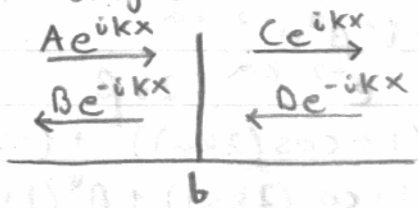
Seth Dorfman   Prelims   December 19, 2005

January 2001 QM

2) Consider a single δ-function at position b:



$V(x) = \alpha\delta(x-b)$

ψ continuous: $Ae^{ikb} + Be^{-ikb} = Ce^{ikb} + De^{-ikb}$ (1)

B.C. on ψ':

$$\int_{b^-}^{b^+} \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\,dx + \int_{b^-}^{b^+} \alpha\delta(x-b)\psi(x)\,dx = \int_{b^-}^{b^+} E\psi(x) = 0$$

$$\frac{-\hbar^2}{2m}\left[\psi'(b^+) - \psi'(b^-)\right] + \alpha\psi(b) = 0$$

$$\psi'(b^+) - \psi'(b^-) = \frac{2m\alpha}{\hbar^2}\psi(b)$$

$$ikCe^{ikb} - ikDe^{-ikb} - (ikAe^{ikb} - ikBe^{-ikb}) = \frac{2m\alpha}{\hbar^2}(Ce^{ikb} + De^{-ikb})$$

$$Ae^{ikb} - Be^{-ikb} = i\frac{2m\alpha}{\hbar^2 k}(Ce^{ikb} + De^{-ikb}) + Ce^{ikb} - De^{-ikb}$$

$$Ae^{ikb} - Be^{-ikb} = Ce^{ikb}\left(1 + i\frac{2m\alpha}{\hbar^2 k}\right) + De^{-ikb}\left(1 - i\frac{2m\alpha}{\hbar^2 k}\right) \quad (2)$$

(1)+(2): $Ae^{ikb} = Ce^{ikb}\left(1 + i\frac{m\alpha}{\hbar^2 k}\right) + De^{-ikb}\left(i\frac{m\alpha}{\hbar^2 k}\right)$

(1)−(2): $Be^{-ikb} = Ce^{ikb}\left(-i\frac{m\alpha}{\hbar^2 k}\right) + De^{-ikb}\left(1 - i\frac{m\alpha}{\hbar^2 k}\right)$

Let $\beta = \frac{m\alpha}{\hbar^2 k}$

Thus for a δ-function at $x=b$ the matrix eqn. is:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1+i\beta & i\beta e^{-2ikb} \\ -i\beta e^{2ikb} & 1-i\beta \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

To find a similar equation for a δ-function at $x=0$ and a second one at $x=a$ we need to muliply the matricies:

$$\underset{\approx}{M} = \begin{pmatrix} 1+i\beta & i\beta \\ -i\beta & 1-i\beta \end{pmatrix}\begin{pmatrix} 1+i\beta & i\beta e^{-2ika} \\ -i\beta e^{2ika} & 1-i\beta \end{pmatrix}$$

Such that $\begin{pmatrix} a_i \\ a_r \end{pmatrix} = \underset{\approx}{M}\begin{pmatrix} a_t \\ 0 \end{pmatrix}$

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#2 (continued)

the transmission coefficient is $\frac{1}{|M_{11}|^2}$

$$M_{11} = (1+i\beta)^2 + \beta^2 e^{2ika}$$
$$= 1-\beta^2 + \beta^2 e^{2ika} + 2i\beta$$
$$|M_{11}|^2 = (1+\beta^2(\cos(2ka)-1))^2 + (2\beta+\beta^2\sin(2ka))^2$$
$$= 1 + \beta^2[2(\cos(2ka)-1) + \beta^2(\cos(2ka)-1)^2 + 4 + 4\beta\sin(2ka) + \beta^2\sin^2(2ka)]$$
$$= 1+\beta^2[-4\sin^2(ka) + 4 + 8\beta\cos(ka)\sin(ka) + \beta^2(1-2\cos(2ka)+1)]$$
$$= 1+4\beta^2[1-\sin^2(ka) + 2\beta\cos(ka)\sin(ka) + \tfrac{1}{2}\beta^2(1-\cos(2ka))]$$
$$= 1+4\beta^2[\cos^2(ka) + 2\beta\cos(ka)\sin(ka) + \beta^2\sin^2(ka)]$$
$$|M_{11}|^2 = 1+4\beta^2(\cos(ka)+\beta\sin(ka))^2$$

the reflection coefficient is $\left|\frac{M_{12}}{M_{11}}\right|^2$

$$M_{12} = -i\beta(1+i\beta) - i\beta e^{2ika}(1-i\beta)$$
$$= -i\beta + \beta^2 - i\beta e^{2ika} - \beta^2 e^{2ika}$$
$$|M_{12}|^2 = (\beta^2 + \beta\sin(2ka) - \beta^2\cos(2ka))^2 + (-\beta-\beta\cos(2ka)-\beta^2\sin(2ka))^2$$
$$= \beta^2[(\beta(1-\cos(2ka)) + \sin(2ka))^2 + (1+\cos(2ka)+\beta\sin(2ka))^2]$$
$$= \beta^2[(2\beta\sin^2(ka) + 2\sin(ka)\cos(ka))^2 + (2\cos^2(ka) + 2\beta\sin(ka)\cos(ka))^2]$$
$$= 4\beta^2[\sin^2(ka)(\beta\sin(ka)+\cos(ka))^2 + \cos^2(ka)(\cos(ka)+\beta\sin(ka))^2]$$
$$|M_{12}|^2 = 4\beta^2(\cos(ka)+\beta\sin(ka))^2$$

$$\therefore\ T = \frac{1}{1+4\beta^2(\cos(ka)+\beta\sin(ka))^2}$$

$$R = \frac{4\beta^2(\cos(ka)+\beta\sin(ka))^2}{1+4\beta^2(\cos(ka)+\beta\sin(ka))^2}$$

where $\beta = \frac{m\alpha}{\hbar^2 K}$, $p = \hbar K$

The particles are completely transmitted when $T=1$, $R=0$

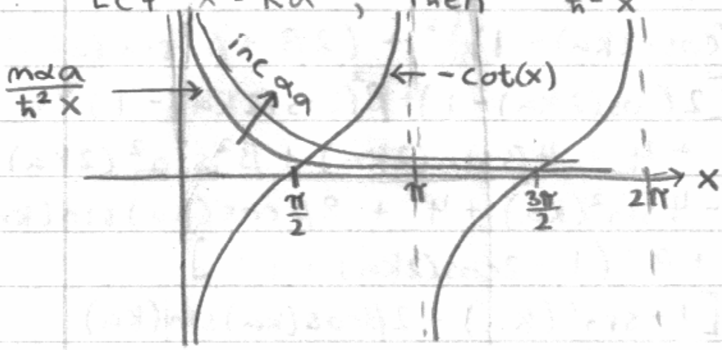
$$\Rightarrow \cos(ka) + \beta \sin(ka) = 0$$

$$\beta = -\cot(ka)$$

$$\frac{m\alpha}{\hbar^2 k} = -\cot(ka)$$

$$a = \frac{1}{k}\,\mathrm{Arccot}\left(\frac{-m\alpha}{\hbar^2 k}\right)$$

Let $x = ka$, then $\frac{m\alpha a}{\hbar^2 x} = -\cot(x)$



As seen from the graph the condition has at least one solution such that $x > \frac{\pi}{2}$

$$ka > \frac{\pi}{2}$$

$$a > \frac{\pi}{2k} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$a > \frac{\pi\hbar}{\sqrt{8mE}}$$

where E is the energy of the scattering particle.