$

Example: As an example, let us examine the Bessel equation.

$$y'' + \frac{1}{x}y' - \left(1 + \frac{\nu^2}{x^2}\right) = 0 \tag{11.27}$$

This equation has a regular singular point at $x = 0$ since none of the terms are more singular than the highest derivative $\sim \frac{1}{x^2}$. We will use the following.

$$y = \sum a_n x^{n+\alpha} \tag{11.28}$$

$$y' = \sum (n+\alpha) a_n x^{n+\alpha-1} \tag{11.29}$$

$$y'' = \sum (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} \tag{11.30}$$

Subbing these into the Bessel equation gives us

$$0 = \sum_{n=0}^{\infty} \left\{ \left[(n+\alpha)(n+\alpha-1) + (n+\alpha) - \nu^2\right] a_n x^{n+\alpha-2} - a_n x^{n+\alpha} \right\} \tag{11.31}$$

Once we cancel terms and start pulling terms out of the sums, we get

$$0 = \underbrace{(\alpha^2 - \nu^2) a_0 x^{\alpha-2}}_{n=0} + \underbrace{\left[(1+\alpha)^2 - \nu^2\right] a_1 x^{\alpha-1}}_{n=1} + \sum_{n=2}^{\infty} \left\{ \left[(n+\alpha)^2 - \nu^2\right] a_n - a_{n-2} \right\} x^{n+\alpha-2} \tag{11.32}$$

We can see from the leading term that $\alpha^2 = \nu^2$, which sets the indicial index at $\alpha = \pm\nu$. Moving to the second term shows us that the only way for it to be zero is if $a_1 = 0$. Moving to the summation, we need to calculate the recursion relation. We can tell that we will iterate from a_0 for a_{even} terms, and from a_1 for the odd terms. As a result, $a_{odd} = 0$. Examination reveals

$$(\alpha^2 - \nu^2 + 4k\alpha + 4k^2)a_{2k} = a_{2k-2}$$

$$4\left(k + \frac{\alpha+\nu}{2}\right)\left(k + \frac{\alpha-\nu}{2}\right)a_{2k} = a_{2k-2}$$

$$a_2 = \frac{a_0}{4\left(1+\frac{\alpha+\nu}{2}\right)\left(1+\frac{\alpha-\nu}{2}\right)}$$

$$a_{2k} = \frac{a_0}{4^k\left(k+\frac{\alpha+\nu}{2}\right)\left(k+\frac{\alpha-\nu}{2}\right)\dots\left(1+\frac{\alpha+\nu}{2}\right)\left(1+\frac{\alpha-\nu}{2}\right)} \tag{11.33}$$

Using $\Gamma(n+1) = n\Gamma(n)$, we can rewrite this as

$$a_{2k} = \frac{a_0\Gamma\left(1+\frac{\alpha+\nu}{2}\right)\Gamma\left(1+\frac{\alpha-\nu}{2}\right)}{4^k\Gamma\left(k+1+\frac{\alpha+\nu}{2}\right)\Gamma\left(k+1+\frac{\alpha-\nu}{2}\right)} \tag{11.34}$$

Thus, for $\alpha = \nu$, our recursion relation is

$$a_{2k} = \frac{a_0\Gamma(\nu+1)}{4^k k!\Gamma(\nu+k+1)} \tag{11.35}$$

At this point, one could choose a_0. If

$$a_0 = \frac{2^-\nu}{\Gamma(\nu+1)} \tag{11.36}$$

is chosen, one gets the modified Bessel function.

$$I_\nu(x) = \sum \frac{\left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(\nu+k+1)} \tag{11.37}$$

Looking for the second solution, it is not enough to simply use $\alpha = -\nu$, since we know that $I_0 = I_{-0}$. Thus we have to find a different solution. See Asymptotics notes (L3.2) for the second solution.

Irregular Singular Points These are more difficult. Points are irregular singular points when they are not either of the previous two types of points. One way to differentiate regular and irregular singular points is to set $x = \frac{1}{t}$ and subbing in. One gets

$$y' = -t^2\dot{y} \tag{11.38}$$

$$y'' = t^4\ddot{y} + 2t^3\dot{y} \tag{11.39}$$

As one examines the differential equation in terms of t, it is often easier to see what term is most singular as $t \to 0$.

One can't use the summation with an indicial exponent since invariably the only way to set the leading term $= 0$ is to set $a_0 = 0$. Thus the whole series equals zero. As a result, we must try something different.

We will use

$$\boxed{y = e^{s(x)}} \tag{11.40}$$

Calculating the derivatives of y gives us the following.

$$\boxed{y' = s'e^s} \tag{11.41}$$

$$\boxed{y'' = \left[s'' + (s')^2\right]e^s} \tag{11.42}$$

Again, we then substitute this into our differential equation and look for dominant behavior near our singularity. Once s is determined, one improves the solution by subbing in $s = s_0 + g$. Dominant behavior is again determined in order to evaluate g. This process is continued until a term is equal to ln( ). At this point, one switches paths and tries to find $W(x)$, a completely divergent series. We let W equal

$$W = \sum a_n x^{\pm n} \tag{11.43}$$

The powers of x should be $\frac{n}{2}$ if the exponents are going in increments of $\frac{1}{2}$. One plugs W in and gets something in terms of W and its derivatives. Then solve for a_n. Stop when the terms of W start to grow.

Our solution will be in the form of

$$y = Cx^{(\,)}e^{x^{(\,)}}W(x) \tag{11.44}$$

The $x^{(\,)}$ term is the ln( ) term.

Example: For example, we will look at

$$x^3 y'' = y \tag{11.45}$$

$x = 0$ is an irregular singular point since $\frac{y}{x^3}$ is more singular than $y'' \sim \frac{y}{x^2}$. We plug in our values for y, y', and y'' and get

$$x^3\left[s'' + (s')^2\right] = 1$$

Looking for dominant behavior leads us to try $s'' \gg (s')^2$. Thus

$$x^3 s'' = 1 \qquad s' = -\frac{1}{2x^2}$$
$$s'' = \frac{1}{x^3} \qquad (s')^2 = \frac{1}{4x^2} \gg s'' \text{ as } x \to 0$$

Clearly, that was not a good balance. Thus we try $(s')^2 \gg s''$, giving us

$$x^3 (s')^2 = 1 \qquad (s')^2 = \frac{1}{x^3}$$
$$s' = \pm x^{-\frac{3}{2}} \qquad s'' = \mp \frac{3}{2} x^{-\frac{5}{2}}$$
$$s = \mp 2x^{-\frac{1}{2}} \qquad y \sim e^{\mp 2x^{-\frac{1}{2}}}$$

Now we need to improve on this by setting $s = \mp 2x^{-\frac{1}{2}} + g$. We calculate the derivatives of s and plug it into the differential equation.

$$\mp \frac{3}{2} x^{-\frac{5}{2}} + g'' + x^{-3} \pm 2x^{-\frac{3}{2}} g' + (g')^2 = x^{-3}$$

We see that some terms cancel. This is a good check of your first order solution. The dominant terms can be determined to be the $\mp 3x^{-\frac{5}{2}}$ and $\pm 2x^{-\frac{3}{2}} g'$. So we set them equal to each other and solve for g' and then get g.

$$g' = \frac{3}{4x} \qquad g = \frac{3}{4} \ln x$$

Here we have found the ln( ) term. If we were to further improve upon our solution from here, we would find that we would bring down a constant term and a term that goes like x to some positive power. Of course, since we are look at where $x \to 0$, this term goes to zero. The constant term is a normalization. This is why we stop here. So our solution for now is

$$y = A x^{\frac{3}{4}} e^{-x^{-\frac{1}{2}}} \qquad (11.46)$$

Now we need to look for the divergent series W. We set

$$y = A x^{\frac{3}{4}} e^{-x^{-\frac{1}{2}}} W(x) = e^s W$$

and plug it into our differential equation.

$$x^3 y'' = x^3 \left[s'' W e^s + 2s' W' e^s_(s')^2 W e^s + W'' e^s \right] = y$$
$$s'' W + 2s' W' (s')^2 W + W'' = \frac{W}{x^3}$$

Then plug in our values for s' and s'' (with the g terms) from the previous step. Lot's of things should cancel.

$$W'' + \left(\frac{3}{2x} - \frac{2}{x^{\frac{3}{2}}}\right) W' - \frac{3}{16x^2} W = 0$$

At this point we try a solution for our series, calculate its derivatives, and plug it in.

$$W = \sum a_n x^{\frac{n}{2}}$$

$$0 = \sum a_n \left\{ \left[\frac{n}{2}\left(\frac{n}{2} - 1\right) + \frac{3n}{4} - \frac{3}{16} \right] - a_{n+1}(n+1) \right\} x^{\frac{n}{2}-2}$$

$$a_{n+1} = \frac{a_n \left(n - \frac{1}{2}\right)\left(n + \frac{3}{2}\right)}{4(n+1)}$$

$$a_1 = \frac{a_0 \left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)}{4}$$

$$a_n = \frac{\left(n - \frac{3}{2}\right) \ldots \left(-\frac{1}{2}\right)\left(n + \frac{1}{2}\right) \ldots \left(\frac{3}{2}\right)}{4^n n!}$$

$$a_n = \frac{-\Gamma\left(n - \frac{1}{2}\right)\Gamma\left(n + \frac{3}{2}\right) a_0}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) 4^n n!}$$

The radius of convergence for this series is 0.

11.6 Stokes Diagrams and Phase Integrals

11.6.1 Derivation of General 2nd Order Differential Equation

For a 2nd order differential equation like

$$y'' + P_1 y' + P_0 y = 0 \tag{11.47}$$

we can eliminate y' by making a change of variables, $x \to z$.

$$y' = \frac{dy}{dz}\frac{dz}{dx} \qquad y'' = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d^2y}{dz^2}\frac{d^2z}{dx^2} \tag{11.48}$$

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left[P_1 \frac{dz}{dx} + \frac{d^2z}{dx^2}\right] + P_0 y = 0 \tag{11.49}$$

We should choose z so that the term in the brackets goes to 0. If we let $f = \frac{dz}{dx}$, then

$$P_1 f + f' = 0 \quad \Rightarrow \quad \frac{f'}{f} = -P_1 \quad \Rightarrow$$

$$\ln f = -\int P_1 \mathrm{d}x \quad \Rightarrow$$

$$\frac{dz}{dx} = \exp\left[-\int P_1 \mathrm{d}x\right] = W(x) \qquad z = \int \mathrm{d}x W(x) \tag{11.50}$$

$$\frac{d^2y}{dz^2} + P_0 \exp\left[2\int P_1 \mathrm{d}x\right] y = 0 \tag{11.51}$$

$$\boxed{\frac{d^2y}{dz^2} + Q(z)y = 0} \tag{11.52}$$

The boxed equation is our general 2nd order equation. We will try our method from equations with irregular singular points now. Subbing in $y = e^s$ and its derivatives into this equation gives us

$$s'' + (s')^2 + Q(z) = 0 \tag{11.53}$$

We are going to assume the 2nd 2 terms are dominant, giving us

$$s' = \pm i\sqrt{Q(z)} \qquad s'' = \pm \frac{i}{2}\frac{Q'}{Q^{\frac{1}{2}}} \tag{11.54}$$

We now will go the the next order $s' = \pm i Q^{\frac{1}{2}} + g'$.

$$\underbrace{\pm\frac{iQ'}{2Q^{\frac{1}{2}}}} + g'' + \left[-Q \underbrace{\pm 2ig'Q^{\frac{1}{2}}} + (g')^2\right] + Q = 0 \tag{11.55}$$

Here we will assume that the underbraced terms are dominant. This leads us to

$$g' = -\frac{Q'}{4Q} \qquad g = -\frac{1}{4}\ln Q \tag{11.56}$$

Thus our asymptotic solution for y is

$$\boxed{y = (z_0, z) = Q^{-\frac{1}{4}} \exp\left[i\int_{z_0}^{z} \sqrt{Q}\mathrm{d}z\right]} \tag{11.57}$$

Notice the new notation (z_0, z) introduced for y. Our solutions are called *subdominant* $(z_0, z)_s$ if they $\to 0$ as $z \to \infty$. They are called *dominant* $(z_0, z)_d$ if they $\to \infty$ as $z \to \infty$.

The general solution to this equation is

$$\psi = a_+(0,z) + a_-(z,0) \tag{11.58}$$

We need a method of relating the a_+, a_- coefficients in one domain to another. Phase integrals are the way to do this.

11.6.2 Rules for Phase Integrals

We start with two definitions of lines in the complex plane. *Anti-Stokes lines* (AS) are when $\sqrt{Q}\mathrm{d}z$ is real. The solution is oscillatory here. We will use *solid* lines to denote anti-Stokes lines. *Stokes lines* (S) are where $\sqrt{Q}\mathrm{d}z = i$. The solutions here are $\pm$ exponential. We will use *dashed* lines to denote Stokes lines.

For example, if $Q = -z$ (and z is real and positive), then the integrand in Eq. (11.57) is

$$i\sqrt{r}e^{\frac{i\theta}{2}}\mathrm{d}re^{i\theta} \tag{11.59}$$

(after going from $z \to re^{i\theta}$). We can see from this that

$$ie^{\frac{i3\theta}{2}} = \text{real} \quad \Rightarrow \quad \theta = \frac{\pi}{3}, \pi, -\frac{\pi}{3} \tag{11.60}$$

Thus our anti-Stokes lines lie at these angles. We can find the π AS line easily by seeing that if $z = (-)$ real, then for $\sqrt{-z}\mathrm{d}z$ to be real, $\mathrm{d}z \to$ real. This is how *1st order zeros* behave.

1st order poles behave differently. In this case, $Q = \frac{a}{z}$. We have AS lines where

$$\sqrt{a}\frac{\mathrm{d}z}{\sqrt{z}} = \text{real} \tag{11.61}$$

If $z = (+)$ real, then $\mathrm{d}z \to$ real. If $z = (-)$ real, then $\mathrm{d}z \to \pm i$. Thus 1st order poles have just one AS line coming out of them where $z > 0$, and where $z < 0$, they have AS lines perpendicular to the z-axis. These lines then wrap around the pole and are more or less parallel to the z-axis where $z > 0$.

We are now ready to calculate phase intergrals. We start off with a solution in a region, typically to the right of the first pole on the real axis. If the real axis here is a Stokes line, we know we either have a dominant or subdominant solution. If it is an anti-Stokes line, since $z = x + iy$, and we usually go around CCW, y is some non-zero quantity. If you look at the form of our solutions, Eq. (11.57), the i outside the integral multiplied by

iy gives us $-y$, which makes the exponential decrease. Thus the solution starts off as subdominant.

We have a couple rules for how we go around the poles. They are:

1. If an AS line is crossed:

$$\boxed{(a,z)_s \leftrightarrow (a,z)_d} \tag{11.62}$$

2. If a cut is crossed CCW (CW):

$$\boxed{(a,z)_{d,s} \rightarrow \mp i(z,a)_{d,s}} \tag{11.63}$$

3. If a S line is crossed CCW (CW):

$$\boxed{(z,0)_d \rightarrow (z,0)_d \pm T(0,z)_s} \tag{11.64}$$

   For isolated singularities, T, the Stokes constant is

$$T = 2i\cos\left(\frac{\pi}{n+2}\right) \tag{11.65}$$

   for an nth order zero. It is $-2i$ for a 1st order pole, i for $n=1$, and doesn't apply for $n=-2$.

   If a S line is stepped on, its $\frac{T}{2}$ and another $\frac{T}{2}$ to step off.

4. To connect from one singularity (a) to another (b), use:

$$\boxed{(z,a) = (z,b)[b,a]} \tag{11.66}$$

   where

$$[b,a] = \exp\left[i\int_b^a \sqrt{Q}\mathrm{d}z\right] \tag{11.67}$$

   If a and b are joined by a S line, connect while on the S line (using the step on/step off part of Rule #3). For these connections

$$\begin{array}{ll} \text{S line} & s \leftrightarrow d \\ \text{AS line} & d,s \rightarrow d,s \end{array} \tag{11.68}$$

A right moving function (wave) $e^{iz} = e^{ix-y}$ is an increasing function as $x \rightarrow \infty$. Left moving is opposite. For example, on the right side of the origin, $e^{ix-i\omega t}$ is outgoing. For $e^{ix^2-i\omega t}$, we see that $x^2 = \omega t + c$. As $t \rightarrow \infty$,

$x^2 \to \infty$, which means if $x < 0$, then it is left moving. If $x > 0$, then it is right moving.

Reflection and transmission coefficients are

$$R = \frac{\text{outgoing}}{\text{incoming}} \qquad T = \frac{1}{\text{incoming}} \tag{11.69}$$

$$|R|^2 + |T|^2 = 1 \tag{11.70}$$

11.7 Perturbation Theory

ϵ is a small quantity. If it is on the highest order derivative, then use boundary layer theory. Otherwise we use perturbation theory, which leads to a solution in the form of a series in ϵ. First, set $\epsilon = 0$ and find the roots of the equation. Once the zero-order solution has been found, let the higher order solution be equal to $x_1 = x_0 + \epsilon a_1 + \epsilon^2 a_2 + \dots$. Then solve for the coefficients a_n.

Example: For example, lets examine the following equation.

$$1 + (x^2 + \epsilon)^{\frac{1}{2}} = e^x \tag{11.71}$$

Setting $\epsilon = 0$ shows us that $1 + x_0 = e_0^x$, with a zero at $x_0 = 0$. We then let $x_1 = 0 + \epsilon a_1 + \epsilon^2 a_2 + \dots$ and sub this in. We get

$$1 + (\epsilon^2 a_1^2 + \epsilon)^{\frac{1}{2}} = e^{\epsilon a_1} \tag{11.72}$$

Since $\epsilon \sim 0$, we can expand the exponential.

$$1 + (\epsilon^2 a_1^2 + \epsilon)^{\frac{1}{2}} = 1 + \epsilon a_1 + \frac{\epsilon^2 a_1^2}{2} + \dots \tag{11.73}$$

Canceling and solving for a_1 shows us that

$$a_1 = \epsilon^{-\frac{2}{3}} \tag{11.74}$$

As a result, we have a value for $x_1 = \epsilon^{\frac{1}{3}}$.

If there is no series (its divergent) that works or if more roots exist then when for $\epsilon \sim 0$, its called singular perturbation theory. In this case, we use the Kruskal graph approach discussed at the beginning of the section.

11.8 Asymptotic Evaluation of Integrals

For the asymptotic evaluation of integrals, we have several paths we can follow. The path chosen depends on the equation.

1. If x is in the limit of the integral we do one of two things depending if x is big or small.
   (a) *Small* x - for small x, we expand the integrand.
   (b) *Big* x - if x is big, we integrate by parts.
$$\int_x^\infty e^{t^2}\mathrm{d}t \rightarrow \frac{d}{dt}(e^{t^2}) = 2te^{t^2}\mathrm{d}t = \mathrm{d}v \qquad u = \frac{1}{2t} \tag{11.75}$$
2. If x is in the integrand, we can do several things. We can use an expansion trick, the Laplace method, or contour deformation.
   (a) *Small* x - Sometimes, x is in the integrand and we look at where it is small. One way to deal with this is to expand the integral into 2 components, one of which evaluates to a constant, while the other one can be evaluated using the above methods (1a) since it has x in the integration limits.
$$\int_1^\infty \frac{\cos xt}{t} \rightarrow \int_x^\infty \frac{\cos xt}{t} = \int_1^x + \int_1^\infty \tag{11.76}$$
   The last term is a constant, and the integrand of the second to last term can be expanded according to case (1a) above.
   (b) *Laplace's method* - The key here is that the equation is in the form of
$$\int_a^b f(t)e^{x\phi(t)}\mathrm{d}t \tag{11.77}$$
   We suppose that ϕ has a max and find out where it is.
$$\phi(t)\Big|_{max} = \phi_0 + \phi'(t-t_0) + \frac{\phi''}{2}(t-t_0)^2 \tag{11.78}$$
   Set $\phi' = 0$ and get t_0. Since $a < t_0 < b$ and most of the contribution to the integral comes from where $t = t_0$, you can expand the integral limits to $\pm\infty$. If there is a local max at one of the end points, one uses $\phi' \neq 0$ instead of a contribution from ϕ'' and only one of the integration limits can be expanded to $\pm\infty$. It is only necessary to expand to the leading order in $f(t)$.
$$y = e^{x\phi_0}\int_{-\infty}^\infty f_0(t)e^{x\phi_0''\frac{(t-t_0)^2}{2}}\mathrm{d}t \tag{11.79}$$

(c) *Contour deformation - Method of steepest descent*[5] - if the exponent of the exponential is imaginary in the above case, we change the contour of integration. We have to use this method since there isn't a stationary point where we can expand about since the function is oscillatory. For example

$$\int_0^1 \mathrm{d}t \rightarrow \int\limits_{C_1} i\mathrm{d}s \int\limits_{C_3} i\mathrm{d}s \tag{11.80}$$

where $C1$ is along the imaginary axis from 0 to ∞ and $t = is$, and $C3$ is a line parallel to the imaginary axis crossing through 1. It is from 0 to ∞ and $t = 1 + is$. $C2$ goes from 0 to 1 at an infinite distance up the imaginary axis, so its contribution is 0.

(d) *Method of Steepest Descent* - ϕ is defined as in Laplace's Method above. I think Laplace's Method is a special case of Steepest Descent. Depending on the endpoints, one usually integrates through the saddle. If this can't be arranged, then usually one goes through perpendicular into the center of the saddle and then finish integrating through the saddle. See Problem 1997, pt. 2, 5.2 solution for a nice example.

i. Saddle points are located at $\phi' = 0$. Find t_0 (what value of t makes $\phi' = 0$.)
ii. Direction of saddle points: If $\phi_0'' < 0$, then $(t - t_0)^2 > 0$ and $(t - t_0) = \mathsf{Re}$, and the saddle is horizontal. If $\phi_0'' > 0$, then then $(t-t_0)^2 < 0$ and $(t-t_0) = \mathsf{Im}$, and the saddle is vertical.
iii. For steepest descent at $t - 1$, $\phi'(t_1)\mathrm{d}z = -\mathsf{Re}$. $\phi(t) - \phi(a) = -u$, where u is real. So if $\phi'(1) = 2 + 4i$, then $\mathrm{d}z = \frac{\mathsf{Re}}{-2+4i} = \frac{1+2i}{c}$. The numerator is the direction of steepest descent.
iv. Usually at this point contribution will come from end points unless you pass through a saddle.

11.9 Constructing Integral Forms of Equations

To construct the integral form of a differential equation, we follow a specific process.

[5] I'm not very confident in the Contour Deformation bit. I never used it. I ended up using Laplace's Method, and long after writing this guide, learned the Steepest Descent technique. Unfortunately, I waited 6 months until after generals to add the Steepest Descent write up that follows.

1. The first step is to plug in the kernel. There are several option available.

   (a) Fourier-Laplace

   $$y = \int e^{xt} f(t) \mathrm{d}t \tag{11.81}$$

   This can also have an indicial x^ν out in front of the integral. Also, it can have a $\frac{1}{2\pi i}$ out front.

   (b) Sommerfeld

   $$y = \int e^{z \sinh(t)} f(t) \mathrm{d}t \tag{11.82}$$

   The indicial form of this kernel has the argument of the exponential as $z \sinh(t) - \nu t$. The sinh( ) can also be a sin( ).

   (c) Euler

   $$y = \int (x-t)^n f(t) \mathrm{d}t \tag{11.83}$$

   (d) Mellin

   $$y = \int x^t f(t) \mathrm{d}t \tag{11.84}$$

2. Plug it into the differential equation.

3. Integrate by parts (xy', xy terms)

4. Make the part under the $\int [\quad] \mathrm{d}t$ go away by setting it equal to 0.

5. Make the other term equal to 0 by seting the limits of integration on the $(\quad)\Big|_{\bullet}^{\bullet}$. Find which endpoints make the expression evaluate to 0. Two paths must be chosen for two solutions.

Example: As an example, we will look at the Airy function.

$$y'' = zy \tag{11.85}$$

We will use the Fourier-Laplace kernel. The 2nd derivative of our kernel is

$$y'' = \frac{1}{2\pi i} \int e^{zt} t^2 f \mathrm{d}t \tag{11.86}$$

Taking our kernel and multiplying it by z gives us

$$zy = \frac{1}{2\pi i} \int \mathrm{d}t f \frac{d}{dt}(e^{zt}) = \frac{1}{2\pi i} f(t) e^{zt}\Big|_{\bullet}^{\bullet} - \frac{1}{2\pi i} \int e^{zt} f' \mathrm{d}t \tag{11.87}$$

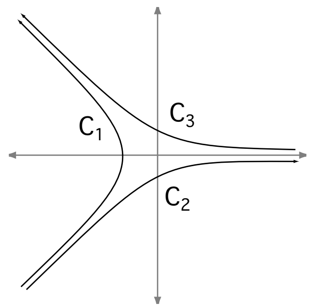

Figure 9: Contours of Integration

If we plug this back into the original equation and separately equate the terms under integrals, and the terms outside of integrals equal to zero, we find

$$t^2 f + f' = 0 \tag{11.88}$$

$$f(t)e^{zt}\Big|_{\bullet}^{\bullet} = 0 \tag{11.89}$$

Working from the first line, we find f.

$$f = e^{-\frac{t^3}{3}} \tag{11.90}$$

Plugging this into the second line and evaluating, we find that there are 3 contours to choose from. See Fig. 9. Since

$$\int\limits_{C_1} = \int\limits_{C_2} + \int\limits_{C_3} \tag{11.91}$$

we can choose any two. The final solution is

$$y = \frac{1}{2\pi i}\int\limits_{C} e^{zt-t^3/3}\mathrm{d}t \tag{11.92}$$

where the contour C is any two of the three contours in Fig. 9.

11.10 Boundary Layers

Boundary layer theory is used when the small parameter ϵ is on the highest order integral. If ϵ is not on y'', then use multiple scale analysis. The general form for these equations is

$$\epsilon y'' + a(x)y' + b(x)y = 0 \tag{11.93}$$

where

$$\underbrace{y(x_1) = A}_{\text{right}} \qquad \underbrace{y(x_2) = B}_{\text{left}} \tag{11.94}$$

The boundary layer (BL) occurs under three circumstances.

Condition	δ	Location
$a(x) = 0$	$\delta = \epsilon^{\frac{1}{2}}$	where $a(x) = 0$
$a(x_2) < 0$	$\delta = \epsilon$	$x = x_2$ on the right
$a(x_1) > 0$	$\delta = \epsilon$	$x = x_1$ on the left

Once it is determined where the BL's are, one then follows the process below. This can be done order by order.

1. Find the outer solution ($\epsilon = 0$).

2. Find the inner solution, where $x = \delta\mathbb{X}$, by dominant balance. Assume $\mathbb{Y}, \frac{\partial \mathbb{Y}}{\partial \mathbb{X}}, \frac{\partial^2 \mathbb{Y}}{\partial \mathbb{X}^2}, \mathbb{X} \sim O(1)$.

3. Match.

$$\mathbb{Y}_{in} \to Y_{match} \text{ as } \frac{x}{\delta} \to \infty$$
$$y_{out} \to Y_{match} \text{ as } x \to 0$$

4. $y_{uniform} = y_{out} + \mathbb{Y}_{in} - y_{match}$

Example: We will find the leading order behavior for the following equation (from (L18.3)).

$$\epsilon y'' - x^2 y' - y = 0 \tag{11.95}$$

where $y(0) = y(1) = 1$. We can see from Eq. (11.93) that $a(x_2 = 1) = -1 < 0$. Thus, according to our conditions, there is a BL here. There is also one at $x = 0$ since $a = 0$ here. Our outer solution is

$$-x^2 y' - y = 0 \tag{11.96}$$

By examination, the solution is

$$y_0 = C_0 e^{\frac{1}{x}} \tag{11.97}$$

We now look at the inner solution for the BL @ $x = 0$. We set $x = \delta z$ and our equation now looks like the following.

$$\frac{\epsilon}{\delta^2}\frac{d^2\mathbb{Y}}{dz^2} - \delta z^2\frac{d\mathbb{Y}}{dz} - \mathbb{Y} = 0 \tag{11.98}$$

We seek our dominant balance now. The last two terms won't do since this sets $\delta = 1$, which is just our outer solution. The first two terms won't work either since it would set $\delta^3 = \epsilon$ and $\mathbb{Y} = 0$, so we end up with the first and the third terms. This gives us the result of $\frac{\epsilon}{\delta^2} = 1 \rightarrow \delta = \sqrt{\epsilon}$. Rearranging our dominant terms on the LHS and the small terms on the RHS, we get

$$\frac{d^2\mathbb{Y}}{dz^2} - \mathbb{Y} = \epsilon^{\frac{1}{2}} z^2\frac{d\mathbb{Y}}{dz} \tag{11.99}$$

The solution to this is $\mathbb{Y} = D_0 e^z + E_0 e^{-z}$, where $z = \frac{x}{\delta} = \frac{x}{\sqrt{\epsilon}}$. We know that $\mathbb{Y}_0(0) = 1$, so $D_0 + E_0 = 1$. If we match our outer solution $y_0 = C_0 e^{\frac{1}{x}}$ to our inner solution at the boundary $x = 0$, we see that $C_0 = D_0 = 0$.

For the BL at $x = 1$, we set $x = 1 - \epsilon\mathbb{X}$ or $\mathbb{X} = \frac{1-x}{\epsilon}$. Dominant balance finds the first two terms of the equation to be dominant. To the leading order

$$\frac{d^2\mathbb{Y}}{d\mathbb{X}^2} + \frac{d\mathbb{Y}}{d\mathbb{X}} = 0 \tag{11.100}$$

The solution to this equation is $\mathbb{Y} = A_0 + B_0 e^{\mathbb{X}}$. We match at $\mathbb{X} \rightarrow \infty$ to 0 and find that $A_0 = 0$. Since $\mathbb{X} = 0$ here, this puts $B_0 = 1$. Thus our uniform solution is

$$y_{uniform} = e^{-\frac{x}{\sqrt{\epsilon}}} + e^{\frac{x-1}{\epsilon}} \tag{11.101}$$

12 Diagnostics

12.1 Langmuir Probes

For Debye shielding, please refer to Section 2.1 on page 8.

12.1.1 Bohm Criterion

We start off with the expressions for electron and sheath density, which are just the Boltzmann distribution. The ion density is then determined from conservation, $n_i V_i = n_{sh} V_{sh}$.

$$n_e = n_0 \exp\left[\frac{eV}{T_e}\right] \tag{12.1}$$

$$n_{sh} = n_0 \exp\left[\frac{eV_{sh}}{T_e}\right] \tag{12.2}$$

$$n_e = n_{sh} \exp\left[\frac{e(V - V_{sh})}{T_e}\right] \tag{12.3}$$

$$n_i = n_{sh} \left[1 - \frac{2e(V - V_{sh})}{mV_{sh}^2}\right]^{-\frac{1}{2}} \tag{12.4}$$

For simplicity's sake, we will call $V - V_{sh} = F$. We now plug these expressions into Laplace's equation.

$$\nabla^2 F = -\frac{en_{sh}}{\epsilon_0}\left[\left(1 - \frac{2eF}{mV_{sh}^2}\right)^{-\frac{1}{2}} - \exp\left(\frac{eF}{T_e}\right)\right] \tag{12.5}$$

We say that $F \ll \frac{mV_{sh}}{2e}, \frac{T_e}{e}$. Thus we can expand the above equation and simplify.

$$\frac{d^2F}{dx^2} = -\frac{en_{sh}}{\epsilon_0}\left[\left(1 + \frac{eF}{mV_{sh}^2}\right) - \left(1 + \frac{eF}{T_e}\right)\right] = -\frac{e^2 n_{sh} F}{\epsilon_0}\underbrace{\left[\frac{1}{mV_{sh}^2} - \frac{1}{T_e}\right]}_{=C^2} \tag{12.6}$$

For non-oscillatory solutions, $C^2 \geq 0$. Thus

$$\frac{1}{T_e} \geq \frac{1}{mV_{sh}^2}$$

$$\boxed{T_e \leq mV_{sh}^2} \tag{12.7}$$

12.1.2 Probe Current

The probe current to Langmuir probe is

$$I_e = \frac{en_e}{4}\left(\frac{8T_e}{\pi m_e}\right)^{\frac{1}{2}} \exp\left[-\frac{e(V_{sp}-V_f)}{T_e}\right] \tag{12.8}$$

$$I_{is} = .6en_e\left(\frac{T_e}{m_i}\right)^{\frac{1}{2}} A_p = en_{sh}V_{sh}A_p \tag{12.9}$$

where

$$V_{sh} = \left(\frac{kT_e}{m_i}\right)^{\frac{1}{2}} \tag{12.10}$$

$$n_{sh} = .6n_0 \tag{12.11}$$

The last equation of importance is the difference between the space potential and the floating potential.

$$V_{sp} - V_f = 3.3\frac{kT_e}{e} + \frac{1}{2}\frac{kT_e}{e}\ln\mu \tag{12.12}$$

12.2 Interferometers

The phase shift between two arms of a Mach-Zehnder interferometer, using a geometrical optics solution (WKB), is

$$\boxed{\Delta\phi = \int (k_{plasma} - k_0)\,\mathrm{d}l = \int (N-1)\,\frac{\omega}{c}\mathrm{d}l} \tag{12.13}$$

Defining the cutoff density n_c as the density of plasma at which our selected wavelength of light is too long to effective penetrate the plasma. This is a constant with a given wavelength of light.

$$n_c \equiv \frac{m_e\omega^2}{4\pi e^2} \tag{12.14}$$

We can now solve for the phase shift as a function of plasma density (Eq. (12.15)).

$$\begin{aligned}\Delta\phi &= \int (N-1)\,\frac{\omega}{c}\mathrm{d}l \\ &= \frac{\omega}{c}\int\left(\left(1-\frac{n_e}{n_c}\right)^{\frac{1}{2}} - 1\right)\mathrm{d}l\end{aligned}$$

Since our cutoff density n_c is so much larger than our value for n_e ($\sim 10^{14}\,\mathrm{cm}^{-3}$), we can make an approximation here.

$$\Delta\phi \approx \frac{\omega}{c}\int\left(1-\frac{1}{2}\frac{n_e}{n_c}-1\right)\mathrm{d}l$$
$$\boxed{\Delta\phi \approx \frac{\omega}{2n_c c}\int n_e \mathrm{d}l} \tag{12.15}$$

12.2.1 Other Interferometry Stuff

The indices of refraction are different depending on whether or not the wave is O or X.

$$N_o^2 = 1 - X \tag{12.16}$$
$$N_x^2 = 1 - \frac{X(1-X)}{1-X-Y^2} \tag{12.17}$$

where

$$X = \frac{\omega_{ps}^2}{\omega^2} \tag{12.18}$$
$$Y = \frac{\Omega_s}{\omega} \tag{12.19}$$

12.2.2 Interferometry Problem #1

We start out with the equations in Section 12.2.1 and Eq. (12.13). The error between the X and the O components will be

$$\left|\frac{\Delta\phi_o - \Delta\phi_x}{\Delta\phi_x}\right| \times 100 = \text{error} \tag{12.20}$$

where

$$\begin{aligned}\Delta\phi_o &= -\frac{\omega}{c}\int\left(1-\frac{X}{2}-1\right)\mathrm{d}l \\ &\propto -\int\frac{X}{2}\mathrm{d}l \propto -\frac{X}{2}l\end{aligned} \tag{12.21}$$
$$\Delta\phi_x = -\frac{1}{2}\left(\frac{X(1-X)}{1-X-Y^2}\right)l \tag{12.22}$$

Putting it all together for the error gives us

$$\text{error} = \left| \frac{-\frac{X}{2} + \frac{X}{2}\frac{1-X}{1-X-Y^2}}{\frac{X}{2}\frac{1-X}{1-X-Y^2}} \right| = \left| \frac{\frac{-1+X+Y^2+1-X}{1-X-Y^2}}{\frac{1-X}{1-X-Y^2}} \right| = \left| \frac{Y^2}{1-X} \right| \tag{12.23}$$

We finish by calculating X and Y.

12.2.3 Interferometry Problem #2

For the He and CO_2 laser problem, we start with a couple phase differences.

$$\Delta\phi_{vib} = \frac{2\pi}{\lambda}\Delta l \tag{12.24}$$

$$\Delta\phi_{\text{He}} = \Delta\phi^p_{\text{He}} + \frac{2\pi}{\lambda_{\text{He}}}\Delta l \tag{12.25}$$

$$\Delta\phi_{\text{CO}_2} = \Delta\phi^p_{\text{CO}_2} + \frac{2\pi}{\lambda_{\text{CO}_2}}\Delta l \tag{12.26}$$

where

$$\begin{aligned} \Delta\phi^p &= \frac{e^2}{4\pi c^2 m_e \epsilon_0}\lambda \int n_e \mathrm{d}l = \frac{\lambda}{\kappa}\int n_e \mathrm{d}l \\ \Rightarrow \int\limits_{\text{He}} n_e \mathrm{d}l &= \kappa \frac{\Delta\phi^p_{\text{He}}}{\lambda_{\text{He}}} \end{aligned} \tag{12.27}$$

Now, going from the second and third equations above

$$\lambda_{\text{He}}(\Delta\phi_{\text{He}} - \Delta\phi^p_{\text{He}}) = 2\pi\Delta l = \lambda_{\text{CO}_2}(\Delta\phi_{\text{CO}_2} - \Delta\phi^p_{\text{CO}_2}) \tag{12.28}$$

$$\lambda_{\text{He}}\Delta\phi_{\text{He}} - \frac{\lambda^2_{\text{He}}}{\kappa}\int n_e \mathrm{d}l = \lambda_{\text{CO}_2}\Delta\phi_{\text{CO}_2} - \frac{\lambda^2_{\text{CO}_2}}{kappa}\int n_e \mathrm{d}l \tag{12.29}$$

$$\frac{\lambda^2_{\text{CO}_2} - \lambda^2_{\text{He}}}{\kappa}\int n_e \mathrm{d}l = \lambda_{\text{CO}_2}\Delta\phi_{\text{CO}_2} - \lambda_{\text{He}}\Delta\phi_{\text{He}} \tag{12.30}$$

$$\boxed{\int n_e \mathrm{d}l = \kappa \frac{\lambda_{\text{CO}_2}\Delta\phi_{\text{CO}_2} - \lambda_{\text{He}}\Delta\phi_{\text{He}}}{\lambda^2_{\text{CO}_2} - \lambda^2_{\text{He}}}} \tag{12.31}$$

The uncertainty in this is

$$\frac{\kappa\pi\lambda_{\mathrm{He}}}{\lambda_{\mathrm{CO_2}}^2 - \lambda_{\mathrm{He}}^2} \sim 6 \times 10^{18}\mathrm{m}^{-2} \tag{12.32}$$

$$\int n_e \mathrm{d}l = 10^{20}\mathrm{m}^{-3} \tag{12.33}$$

$$\Downarrow$$

$$\text{Fractional error} = \frac{\text{uncertainty}}{10^{20}\mathrm{m}^{-3}} \sim 6\% \tag{12.34}$$

12.3 Simple Magnetic Diagnostics

12.3.1 Magnetic Pickup Loops

The voltage out on a magnetic pickup loops is simply

$$\boxed{V_{out} = -N \int_{\text{loop}} \underline{\dot{\boldsymbol{B}}} \cdot \mathrm{d}\underline{\boldsymbol{l}} + \int_{\text{leads}} \underline{\dot{\boldsymbol{B}}} \cdot \mathrm{d}\underline{\boldsymbol{l}}} \tag{12.35}$$

where N is the number of loops. There are two ways to get something meaningful out of this.

$$\underline{\boldsymbol{B}} = -\frac{1}{NA} \int \mathrm{d}t V_{out} \tag{12.36}$$

$$\underline{\boldsymbol{E}} = \frac{V_{out}}{NL} \tag{12.37}$$

where A is the coil area and L is the coil length.

12.3.2 Ragowski Coil

The derivation of how a Ragowski Coil works follows. We know from Ampere's Law that

$$\int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = \mu_0 I_{enc} \tag{12.38}$$

Since we know that the flux through one loop is $\oint \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{a}} = \Phi$, for the n loops around the contour of our line integral, our total flux is

$$\Phi = n \int \oint \mathrm{d}a \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} = nA \int \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \tag{12.39}$$

$$\Phi = An\mu_0 I_{enc} \tag{12.40}$$

Since $\frac{\partial \Phi}{\partial t} = -V$, the voltage out of our Ragowski coil is

$$V = An\mu_o \dot{I} \tag{12.41}$$

Finally, to get the current through the coil, one must integrate by time.

$$I = \frac{1}{\mu_0 n A} \int V \mathrm{d}t \tag{12.42}$$

12.3.3 Toroidal Loop

$$V_{loop} = 2\pi R E_\varphi \tag{12.43}$$

If one looks at the Poynting flux through a surface

$$\int \underline{\boldsymbol{P}} \cdot \mathrm{d}\underline{\boldsymbol{s}} = \int \frac{1}{\mu_0} \underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{s}} \tag{12.44}$$

$$= \int \frac{1}{\mu_0} 2\pi R E_\varphi \underline{\boldsymbol{B}} \cdot \mathrm{d}\underline{\boldsymbol{l}} \tag{12.45}$$

$$= I_p V_{loop} \tag{12.46}$$

Also, average resistivity can be found by using

$$E_\varphi = \frac{V_{loop}}{2\pi R} = \frac{I_p}{\pi a^2}\eta \tag{12.47}$$

12.4 Thomson Scattering Cross Section

$\underline{\boldsymbol{E}}_i$ is the incident wave, and $\underline{\boldsymbol{E}}_s$ is the scattered wave. R is the distance to the observer.

$$\dot{\underline{\boldsymbol{v}}} = -\frac{e}{m} \underline{\boldsymbol{E}} \tag{12.48}$$

$$\underline{\boldsymbol{E}}_s = -\frac{e}{4\pi\epsilon_0} \frac{1}{Rc^2} \big(\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \dot{\underline{\boldsymbol{v}}}) \big) \tag{12.49}$$

$$\frac{dP}{d\Omega_s} = R^2 c\epsilon_0 \left| E_s \right|^2 = r_e^2 \sin^2\theta c\epsilon_0 \left| E_i \right|^2 \tag{12.50}$$

$$\frac{d\sigma}{d\Omega_s} = \frac{\frac{dP}{d\Omega_s}}{c\epsilon_0 \left| E_i \right|^2} = r_e^2 \sin^2\theta \tag{12.51}$$

since

$$\mathrm{d}\Omega_s = 2\pi \sin\theta \mathrm{d}\theta \tag{12.52}$$

$$\mathrm{d}\sigma = 2\pi r_e^2 \sin^3\theta \mathrm{d}\theta \tag{12.53}$$

and we get

$$\boxed{\sigma = \frac{8\pi}{3} r_e^2} \tag{12.54}$$

The calculation of the fraction of photons incoherently scatter from a 1 cm path length of laser beam from a plasma with an electron density of 1^{20} m^{-3} with a solid angle of detection of 0.01 sr is below.

$$\frac{N_s}{N_i} = n_e L \Delta\Omega \frac{\partial\sigma}{\partial\Omega}(\theta) \tag{12.55}$$

$$= (1 \times 10^{20}/\mathrm{m}^3)(.01\mathrm{m})(.01\mathrm{sr}) \frac{\partial\sigma}{\partial\Omega}(90^\circ) \tag{12.56}$$

$$= (1 \times 10^{20}/\mathrm{m}^3)(.01\mathrm{m})(.01\mathrm{sr}) r_e^2 \tag{12.57}$$

$$= 7.8 \times 10^{-14} \tag{12.58}$$

12.5 List of Diagnostics

Cheers Neutral beam injection (NBI) exchanges electrons with impurities, causing the ions to get excited and radiate. One can then determine T_i, n_i, and the amount of impurities by looking at the radiation.

BES NBI atoms get excited and radiate. Shows density fluctuations.

MSE Motional Stark Effect. The NBI beems feels the $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_{plasma} + \underline{\boldsymbol{v}} \times \underline{\boldsymbol{B}}$ so one can back out the $\underline{\boldsymbol{B}}$ field and direction.

Neutron Detectors Count neutrons exciting the plasma.

- **Gas Proportional** - Fast charged particles going through the detector can ionize the gas. An applied $\underline{\boldsymbol{E}}$ then causes and avalanche which one can pick up.
- **BF$_3$** - Detects slow neutrons. BF_3 is the gas proportional counter and a source of ^{10}B which has nuclear reactions with the neutron.
- **^{235}U** - Fission detector detects slow neutrons. It picks up n + ^{235}U reactions with the n being in the 3.5 MeV range.
- **^{238}U** - Fission detector for fast neutrons. Picks up both fast and slow neutrons.

X-rays One can get T_i from the Doppler broadened line and $\underline{\boldsymbol{v}}_{flow}$ from the Doppler shift. There are three sources of x-rays.

- *Bremsstrahlung* is a free-free reaction. The electron scatters off a nucleus and loses energy, emitting 1 $h\nu$ (a photon).
- *Recombination* is a free-bound reaction. The electron is captured by a nucleus, giving up energy and emitting a photon.
- *Line radiation* is a bound-bound reaction. The electron inhabits a high energy orbital and after emitting a photon, drops to a lower energy orbital.

12.6 Miscellaneous Diagnostics

12.6.1 Blackbody

The intensity of radiation given off by a blackbody is

$$\boxed{I = \frac{\omega^2 T k}{8\pi^3 c^2} \propto T} \tag{12.59}$$

For a plasma to be considered to be a blackbody, the optical thickness τ must be

$$\tau \geq 2 \tag{12.60}$$

where

$$\boxed{\tau = \frac{\alpha_m(t)}{m\left|\frac{d\Omega_s}{dr}\right|}} \tag{12.61}$$

The 2nd harmonic X wave and the fundamental O wave are the only ones that are really absorbed. The $\frac{d\Omega_s}{dr}$ goes like $\frac{\Omega_0 R_0}{R_0^2}$.

12.6.2 Fusion Power

Fusion power P_F is defined as

$$P + F = 1.4(\beta_T B_T^2)^2 V \tag{12.62}$$

The power dissipated on the center stack is

$$P_c = \frac{\eta_c h_c I_c^2}{\lambda\pi(fR_c)^2} \tag{12.63}$$

The ratio of the two gives us our efficiency.

$$\frac{P_F}{P_c} = \frac{10\lambda f^2}{\eta_c} J_c^2 \beta_T^2 \frac{R_C^4 (A-1)^2}{A^3} \tag{12.64}$$

$$\beta_T \beta_P = 25 \left(\frac{1+\kappa^2}{2}\right) \left(\frac{\beta_N}{100}\right)^2 \tag{12.65}$$

$$\boxed{\frac{P_F}{P_c} \propto \beta_N^4 \frac{(A-1)^2}{A^3}} \tag{12.66}$$

To maximize this with respect to the aspect ratio A, we simply take the derivative of the boxed equation with respect to A, set it equal to 0, and solve for A_0.